



Human capital and welfare Human capital and welfare

Stefano Bosi, Carmen Camacho, David Desmarchelier

► To cite this version:

Stefano Bosi, Carmen Camacho, David Desmarchelier. Human capital and welfare Human capital and welfare. 2020. halshs-03920429v1

HAL Id: halshs-03920429

<https://shs.hal.science/halshs-03920429v1>

Preprint submitted on 18 Feb 2020 (v1), last revised 19 Apr 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



WORKING PAPER N° 2020 – 04

Human capital and welfare

**Stefano Bosi
Carmen Camacho
David Desmarchelier**

JEL Codes: I00, O11, C61.

Keywords: human capital, unbalanced growth, transversality condition



Human capital and welfare*

Stefano BOSI[†], Carmen CAMACHO[‡], David DESMARCHELIER[§]

February 15, 2020

Abstract

We consider a growth model à la Lucas (1988), where human capital has an additional positive effect on social welfare through the quality of individual health and education. Considering a simple example with isoelastic technology and preferences, we show that, differently from Lucas' contribution, the BGP is optimal only in the specific case in which consumption and human capital have equal weight in total utility. In general, the BGP fails to be optimal. Indeed, when the weight of consumption exceeds that of human capital, the household chooses to spend all her time working from a critical date on. As a result, if individuals do not appreciate enough human capital, then they fix a ceiling level of human capital. On the contrary, the working time decreases to zero when the weight of human capital exceeds that of consumption. On methodological ground, very interestingly, we obtain a strictly positive transversality condition. Furthermore, we compute the explicit optimal trajectories for human capital, labor supply and consumption demand.

JEL codes: I00, O11, C61.

Keywords: human capital, unbalanced growth, transversality condition.

1 Introduction

When Mahbub ul Haq introduced human development as a compound process which included education and life expectancy, he paved the way for a new avenue of research in development economics. The introduction of the Human Development Index (HDI) in ul Haq (1995) led to a new paradigm and to a modern theory of human development. However, although his works have influenced prominent economists like Amartia Sen, most growth theorists still focus on the utility of consumption. Stemming from the common belief that educated and healthy people enjoy life differently, the present paper considers that human

*The authors acknowledge Cuong Le Van for his helpful comments and suggestions.

[†]EPEE, University Paris-Saclay. E-mail: stefano.bosi@univ-evry.fr.

[‡]PJSE UMR8545, PSE. E-mail: carmen.camacho@psemail.eu.

[§]BETA, University of Lorraine, AgroParisTech, CNRS, INRA. E-mail: david.desmarchelier@gmail.com.

capital is made of both education and health. Then we incorporate the HDI into the utility function defining a new composite good, which generalizes the consumption good. The new good embodies the notion that human capital is crucial to appreciate consumption, and it allows us to study the cross effect of human capital on the marginal utility of consumption.

While the seminal notion of human capital was introduced by Adam Smith in 1776 and later by Arthur Cecil Pigou in 1928, the modern theory of human capital can be traced back to Schultz (1961) and Becker (1964). Uzawa (1965) was first to incorporate human capital as an engine of growth in a theoretical model. The emergence of a new endogenous growth literature stimulated the interest of economists in the role of human capital. Building on Rosen (1976), Lucas (1988) underlined that the accumulation of human capital can trigger a mechanism of perpetual growth. In particular, he shows that the growth rate of income per capita depends on the growth rate of human capital, which in turn depends on the time individuals use to acquire skills and to protect their health. After Lucas (1988), the literature developed fast, scrutinizing all factors underlining the formation of human capital: education, transmission, health, culture,...¹ Nevertheless, the literature becomes thinner regarding the other various roles of human capital in an economy. Although sociologists report that as a matter of fact, education does affect life enjoyment, research in economics has for the most part neglected the role of human capital in welfare. Among the studies in sociology let us mention Ross and Wu (1995), who find that well educated individuals have a more fulfilling job and a better control over their lives. Moreover, they are in a healthier condition since they smoke and drink less. Similarly, Finkelstein et al. (2013) point out a complementarity between health and consumption demand: an increase of 1% in the number of chronic diseases reduces the marginal utility of consumption from 10% to 25%. From an economist's viewpoint, this empirical evidence suggests that human capital does increase the marginal utility of consumption.

In order to consider the effect of education and health on wellbeing, it is necessary to enlarge the standard view of household's preferences. Nevertheless, introducing human capital in the utility function requires careful attention. Indeed, from a theoretical perspective, the effects of human capital on the marginal utility of consumption are potentially ambiguous. On the one hand, human capital increases the marginal utility of consumption when, for instance, a well educated individual watches a movie and fully grasps all cultural references; on the other, human capital decreases the marginal utility of consumption when she becomes aware about the environmental damages of consumption. Empirical studies are not definite about the cross effect of human capital on consumption demand.

To our knowledge, Chakraborty and Gupta (2006) is the only model à la Lucas (1988) where human capital has the two essential roles we have underlined.

¹Let us mention some of the most notorious works. Regarding education and human capital formation, see De la Croix and Doepke (2003), Tamura (2001) or Cervellati and Sunde (2005). Concerning health and human capital formation, the interested reader can refer to De la Croix and Licandro (1999) or Kalemli-Ozcan et al. (2000).

There, human capital is, at the same time, a production factor and a source of wellbeing, and it enters both in the production and in the utility functions. When human capital affects more the agent's preferences than consumption, then they find that the growth rate is higher along the Balanced Growth Path (BGP). When it affects less the production function, then multiple steady states coexist. Note however that the authors do not study the optimality of their BGP solution. Our objective is to address the important question of the role of human capital in wellbeing by considering a growth model where, as in Lucas (1988), the household chooses the working time and the time devoted to acquire skills. For simplicity, there is no physical capital but, as in Chakraborty and Gupta (2006), human capital affects both the production and the utility function.

Despite the absence of an optimality analysis in Chakraborty and Gupta (2006), intuition suggests that the optimality of the BGP is a relevant question when human capital enters preferences. Indeed, along the BGP, human capital grows at a constant rate by definition. Now, suppose human capital enters the utility function with standard properties, that is, a decreasing marginal utility. Then, depending on preferences, there may exist a critical level of human capital beyond which a further increase does not confer any extra utility. Beyond the threshold, the household stops acquiring skills and devotes all her time to work, in contradiction with the perpetual growth of human capital along a BGP. Therefore, when the human capital enters the utility function, the BGP could be no longer optimal.

The present paper presents a simple framework where preferences and technology are isoelastic, and where the HDI is a geometrical average of income and human capital (Cobb-Douglas). We find the explicit optimal solution for the economic transition and in the spirit of Borissov et al. (2018), we prove the inefficiency of the BGP. More precisely, we prove that (1) when consumption matters less than human capital, then the individual's human capital and consumption grow together over time, although her labor decreases; (2) when their weights in preferences are identical, the BGP is optimal, and (3) when the consumption goods matters more, then the individual accumulates human capital until a given date and, after that date, she spends the rest of her life to work and to consume. Building on the optimal trajectory, we obtain a full description of the dynamics of the HDI.

One of our most striking results is that if households do not care more about human capital than about consumption, then they fix a maximum target value for human capital. All households, from the most educated to the least educated, surrender human capital to the sake of consumption. On the contrary, when households enjoy better human capital than consumption, then they will accumulate human capital forever. Our results underline the need to inquire about households' preferences before undertaking any policy regarding human capital.

Finally, an important contribution of our paper is about the role of the transversality condition in intertemporal optimization. Many papers introduce a sufficient transversality condition like $\lim_{t \rightarrow \infty} \lambda(t) x(t) = 0$, where $x(t)$ and $\lambda(t)$ are the state and the co-state variable, but few papers explain why. Con-

versely, we show that this condition fails along the optimal path. More precisely, we find a more general condition: $\lim_{t \rightarrow \infty} \lambda(t) x(t) \in \mathbb{R}$, which is necessary but not sufficient, and we prove by way of example that, along the optimal path, $\lim_{t \rightarrow \infty} \lambda(t) x(t) > 0$. To the best of our knowledge, this is new in the economic literature and hinges on the fact that human capital enters the utility function through the HDI.

The rest of the paper is organized as follows. The model is presented in Section 2. Section 3 considers the optimal solution when both utility and technology are isoelastic functions. Assuming a unit elasticity of substitution both in preferences and in the HDI, Section 4 provides the explicit optimal solutions for human capital and labor. Finally, Section 5 concludes. All the proofs are gathered in the Appendix.

2 The model

In the spirit of Chakraborty and Gupta (2006), we assume that human capital (health and education) increases workers' productivity and households' utility. There exists a representative household, who lives forever and maximizes the infinite-horizon utility:

$$\max \int_0^\infty e^{-\theta t} u(\tilde{x}(t)) dt \quad (1)$$

choosing the optimal trajectories of consumption, c , and labor, l , and where the Human Development Index (HDI)², $\tilde{x}(t) \equiv x(c(t), h(t))$, is an increasing function of consumption and human capital. Optimal decisions are subject to the human capital accumulation law:

$$\frac{\dot{h}(t)}{h(t)} \leq B[1 - l(t)] \quad (2)$$

the resource constraint:

$$c(t) \leq y(t) \equiv h(t)l(t) \quad (3)$$

and the initial condition $h(0) \equiv h_0$. Output $y(t)$ is a linear function of labor, whose productivity is precisely human capital.

