



## Introduction – A critical approach to the opposition between “concrete” and “abstract” numbers

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## **“Concrete numbers” versus “abstract numbers”: an anthropological, historical, historiographical and didactical approach**

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## Introduction—A critical approach to the opposition between “concrete” and “abstract” numbers

**Christine Proust and Eric Vandendriessche** (SPHERE, CNRS & Paris Cité University, France)

This special issue is part of a recent historiographical trend that focuses on the entwined histories of calculation and measurement practices. On the philosophical level, we can mention, for example, the revival of philosophical studies of measurement as witnessed by the *Measurement at the Crossroads Conference* held in Paris in 2018.<sup>1</sup> In the field of the history of mathematics, this revival has come mainly from scholars who have been interested in the various ways numbers and measurement values have been shaped in the ancient worlds<sup>2</sup>. Finally, the developments of the fledgling field of ethnomathematics have enlightened the diversity of practices of calculation and measurement (Vellard 1988; Trouré and Bednarz 2010, Owens, Lean, Paraide and Muke 2018; Vandendriessche 2016, 2018).

These lines of research have raised historical and historiographical issues on which, in this volume, we would like to bring a particular light. To what extent has the history of numbers and calculation, on the one hand, and the history of metrology and measurement on the other, have been approached separately by historians and philosophers of mathematics?<sup>3</sup> How have ethnographic and anthropological studies of the 19th and 20th centuries impacted analytical works on numbers, up to now? We have chosen to address these questions by focusing on the opposition between “concrete numbers” and “abstract numbers”, whatever this opposition may have been conceptualized by ancient authors themselves or theorized by modern commentators.

The present special issue draws on several disciplinary perspectives in order to critically analyze the historiographical opposition between “concrete numbers” and “abstract numbers”. Some historians, philosophers, anthropologists or researchers in education have theorized a separation between “numbers” and the entities enumerated or counted with these numbers, and more particularly, between numbers and measurement units attached to them in the expression of measurement values. To what extent does this separation reflect on the practices carried out in societies or social groups under scrutiny by these scholars? How the notion of “abstract numbers”—as opposed to those described as “concrete numbers”—has shaped the history of numerations?

The chapters in this special issue confront common historiography with the great diversity of numeration and measurement systems attested to by the various textual and ethnographic sources available to us. This volume—chapters 3 and 4 in particular—shows how a conceptualization of numbers consisting in isolating/extracting numbers from their context of

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<sup>1</sup> The *Measurement at the Crossroads Conference* was part of the events organized around the redefinition of the International System of Measurement (De Courtenay, Gregis, Lacki, and Proust, 2021). It was the third of a conference series devoted to the philosophy of measurement. The previous events were the *Dimensions of Measurement Conference*, Bielefeld 2013, and *The Making of Measurement Conference*, Cambridge 2015.

<sup>2</sup> The “Science in Ancient World” European Project (ERC-Advanced Grant, 2011-2016), led by Pr. Karine Chemla, was largely devoted to this issue (Michel and Chemla, 2020; Chemla, Keller, and Proust, 2022). See also (Bernard, Chambon, and Ehrhardt, 2010).

<sup>3</sup> Chemla, 2022

practice, gave rise to a linear history of oral and written numerations rooted in evolutionary theories and classifications.

The chapters devoted to modern arithmetic textbooks (6, 7, 8, 9), beyond the notion of “concrete numbers”, address the issue of the treatment, in mathematical practices, of the quantities (in the sense of “magnitude values” according to chapter 6’s terminology). How was the problem of multiplying quantities of different dimensions (for example a price by a length) tackled in modern arithmetic textbooks? As rightly pointed out by Ferreira and Schubring (chapter 7), “It is precisely this blind spot that [their] article, and the [present] special issue as a whole, seeks to highlight”.

The initial international call for papers for this special issue on “concrete” versus “abstract” numbers was not limited to any period or region of the world. Nevertheless, we (the Editors) received proposals essentially related to three types of historical or historiographical documentation: first, ethnographic observations on numerations; second, historical sources on numbers from the Ancient Near East; and third, elementary arithmetic treatises and textbooks produced between the end of the 18<sup>th</sup> century and the beginning of the 20<sup>th</sup> century.

