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On Future Allocations of Scarce Resources without Explicit Discounting Factors*

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Abstract

In this paper, we show that by merely fixing upper bounds and lower bounds for the stream of consumptions, we can compute the optimal planning of consumptions independently from an explicit sequence of discounting factors as soon as they are assumed to be strictly decreasing. The optimal solution is unique and exhibits two regimes with a pivotal period in the middle. The same principle applies to future reimbursements of a debt as soon as we assume discounting factors to be strictly increasing. Therefore, one gets plans satisfying some kind of intergenerational fairness since: the highest effort is supported by the first generations and then it decreases for the remaining ones. Furthermore, we show that the solution is time consistent and we study the link with the standard discounted utilitarian model.

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1 Introduction

Since Ramsey [18], Samuelson[19] and Koopmans [12], there is a large tradition to value infinite streams of payoffs (consumptions, ...) using the discounted utilitarian model. But, and as mentioned in the seminal paper of Chambers and Echenique [4], there are substantial disagreements among economists about the proper discount rate for discounting long-term streams.

Deviating from the introduction of axioms aiming at resolving conflicting discount rates via a social choice approach as in [4] (see also among many others contributions [8]), we first follow the traditional route for evaluating infinite streams of consumptions by merely assuming the existence of a sequence of strictly decreasing discounting factors, which in our framework need not to be explicit. But the key difference comes from the fact that we impose a minimal and a maximal consumption for each year in order to allow a minimal surviving consumption with the lower bound and impose limitation to over consumption of the resource through the upper bound. It turns out that there exists a unique optimal plan, independently from strictly decreasing discounting factors.

This model is motivated by the optimal consumption or extraction of a scarce resource, which is indispensable for the survival of generations but detrimental for the environment. Then lower bounds are minimal survival levels and upper bounds are driven by environmental concern aiming at reducing the environmental damages or mitigating global warming. The dynamics of these bounds is driven by the development of alternative clean technologies allowing to reduce the use of this polluting resource.

With the same methodology, we also tackle the minimisation of the present value of the reimbursement of a debt considering again lower and upper bounds at each period for the annuity. Once again, the unique solution does not depend on the sequence of strictly increasing discounting factors. The economic motive is then coming from a debt incurred to finance alternative clean technologies. Therefore, one gets an optimal allocation satisfying some kind of intergenerational fairness since the highest effort is supported by the first generations, who are mainly responsible of the pollution, and then it decreases for the remaining ones. Note that the idea of a strictly increasing time discounting is already studied in the literature, see for instance Fleurbaey and Zuber [9].

In both cases, the optimal solution exhibits two regimes with a pivotal period at the middle. We briefly study the sensibility of this pivotal period to the lower and upper bounds. In particular, we show that it is not affected by the asymptotic behaviour of the upper (resp. lower) bounds in the first (resp. second) case.

We complete the paper by discussing some related issues. First we show that considering a finite horizon or an infinite horizon leads to similar results. Then, we prove that the optimal solution is time consistent in the sense that the truncated stream of consumptions after a given period is still the optimal path starting from the current stock of the resource for the truncated maximisation problem.

We also imagine the situation where a future generation would like to revise the lower and upper bounds according to a better knowledge of the evolution of the technologies. Then, we show that this future generation will not blame the previous choice. Indeed if this new bounds would have been taken into account at the initial period, then the remaining available quantity of the resource for this future generation would be smaller or equal to the one decided initially. So, there is no regret against the initial decision since this future generation benefits from more possibilities.

Finally, we are comparing our work with the traditional discounted utilitarian model. We prove that given an instantaneous utility function, there exists unique normalized discounting factors such that the optimal solution of our problem is a solution of the discounted utility maximisation. Conversely, for given normalized discounting factors, we exhibit the smallest and the greatest compatible utility function for which we have the coincidence of the optimal solutions. Nevertheless, we also show that some discounting factors are incompatible with some lower bounds if we require the utility function to be continuous at 0. This condition means that some intertemporal substitutions are possible even if one consumption level at a given period is very small.

At this stage, it would be very interesting to investigate for a real problem like oil extraction, the optimal solution given by the discounted utilitarian model with lower and upper bounds aggregating engineering experts opinion.

Surprisingly, these results are strongly related to the literature on the core of convex capacities. Indeed, in Campos et al. [3], it is shown that the so-called probability intervals, which is the set of probabilities satisfying an upper bound and a lower bound for each state, is the core of a convex capacity for which we have an explicit formula. We show that our problem becomes a maximisation or a minimisation of a linear mapping on the core of a capacity. It is known, see for example Schmeidler [20], Marinacci and Montrucchio [15], that the value of the problem is the Choquet's integral with respect to the capacity and the solution is an extreme point of the core, which are described in Delbaen [6].