The representative household arbitrates between the working time in the firm, $l(t)$, and the time spent to acquire skills, $1 - l(t)$. As in Lucas (1988), the growth rate of human capital is a linear function of $1 - l(t)$, and it attains its maximum value B when $l(t) = 0$, that is, when the household's labor is zero.

For simplicity, we will omit the time argument in the following.

Assumption 1 *Function $x : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is C^2 , strictly increasing and homogeneous of degree 1. u is C^2 , strictly increasing and strictly concave.*

²If $y(t) \equiv h(t)l(t)$ denotes the per capita income (production) and $c(t) = y(t)$, we recover ul Haq's definition of HDI.

In order to solve the household's program, let us introduce the Hamiltonian function:

$$H(h, l, \lambda, t) \equiv e^{-\theta t} u(x(hl, h)) + \lambda B(1-l)h \quad (4)$$

The following proposition characterizes the optimal behavior of human capital, h , and of $\mu \equiv \lambda e^{\theta t}$, its shadow price.

Proposition 1 (necessary conditions) *Under Assumption 1, the solution of program (1) satisfies the following necessary conditions:*

$$\mu = \frac{u'(x)}{B} \frac{\partial x}{\partial c} \quad (5)$$

$$\frac{\dot{\mu}}{\mu} = \theta - B(1-l) - \frac{u'(x)}{\mu} \left(\frac{\partial x}{\partial c} l + \frac{\partial x}{\partial h} \right) \quad (6)$$

$$\frac{\dot{h}}{h} = B(1-l) \quad (7)$$

$$\lim_{t \rightarrow \infty} e^{-\theta t} \mu h \in \mathbb{R} \quad (8)$$

Proposition 2 (sufficient conditions) *Conditions (5) to (7) are not only necessary but also sufficient if the Arrow-Kurz sufficient condition holds*

$$\frac{\partial^2 x}{\partial c^2} [l(h) + hl'(h)]^2 + 2 \frac{\partial^2 x}{\partial c \partial h} [l(h) + hl'(h)] + \frac{\partial^2 x}{\partial h^2} < 0 \quad (9)$$

jointly with a stronger transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\theta t} \mu h = 0$$

Proof. See the Appendix. ■

Remark 3 *The transversality condition $\lim_{t \rightarrow \infty} e^{-\theta t} \mu h \in \mathbb{R}$ is necessary for optimization (see the proof of Proposition 1), while $\lim_{t \rightarrow \infty} e^{-\theta t} \mu h = 0$ is a sufficient condition according to Pontryagin et al. (2018, page 49). Many economic papers introduce $\lim_{t \rightarrow \infty} e^{-\theta t} \mu h = 0$ as a sufficient condition without giving proof, while few show the sufficiency (see Bosi et al. (2017) among others). We will provide an economic example where the optimal solution, explicitly computed, satisfies (5) to (8) and (9) with $\lim_{t \rightarrow \infty} e^{-\theta t} \mu h > 0$. To the best of our knowledge, only Halkin provides a mathematical example, but meaningless in economic terms (see the footnote at page 46 in Arrow and Kurz (1970) and the counterexample at page 271 in Halkin (1974)). In our case, the economic variables μ and h are always non-negative and the transversality condition is nonzero, that is positive. This is due to the presence of the state variable (human capital) in the objective function (utility): it is rational to remain with a positive value of human capital at the "end of life" if capital gives us a utility not because we consume it but because we own it.*

The following proposition provides the optimal trajectories for human capital, labor supply and consumption demand.

Proposition 4 (dynamic system) *The optimal solution (h, l) satisfies the following dynamical system:*

$$\frac{\dot{h}}{h} = B(1-l) \quad (10)$$

$$\frac{\dot{l}}{\bar{l}} = \frac{\theta - B - B \frac{\partial x / \partial h}{\partial x / \partial c} - B(1-l) \left[\frac{x u''(x)}{u'(x)} \left(\frac{c}{x} \frac{\partial x}{\partial c} + \frac{h}{x} \frac{\partial x}{\partial h} \right) + \frac{c}{\partial x / \partial c} \frac{\partial^2 x}{\partial c^2} + \frac{h}{\partial x / \partial c} \frac{\partial^2 x}{\partial c \partial h} \right]}{\frac{c}{x} \frac{\partial x}{\partial c} \frac{x u''(x)}{u'(x)} + \frac{c}{\partial x / \partial c} \frac{\partial^2 x}{\partial c^2}} \quad (11)$$

where $c = hl$.

Proof. See the Appendix. ■

The computation of the optimal solution in the general case presented in Proposition 4 turns out to be impossible. However, in the case of isoelastic functional forms, the two-dimensional system (10)-(11) comes down to a single differential equation, easier to study.

3 The case of isoelastic functional forms

Consider the isoelastic functions:

$$u(x) \equiv \frac{\varepsilon}{\varepsilon - 1} x^{\frac{\varepsilon - 1}{\varepsilon}} \quad (12)$$

$$x(c, h) \equiv \left(ac^{\frac{\sigma - 1}{\sigma}} + bh^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}} \quad (13)$$

where ε is the (dynamic) elasticity of substitution between the HDI x today and tomorrow, and σ is the (static) elasticity of substitution in the HDI between the consumption c and the human capital h today.

Proposition 5 (constant elasticities of substitution) *If the utility function and the HDI are defined by (12) and (13), the Arrow-Kurz criterion (9) is satisfied and, thus, the first-order conditions (5) to (7) jointly with the transversality condition are necessary and sufficient for utility maximization. Optimal labor supply is driven by a single differential equation:*

$$\dot{l} = Bl \left(l + \varepsilon \frac{b}{a} l^{\frac{1}{\sigma}} - 1 - \varepsilon \frac{\theta - B}{B} \right) \frac{al^{\frac{\sigma - 1}{\sigma}} + b}{al^{\frac{\sigma - 1}{\sigma}} + b \frac{\varepsilon}{\sigma}} \equiv \varphi(l) \quad (14)$$

Proof. See the Appendix. ■

Proposition 6 (Balanced Growth Path) *There exists a non-zero stationary solution \bar{l} to (14), that is a solution to the following equation:*

$$l + \varepsilon \frac{b}{a} l^{\frac{1}{\sigma}} = 1 + \varepsilon \frac{\theta - B}{B} \quad (15)$$

The steady state \bar{l} for labor supply determines the human capital growth rate $g = B(1 - \bar{l})$. More explicitly, the growth rate is balanced, that is the same for human capital and consumption:

$$\bar{h}(t) = e^{B(1-\bar{l})t} h_0 \quad (16)$$

$$\bar{c}(t) = e^{B(1-\bar{l})t} h_0 \bar{l} \quad (17)$$

The Balanced Growth Path (BGP) (16)-(17) is globally unstable.

Proof. See the Appendix. ■

One of the main issues of this paper is the analysis of the impact of human capital on consumption demand. More precisely, we are interested in the impact of human capital on the marginal utility of consumption. As it follows from (13), this effect is negative (positive) if consumption and human capital are substitutes (complements). Hence, the elasticity of consumption-human capital substitution affects the optimal solution to the initial program, which generically differs from the BGP.

According to (14), when the initial condition $l_0 < \bar{l}$, labor supply $l(t)$ decreases, while, when $l_0 > \bar{l}$, labor supply $l(t)$ increases. Note that the dynamics of labor are not necessarily monotonic nor symmetric on either side of \bar{l} . For instance, in the simple example of Section 4, we show that $l(t)$ decreases to zero if $l_0 < \bar{l}$, while, if $l_0 > \bar{l}$, it reaches 1 in a finite lapse of time T . After date T , human capital stops growing.

The elasticity of consumption-human capital substitution (σ) affects the critical value \bar{l} and, thus, it plays a role in the dynamics of labor supply and human capital accumulation. Indeed, totally differentiating (15), we get

$$\frac{\sigma}{\bar{l}} \frac{\partial \bar{l}}{\partial \sigma} = \frac{B(1 - \bar{l}) + \varepsilon(\theta - B)}{\sigma a \bar{l} + B(1 - \bar{l}) + \varepsilon(\theta - B)} \ln \bar{l} = \frac{\varepsilon b \bar{l}^{\frac{1}{\sigma}}}{\sigma a \bar{l} + \varepsilon b \bar{l}^{\frac{1}{\sigma}}} \ln \bar{l} < 0$$

Hence, σ has a clear negative impact on \bar{l} :

(1) the more substitutable are consumption and human capital, the lower is \bar{l} and, hence, the more likely is that $l_0 > \bar{l}$. That is, the more plausible is the situation in which labor supply increases over time until the critical date T beyond which the household stops investing in human capital;

(2) the less substitutable are consumption and human capital, the larger is \bar{l} . Thus, the more likely is that $l_0 < \bar{l}$, that is, the more plausible a decrease of labor supply to zero ensuring, asymptotically, the largest human capital (and consumption) growth rate $g = B$.

In other words, when human capital lowers the marginal utility of consumption (substitutability case), the individual no longer invests in human capital in the long run (beyond the critical date T). Conversely, when human capital

raises the marginal utility of consumption (complementarity case), the individual wants to invest in human capital at the maximal rate (B) in the long run.