What would such focalization mean? Although it’s difficult to know, one may observe that the chapters devoted to either the first or second types of documentation focus mainly on historiographical matters, i.e. the way in which some scholars have analyzed the use of numbers and measurement units in ancient or ethnographic sources (chapters 2, 3, 4). In these cases, the opposition between “abstract” and “concrete” numbers reflect the observers’ categorizations. Indeed, the ethnographic sources, on the one hand, and the archaeological and epigraphic sources, on the other hand, relating to the birth of writing in various regions of the world, have stimulated many observers’ considerations on the alleged emergence of “abstract” numbers, and more generally on the concept of abstraction. On the basis of such considerations, value judgments have been made about the ancient or native actors’ capacity for being able (or not) to deal with abstract thought. This paved the way for an evolutionary conception of the history of mathematics, leading to the historical pattern of a unilinear development of numbers from the concrete to the abstract (chapters 2, 3, 4). By contrast, the chapters on modern arithmetic treatises and textbooks (chapter 6, 7, 8, 9) analyze an opposition between “abstract” and “concrete” numbers, or similar dichotomies, that the authors of these arithmetic books explicitly theorize themselves.

French mathematician Etienne Bézout’s (1730–1783) definition of “concrete” and “abstract” numbers is certainly the most prevalent and still in use today. This definition is therefore worth considering, but it should be kept in mind that other approaches may have deviated from it (see for example Thureau-Dangin’s use of the notion of “abstract numbers” in chapter 4). Bézout defines “abstract” and “concrete” numbers as follows:

A number that is expressed without mentioning the kind of units, as when one simply says *three* or *three times*, *four* or *four times*, is called an “abstract number”; and when the kind of units is concurrently specified, as when one says *four pounds*, *one hundred*

*barrels*, it is called a “concrete number” (Bézout, [1764] 1775, 2–3, Section 6, our translation).<sup>4</sup>

Similar but slightly different categorizations have been introduced before and after him. For instance, in his textbooks on elementary arithmetic, Polish mathematician Stefan Banach (1892–1945) relies on a comparable dichotomy that he introduces as an opposition between “denominate numbers” and “non-denominate numbers” (chapter 9). Although absent from this special issue, similar forms of opposition between different classes of numbers can be found in earlier arithmetic treatises and early modern documents.

In histories of arithmetic from the 19<sup>th</sup> century onward, the opposition between “abstract” and “concrete” numbers has played a more or less important role, depending on the author. This opposition definitely structures the *Arithmetic* published by Cambridge algebraist Rev. George Peacock (1791–1858) in 1845. Indeed, Peacock divides the historical section (entitled “History of the science”) of the paper into *Part I Arithmetic of abstract numbers* and *Part II Concrete numbers*. Furthermore, as shown by the historian of mathematics Maris-José Durand Richard (2011), Peacock’s method is groundbreaking in that he develops a genuine algebraic approach to measurement units.<sup>5</sup>

In contrast, this opposition is completely absent from *A History of Mathematics* (1893) by Florian Cajori (1859–1930), which only deals with so-called pure numbers (that is, “abstract” numbers according to Bézout’s definition) and ignores arithmetic with measurement units.

American mathematician David Eugene Smith (1860–1944) for his part does mention the opposition between concrete and abstract numbers in his book *A History of Mathematics* (1925). However, he considers it as uninteresting, and actually not useful for mathematicians. According to him the latter categorization of numbers could be of interest only to elementary mathematics teachers:

## 5. ELEMENTARY CLASSIFICATIONS OF NUMBERS

Abstract and Concrete. The distinction between abstract and concrete numbers is modern. The Greek arithmeticians were concerned only with the former, while writers on logistic naturally paid no attention to such fine distinctions. It was not until the two streams of ancient numbers joined to form our modern elementary arithmetic that it was thought worthwhile to make this classification, and then only in elementary school.

The terms “abstract” and “concrete” were slow in establishing themselves. The mathematicians did not need them, and the elementary school teachers did not have enough authority to standardize them. In the 16<sup>th</sup> century, textbook writers began to make the distinction between pure numbers and numbers to which some denomination was attached, and so we find Trenchant (1566), for example, speaking of absolute and denominate numbers, the latter including not only 3 feet but also 3 fourths. (note 1: quotation excerpted from a 1578 edition).

