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Is Descartes's Geometry an extension of Euclid's?

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1. Introduction

Descartes's *Géométrie* ([4]), as well as his whole mathematical production, manifest little attention for theorems, and mainly focus on solving problems. This has brought many commentators to describe Descartes's geometry as mainly concerned with problem-solving. Henk Bos has gone so far as to infer from this that Descartes's "primary aim" in doing geometry "was to provide a general method for geometrical problem solving" ([2], p. 228).

The description is irrefutable. But is it the same for the consequence that Bos (but not only him) has drawn from it? I resist to assent. I argue, instead, that there is a plausible reading of Descartes's *Géométrie* according to which its main aim is rather that of expounding a genuine mathematical theory, in which the crucial importance of problems and their solution, even their absolute centrality, do not go together with a primary attention for problem-solving *per se*, that is, with the idea that the ultimate task of geometry is to provide a(n appropriate) solution for as many problems as possible. In other terms, I do not believe that Descartes's attention for solution of problems prefigure a conception of geometry as a collection of methods for solving problems, rather than as a genuine mathematical theory.

According to this reading, Descartes’s “primary aim” was rather to extend the geometrical theory expounded in the *Elements*, namely in its first six books (though the fifth concerns, in fact, a tool—the so called theory of proportions—that, despite of being applied in the sixth to the objects studied in the first four ones, also admits a more general application), which I will call, for short, ‘Euclid’s Plane Geometry’. (As a matter of fact, many of the things I will say here about Euclid’s Plane Geometry also apply to solid one. Still, many others either only pertain to the former, or apply to the latter under condition of appropriate adjustments that I would not have room here to detail; also considering that, in the *Géométrie*, Descartes only deals with plane geometry, I prefer, then, to focus on the former.) The mathematical theory expounded in the *Géométrie* is just that which results from such an extension. Call it ‘Descartes’s Geometry’. Promoting it went together with advocating a wide-ranging reform in geometry, and suggesting a fresh foundation of it, both faithful to the spirit of, and deeply innovative with respect to, Euclid’s Plane Geometry: a reform whose import was much larger than making a large class of problems solvable with licensed means.

My present purpose is of shortly accounting for this reform. The main claims I will advance here are better argued for in my [11] and [10]. Here, I’ll come back to them in a less detailed, but, I hope, more incisive and perspicuous way. I also refer the reader to these papers for a number of cases-studies illustrating with appropriate examples the circumstances, distinctions, and options I will mention below.

2. The role of problems in Euclid’s Plane Geometry

2.1. The ontology of Euclid’s Plane Geometry

In Euclid’s Plane Geometry, problems play a crucial and fundamental role: Solving them results in providing this Geometry with its own ontology. Insofar as extending Euclid’s Plane Geometry could but have been a matter of extending its ontology, anybody who had aspired to

extend it, though remaining faithful to its spirit, should have necessarily looked for new ways for solving problems resulting in the introduction of new (sorts of) geometrical objects. This was just what Descartes did, in my view.

To explain this, I begin with making the following claim: Euclid's Plane Geometry is not about a fixed domain of objects, consistently defined, and supposed or proved to exist once for all; it is rather about some sorts of transient objects available through a licensed construction (where 'available' applies to 'sorts', not to 'objects').

To appreciate the difference, remark that 'to be available' is here intended as a modal verb. By saying that a sort of objects is available through a licensed construction—I will later explain what makes this different from saying both that an object, or some objects are so, and that (all) the objects, or any object of each of the relevant sorts are so—I do not mean that this construction makes some objects of this sort—or, *a fortiori*, (all) the objects, or any object of this sort—be brought to existence, or come somehow to be there. I rather mean that objects of this sort are susceptible of being got through such a construction, can result from it. Hence, the first part of the claim is that Euclid's Plane Geometry is about objects, classified as being of different sorts, that could be got through a licensed construction.

Consider equilateral triangles as an example. This part of my claim is that Euclid's Plane Geometry is, among others, about equilateral triangles, insofar as equilateral triangles are susceptible of being got through a licensed construction, that is, are available through a licensed construction, which I take to be essentially different from claiming that (all) the equilateral triangles, or any equilateral triangles are so. As I have just said, I will come back later on this difference. For the time being, it is enough to notice that claiming that equilateral triangles are susceptible of being got through a licensed construction, is another way for claiming that a construction-type is licensed, each instance of which results in an equilateral triangle: a claim whose understanding is perfectly independent of the understanding of what one could mean by speaking of (all) the equilateral triangles, namely of the totality of them.