For the sake of precision, let us observe that the optimal starting point l_0^* is endogenous. Consequently, even a very narrow (large) interval $(0, \bar{l})$ does not ensure that $l_0^* > \bar{l}$ (neither that $l_0^* < \bar{l}$). In order to avoid any heuristic interpretation and to determine unambiguously whether the optimal initial labor supply l_0^* is lower or higher than \bar{l} , in the following example, we fix $\sigma = 1$.

4 An example with a positive transversality condition

In order to obtain a complete description of the optimal solution, let us consider logarithmic preferences: $u(x) \equiv \ln x$, with a Cobb-Douglas HDI: $x(c, h) \equiv c^\alpha h^{1-\alpha}$. These functions have unit elasticities of substitution: $\varepsilon = \sigma = 1$, and allow us to compute explicitly the optimal initial value for labor supply. The method we employ here to obtain the optimal value of l_0 follows Borissov et al. (2018), and it consists in remaximizing (1) with respect to l_0 under the system of constraints (10)-(11). We are able to prove that, generically, the starting point $l_0 = \bar{l}$ corresponding to the BGP is not optimal. As a result, the BGP is not optimal. Furthermore, once the optimal l_0 is determined, we also obtain the complete characterization of the HDI.

α is a new important incoming parameter. When the HDI is given by (13) and $\sigma = 1$, the HDI x is defined as a geometric average of income (consumption demand) and human capital with weights α and $1 - \alpha$ respectively. Thus, α captures the propensity to consume, while $1 - \alpha$ represents the propensity to invest in education and health.

The following proposition provides the optimal paths for labor supply, human capital and consumption demand depending on the value of α . Interestingly, this example shows the possibility of a strictly positive transversality condition. To the best of our knowledge, there are no economic models with positive transversality condition. However, Halkin (1974) provides a mathematical example of non-zero transversality condition.

Proposition 7 *Let $a = b$, $B = \theta = 1/2$ and $\varepsilon = \sigma = 1$.*

(1) If $\alpha < 1/2$, the optimal growth path is given by

$$\begin{aligned} l(t) &= \frac{l_0^*}{2l_0^* + (1 - 2l_0^*)e^{\frac{t}{2}}} \\ h(t) &= h_0 e^{\frac{t}{2}} \left(2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^* \right)^{\frac{1}{2}} \end{aligned} \tag{18}$$

$$c(t) = \frac{h_0 l_0^*}{\left(2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^* \right)^{\frac{1}{2}}} \tag{19}$$

for any $t \geq 0$, where $l_0^* \in (0, 1/2)$ is the optimal initial labor supply.

In this case, the transversality condition is positive:

$$\lim_{t \rightarrow \infty} e^{-\frac{t}{2}} \mu(t) h(t) = \mu_0^* h_0 (1 - 2l_0^*) > 0$$

(2) If $\alpha = 1/2$, the optimal growth path is the BGP:

$$\begin{aligned} l(t) &= \bar{l} = 1/2 \\ h(t) &= h_0 e^{\frac{t}{4}} \end{aligned} \tag{20}$$

$$c(t) = \frac{1}{2} h_0 e^{\frac{t}{4}} \tag{21}$$

for any $t \geq 0$.

(3) If $\alpha > 1/2$, optimal initial labor supply l_0^* lies within $(1/2, \alpha)$. The optimal growth path has two regimes. There exists a critical date

$$T \equiv 2 \ln \frac{l_0^*}{2l_0^* - 1} \tag{22}$$

such that for any $t \in [0, T)$, the optimal growth path is given by

$$\begin{aligned} l(t) &= \frac{l_0^*}{2l_0^* - (2l_0^* - 1)e^{\frac{t}{2}}} \\ h(t) &= h_0 e^{\frac{t}{2}} \left(2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^* \right)^{\frac{1}{2}} \\ c(t) &= \frac{h_0 l_0^*}{\left(2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^* \right)^{\frac{1}{2}}} \end{aligned}$$

and, for any $t \geq T$,

$$\begin{aligned} l(t) &= 1 \\ h(t) &= h_T \equiv \frac{h_0 l_0^*}{(2l_0^* - 1)^{\frac{1}{2}}} \end{aligned} \tag{23}$$

$$c(t) = h_T \tag{24}$$

Proof. See the Appendix. ■

Remark 8 (positive transversality condition) The existence of a positive transversality condition $\lim_{t \rightarrow \infty} e^{-\theta t} \mu(t) h(t) > 0$ allows us to raise a fundamental economic question. Indeed, here, we do not introduce the zero transversality condition: $\lim_{t \rightarrow \infty} e^{-\theta t} \mu(t) h(t) = 0$, as an additional constraint to dynamic optimization. Conversely, many economic papers impose the zero transversality condition to solve the model backwards without justifying the restriction in the space of feasible trajectories. Notice that, strictly speaking, imposing the

zero transversality condition is equivalent to solving a constrained optimization problem. Hence, in general, nothing ensures that the trajectory obtained with $\lim_{t \rightarrow \infty} e^{-\theta t} \mu(t) h(t) = 0$ is the optimal trajectory if we remove this asymptotic restriction. As a matter of fact, this trajectory could be dominated by another trajectory for which the transversality condition takes a non-zero real value. Our constructive example in case (1) shows indeed that an alternative feasible path exists which violates the zero transversality condition and dominates any other feasible path in terms of welfare. Moreover, the positive transversality condition we obtain in this case, is compatible with the convergence of the utility functional along the optimal path: $\int_0^\infty e^{-\theta t} u(\tilde{x}^*(t)) dt < \infty$.

Using the trajectories provided in Proposition 7, we compute next the growth rates along the optimal trajectories. As corollary 9 shows, the growth rates of h and l are different from each other, that is, unbalanced, except in the case $\alpha = 1/2$.

Corollary 9 (unbalanced growth rates) (1) If $\alpha < 1/2$, the optimal growth rates for human capital and consumption are positive but unbalanced:

$$\begin{aligned} \frac{h'(t)}{h(t)} &= \frac{1}{2} \frac{l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*}{2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*} \in \left(\frac{1-l_0^*}{2}, \frac{1}{2} \right) \\ \frac{c'(t)}{c(t)} &= \frac{1}{2} \frac{l_0^* e^{-\frac{t}{2}}}{2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*} \in \left(0, \frac{l_0^*}{2} \right) \end{aligned}$$

because $l_0^* < 1/2$.

(2) If $\alpha = 1/2$, the growth rates are positive and balanced:

$$\frac{h'(t)}{h(t)} = \frac{c'(t)}{c(t)} = \frac{1}{4} \quad (25)$$

but smaller than the maximal growth rate ($B = 1/2$). Here, $l_0^* = \bar{l} = 1/2$.

(3) Let $\alpha > 1/2$. For $t \leq T$, the optimal growth rate of human capital is positive:

$$\frac{h'(t)}{h(t)} = \frac{1}{2} \frac{l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*}{2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*} \in \left(0, \frac{1-l_0^*}{2} \right)$$

because $l_0^* \in (1/2, \alpha)$. The growth rate of h decreases from its initial value $(1-l_0^*)/2 > 0$ to 0 when $t = T$. Then, it remains equal to 0 for any $t > T$. Note that the growth rate is continuous at $t = T$.

For $t \leq T$, the optimal growth rate of consumption is positive:

$$\frac{c'(t)}{c(t)} = \frac{1}{2} \frac{l_0^* e^{-\frac{t}{2}}}{2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*} \in \left(\frac{1}{2}, \frac{l_0^*}{2} \right)$$

because $l_0^* \in (1/2, \alpha)$. The growth rate of c decreases from its initial value $l_0^*/2 > 0$ to $1/2$ when $t = T$. Then, it becomes 0 for any $t > T$. It is discontinuous at $t = T$.

Interestingly, the BGP is optimal only in the special case $\alpha = 1/2$, while in all the other cases, it is never optimal. This result sharply contrasts with Lucas (1988), where human capital only enters the production function. In our model, human capital also enters the utility function with decreasing marginal utility. Recall that the relative contribution of c and h to utility through the HDI $\tilde{x}(c, h) \equiv c^\alpha h^{1-\alpha}$ is precisely given by α and $1 - \alpha$. When $\alpha < 1/2$, an extra unit of human capital has a larger effect on welfare than an extra unit of consumption. In this case, the household devotes initially less time to labor than to acquire human capital, and it is always optimal to increase the time spent in human capital accumulation with $\lim_{t \rightarrow \infty} l(t) = 0$. Human capital and consumption grow at positive but distinct rates, proving that the BGP is not the optimal solution when $\alpha < 1/2$.

Conversely, when $\alpha > 1/2$, an extra unit of consumption has a larger effect on welfare than an extra unit of human capital. Because of the decreasing marginal utility, there exists a critical date beyond which an increase in human capital does not confer a sufficiently large additional utility to encourage the household to acquire more skills. She prefers instead, beyond that date, to work and consume. In other words, it is not optimal to accumulate human capital forever at a constant rate (along a BGP). Proposition 7 shows precisely how this mechanism takes place for $\alpha > 1/2$: whatever the value of h_0 , human capital increases from $t = 0$ until $t = T$. At the beginning it increases sharply, gaining less human capital with time. h_T is the ceiling of human capital (h_T corresponding to $l = 1$ in the long run).