“From that time on the distinction is found with increasing frequency in elementary works. Such refinements, however, as required the product to be of the same denomination as the multiplicand are, in general, 19<sup>th</sup> century creations of the schools” (Smith 1925, 12).

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<sup>4</sup> French original text: « Un nombre qu'on énonce sans désigner l'espèce des unités, comme quand on dit simplement *trois* ou *trois fois*, *quatre* ou *quatre fois*, s'appelle un *nombre abstrait* ; et lorsqu'on énonce en même temps l'espèce des unités, comme quand on dit *quatre livres*, *cent tonneaux*, on l'appelle *nombre concret*. »

<sup>5</sup> Regarding Peacock’s contribution see also chapter 2 of the present volume.

The importance afforded to the opposition between “abstract” and “concrete” numbers by these different authors is correlated to the way they consider (or do not consider) measurement units as worthy of interest from a mathematical point of view. The controversial role of this opposition is further discussed in chapter 6.

Beyond the opposition between “concrete” and “abstract” numbers, what is the meaning of the concepts of “abstraction” and “concreteness” often put forward in the context of studies devoted to numbers and numerations? What is striking, through the chapters offered in the present collection of papers, is that anthropologists, historians and philosophers of mathematics, as well as mathematicians—at least from the 18<sup>th</sup> century onward—use differently the concept of “abstraction”—as well as “concreteness”—when these concepts are applied to the elaboration of numbers. This point is first brought to light in the present special issue by studying how some 19<sup>th</sup> until mid-20<sup>th</sup> century mathematicians and philosophers, like Levi Leonard Conant (1857–1916) and Lucien Lévy-Bruhl (1857–1939) in particular, analyzed the numerical systems in use in the so-called primitive societies, documented by 19<sup>th</sup>-century ethnographers. These scholars often claimed that these systems would have a more “concrete”—or less “abstract”—character than those developed in more “advanced” societies, due to their close relation to sensible reality (digital counting, use of classifiers, etc.) as well as their mystical aspects specifically (chapters 2, 3). Regarding 19<sup>th</sup> and 20<sup>th</sup> century historical studies, the “abstract” character of numbers is also put forward in various notions. One of these is Bézout’s (chapters 6, 7), but a slightly different conception is used by Thureau-Dangin (chapter 4).

The present special issue consists of a collection of eight chapters divided into three parts.

### **Part I *What is concrete in “concrete numbers”?***

This part examines different meanings given to “abstraction” and “concreteness” in the historiography of numbers. How the notion of abstraction, as applied to numbers, has been understood and used in historiography? This historiographical approach is contrasted with evidence from ancient and ethnographic sources.

**Eric Vandendriessche** (chapter 2) opens the present volume with a chapter aiming at highlighting the scholarly discussions that occurred from the 19<sup>th</sup> century until mid-20<sup>th</sup> century on the presumed concreteness of numerations elaborated in the so-called primitive-inferior-barbarous societies. These discussions took place within different disciplinary fields (history of mathematics, philosophy of science, ethnology/anthropology) on the basis of ethnographical sources collected either by missionaries, explorers, or ethnologists, within these societies. Adopting a historiographical approach, and by focusing more particularly on the contributions of some influential 19<sup>th</sup> and early 20<sup>th</sup>-century scholars—George Peacock (1845), Levi Leonard Conant (1896), Lucien Lévy-Bruhl (1910), Léon Brunschvicq (1912)—this chapter highlights how these new sources on (non-written) numbers/numerations have been an opportunity for these scholars to reflect on the supposed abstraction process related to the elaboration of numbers. According to these authors, this process, which would consist in detaching/distancing numbers from what is counted, would have rarely occurred, in their views, with the notable exception of Peacock’s more nuanced approach, among “primitive” peoples.

This chapter brings us to the context of colonialism, and the predominant late 19<sup>th</sup> century anthropological evolutionist theories, as well as philosophical theories on the “primitive” mentalities (steeped in a profound racism). Indeed, these studies of numerations, as elaborated by “primitive” peoples, have been carried out as part of a more general discussion on the characteristics of the “primitive” mind, and the ability of these people to use abstract and rational concepts. Thus, tracing a red line between humans whose minds are governed by abstraction/rationality and the others. In spite of the criticisms to which they have been subjected, these evolutionary theories are still prevalent today, as the following chapters show.