The second part of my claim pertains to the adjective 'transient'.

What I mean by it is that, once constructed, an object that Euclid's Plane Geometry is about is not kept in stock: it is neither ascribed to a cumulative totality, so as to make it possible, in future, to refer both to it or to the totality itself, nor it is stored for further uses. Keep attention: here I'm speaking of an object, not of a sort of objects. What I just said concerns single objects of a certain sort, not sorts of objects. It concerns, for instance, neither triangles or squares as such, nor triangles or squares so and so, but rather single triangles or squares. What a licensed construction allows to get are single objects of a given sort: a single triangle, or a single square, for example. And what I take to be transient are just these single objects. More than that: typically, once some assumptions are made, relative to the objects which are taken as given at the beginning of the construction, these single objects are determined as such. They are neither arbitrary or generic triangles or squares, nor arbitrary or generic triangles or squares so and so. They are those determined triangles or squares. For example, what the construction expounded in Euclid's solution of *Elements* I.1 allows to get is just one of the two determined equilateral triangles having the given segment as one of their sides (or possibly both).

But then, why did I say that Euclid's Plane Geometry is about some sorts of transient objects? Here comes the third part of my claim. Though a licensed construction cannot but result in single transient objects, what is relevant for Euclid's Plane Geometry is generality. And generality cannot hinge on the construction of single transient objects. It rather hinges on the possibility of permanently constructing single (transient) objects of a certain sort in a licensed way, that is, not on some single objects of that sort (having) being transiently constructed, but on the permanent availability of that sort of objects.

This means that fixing the ontology of Euclid's Plane Geometry depends on answering the following question: Which sorts of objects are available through a licensed construction? That is, which sorts of objects can be got though such a construction? Solving problems just allows answering this question.

This is not the only, general, crucial question that doing Euclid's Plane Geometry hinges on answering to. Another is the following: How objects of a certain sort available through a licensed construction

are, precisely? That is, how these objects actually are, besides being objects of the relevant sort? For example: how are equilateral triangles, besides being equilateral triangles? How are their internal angles, besides being internal angles of equilateral triangles?

What allows answering this second question is proving theorems. This is as much as important as answering the first question, or possibly even more, since it is only insofar as this second question is answered that Euclid's Plane Geometry is able to tell something of the objects it is about. Still, this second question cannot be answered within Euclid's Plane Geometry, and, in a sense, can neither be soundly stated, there, if the first is not answered. This makes the first question more fundamental, in a sense. For example, it is only insofar as the sort of equilateral triangles is available through a licensed construction that it is possible to answer, and, in a sense, soundly ask, within Euclid's Plane Geometry any question concerning their internal angles, and then answering by proving, for example, that these angles are all equal to each other.

In spite of the importance of the second question, for my present purpose it is more important to focus on the first one, and on the way it is answered. Still, it is also important to remark that answering neither the first, nor the second question requires the single objects got through a licensed construction be permanent, that is, kept in stock, once they have been constructed. It does not require these objects be ascribed to a cumulative totality, so as to make it possible, in future, to refer both to them, or to the totality itself, since, in Euclid's Plane Geometry, generality does concern no totality of objects, either fixed or cumulative, but rather concerns availability, that is, susceptibility of being got through a licensed construction. It does not require these objects to be stored for further uses, since, if a certain sort of objects is available through a licensed construction, an object of this sort can always be constructed in a licensed way any time this is required.

It follows that, for the solution of a problem to be able to answer the question it is required to answer—and to make then possible to soundly state other questions that proofs of theorems can answer to—, there is no need to ensure any kind of permanent existence to any object. All what solving a problem is required to make is showing that

the relevant sort of objects is available through a licensed construction. And it does it by actually constructing a single object of this sort through such a construction.

This makes a question arise: how can constructing a single object of a certain sort through a licensed construction provide a proof that this sort of objects is available through such a construction?