The relative contributions of c and h to utility through the composite good x become equal when $\alpha = 1/2$. In this case, the household continues spending time in both the activities (working in the firm and acquiring skills) in the long run. Only in this special (nongeneric) case, the BGP turns out to be optimal.

The explicit optimal paths for consumption and human capital provided in Proposition 7 prepare the way to compute the optimal path for the HDI.

Proposition 10 (HDI dynamics) *Let us assume that $a = b$, $B = \theta = 1/2$ and $\varepsilon = \sigma = 1$, and let us denote by l_0^* the optimal initial labor supply.*

(1) *If $\alpha < 1/2$, the Human Development Index (HDI) is given by*

$$\tilde{x}(t) = h_0 l_0^{*\alpha} e^{\frac{1-\alpha}{2}t} \left(2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^* \right)^{\frac{1}{2}-\alpha} \quad (26)$$

where $l_0^ < 1/2$. The HDI grows forever at the rate:*

$$\frac{\tilde{x}'(t)}{\tilde{x}(t)} = \frac{1}{2} \frac{l_0^* e^{-\frac{t}{2}} + (1-\alpha)(1-2l_0^*)}{2l_0^* e^{-\frac{t}{2}} + 1 - 2l_0^*} \quad (27)$$

The HDI's growth rate increases from its minimum level $[l_0^ + (1-\alpha)(1-2l_0^*)]/2 > 0$ attained at $t = 0$, to $(1-\alpha)/2$ (asymptotically).*

(2) If $\alpha = 1/2$, the HDI is given by

$$\tilde{x}(t) = \frac{h_0}{2^\alpha} e^{\frac{t}{4}} \quad (28)$$

and it grows forever at the constant rate

$$\frac{\tilde{x}'(t)}{\tilde{x}(t)} = \frac{1}{4}$$

In this case, the HDI's growth rate is balanced (compare with (25)), but lower than the largest possible growth rate $B = 1/2$ given by (2) with $l(t) = 0$.

(3) Let $\alpha > 1/2$. For $t \leq T$ (defined by (22)), the HDI $\tilde{x}(t)$ is still given by (26) and grows at the rate (27), which also increases from

$$\frac{1}{2} [l_0^* + (1 - \alpha)(1 - 2l_0^*)] > 0$$

(when $t = 0$), to $\alpha/2$ (when $t = T$) because $l_0^* \in (1/2, \alpha)$.

At $t = T$, the HDI stops growing and remains constant thereafter:

$$\tilde{x}(t) = h_T \quad (29)$$

where h_T is given by (23). Noteworthy, in this last case, the growth rate is discontinuous at $t = T$ and it falls to zero suddenly.

Summing up, Proposition 10 has proven that the HDI grows forever when $\alpha \leq 1/2$ because households prefer human capital and, thus, human capital grows perpetually. When $\alpha > 1/2$, the HDI grows initially although it stops growing at T and it remains constant thereafter because the household's stronger preference for labor and consumption prevails over education and health.

5 Conclusion

In this paper, we have revisited the model of human capital accumulation published by Lucas in 1988. In our model, human capital not only enters the production but also the utility function. Differently from Lucas' seminal contribution, focusing on a simple example with isoelastic technology and preferences, we prove that the BGP is inefficient, except in the very special (nongeneric) case where the weights of consumption and human capital in HDI are exactly equal.

In all the other cases, the BGP fails to be optimal.

When the weight of consumption exceeds that of human capital, the household chooses to spend all the time at her disposal to work from a critical date on. The HDI increases until a critical level of human capital is reached.

On the contrary, the working time decreases to zero when the weight of human capital exceeds that of consumption. Here, human capital and consumption increase forever, and so does the HDI. Interestingly, in this case, the transversality condition is positive because of the human capital in the utility function.

6 Appendix

Proof of Proposition 1

We define a feasible solution to be a trajectory (h, l) which satisfies the initial condition for h , $h(0) = h_0$ and the law of motion (2).

To apply Ekeland's variational principle, we build a value function $V(h, l, \lambda)$ as follows:

$$\begin{aligned} V(h, l, \lambda) \equiv & \int_0^\infty e^{-\theta t} u(x(h(t)l(t), h(t))) dt \\ & + \int_0^\infty \lambda^*(t) \left(h(t) B[1 - l(t)] - \dot{h}(t) \right) dt \end{aligned} \quad (30)$$

where λ^* is the multiplier function associated to the optimal solution $(h, l)^*$.

We can rearrange the last term in the second integral using integration by parts:

$$\begin{aligned} \int_0^\infty \lambda^*(t) \dot{h}(t) dt &= [\lambda^*(t) h(t)]_0^\infty - \int_0^\infty \dot{\lambda}^*(t) h(t) dt \\ &= \lim_{T \rightarrow \infty} \lambda^*(T) h(T) - \lambda^*(0) h(0) - \int_0^\infty \dot{\lambda}^*(t) h(t) dt \end{aligned}$$

Replacing this into (30), we obtain

$$\begin{aligned} V(h, l, \lambda) &= \int_0^\infty e^{-\theta t} u(x(h(t)l(t), h(t))) dt \\ &+ \int_0^\infty \left(\dot{\lambda}^*(t) h(t) + \lambda^*(t) h(t) B[1 - l(t)] \right) dt \\ &+ \lambda^*(0) h(0) - \lim_{T \rightarrow \infty} \lambda^*(T) h(T) \end{aligned} \quad (31)$$

According to Ekeland's variational principle, if there exists an optimal solution $(h, l)^*$, then any other trajectory (h, l) can be written as a deviation from the optimal.

Hence, for every feasible pair $(h, l) \neq (h, l)^*$ and a constant $\varepsilon \in \mathbb{R}/\{0\}$, there exist functions $H, L : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\begin{aligned} H(t) &\equiv [h(t) - h^*(t)] / \varepsilon \\ L(t) &\equiv [l(t) - l^*(t)] / \varepsilon \end{aligned}$$

that is

$$(h(t), l(t)) = (h^*(t), l^*(t)) + \varepsilon (H(t), L(t))$$

(H, L) represents a given direction in the function space. Any feasible solution can be written as an ε -deviation from the optimal along a direction (H, L) . We can find the optimal solution by maximizing the value function with respect

to ε . Indeed, given (H, L) and $(h, l)^*$, the value V computed along the feasible trajectory (h, l) can be written as a function of ε . More precisely, we have

$$\tilde{V}(\varepsilon) \equiv V(h^* + \varepsilon H, l^* + \varepsilon L, \lambda^*)$$

where $(h, l)^*$, (H, L) and λ^* are given.

Under Assumption 1, the necessary condition for $(h, l)^*$ to be an optimal solution is $\tilde{V}'(0) = 0$.

First, we observe that

$$\begin{aligned} & \lambda^*(0) h(0) - \lim_{T \rightarrow \infty} \lambda^*(T) h(T) \\ &= \lambda^*(0) h^*(0) - \lim_{T \rightarrow \infty} \lambda^*(T) h^*(T) - \varepsilon \lim_{T \rightarrow \infty} \lambda^*(T) H(T) \end{aligned}$$

so that

$$\frac{\partial}{\partial \varepsilon} \left[\lambda^*(0) h(0) - \lim_{T \rightarrow \infty} \lambda^*(T) h(T) \right] = - \lim_{T \rightarrow \infty} \lambda^*(T) H(T)$$

because $\lambda^*(0) h(0) - \lim_{T \rightarrow \infty} \lambda^*(T) h^*(T)$ does not depend on ε .

Next, let us obtain the optimal conditions. We obtain the derivative (31) w.r.t. ε :

$$\begin{aligned} \tilde{V}'(\varepsilon) &= \\ & \int_0^\infty e^{-\theta t} u'(x(h(t)l(t), h(t))) \left(\frac{\partial x}{\partial c} [H(t)l(t) + h(t)L(t)] + \frac{\partial x}{\partial h} H(t) \right) dt \\ & + \int_0^\infty \left(\dot{\lambda}^*(t) H(t) + \lambda^*(t) H(t) B[1 - l(t)] - \lambda^*(t) h(t) BL(t) \right) dt \\ & - \lim_{T \rightarrow \infty} \lambda^*(T) H(T) \end{aligned} \tag{32}$$

The first-order condition is given by $\tilde{V}'(0) = 0$, that is by³

$$\begin{aligned} & \int_0^\infty e^{-\theta t} u'(x(h^*(t)l^*(t), h^*(t))) \left(\frac{\partial x}{\partial c} [H(t)l^*(t) + h^*(t)L(t)] + \frac{\partial x}{\partial h} H(t) \right) dt \\ & + \int_0^\infty \left(\dot{\lambda}^*(t) H(t) + \lambda^*(t) H(t) B[1 - l^*(t)] - \lambda^*(t) h^*(t) BL(t) \right) dt \\ & - \lim_{T \rightarrow \infty} \lambda^*(T) H(T) = 0 \end{aligned} \tag{33}$$