**Miguel Valério and Silvia Ferrara**’s chapter 3 offers a critical analysis of the historiography of numeracy, in Mesopotamia above all, but also in Egypt, the Aegean world, China and Mesoamerica. They criticize anthropological literature and examples collected by authors such as Conant, who consider that numbers are generally context-dependent. They object to the linear and evolutionist conception of the history of numbers adopted by most historians. In particular, they refute the idea that the history of numbers is that of an evolution from concreteness to abstractness. Rejecting the evolutionist scheme, they differ from Denise Schmandt-Besserat’s thesis. Indeed, according to Schmandt-Besserat, the history of numbers in the Ancient Near East goes through three stages: (1) One-to-one correspondence (2) Concrete counting (3) Abstract counting. Schmandt-Besserat opposes proto-literate (ca. before 3500 BCE) evidence, essentially tokens and signs marked on the external surfaces of bullae or flat pieces of clay, from later written sources on tablets. Tokens and the like would fuse the notions of quality and quantity, while 3<sup>rd</sup> millennium written sources would exhibit “abstract” numbers, that is, represented by signs separated from the items quantified, and which has the same form independently of the items quantified. Valério and Ferrara insist that the function of tokens is much more obscure than what is claimed by Schmandt-Besserat because we have very little—or no—information on their archaeological context. Against Schmandt-Besserat, they argue that, “all numbers have an inherent relation to abstraction, even when things are materially counted”. As an alternative to the linear evolution from concreteness to abstractness, they claim that the crucial shift in the history of numbers—at least in Mesopotamia—was the introduction of power-based notations. The history of numeracy in Mesopotamia is envisaged in the broader context of the history of writing, which includes two paths, numeracy and literacy as separate (albeit interconnected) phenomena. Valério and Ferrara conclude by making a connection between the three aspects they analyze in their chapter: 1) the specific history of numeracy, different from that of literacy, 2) the criticism of the dichotomy “abstract” *versus* “concrete”, and 3) the appearance of power-based systems of numbers.

**Christine Proust** (chapter 4) shows that the “abstract” character of numbers was understood in quite different ways by historians. In Schmandt-Besserat’s scheme, described in chapter 3, the abstract numbers are those which have an autonomous identity, and are represented by signs distinct from those which represent quantified items. In Damerow and Englund’s scheme, numbers written in sexagesimal place-value notation are abstract in the sense that they represent a progress in the path toward mathematical abstraction (Nissen, Damerow and Englund 1993). For Thureau-Dangin, numbers written in sexagesimal place-value notation are abstract because of their floating character. Unlike Damerow and Englund, he draws a clear link between the numeration systems and the nature of operations performed with the

numbers. Interestingly enough, Thureau-Dangin was the first to introduce the notion of “abstract numbers” in relation to the use of a floating sexagesimal place-value notation, but abandoned this notion later, when he abandoned the idea that these were floating numbers. In his publications, the expression “abstract numbers” disappears together with the introduction of the orders of magnitudes of numbers in translations. This confirms that, for Thureau-Dangin, the “abstract” character of numbers was linked, in his earliest publications, to their “floating” nature, and not to the presence of measurement units after the numbers. Turning to a close examination of cuneiform sources, Proust shows that a diversity of numerical systems coexisted in the same context, or texts, and that these different numerical systems do not reflect different stages of an evolution, but different functions in the same context. Similar observations emanate from documents produced in other parts of the world, for example in India and China (Chemla, Keller, and Proust 2022).

## **Part II Working on numbers with and without measurement units**

This part analyzes the meaning of the presence of measurement units in ancient administrative texts or modern teaching practices. How is the mathematical work on measurement units an integral—and sometimes essential—part of the mathematical elaborations? How does the inclusion of measurement units shape our understanding of numerical systems in past or present treaties and textbooks? How could focusing on measurement units open up new prospects for discussion on mathematical practices and teaching?