What is important to notice, here, is that this question is not concerned with (all) the objects, any object, or the totality of objects of this sort. What is asked here is not how can constructing a single object of a certain sort through a licensed construction provide a proof that (all) the objects, any object, or the totality of objects of this sort are available through such a construction.

This latter question would be sensible only if it were possible to give a clear sense to the notion of a totality of objects of the relevant sort. But this is just what is not possible to do within Euclid's Plane Geometry, for the appropriate identity conditions that would make this possible are lacking.

Imagine you are teaching Euclid's Plane Geometry, and you consider an equilateral triangle, represent it by an appropriate diagram on a blackboard, and prove that all its internal angles are equal. Imagine you, then, erase the blackboard, go to another room, face to other students, and make the same: consider an equilateral triangle, represent it by an appropriate diagram on a blackboard, and prove that all its internal angles are equal. Have you dealt twice with the same equilateral triangle, or have you dealt with two distinct triangles? The question is hardly sensible. And being able to answer it is certainly not required for making and teaching Euclid's Plane Geometry. But if there is no way to fix the conditions according to which a question like this can be answered in a way or another, within Euclid's Plane Geometry, there is no more anything as a totality of equilateral triangles that Euclid's Plane Geometry is about.

This having being said, let us come back to the good question: how can constructing a single object of a certain sort through a licensed construction provide a proof that this sort of objects is available through such a construction? This is a question about the universality of proofs in Euclid's Plane Geometry. More in general, the question is

this: how can an argument concerned with a single object, or a single configuration of objects, have an universal import? The question is quite difficult, and many answers have been suggested to it (though it has been rarely stated in the appropriate way, in my view). This is not the place for advancing a new answer, or for arguing in favor of one or another of those that have been already offered. All what is important is to notice that the universality of Euclid's Plane Geometry just depends on the possibility of answering to it in a way or another. Here, I merely admit that an appropriate answer is possible, which is the same as recognizing that Euclid's Plane Geometry is endowed with the appropriate sort of universality, though being about some sorts of transient objects, as I have said.

The point that I mainly want to focus on is another. It concerns the notion of a licensed construction, and the role that the identification of the licensed constructions has within Euclid's Plane Geometry. On this matter, my claim is that (appropriately) fixing these constructions complies with a double purpose within this geometry. On one side, it ensures that any such construction is repeatable, with the result that constructing a single object according to such a construction can be taken as a proof that the same can be done in other occasions, which certainly contributes (though in a way that it is not my intention to investigate, here) to make Euclid's Plane Geometry universal. On the other side, it contributes to make Euclid's Plane Geometry a closed theory, that is, a deductive system with sharp limits. In particular, it makes its ontology have sharp limits, though it is not provided by any given totality of objects. This is because Euclid's Plane Geometry is about a certain sort of objects if and only if this sort of objects is available through a construction that is licensed within it. If this is so, we could, then, also say, that this sort of objects is available within this geometry.

All this having been said, we can now simplify our language: provided that a certain sort of objects S is available through a construction licensed within Euclid's Plane Geometry, and, then, within this geometry itself, and t_S 's are objects of this sort, I will say that t_S 's are in turn available within Euclid's Plane Geometry. Insofar as it does make no sense to speak of (all) the objects, or any object of S (and I will not

speak of them, indeed), this does not entail that (all) the t_S 's are, or any t_S is available within Euclid's Plane Geometry. For example, whereas I will feel free to say that equilateral triangles are available there, I will neither say nor imply that this is for (all) the equilateral triangles or any one of them.

2.2. *Geometrical Exactness*

There should be no need to say that the licensed constructions in Euclid's Plane Geometry are those by rule and compass. What is needed, instead, is to remember that Euclid's never described these constructions this way. This description rather pertains to the way Descartes thought at them, in the framework of his reform. To avoid any risk of explanatory circularity, for the time being, let us call them 'elementary constructions'. They are fixed by a number of clauses, some of which are explicit—namely those stated by postulates I.1-3—, while others are tacit, or only implicitly advanced. This is, for example, the case of the clause that allows a construction to start by taking a certain (finite) number of unrelated segments as given. A construction is licensed if and only if any elementary step of it agrees with one of these clauses.