³With some notational abuse, we write

$$\left(\frac{\partial x}{\partial c}, \frac{\partial x}{\partial h} \right)$$

instead of

$$\left(\frac{\partial x}{\partial c}(h(t)l(t), h(t)), \frac{\partial x}{\partial h}(h(t)l(t), h(t)) \right)$$

because the optimal solution corresponds to $\varepsilon = 0$:

$$(h(t), l(t)) = (h^*(t), l^*(t)) + 0 * (H(t), L(t)) = (h^*(t), l^*(t))$$

Focus again on (33). We gather the terms that multiply $H(t)$ and $L(t)$:

$$\begin{aligned} & \int_0^\infty H(t) \left(e^{-\theta t} u' (x(h^*(t)l^*(t), h^*(t))) \left[\frac{\partial x}{\partial c} l^*(t) + \frac{\partial x}{\partial h} \right] + \dot{\lambda}^*(t) + \lambda^*(t) B [1 - l^*(t)] \right) dt \\ & + \int_0^\infty L(t) \left[e^{-\theta t} u' (x(h^*(t)l^*(t), h^*(t))) \frac{\partial x}{\partial c} h^*(t) - \lambda^*(t) h^*(t) B \right] dt \\ & - \lim_{T \rightarrow \infty} \lambda^*(T) H(T) = 0 \end{aligned} \quad (34)$$

Consider the subclass S of functions H such that

$$\lim_{T \rightarrow \infty} \lambda^*(T) H(T) = 0$$

or, equivalently, such that

$$\lim_{T \rightarrow \infty} \lambda^*(T) h(T) = \lim_{T \rightarrow \infty} \lambda^*(T) h^*(T) \in \mathbb{R} \quad (35)$$

In this case, condition (34) is equivalent to

$$\begin{aligned} & \int_0^\infty H(t) \left(e^{-\theta t} u' (x(h^*(t)l^*(t), h^*(t))) \left[\frac{\partial x}{\partial c} l^*(t) + \frac{\partial x}{\partial h} \right] + \dot{\lambda}^*(t) + \lambda^*(t) B [1 - l^*(t)] \right) dt \\ & + \int_0^\infty L(t) \left[e^{-\theta t} u' (x(h^*(t)l^*(t), h^*(t))) \frac{\partial x}{\partial c} h^*(t) - \lambda^*(t) h^*(t) B \right] dt = 0 \end{aligned}$$

whatever the direction (H, L) with $H \in S$ you consider. Because (H, L) identifies an arbitrary direction in the function space, we require

$$\begin{aligned} e^{-\theta t} u' (x(h^*(t)l^*(t), h^*(t))) \left[\frac{\partial x}{\partial c} l^*(t) + \frac{\partial x}{\partial h} \right] + \dot{\lambda}^*(t) + \lambda^*(t) B [1 - l^*(t)] &= 0 \\ e^{-\theta t} u' (x(h^*(t)l^*(t), h^*(t))) \frac{\partial x}{\partial c} h^*(t) - \lambda^*(t) h^*(t) B &= 0 \end{aligned}$$

These are necessary conditions for (h^*, l^*) to be optimal. But, according to (34), these conditions also imply that $\lim_{T \rightarrow \infty} \lambda^*(T) H(T) = 0$, that is $\lim_{T \rightarrow \infty} \lambda^*(T) h(T) \in \mathbb{R}$ for whatever feasible h and, in particular, for h^* . The (transversality) condition

$$\lim_{T \rightarrow \infty} \lambda^*(T) h^*(T) \in \mathbb{R} \quad (36)$$

is also necessary.

We obtain then the system of necessary optimal conditions:

$$\lambda^*(t) = e^{-\theta t} u'(x(h^*(t)l^*(t), h^*(t))) \frac{1}{B} \frac{\partial x}{\partial c} \quad (37)$$

$$\dot{\lambda}^*(t) + \lambda^*(t) B [1 - l^*(t)] + e^{-\theta t} u'(x(h^*(t)l^*(t), h^*(t))) \left[\frac{\partial x}{\partial c} l^*(t) + \frac{\partial x}{\partial h} \right] = 0 \quad (38)$$

jointly with the law of motion, $\dot{h}^*(t) = h^*(t) B [1 - l^*(t)]$, and the transversality condition $\lim_{t \rightarrow \infty} \lambda^*(t) h^*(t) \in \mathbb{R}$.

Since $\mu \equiv \lambda e^{\theta t}$ and $\dot{\lambda}/\lambda = \dot{\mu}/\mu - \theta$, (37) and (38) become:

$$\begin{aligned} \mu^*(t) &= \frac{u'(x(h^*(t)l^*(t), h^*(t)))}{B} \frac{\partial x}{\partial c} \\ \frac{\dot{\mu}^*(t)}{\mu^*(t)} &= \theta - B [1 - l^*(t)] - \frac{u'(x(h^*(t)l^*(t), h^*(t)))}{\mu^*(t)} \left[\frac{\partial x}{\partial c} l^*(t) + \frac{\partial x}{\partial h} \right] \end{aligned}$$

that is (5) and (6). Since $\mu^*(t) > 0$, (2) holds with equality and we obtain (7). Finally, (8) is given by (36). ■

Proof of Proposition 2

The Arrow-Kurz sufficiency theorem in Arrow and Kurz (1970) states that the first-order conditions (5) to (7) are not only necessary but also sufficient if the Hamiltonian $H(h, l, \lambda, t)$ maximized with respect to the control variable l (given h , λ and t), that is $H^*(h, \lambda, t) = H(h, l^*(h, \lambda, t), \lambda, t)$, is concave in h (state variable), given λ and t , that is

$$\frac{\partial^2 H^*}{\partial h^2}(h^*, \lambda, t) < 0$$

jointly with the transversality condition $\lim_{t \rightarrow \infty} \lambda h^* = 0$.

In our case, (4) becomes $H^*(h, \lambda, t) = e^{-\theta t} u\left(x\left(h\tilde{l}^*(h), h\right)\right) + \lambda B \left[1 - \tilde{l}^*(h)\right] h$, where $\tilde{l}^*(h) \equiv l^*(h, \lambda, t)$ given λ and t . We compute the first derivative:

$$\begin{aligned} \frac{\partial H^*}{\partial h}(h, \lambda, t) &= e^{-\theta t} u'\left(x\left(h\tilde{l}^*(h), h\right)\right) \left(\frac{\partial x}{\partial c} \left[\tilde{l}^*(h) + h\tilde{l}^{*'}(h) \right] + \frac{\partial x}{\partial h} \right) \\ &\quad + \lambda B \left[1 - \tilde{l}^*(h) \right] - \lambda B \tilde{l}^{*'}(h) h \end{aligned}$$

then the second derivative:

$$\begin{aligned}
\frac{\partial^2 H^*}{\partial h^2}(h, \lambda, t) &= e^{-\theta t} u'' \left(x \left(h \tilde{l}^*(h), h \right) \right) \left(\frac{\partial x}{\partial c} \left[\tilde{l}^*(h) + h \tilde{l}^{*'}(h) \right] + \frac{\partial x}{\partial h} \right)^2 \\
&+ e^{-\theta t} u' \left(x \left(h \tilde{l}^*(h), h \right) \right) \\
&* \left(\frac{\partial^2 x}{\partial c^2} \left[\tilde{l}^*(h) + h \tilde{l}^{*'}(h) \right]^2 + 2 \frac{\partial^2 x}{\partial c \partial h} \left[\tilde{l}^*(h) + h \tilde{l}^{*'}(h) \right] + \frac{\partial x}{\partial c} \left[2 \tilde{l}^{*'}(h) + h \tilde{l}^{*''}(h) \right] + \frac{\partial^2 x}{\partial h^2} \right) \\
&- 2 \lambda B \tilde{l}^{*'}(h) - \lambda B \tilde{l}^{*''}(h) h
\end{aligned} \tag{39}$$

Using (37), we write

$$e^{-\theta t} u'(x) = \lambda B \left(\frac{\partial x}{\partial c} \right)^{-1}$$

and replacing it in (39), we obtain

$$\begin{aligned}
&\frac{\partial^2 H^*}{\partial h^2}(h, \lambda, t) \\
&= n + \lambda B \left(\frac{\partial x}{\partial c} \right)^{-1} \left(\frac{\partial^2 x}{\partial c^2} \left[\tilde{l}^*(h) + h \tilde{l}^{*'}(h) \right]^2 + 2 \frac{\partial^2 x}{\partial c \partial h} \left[\tilde{l}^*(h) + h \tilde{l}^{*'}(h) \right] + \frac{\partial^2 x}{\partial h^2} \right)
\end{aligned}$$

where n is given by

$$n \equiv e^{-\theta t} u'' \left(x \left(h \tilde{l}^*(h), h \right) \right) \left(\frac{\partial x}{\partial c} \left[\tilde{l}^*(h) + h \tilde{l}^{*'}(h) \right] + \frac{\partial x}{\partial h} \right)^2$$