The agrarian yield rates, or “*šukunnûm*” numbers, analyzed in chapter 5 by **Baptiste Fiette** provide excellent material for a discussion of the notion of “abstraction” conveyed by the expression “abstract number”. Numbers representing agrarian yield rates appear, under the name of “*šukunnûm*”, in a corpus of administrative texts from the Larsa region (South Mesopotamia) and dated to the reign of King Hammurabi (ca. 1792-1750 BCE). The particularity of these numbers is that, contrary to the general rule in administrative texts, they are not accompanied by measurement units. What is the meaning of the lack of measurement units after yield rates in the texts considered? In which sense are the *šukunnûm* numbers “abstract” in this context? Fiette elaborates a complex and nuanced answer to this question. He analyzes in detail the texts containing yield rates, examining their function and use as well as the epigraphy of these numbers. The texts studied (yield tables and harvest registers) were designed by a provincial governor (*Šamaš-hazir*) to assess the amount of taxes on barley production owed to King Hammurabi. These texts contain data on surfaces, yields and corresponding quantities of grain in tabular format. Other texts such as letters between the governor and the king, a list of crops, and a land sale contract, allow Fiette to specify the function and use of these tables and registers. Evaluating yields requires an estimate of the surfaces and production of these fields. Fiette details the methods used by the governor’s surveyors to measure field dimensions, in order to calculate their surfaces. Interestingly, he shows that the adopted metrology of surfaces differs according to the nature and size of the field: the evaluation of large domains providing enormous amounts of barley to the Royal Palace is based on the unit *bur* (ca. 6.5 ha) and expressed roughly, while that of small individual fields are based on the unit *sar* (ca. 36 m<sup>2</sup>) and expressed with high precision values. He suggests that these two approaches reflect different conceptions of surfaces. These conceptions are connected with different uses of estimated yields: to evaluate the taxes to be



paid to the king, or “to calculate the agricultural productivity of a team of harvesters in a small area”. Various pieces of evidence, such as the content of letters and the dates of the forecast tables, show that the yields have been estimated by “the examination of the first visible shoots of barley [...] in early spring”. For this reason, the translation of the “*šukunnum*” number adopted by Fiette is “estimated yield”. As we can see, the estimated yields present two facets: they are shaped by incorporating empirical observations of shoots of barley and a rough estimation of the yield (by dividing the surface sample by the estimated production in early spring), and they are used for calculation in advance of the final production (multiply the actual surface by the estimated yield). In conclusion, these tables and registers, in which the yield rates appear, are intended to estimate the expected harvest, and thus to determine the quantities of barley owed to the king. Fiette shows how this context explains the particular form that yield rates take. The author concludes that estimated yields are “abstract” in the sense that they are predictive by nature, and not so abstract in the sense that they result from empirical estimations.

The last contribution (chapter 6) of this second part examines the mathematics education field, highlighting the use—or not—of magnitudes and measurement units in the teaching of arithmetic in French elementary schools, from 1845 until the 2000s, focusing more specifically on the teaching of proportionality. Two major previously introduced sets of conceptual/methodological tools (serving as reading grids) allow the authors, **Christine Chambris** and **Jana Visnovska**, to analyze in great detail the various and successive syllabi/curricula that have structured elementary mathematics teaching during this period. The first one is known as the concept of “praxeologies” introduced by a didactician of mathematics Yves Chevallard (1992), in an attempt to study jointly practices (*praxis*/know-how) and the discourse (*logos*) about them. The second set is borrowed from mathematician Hans Freudenthal (1983), who intended to define the different concepts of “quantity”, “magnitude”, “magnitude value”, and “measuring number”, while introducing the extremely useful distinction between two types of ratios defined as “internal” (*i.e.* ratio between values of the same magnitude) and “external” (*i.e.* ratio between values of different magnitudes).