It follows that extending Euclid's Plane Geometry by extending its ontology, though remaining faithful to its spirit, requires licensing further constructions besides elementary ones. This results, indeed, in making further (sorts of objects) available in geometry, besides those available within Euclid's Plane Geometry. This could be done in different ways: either by relaxing Euclid's clauses in some way, or by jettisoning the 'only if' implication in the previous condition, and locally allowing some constructive step which do not agree with these clauses, or by adding new clauses, or, finally, by replacing Euclid's clauses with a more general precept, entailing them as particular cases.

The first three strategies have all been followed before Descartes, which rather chose the fourth. The question that Bos refers to as that of geometrical exactness in early modern geometry was just that of settling the appropriate strategy to be followed for making new (sorts of) geometrical objects available in geometry besides those available within Euclid's Plane Geometry, so as to make one granted to deal

with objects of these sorts, namely either to appeal to them in further constructions and, then, in the solution of problems insolvable within Euclid's Plane Geometry, or to study them as such. Let us try to better grasp the nature of the question.

I begin with an important distinction. Euclid's Plane Geometry could have been extended, and was, indeed, extended, either locally, or globally. By local extension, I mean an extension merely allowing to make available in geometry some particular new (sorts of) objects, with the purpose of appealing to objects of these sorts in solving problems which are (or, at least, appeared to be) insolvable within Euclid's Plane Geometry. By global extension, I mean an extension that not only allows this, but also results in a new organic theory, larger than Euclid's Plane Geometry, but equally closed, in its being, as well, a deductive system with sharp limits (better, a reform which allows the former just insofar as it results in the latter). By saying that Descartes advocated a wide-ranging reform of geometry, and suggested a fresh foundation of it, I mean that he pursued such a global extension of Euclid's Plane Geometry.

What I have said about local extensions could make a puzzle arise: if showing that some (sorts of) objects are available within Euclid's Plane Geometry depends on solving problems, how should one intend the purpose of appealing to new (sorts of) objects, freshly made available in geometry through a local extension of this geometry, for solving problems otherwise insolvable (or apparently so) within it? There is no some kind of circularity here? To see why this is not so, we need to go a little bit farther in inspecting the nature and role of problems in Euclid's Plane Geometry. This is the task of the next sub-section.

2.3. *Unconditional and Conditional E-objects*

The crucial point pertains to the distinction between two quite different ways of specifying a certain sort of geometrical objects (by using the language of Euclid's Plane Geometry, or an appropriate extension of it).

The first way provides an intrinsic specification of the relevant objects considered *per se* (for the objects specified using the language of

Euclid's Plane Geometry, this generally results from a definition explicitly signaled in the *Elements*, or from a particularization of it). As it is well known, no other objects than points, segments, circles, polygons, and angles—both rectilinear and horn—are available within Euclid's Plane Geometry. Hence, limited to objects available within this geometry, saying that an object is, in this sense, an object of a certain sort results in saying that it is a point, a segment, a circle, a polygon of a certain genus or species, like a triangle, or an equilateral one, a parallelogram, a square, a gnomon, a pentagon, etc., or an angle—possibly an obtuse, an acute or a right one, or a rectilinear or a horn one.

The second way provides an extrinsic specification of objects already specified in the first way, relatively to other such objects. It depends on fixing some appropriate relations that the former objects are required to bear with the latter. Saying that an object is, in this sense, an object of a certain sort results in saying, for example, that it is a point cutting a given segment in extreme and mean ratio, or a square equal to a given triangle, or to a given circle.

For short, I call an object: 'unconditional', insofar as its sort is specified in the former way, and 'conditional', insofar as its sort is specified in the latter way. According to this liberal parlance, a triangle, or also an equilateral or a right-angled one, are unconditional objects, whereas a right-angled triangle having an angle equal to a certain acute given angle is a conditional one, for example. It follows that, for a certain object, to be unconditional or conditional is not to be, as such, so and so, but to be regarded as being so and so. In others terms, the distinction concerns our specification of the relevant objects, not their intrinsic nature.

Let me, now, introduce a further terminological convention: I call 'E-objects' (for 'Euclid's objects') either unconditional points, segments, circles, polygons of any genus and species, and angles—either rectilinear or horn—, or conditional such objects specified, in the language of Euclid's Plane Geometry, relatively to other such objects, only. In short, an E-object is then an object that Euclid's Plane Geometry is concerned with.