Under Assumption 1, $n < 0$ and $\partial x / \partial c > 0$. Then,

$$\frac{\partial^2 x}{\partial c^2} \left[\tilde{l}^*(h^*) + h^* \tilde{l}^{*'}(h^*) \right]^2 + 2 \frac{\partial^2 x}{\partial c \partial h} \left[\tilde{l}^*(h^*) + h^* \tilde{l}^{*'}(h^*) \right] + \frac{\partial^2 x}{\partial h^2} < 0 \tag{40}$$

that is (9), implies

$$\frac{\partial^2 H^*}{\partial h^2}(h^*, \lambda, t) < 0$$

■

Proof of Proposition 4

Taking the logarithm of (5) and derivating the result with respect to t , we obtain

$$\begin{aligned}
\frac{\dot{\mu}}{\mu} &= \frac{x u''(x)}{u'(x)} \left[\left(\frac{c}{x} \frac{\partial x}{\partial c} + \frac{h}{x} \frac{\partial x}{\partial h} \right) \frac{\dot{h}}{h} + \frac{c}{x} \frac{\partial x}{\partial c} \frac{\dot{l}}{l} \right] \\
&+ \left(\frac{c}{\partial x / \partial c} \frac{\partial^2 x}{\partial c^2} + \frac{h}{\partial x / \partial c} \frac{\partial^2 x}{\partial c \partial h} \right) \frac{\dot{h}}{h} + \frac{c}{\partial x / \partial c} \frac{\partial^2 x}{\partial c^2} \frac{\dot{l}}{l}
\end{aligned} \tag{41}$$

Replacing (5) in (6), we get

$$\frac{\dot{\mu}}{\mu} = \theta - B - B \frac{\partial x / \partial h}{\partial x / \partial c} \quad (42)$$

Substituting (7) and (42) in (41), we find system (10)-(11). ■

Proof of Proposition 5

From (5), using the functional forms (12) and (13), and noticing that $c = hl$, we obtain

$$h = l^{-\frac{\varepsilon}{\sigma}} \left(al^{\frac{\sigma-1}{\sigma}} + b \right)^{\frac{\varepsilon-\sigma}{\sigma-1}} \left(\frac{a}{\mu B} \right)^{\varepsilon}$$

Hence, $l = l(h)$ is an implicit function of h with the following elasticity:

$$\omega(h) \equiv \frac{hl'(h)}{l(h)} = - \frac{\sigma al(h)^{\frac{\sigma-1}{\sigma}} + \sigma b}{\sigma al(h)^{\frac{\sigma-1}{\sigma}} + \varepsilon b} \quad (43)$$

Given λ and t , let $H^*(h) \equiv u(x^*(h)) + \mu B[1 - l(h)]h$ be the maximum of H with respect to l , with $x^*(h) \equiv x(hl(h), h)$.

According to Arrow and Kurz (1970), we require H^* to be strictly concave: $H^{*''}(h) < 0$.

In the isoelastic case, we obtain

$$\left(\frac{x}{c}, \frac{x}{h} \right) = \left(al^{\frac{\sigma-1}{\sigma}} + b \right)^{\frac{\sigma}{\sigma-1}} \left(\frac{1}{l}, 1 \right) \quad (44)$$

$$\left(\frac{\partial x}{\partial c}, \frac{\partial x}{\partial h} \right) = \left(al^{\frac{\sigma-1}{\sigma}} + b \right)^{\frac{1}{\sigma-1}} \left(al^{-\frac{1}{\sigma}}, b \right) \quad (45)$$

$$\begin{bmatrix} \frac{\partial^2 x}{\partial c^2} & \frac{\partial^2 x}{\partial h \partial c} \\ \frac{\partial^2 x}{\partial c \partial h} & \frac{\partial^2 x}{\partial h^2} \end{bmatrix} = \frac{ab}{c\sigma} \left(al^{\frac{\sigma-1}{\sigma}} + b \right)^{\frac{\sigma}{\sigma-1}-2} l^{\frac{\sigma-1}{\sigma}} \begin{bmatrix} -\frac{1}{l} & 1 \\ 1 & -l \end{bmatrix} \quad (46)$$

As discussed in the proof of Proposition 1, the first-order conditions of the maximization program are not only necessary but also sufficient if (40) holds. That is, if

$$(l[1 + \omega(h)])^2 \frac{\partial^2 x}{\partial c^2} + 2l[1 + \omega(h)] \frac{\partial^2 x}{\partial c \partial h} + \frac{\partial^2 x}{\partial h^2} < 0$$

Replacing (43) and (46), we obtain

$$-\frac{ab}{c\sigma} \left[l^{\frac{2\sigma-1}{\sigma}} \left(al^{\frac{\sigma-1}{\sigma}} + b \right)^{\frac{\sigma}{\sigma-1}-2} \right] \left(\frac{\sigma al^{\frac{\sigma-1}{\sigma}} + \sigma b}{\sigma al^{\frac{\sigma-1}{\sigma}} + \varepsilon b} \right)^2 < 0$$

which is always true. Surprisingly, we do not need any restriction on σ or ε .

Reconsidering (12) and (13), and using (44), (45) and (46), we find

$$\frac{\partial x / \partial h}{\partial x / \partial c} = \frac{b}{a} l^{\frac{1}{\sigma}} \quad (47)$$

and the elasticities

$$\left(\frac{c}{x} \frac{\partial x}{\partial c}, \frac{h}{x} \frac{\partial x}{\partial h} \right) = \frac{1}{al^{\frac{\sigma-1}{\sigma}} + b} \left(al^{\frac{\sigma-1}{\sigma}}, b \right) \quad (48)$$

$$\left(\frac{c}{\partial x / \partial c} \frac{\partial^2 x}{\partial c^2}, \frac{h}{\partial x / \partial c} \frac{\partial^2 x}{\partial c \partial h} \right) = \frac{1}{\sigma} \frac{b}{al^{\frac{\sigma-1}{\sigma}} + b} (-1, 1) \quad (49)$$

Replacing $c = hl$, $u'(x) / [xu''(x)] = -\varepsilon$ and expressions (48) and (49) in (11), we obtain equation (14). ■

Proof of Proposition 6

Let \bar{l} be a steady state solution of $\varphi(l) = 0$. Thus, the nonzero steady state \bar{l} is solution of (15). Integrating $\dot{h}/h = B(1 - \bar{l})$ over time, we get (16) and, replacing \bar{l} and \bar{h} in $c = hl$, we obtain (17). Reconsidering (14), we find that l decreases if $l_0 < \bar{l}$, while l increases if $l_0 > \bar{l}$. Thus the BGP is globally unstable. ■

Proof of Proposition 7

Under the parametric specification, the differential equation (14) becomes

$$\dot{l} = l \left(l - \frac{1}{2} \right) = \varphi(l) \quad (50)$$

The explicit solution of equation (50) is the following:

$$l(t) = \frac{l_0}{2l_0 + (1 - 2l_0)e^{\frac{t}{2}}} \quad (51)$$

where $l_0 \equiv l(0) \in [0, 1]$, with

$$l'(t) = -\frac{1}{2} \frac{l_0 (1 - 2l_0) e^{\frac{t}{2}}}{\left[2l_0 + (1 - 2l_0) e^{\frac{t}{2}} \right]^2}$$

Then, depending on the initial labor allocation we observe three different cases

- (1) If $l_0 < 1/2$, then $l'(t) < 0$.
- (2) If $l_0 = 1/2$, then $l'(t) = 0$, that is $l(t) = \bar{l} = 1/2$ for any t , and the economy grows at the balanced growth rate $g = B(1 - \bar{l}) = B/2$ (BGP).
- (3) If $l_0 > 1/2$, then $l'(t) > 0$.

Hence, the steady state $\bar{l} = 1/2$ is globally unstable.

We observe that h_0 is a given initial condition, while l_0 is a choice variable.

- (1) Notice that, if $l_0 < 1/2$, then $\lim_{t \rightarrow \infty} l(t) = 0$.
- (2) If $l_0 = 1/2$, then $l(t) = 1/2$ forever.

(3) If $1/2 < l_0 \leq 1$, then

$$l(t) = \frac{l_0}{2l_0 + (1 - 2l_0)e^{\frac{t}{2}}}, \quad \text{if } 0 \leq t < 2 \ln \frac{l_0}{2l_0 - 1} \equiv T \quad (52)$$

$$l(t) = 1, \quad \text{if } t \geq 2 \ln \frac{l_0}{2l_0 - 1}$$

We compute the value of the functional $\int_0^\infty e^{-\theta t} u(x) dt$ in these three cases with $u(x) \equiv \ln x$ and $x(c, h) \equiv c^\alpha h^{1-\alpha}$. We observe that these functions have a unit elasticity of substitution: $\varepsilon = \sigma = 1$.

We organize the proof in three parts, computing $c(t)$ and $h(t)$ in each of the three cases and evaluating the utility functional

$$\int_0^\infty e^{-\frac{t}{2}} \ln(c^\alpha h^{1-\alpha}) dt \quad (53)$$

Case (1): $l_0 < 1/2$

Replacing (51) in (10), we obtain

$$(\ln h)' = \frac{\dot{h}}{h} = \frac{1}{2} \left[1 - \frac{l_0}{2l_0 + (1 - 2l_0)e^{\frac{t}{2}}} \right]$$

Integrating both the sides, we find

$$h(t) = h_0 e^{\frac{t}{2}} \left(2l_0 e^{-\frac{t}{2}} + 1 - 2l_0 \right)^{\frac{1}{2}} \quad (54)$$

We observe that $h(0) = h_0$.