With this conceptual and methodological framework in mind, Chambris and Visnovska question the “roles of magnitudes and units in the French elementary syllabi” from the mid-19th century onward, and “how these changes were orchestrated”, as well as their impact on the “teaching and learning of proportionality”. The article brings forward a “long period of relative stability” regarding the latter issue—from the mid-19th century until the 1940s—where the notion of concrete *versus* abstract numbers was conceptualized in the syllabi as well as in textbooks, while considering magnitudes as a central concept in arithmetic. Nevertheless, it is shown that the teaching of proportionality (and the “rule of three” in particular) has entailed pedagogical debates and disagreements, introducing different approaches, from one period to another, involving the use of either “internal ratios”, rates as “external ratios of magnitudes”, and rates with “quotient units”. The latter approach has thereafter been the most commonly used, until the 1970s, and the setup of the French New Math Reform (1955–1975). Consequently, in addition to the disappearance of the notion of “concrete numbers”, the introduction of “linear functions” became the unique conceptual tool for teaching proportionality. As the authors demonstrate, French mathematics education will take a big step backwards, from 2000 onward, reintroducing measurement units, either in formal, or informal calculations. Indeed, all of these pedagogical changes regarding the

teaching of basic arithmetic in elementary schools, suggests that the issue of dealing with quantities has not been tackled in a satisfactory way in mathematics in general, in contrast with what has been achieved through theories of measurement in physics. The issue of multiplication and division of quantities (in the sense of “magnitude values”) is again addressed in depth in Part III.

### **Part III *Complex and denominate numbers***

As an extension of Chambris and Visnovska’s analysis, this part delves deeper into the issue of operating with “concrete” numbers. The fundamental question that appears behind the opposition between abstract and concrete numbers, or denominate and non-denominate, or other similar dichotomies, is the distinction between number and quantity of something (or “magnitude values”). Indeed, this distinction plays an essential role in the implementation of multiplications and divisions performed on these entities. In addition, the treatises and textbooks examined in this part highlight the impact of the French Revolution and the Reform of weights and measures on the conceptualizations of numbers. More precisely, it shows how the new decimal system has led to redefining, marginalizing or questioning the notion of “concrete” numbers first in France, then in Europe and Brazil.

**Débora Ferreira and Gert Schubring** (chapter 7) focus on a specific sort of concrete number, the “complex numbers” as defined by Camus (1749), Bézout and other French authors quoted in chapter 8. According to Bézout, among the concrete numbers, some are “complex” (*i.e.* composed of several juxtaposed elements, such as “2 hours and 20 minutes”), others are “uncomplex” (*i.e.* composed of a single element, such as “2 hours”). The importance of this distinction for Bézout appears through his reflections on the commutativity of the multiplication of concrete numbers. He shows that the multiplication of a complex number by a concrete number does not provide the same results as the inverse multiplication. Ferreira and Schubring critically discuss Bézout’s approach of the commutativity of the multiplication and detect a “trick” in Bézout’s reasoning. In order to solve the problems posed by Bézout, they open the broader issue of the multiplication of quantities, hence the dimension of the product. Ferreira and Schubring thus analyze in depth the links between numbers, quantities and operations in modern arithmetic textbooks.

Bézout exhibits an example showing that the product of two concrete numbers, one representing a length and the other a price, is different if the factors are switched. How does Bézout explain this phenomenon? His explanation relies mainly on the fact that for him, the product must be of the same nature as the multiplicand. Ferreira and Schubring highlight that Bézout’s argument is directly related to his definition of multiplication. Indeed, multiplication is defined “canonically” as an iterated addition, in Euclid’s style. In this dissymmetric definition, one of the factors acts as a multiplicand (the quantity to be iterated), the other as a multiplier (the number of times the quantity is added to itself), which implies that the dimension of the product is that of the multiplicand. However, this definition concerns only the product of two numbers (integers, fractions and even irrationals by extension), and the product of a quantity of something by a number. It does not work for the product of two quantities of something, which raises the problem of the result’s dimension.

Yet, according to Ferreira and Schubring, in his examples, Bézout multiplies two quantities of different dimensions (a length with a price). When he claims that the product is of the same