By definition, any unconditional E-object can be defined in the lan-

guage of this geometry. But some unconditional objects that can be so described are not E-objects. For example, this is so for non-degenerate conics: ellipses, parabolas and hyperbolas. What is more relevant, is not this, however, but rather that not any sort of unconditional E-objects is available within Euclid's Plane Geometry. For example heptagons are not. Moreover, showing that a sort of unconditional E-objects is available within Euclid's Plane Geometry can be far from simple. For example, proving this for heptadecagons is quite complex, and was firstly proved only by Gauss ([5] and [6], Sect. VII). A first crucial purpose of solving problems in Euclid's Plane Geometry is, then, to prove that a certain sort of unconditional E-objects is available within it. The solution of Proposition I.1 shows this for equilateral triangles, for example.

For conditional E-objects the dyscrasia is even more evident. Any sort of conditional E-object can be defined in the language of this geometry, but many sorts of conditional E-objects are not available within it. For example, pairs of segments providing two mean proportionals between two given segments, angles trisecting a given rectilinear angle, or squares equal to a given circle, are not. Another crucial purpose of solving problems in Euclid's Plane Geometry is to prove that a certain sort of conditional E-objects is available within Euclid's Plane Geometry, which means that single objects of this sort can be got through an elementary construction, starting from the objects relatively to which they are specified. The solution of Proposition I.2 shows this, for example, for segments equal to given ones, and having one of their extremities in a given point.

It should now not be difficult to see why what I have said above generates no circularity. One could appeal to some (sort of) unconditional objects (either E-ones or not), freshly made available in geometry through an extension of Euclid's Plane Geometry, for solving problems otherwise insolvable (or apparently so) within this geometry, and so making some (sorts of) conditional objects (typically E-ones), which are not available within it, become available in geometry. For example, Menaechmus showed how to appeal to appropriate conics in order to construct a pair of segments providing two mean proportionals between two given segments ([1], vol.III, pp. 92–97; [2], pp. 38-

40), which suggested to locally extend this geometry by licensing the construction of conics.

3. Extending Euclid's Plane Geometry Before Descartes

Tough quite broad and general, what I have said up to now should be enough, considering the quite limited purpose of the present paper, for fixing those aspects of Euclid's Plane Geometry that are relevant, in my view, for understanding the nature of Descartes's reform. I can, then, stop here concerning this matter, and coming to another matter, also useful for understanding the nature of this reform, namely to the state that the question of geometrical exactness had reached before Descartes.

Euclid's Plane Geometry is far from exhausting pre-Cartesian, and even Greek geometry. Still, with the only possible, partial exception of Apollonius's theory of conics, no other portion of pre-Cartesian geometry looks like it, that is, as a closed organic theory. Before Descartes, and with this only partial exception, going beyond Euclid's Plane Geometry was the same as entering a much more slipping area.

Moving in this area required handling non-elementary constructions: on the one side, these were constructions of E-objects non available within Euclid's Plane Geometry, especially of conditional ones, like segments providing two mean proportionals between two given segments, angles trisecting given angles, or squares equal to given circles; on the other side, these were constructions of geometrical objects other than E-ones, typically curves other than circles.

The most part of these curves were defined by describing possible constructions of them. Admitting these constructions came then, *ipso facto*, together with warranting for their availability in geometry. Hence, concerning them, the relevant question was less that of looking for appropriate constructions, than that of arguing in favor or against the admissibility of the constructions of which they were described as being the outcomes. A natural way for arguing in favor of it was by emphasizing that admitting these constructions (and taking, then, the sorts of curves constructed by them to be available in geometry) al-

lowed to construct (in a suitable way) E-objects non available within Euclid's Plane Geometry. These made the question of the admissibility of (non-elementary) constructions geometrical objects other than E-ones an aspect of the question of the admissibility of (non-elementary) constructions of E-objects non available within Euclid's Plane Geometry.