Replacing (51) and (54) in $c(t) = h(t)l(t)$, we get

$$c(t) = \frac{h_0 l_0}{\left(2l_0 e^{-\frac{t}{2}} + 1 - 2l_0 \right)^{\frac{1}{2}}} \quad (55)$$

Using (54) and (55), we evaluate the utility functional:

$$\begin{aligned} & \int_0^\infty e^{-\frac{t}{2}} \ln \left[c(t)^\alpha h(t)^{1-\alpha} \right] dt \\ &= \ln(h_0 l_0^\alpha) \int_0^\infty e^{-\frac{t}{2}} dt + \frac{1}{4} \int_0^\infty t e^{-\frac{t}{2}} dt \\ & \quad + \left(\frac{1}{2} - \alpha \right) \int_0^\infty e^{-\frac{t}{2}} \ln \left[2l_0 + (1 - 2l_0)e^{\frac{t}{2}} \right] dt \\ &= 2 \ln(h_0 l_0^\alpha) + 1 - \left(\frac{1}{2} - \alpha \right) \frac{1 - 2l_0}{l_0} \ln(1 - 2l_0) \equiv v_1(l_0) \end{aligned}$$

For any $l_0 \in (0, 1/2)$, $v_1(l_0)$ and $v'_1(l_0)$ are given by

$$\begin{aligned} v_1(l_0) &= 2 \ln(h_0 l_0^\alpha) + 1 + \left(\alpha - \frac{1}{2}\right) \frac{1 - 2l_0}{l_0} \ln(1 - 2l_0) \\ v'_1(l_0) &= \frac{1}{l_0^2} \left[l_0 - \left(\alpha - \frac{1}{2}\right) \ln(1 - 2l_0) \right] \end{aligned}$$

Hence, from the above expressions,

$$v'_1(l_0) > 0 \Leftrightarrow l_0 > \left(\alpha - \frac{1}{2}\right) \ln(1 - 2l_0) \equiv w(l_0)$$

Since $\ln(1 - 2l_0) < 0$, $\alpha \geq 1/2$ implies $v'_1(l_0) > 0$ and

$$\lim_{l_0 \rightarrow (1/2)^-} v_1(l_0) = 2 \ln h_0 - 2\alpha \ln 2 + 1$$

Let $\alpha < 1/2$. Observe that $w(0) = 0$ and that

$$\begin{aligned} w'(l_0) &= \frac{1 - 2\alpha}{1 - 2l_0} > 0 \\ w''(l_0) &= 2 \frac{1 - 2\alpha}{(1 - 2l_0)^2} > 0 \end{aligned}$$

We compare the slope of l_0 and $w(l_0)$ at 0: $1 > w'(0) = 1 - 2\alpha$, and we notice that $\lim_{l_0 \rightarrow (1/2)^-} w'(l_0) = +\infty$. Then, there exists a unique $l_0^* \in (0, 1/2)$ such that $l_0 > w(l_0)$ if $l_0 < l_0^*$ and $l_0 < w(l_0)$ if $l_0 > l_0^*$, that is

$$v'_1(l_0) > 0 \Leftrightarrow l_0 < l_0^*$$

Therefore,

$$l_0^* = \arg \max_{l_0 \in (0, 1/2)} v_1$$

Let us show that l_0^* is the optimal solution of the initial program (1) with $\alpha < 1/2$. We need to check the transversality condition: $\lim_{t \rightarrow \infty} e^{-\theta t} \mu(t) h(t) \in \mathbb{R}$. From equations (42) and (47), we get

$$\frac{\dot{\mu}(t)}{\mu(t)} = \theta - B - B \frac{\partial x / \partial h}{\partial x / \partial c} = \theta - B - B \frac{b}{a} l^{\frac{1}{\sigma}} = -\frac{1}{2} l(t) \quad (56)$$

Using (51), we get

$$[\ln \mu(t)]' = \frac{\dot{\mu}(t)}{\mu(t)} = -\frac{1}{2} \frac{l_0^*}{2l_0^* + (1 - 2l_0^*) e^{\frac{t}{2}}}$$

with solution

$$\mu(t) = \mu_0^* \left(1 - 2l_0^* + 2l_0^* e^{-\frac{t}{2}} \right)^{\frac{1}{2}} \quad (57)$$

From (54) and (57), we obtain the transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\frac{t}{2}} \mu(t) h(t) = \mu_0^* h_0 \lim_{t \rightarrow \infty} \left(1 - 2l_0^* + 2l_0^* e^{-\frac{t}{2}}\right) = \mu_0^* h_0 (1 - 2l_0^*) > 0$$

Therefore, the necessary transversality condition (8) is satisfied.

Notice that, interestingly, the transversality condition is positive and, therefore, different from the usual one: $\lim_{t \rightarrow \infty} e^{-\frac{t}{2}} \mu(t) h(t) = 0$.

Case (2): $l_0 = 1/2$.

Replacing (16) and (17) with $l_0^* = 1/2$ in (53), we obtain the value of the utility functional along the BGP. More explicitly, we find

$$\begin{aligned} h^*(t) &= h_0 e^{\frac{t}{4}} \\ c^*(t) &= \frac{1}{2} h_0 e^{\frac{t}{4}} \end{aligned} \tag{58}$$

and

$$\begin{aligned} & \int_0^\infty e^{-\frac{t}{2}} \ln \left[c^*(t)^\alpha h^*(t)^{1-\alpha} \right] dt \\ &= (\ln h_0 - \alpha \ln 2) \int_0^\infty e^{-\frac{t}{2}} dt + \frac{1}{4} \int_0^\infty t e^{-\frac{t}{2}} dt \\ &= 1 + 2 \ln h_0 - 2\alpha \ln 2 \equiv v_2^* \end{aligned}$$

We check the transversality condition. According to (57), we find

$$\mu(t) = \mu_0^* \left(1 - 2l_0^* + 2l_0^* e^{-\frac{t}{2}}\right)^{\frac{1}{2}} = \mu_0^* e^{-\frac{t}{4}}$$

Using (58), we obtain

$$\lim_{t \rightarrow \infty} e^{-\frac{t}{2}} \mu(t) h(t) = \mu_0^* h_0 \lim_{t \rightarrow \infty} e^{-\frac{t}{2}} = 0$$

Therefore, the transversality condition is satisfied.

Case (3): $l_0 > 1/2$

As seen above, if $l_0 > 1/2$, then $l'(t) > 0$ and l increases over time. At time T , l reaches the upper bound: $l(T) = 1$.

We evaluate the utility functional (53) considering the integrals before and after T :

$$\int_0^\infty e^{-\frac{t}{2}} \ln (c^\alpha h^{1-\alpha}) dt = \int_0^T e^{-\frac{t}{2}} \ln (c^\alpha h^{1-\alpha}) dt + \int_T^\infty e^{-\frac{t}{2}} \ln (c^\alpha h^{1-\alpha}) dt \tag{59}$$

where T is given by (52).

If $0 \leq t < T$, as in case (1), (51) holds and the trajectories (54) and (55) for human capital and consumption hold as well. If $t \geq T$, we obtain $l(t) = 1$ and, using the definition of T in (52),

$$h(t) = h(T) = h_0 e^{\frac{1}{2}T} \left(2l_0 e^{-\frac{T}{2}} + 1 - 2l_0 \right)^{\frac{1}{2}} = \frac{h_0 l_0}{(2l_0 - 1)^{\frac{1}{2}}} \equiv h_T \quad (60)$$

$$c(t) = h(t) l(t) = h_T \quad (61)$$

Replacing (54), (55), (60) and (61) in (59) and observing that

$$T \equiv 2 \ln \frac{l_0}{2l_0 - 1} \text{ and } h_T \equiv \frac{h_0 l_0}{(2l_0 - 1)^{\frac{1}{2}}}$$

we obtain that the utility functional becomes a function v_3 of l_0 :

$$\begin{aligned} & \int_0^\infty e^{-\frac{t}{2}} \ln \left[c(t)^\alpha h(t)^{1-\alpha} \right] dt \\ &= \int_0^T e^{-\frac{t}{2}} \ln \left[c(t)^\alpha h(t)^{1-\alpha} \right] dt + \int_T^\infty e^{-\frac{t}{2}} \ln h_T dt \\ &= \ln h_T \int_T^\infty e^{-\frac{t}{2}} dt + \ln(h_0 l_0^\alpha) \int_0^T e^{-\frac{t}{2}} dt + \frac{1}{4} \int_0^T t e^{-\frac{t}{2}} dt \\ & \quad + \left(\frac{1}{2} - \alpha \right) \int_0^T e^{-\frac{t}{2}} \ln \left[2l_0 + (1 - 2l_0) e^{\frac{t}{2}} \right] dt \\ &= 2e^{-\frac{T}{2}} \ln h_T + 2 \left(1 - e^{-\frac{T}{2}} \right) \ln(h_0 l_0^\alpha) + 1 - \frac{2+T}{2} e^{-\frac{T}{2}} \\ & \quad + \left(\frac{1}{2} - \alpha \right) \left[\frac{1-2l_0}{2l_0} t - 2 \left(e^{-\frac{t}{2}} + \frac{1-2l_0}{2l_0} \right) \ln \left[2l_0 + (1-2l_0) e^{\frac{t}{2}} \right] \right]_0^T \\ &= 2 \ln h_0 - 1 + 2\alpha \ln l_0 + \frac{1}{l_0} - (1-2\alpha) \frac{1-2l_0}{2l_0} \ln(2l_0 - 1) \equiv v_3(l_0) \quad (62) \end{aligned}$$

Notice that v is continuous in $(1/2, 1]$ with $\lim_{l_0 \rightarrow (1/2)^+} v_3(l_0) = 1 + 2 \ln h_0 - 2\alpha \ln 2$ and $v_3(1) = 2 \ln h_0$.