nature as the multiplicand, he does not question his definition of the multiplication and the dimension of the product. In his *Géométrie* (1770, p. 89), Bézout refuses to multiply a line by a line, but, according to Ferreira and Schubring, he tries to multiply geometrical quantities by a method which “revives the notion of indivisibility”. The question of the multiplication of geometric quantities had been addressed before Bézout, notably by Camus, who adopts another point of view. For Camus, the units of the product are products of the units: *toise-toise*, *toise-foot*, etc. “Camus was well aware that the dimension of the product has to be different from those of the factors”. But, unlike Bézout, Camus does not consider the problem of commutativity. The authors of arithmetic textbooks after Bézout, in France as well as in other regions of Europa and in Brazil, point out the fact that, in the multiplication of complex numbers, one cannot switch the factors. Two reasons are generally given to explain this non-commutativity: the first and most frequent one is, according to Bézout, that the dimension (“specie”) of the product is that of the multiplicand. The second one (minor except in a few cases as in Ottoni’s arithmetic textbooks where it occupies the first place) is that, if the metrologies of the two factors are different, the rules of calculation are different. Only in the 20<sup>th</sup> century, some physicists and mathematicians, notably Hans Freudenthal (1905–1990), did “solve the riddle” by considering quantities as “tensors”, thus regarding quantities as “proper domains of mathematical entities”.

**Xiaofei Wang**’s chapter 8 concerns the context of the establishment of the decimal system reform initiated during the French Revolution (1789–1799) and onward during the First Empire (1804–1814), by mathematicians/astronomers of the time, at the *École Normale de l’an III* in particular, which thereafter became the *École Polytechnique*. The chapter begins by highlighting the fundamental role played by mathematician Joseph-Louis Lagrange (1736–1813), at the turn of the 19th century, in the promotion and dissemination of the decimal arithmetic that allows—in Lagrange’s view—“regularity and uniformity to the operations of arithmetic.” As Wang brings to light, Lagrange conceived of Arithmetic as the “science of calculation” (in contrast to a “science of numbers”). This approach led him to reflect on and compare different calculation methods in the history of mathematics—methods obviously linked to the different systems of numeration. Consequently, he notably contributed to a renewed interest in the arithmetic of the Ancient Greeks.

Another important scholar of the time, the astronomer and mathematician Jean-Baptiste Delambre (1749–1822), shared with Lagrange the latter’s interest in ancient arithmetic. As shown by Wang, his contribution on the topic is of fundamental importance, affording an original approach (and reconstruction) to calculation as it was presumably carried out by the Ancient Greeks. To achieve this, the chapter takes a detour by the description of the so-called complex numbers, defined as “concrete numbers composed of several parts related respectively to different [measurement] units”. Commonly used before the French Revolution (see chapter 7), these complex numbers (and the sophisticated calculation methods that their use entails) have coexisted with the weights and measurements decimal system, particularly in textbooks, over several decades. In a very original way, Delambre used the calculation on complex numbers (*cf.* Bezout), as a modeling tool to explore the arithmetic of the Ancient Greeks. He argued that Greek numbers can be rewritten/translated into numbers in a form similar to “complex numbers”, based on regular (and decimal) subdivisions of numbers. According to him, such rewriting of Greek numbers allows us to hypothesize on the arithmetical methods carried out in Ancient Greece.

The last chapter of this volume offered by historian of mathematics **Karolina Karpińska**, highlights a little-known part of the famous Polish mathematician Stefan Banach's (1892–1945) professional life. In the interwar period, Banach—facing some financial difficulties—wrote several textbooks (as author or co-author) for secondary schools. This occurred in the context of the reform of the curriculum that followed Poland's independence (from Prussia, Austria, and Russia, in 1918). Some of these publications are devoted to the teaching of arithmetic, and became some of the most popular mathematics textbooks used in Polish schools during this period. Karpińska analyzes the latter publications in comparison to other textbooks published at the same period, as well as to the textbooks that Banach used as a student of the “4th Gymnasium” in Kraków. This affords us a better understanding of Banach's influence, as well as his original contributions to the field of elementary mathematics education. In addition to the introduction of the different kinds of numbers (integer, fractional, relative, rational, irrational), Banach presents the concept of *liczby mianowan*—literally “denominate numbers”—aiming at expressing measures of lengths, weights, volumes, and other “magnitude values” to use chapter 6's terminology. He considers these denominate numbers as singular numerical expressions, elaborated from numbers (sometimes also defined as *liczba niemianowana*—literally “non-denominate numbers”). In contrast to other Polish authors of the time, Banach's attempt is to make his readers perceive the very nature of these numerical expressions through the operations that can be performed on (simple or compound) denominate numbers. As shown by Karpińska, the latter numerical expressions are indeed of fundamental importance to Banach's approach of numeracy, in its links to other disciplines like physics, geography, among others.

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