Asking for the construction of an E-object non available within Euclid's Plane Geometry was to state a problem, namely one that I suggest to call a 'quasi-Euclidean problem'. For its own nature, such a problem required a solution outside Euclid's Plane Geometry, that is, a solution depending on a non-elementary construction. Understanding Descartes's reform of this geometry is mainly helped by looking at the different ways suggested, before him, for solving quasi-Euclidean problems.

I have no space here for parsing examples. I confine myself to say that the angle-trisecting, the two-mean proportionals, and the circle-squaring problems were among the most relevant and usual quasi-Euclidean problems that mathematicians dealt with, before Descartes. Plenty of solutions of these problems (and other similar) were offered, differing to each other not only in their details, but also, and overall, for the general strategy they were following and the very nature of the non-elementary constructions involved in them. I suggest distinguishing six different families of non-elementary constructions that one can find in these solutions.

Firstly, come two families of what I call 'quasi-elementary constructions': constructions relying only on E-objects, though applying constructive clauses not included among those of elementary constructions.

The first -family includes constructions using some instruments for getting some points under the condition that some elements of these instruments coincide with (appropriate elements of the diagrams representing) some given geometrical objects. In my parlance, these are constructions using mechanical instruments in the pointing way. An example is provided by Eratosthenes's solution to the two-mean proportionals problem related by Pappus ([12], vol. I, pp. 56–59; [8], p. 211; [9], pp. 64–65), involving three rectangular plates, sliding to each

other up to assume a position in which a point on one of them falls on the right line passing through a vertex of another and the intersection point of a side of the latter and the diagonal of the third.

The second family includes constructions appealing to some explicit stipulations or tacit admissions working as new constructive clauses. Viète's *neusis* postulate ([14], prop IX; [15], p. 398; [2], pp. 167-173) provides an example.

Then come four families of what I call 'strictly non-elementary constructions': constructions relying on some objects other than E-objects, namely same curves other than circles. They differ to each other for the way these curves are constructed.

The first family includes constructions involving conics which were supposed to be *ipso facto* constructed for their being univocally determined. Menaechmus's solution of the two-mean proportionals problem mentioned above provides an example. Another is provided by a solution of the angle-trisecting problem offered by Pappus ([12], vol. I, pp. 271–277; [9], pp. 213–216; [2], pp. 53-56), in which a *neusis* is not merely granted according to a suitable postulate (as later suggested by Viète), but rather constructed by intersection of a circle and a hyperbola.

The second family includes constructions involving curves traced by instruments used to trace them, that is, as I say, in the tracing way. An example is provided by the solution of the two-mean proportionals problem involving a conchoid traced through an appropriate instrument, as suggested by Nicomedes and related by Eutocius ([1], vol. III, pp. 114-127; [2], pp. 3133), or by any constructions involving conics that are supposed to be traced by a "perfect compass" ([16]).

The third family includes constructions involving loci that were supposed to have been constructed if a generic point of them had been so, starting from the supposition that their generating point is given. An example is provided by the solution of the two-mean proportionals related by Villalpando ([13], vol. III, pp. 289-290; [2], pp. 75-78).

Finally, the fourth family includes constructions involving curves resulting by interpolation from countably many points previously constructible through an indefinitely protractible (licensed) procedure. An example is provided by the solution of the circle-squaring problem in-

volving a quadratrix constructed by points, according to Clavius's suggestion ([3], vol I, pp. 895–896; [2], pp. 160-166).

4. Descartes's Reform

A rapid way to account for Descartes's reform is by accounting for his attitude with respect to non-elementary constructions.

The first thing to be said is that his aim was not merely to accept some of them while rejecting the others, by relying on some good reasons. It was rather to integrate some of them within a new theory extending Euclid's Plane Geometry, and being as close and organic as it. In other terms, what he was looking for was not some scattered sorts of admissible non-elementary constructions, but a global extension of this geometry grounded on a unique principle to be used to license a unique sort of constructions, including elementary ones, but going largely beyond them, so as to allow the solution of the most part of quasi-Euclidean problems.