From (62), we can compute $v'_3(l_0)$:

$$v'_3(l_0) \equiv \frac{1}{l_0^2} \left[l_0 - 1 + \left(\frac{1}{2} - \alpha \right) \ln(2l_0 - 1) \right]$$

If $\alpha < 1/2$, then $v'_3(l_0) < 0$.

If $\alpha > 1/2$, then

$$\begin{aligned} v'_3(l_0) &\equiv \frac{1}{l_0^2} \left[l_0 - 1 + \left(\frac{1}{2} - \alpha \right) \ln(2l_0 - 1) \right] > 0 \\ \Leftrightarrow \delta(l_0) &\equiv l_0 - 1 + \left(\frac{1}{2} - \alpha \right) \ln(2l_0 - 1) > 0 \end{aligned}$$

We notice that δ is a convex function in $(1/2, 1]$, strictly decreasing in $(1/2, \alpha)$ and strictly increasing in $(\alpha, 1)$ with $\lim_{l_0 \rightarrow (1/2)^+} \delta(l_0) = +\infty$ and $\delta(1) = 0$. Then, there exists a value $l_0^* \in (1/2, \alpha)$ such that $\delta(l_0) > 0$ if $l_0 < l_0^*$ and $\delta(l_0) < 0$ if $l_0 > l_0^*$, that is such that $v_3'(l_0) > 0$ if $l_0 < l_0^*$ and $v_3'(l_0) < 0$ if $l_0 > l_0^*$. Then, v_3 attains the global maximum in $(1/2, 1]$ at l_0^* .

Eventually, we check the transversality condition. According to (56), we know that

$$\frac{\dot{\mu}(t)}{\mu(t)} = -\frac{1}{2}l(t) = -\frac{1}{2}$$

and, therefore,

$$\mu(t) = \mu_T^* e^{-\frac{t-T}{2}}$$

Moreover, if $t \geq T$, we have

$$h(t) = h_T^* \equiv \frac{h_0 l_0^*}{(2l_0^* - 1)^{\frac{1}{2}}}$$

and, thus,

$$\lim_{t \rightarrow \infty} \left[e^{-\frac{t}{2}} \mu(t) h(t) \right] = \mu_T^* h_T^* e^{\frac{T}{2}} \lim_{t \rightarrow \infty} e^{-t} = 0$$

Hence, the transversality condition is satisfied.

Global case

Let now

$$v(l_0) \equiv \int_0^\infty e^{-\frac{t}{2}} \ln(c^\alpha h^{1-\alpha}) dt$$

be the utility functional with

$$\begin{aligned} v(l_0) &= v_1(l_0), & \text{if } 0 < l_0 < \frac{1}{2} \\ v(l_0) &= v_2^*, & \text{if } l_0 = \frac{1}{2} \\ v(l_0) &= v_3(l_0), & \text{if } \frac{1}{2} < l_0 \leq 1 \end{aligned}$$

$v(l_0)$ is a continuous function in $l_0 \in (0, 1]$, with $\lim_{l_0 \rightarrow 0^+} v(l_0) = -\infty$,

$$\lim_{l_0 \rightarrow (1/2)^-} v_1(l_0) = 1 + 2 \ln h_0 - 2\alpha \ln 2 \equiv v_2^* = \lim_{l_0 \rightarrow (1/2)^+} v_3(l_0)$$

and $v_3(1) = 2 \ln h_0$.

If $\alpha < 1/2$, then $l_0^* = \arg \max_{l_0 \in (0, 1/2)} v_1 \in (0, 1/2)$ and $v_3'(l_0) < 0$. Then, v attains the global maximum at $l_0^* \in (0, 1/2)$. Since the transversality condition is satisfied, the original program (1) has a unique solution.

If $\alpha \geq 1/2$, then $v'_1(l_0) > 0$ and there exists $l_0^* \in [1/2, 1]$ that is a global maximum for v in $[1/2, 1]$. If $\alpha > 1/2$, then $l_0^* \in (1/2, \alpha)$ with $\lim_{\alpha \rightarrow 1/2} l_0^* = 1/2$. If $\alpha = 1/2$, we have $v'_1(l_0) > 0$ with $0 < l_0 < 1/2$, $v'_3(l_0) < 0$ with $1/2 < l_0 < 1$ and $v'_3(1) = 0$, that is $l_0^* = \bar{l} = 1/2$, the BGP. ■

Proof of Proposition 10

(1) Let $\alpha < 1/2$. Replacing (18) and (19) in $x(c, h) \equiv c^\alpha h^{1-\alpha}$, we find (26) for any $t \geq 0$. Taking the time derivative of $\ln \tilde{x}(t)$, we find the growth rate (27). Taking the derivative of $\tilde{x}'(t)/\tilde{x}(t)$, we see that the growth rate increases from $\tilde{x}'(0)/\tilde{x}(0) = [l_0^* + (1 - \alpha)(1 - 2l_0^*)]/2$ to $\lim_{t \rightarrow \infty} [\tilde{x}'(t)/\tilde{x}(t)] = (1 - \alpha)/2$.

(2) Let $\alpha = 1/2$. Replacing (20) and (21) in $x(c, h) \equiv c^\alpha h^{1-\alpha}$, we find (28).

(3) Let $\alpha > 1/2$. If $t \leq T$, the HDI is given by (26) and the growth rate by (27). Taking the derivative of $\tilde{x}'(t)/\tilde{x}(t)$, we see that the growth rate increases from $\tilde{x}'(0)/\tilde{x}(0) = [l_0^* + (1 - \alpha)(1 - 2l_0^*)]/2$ to $\tilde{x}'(T)/\tilde{x}(T) = \alpha/2$.

If $t \geq T$, replacing (23) and (24) in $x(c, h) \equiv c^\alpha h^{1-\alpha}$, we obtain (29). ■

References

- [1] Arrow K. and M. Kurz (1970). Optimal growth with irreversible investment in a Ramsey model. *Econometrica* 38, 331-344.
- [2] Barro R.J. and X. Sala-i-Martin (2004). *Economic Growth*. MIT Press.
- [3] Becker G.S. (1964). *Human Capital: A Theoretical and Empirical Analysis, with Special Reference to Education*. University of Chicago Press, Chicago.
- [4] Borissov K., S. Bosi, T. Ha-Huy and L. Modesto (2018). Heterogeneous human capital, inequality and growth: The role of patience and skills. Forthcoming in the *International Journal of Economic Theory*.
- [5] Bosi S., Le Van C. and N.-S. Pham (2017). Rational land and housing bubbles in infinite-horizon economies. In Nishimura K., A. Venditti and N.C. Yannelis eds., *Sunspots and Non-Linear Dynamics: Essays in Honor of Jean-Michel Grandmont*, Studies in Economic Theory 31, Springer.
- [6] Cervellati M. and U. Sunde (2005). Human capital formation, life expectancy, and the process of development. *American Economic Review* 95, 1653-1672.
- [7] Chakraborty B. and M.R. Gupta (2006). A note on the inclusion of human capital in the Lucas model. *International Journal of Business and Economics* 5, 211-224.
- [8] De la Croix D. and M. Doepke. (2003). Inequality and growth: Why differential fertility matters. *American Economic Review* 93, 1091-1113.

- [9] De la Croix D. and O. Licandro (1999). Life expectancy and endogenous growth. *Economics Letters* 65, 255-263.
- [10] Finkelstein A., E.F.P. Luttmer and M. Notowidigdo (2013). What good is wealth without health? The effect of health on the marginal utility of consumption, *Journal of the European Economic Association* 11, 221-258.
- [11] Halkin H. (1974). Necessary conditions for optimal control problems with infinite horizons. *Econometrica* 42, 267-272.
- [12] Kalemli-Ozcan S., H.E. Ryder and D. Weil (2000). Mortality decline, human capital investment, and economic growth. *Journal of Development Economics* 62, 1-23.
- [13] Lucas R.E. (1988). On the mechanics of economic development. *Journal of Monetary Economics* 22, 3-42.
- [14] Pigou A.C. (1928). *A Study in Public Finance*. Macmillan, London.
- [15] Pontryagin L.S., V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko (2018). *Mathematical Theory of Optimal Processes*. Routledge.
- [16] Rosen S. (1976). A theory of life earnings. *Journal of Political Economy* 84, 545-567.
- [17] Ross C.E. and C.-L. Wu. (1995). The links between education and health. *American Sociological Review* 60, 719-745.
- [18] Schultz T.W. (1961). Investment in human capital. *American Economic Review* 51, 1-17.
- [19] Smith A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations*. Strahan and Cadell, Londres.
- [20] Takahashi H. (2008). Optimal balanced growth in a general multi-sector endogenous growth model with constant returns. *Economic Theory* 37, 31-49.
- [21] Tamura R. (2001). Teachers, growth, and convergence. *Journal of Political Economy* 109, 1021-1059.
- [22] Ul Haq Mahbub (1995). *Reflections on Human Development*. Oxford University Press.
- [23] Uzawa H. (1965). Optimum technical change in an aggregative model of economic growth. *International Economic Review* 6, 18-31.