I suggest referring to this unique sort of construction as to constructions by rule, compass and reiteration. The reason is that they obey the following idea: one begins by admitting that whatever is constructible by an elementary construction is also constructible by such a construction; one can then appeal to an elementary construction to construct appropriate configurations of E-objects available within Euclid's Plane Geometry, such that conceiving of them as movable makes a point of them describe a curve; any curve so described is also taken to be constructible by such a construction; by appealing to whatever such curves and acting on them according to the clauses of an elementary construction (in which these curves are taken as given), one can then construct new configurations of objects (including these very curves), which are also such that conceiving them as movable makes a point of them describe a new curve, which is, in turn, taken to be constructible by such a construction; one can then continue this way indefinitely, by constructing new and new sorts of curves, and new and new configurations of objects (including these new curves), all such that conceiving them as movable makes a point of them describe a further curve,

which is both taken as constructible by such a construction, and possibly used, in the same way as the previous ones, to construct other curves.

It is clear that the movable configurations entering such a construction are supposed to play the role that Descartes ascribes to his tracing instruments. What I argue for is, then, that these instruments have to be regarded as *bona fide* geometric objects, though movable ones; namely objects available within the extended geometry, i. e. Descartes's Geometry, in the same sense as E-objects constructible by an elementary construction are available within Euclid's Plane Geometry: they are both so for they are susceptible of being got through a licensed construction, in Descartes's and within Euclid's Plane Geometry, respectively.

Still, for it to be appropriate for entering a construction by rule, compass and reiteration, a movable configuration has to be so determined by this very construction that it be enough to chose a point on it and to identify its movable and fixed -parts (with respect to the surface on which the curve is to be described), for entirely determining a unique curve that it traces through its motion. Of course, one can suppose that such a configuration is moved by some external mover, for example by a human hand, but the way this mover acts on it, and, in particular, the speed it imparts to it, have to have no influence on the trajectory of its movable parts, and, then, on the curve that it describes by its motion. This mover is then to be supposed to act on a unique (movable) point of the configuration, and whatever this point might be, and whatever the mover acts on it, the curve traced by the configuration is to be the same. Moreover, the transmission of the motion from the point on which the mover acts to the other movable points of the configuration —and, in general, from any its movable point to any other—has to merely depend on the requirement that the conditions fixed by construction be conserved through the motion, and to be, in particular, independent of any internal force or mechanical constraint. To say it shortly, a movable configuration entering a construction by rule, compass and reiteration is required to be a purely kinematical devise (having only one degree of freedom), not a dynamical one.

Consider a circle wheeling on a straight line without sliding on it.

A fixed point of it describes a curve, namely a cycloid, which is entirely determined by the circle itself and the mere admission that it wheels on a straight line without sliding on it. Still, this is not a movable configuration appropriate for entering a construction by rule, compass and reiteration, and cycloids are not curves constructible by such a construction, since a circle that wheels on a straight line without sliding on it is a dynamic devise, not a pure kinematical one: for it to move this way, its rotation has to be coupled with a rectilinear motion of its center, which, if it not independently imparted to it, is to induced by some friction. A similar argument also applies to tracing instruments involving strings and pulleys, if the way they work depends on the fact that their strings and pulleys transmit motion by exerting a force, as in the case of Huygens's instrument for tracing spirals: [7], vol. 11, p. 216; [2], pp. 347-349).

Imposing these conditions on movable configurations entering a construction by rule, compass and reiteration is quite natural, if one wants these configurations be regarded as genuinely geometric devises. These conditions are, moreover, just those that are met by a tracing rule (a fixed straight line with a tracing point moving on it) and a usual compass (a segment rotating around one of its extremities, whose other extremity is a tracing point). This makes these configurations natural extensions of the tracing devises involved in elementary constructions, conceived as constructions by rule and compass. In my view, this perfectly explains why Descartes call 'geometrical' the curves traced by them and 'mechanical' those that cannot be so traced (like the cycloid), and insists that the former have to be admitted in geometry, while the latter have to be excluded by it. Though this is not Descartes's use, it is then also natural to call, by extension, 'geometrical linkage' a movable configuration involved in a construction by rule, compass and reiteration.

Just like E-objects, geometrical linkages, and, more generally, any objects constructible by rule, compass and reiteration, can be either unconditional or conditional, according whether they are specified *per se*, or in relation to other given objects. By definition, both geometrical linkages and, more generally, objects constructed by rule, compass and reiteration are available within Descartes's Geometry. Un-

conditional ones are so insofar as they are constructible by rule, compass and reiteration starting from any number of unrelated segments taken as given (just as for E-objects within Euclid's Plane Geometry). Conditional ones are so insofar as they are constructible by rule, compass and reiteration starting by the objects relatively to which they are specified, which have to be, in turn, available within Descartes's Geometry. A conditional geometrical linkage is supposed to enter the construction (by rule, compass and reiteration) of a conditional geometric curve (other than a circle), if this it is required to stay in a particular position (relatively to given objects), which can, in turn, be required in order to solve a quasi-Euclidean problem or to construct another conditional curve by rule, compass and reiteration.

Many other details and specifications could be given (an should be so, in a more comprehensive exposition), and many examples offered. But the little I have said should be enough to make clear how licensing constructions by ruler, compass and reiteration extends Euclid's Plane Geometry. The only further point that I want to make concerns Descartes's attitude with respect to the six families of non-elementary constructions considered above. This attitude comes naturally together with the choice of globally extending Euclid's Plane Geometry by admitting, besides elementary constructions, also (and only) constructions by rule, compass and reiteration. In my view, the latter is, even, a natural outcome of the former, and can then be explained by it.

It seems quite clear, firstly, that Descartes rejected, from the very beginning, quasi-elementary constructions. He possibly rejected those based on specific stipulations, like Viète's *neusis* postulate, for he considered these stipulations *ad hoc*, and those involving instruments used in the pointing way, for this way of using instruments is not only essentially different from the way rules and compasses can be taken to be used in elementary constructions, but also since it induces a use of diagrams which is, also, essentially different from the use they have in Euclid's Plane Geometry. This last point deserves an explanation. The difference pertains to this: using instruments in the pointing way allows performing the relevant constructions only if these instruments are conceived as physical objects that, by an appropriate (typical visual) inspection, are recognized to take some required positions with re-

spect to the relevant diagrams; rules and compasses can, instead, be taken to enter an elementary construction at most insofar as they are used as devices for tracing (diagrams representing) segments and circles in positions that they are supposed to take (without any need of verifying that they do, through any sort of inspection, either visual or of any other kind). The latter is, by the way, also what happens, *mutatis mutandis*, for geometrical linkages in constructions by rule, compass, and reiteration.

Concerning strictly non-elementary constructions, Descartes seems to have, *mutatis mutandis*, the same attitude toward all of them: taking them as licensed if and only if that can be regarded as constructions by rule, compass and reiteration. Since a conic univocally determined relatively to given objects can always be traced through an appropriate geometrical linkage, he straightforwardly admitted those involving conics taken as *ipso facto* given for their being univocally determined. He admitted, then, those involving curves traced by instruments used in the traced way, if and only if these instruments can be re-conceived as (or replaced by) geometrical linkages, and those involving loci that are supposed to have been constructed if a generic point of them has been so, if and only if these loci can be traced by such a linkage (whose construction is often directly suggested by the very construction of this generic point). Finally, insofar as, in general, the identification of an indefinitely protractible (licensed) procedure apt to construct countably many points provides no suggestion for constructing a geometrical linkage tracing a curve interpolating these points, Descartes rejected those strictly non-elementary constructions that involve such a curve, unless it were not independently shown how to construct this curves through a construction by ruler, compass and reiteration.

To complete the account of Descartes's reform, one should of course, besides adding a lot of details, explaining how he suggested to associate geometrical curves with polynomial equations, and to chose, among different solutions of the same problem, all depending on a construction by ruler, compass and reiteration, the one regarded as the simplest one, according to an algebraic criterion of simplicity pertaining to these equations. This is not something I can do here, however. My aim was only to show how Descartes's Geometry directly stems

from extending Euclid's Plane one. The importance of his associating geometrical curves with polynomial equations can hardly be overestimated, also from the point of view from which I have suggested to look at the former geometry, since it suggested the path that mathematicians followed over the further two centuries, and that finally brought to mathematical theories essentially different from the latter, insofar as they are based on a structurally different ontology. In my view, the fact remains, however, that the introduction of polynomial equations for expressing geometrical curves was a posterior step in the fashioning of Descartes's Geometry, a step which was made possible, among other things, by the way this geometry stemmed from extending Euclid's Plane one.

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