The rest of this article is organized as follows. The problem under consideration, the main theorem and the properties of the optimal solution as well as an alternative minimisation problem with an increasing sequence of discounting factors are stated in Section 2. Various issues are described in Section 3, namely the role of the horizon length, the intertemporal consistency of the optimal solution, the scope for regret after a revision of the bounds and the understanding of implicit discount factors that would relate the current approach to a standard discounted utilitarian model.

2 The main Theorems

2.1 Extraction of a non-renewable resource

We consider the following problem of optimal extraction for a non-renewable resource depending on q_0, the initial quantity of the non-renewable resource, the lower and upper bounds of extraction per period, (α_n) and (β_n) and a sequence of strictly decreasing discounting factors, $\mu = (\mu_n)$:

$$(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{n=1}^{\infty} \mu_n x_n \\ \forall n \geq 1,\ \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{\infty} x_n = q_0 \end{cases}$$

We posit the following assumption on the parameters:

Assumption 1

a) $q_0 = 1$;

b) $(\alpha_n) \in \ell_1^+$ and $(\beta_n) \in \ell_\infty^+$, $\alpha_n < \beta_n$ for all n and $\sup\{\sum_{n=1}^m \beta_n \mid m \geq 1\} > 1 > \sum_{n=1}^\infty \alpha_n$;

c) $\mu = (\mu_n) \in \ell_\infty^+$ is a strictly decreasing sequence.

Remark 1 *The results presented below show that the asymptotic behaviour of the upper-bound β has no influence on the optimal solution. So, we could as well assume that β is in ℓ_1.*

We could consider that the feasibility condition $\sum_{n=1}^\infty x_n = 1$ should be replaced by an inequality constraint $\sum_{n=1}^\infty x_n \leq 1$ to leave more opportunity to the optimal allocation. Nevertheless, this has no effect on the solution since the inequality constraint is necessarily binding.

Our first result states that the solution is unique and does not depend on the sequence of discounting factors μ.

Theorem 1 *The problem $(\mathcal{P})$ has a unique solution $(\hat{x}_n)$ given by the explicit formula: $\hat{x}_1 = 1 - \max\{1 - \beta_1, \sum_{k\geq 2} \alpha_k\}$ and*

$$\hat{x}_n = \max\left\{1 - \sum_{k=1}^{n-1} \beta_k, \sum_{k=n}^{\infty} \alpha_k\right\} - \max\left\{1 - \sum_{k=1}^{n} \beta_k, \sum_{k=n+1}^{\infty} \alpha_k\right\} \text{ for } n \geq 2$$

Proof. Let

$$K = \{x \in \ell_1 \mid \forall n \geq 1, \alpha_n \leq x_n \leq \beta_n, \sum_{n=1}^{\infty} x_n = 1\}$$

be the set of feasible extraction plans. The proof relies on the fact that K is the core of a convex capacity. Indeed, K is nonempty thanks to the assumptions on (α_n) and (β_n). Indeed, if (β_n) belongs to ℓ_1, let $f(\gamma) = \gamma \sum_{n=1}^\infty \beta_n + (1-\gamma) \sum_{n=1}^\infty \alpha_n$ with $\gamma \in [0, 1]$. f is clearly continuous on $[0, 1]$. Since $f(1) = \sum_{n=1}^\infty \beta_n \geq 1$ and $f(0) = \sum_{n=1}^\infty \alpha_n \leq 1$, there exists $\hat{\gamma}$ such that $f(\hat{\gamma}) = 1$. So $x_n = \hat{\gamma}\beta_n + (1 - \hat{\gamma})\alpha_n$ satisfies the conditions and K is nonempty.

If (β_n) does not belongs to ℓ_1, we consider an auxiliary upper bounds $(\tilde{\beta}_n)$ in ℓ_1 such that for all $n \geq 1$, $\alpha_n < \tilde{\beta}^n \leq \beta^n$ and $\sum_{n=1}^{\infty} \tilde{\beta}_n > 1$. Then we merely apply the above argument to show that K is nonempty[1].

Note that if K is nonempty, then, obviously, $\alpha_n \leq \beta_n$ for all n, $\sup\{\sum_{n=1}^{m} \beta_n \mid m \geq 1\} \geq 1$ and $\sum_{n=1}^{\infty} \alpha_n \leq 1$.

Let v be the capacity defined for all $A \in \mathcal{A} = \mathcal{P}(\mathbb{N}^*)$ by

$$v(A) = \max\left\{\sum_{k\in A} \alpha_k, 1 - \sum_{k\notin A} \beta_k\right\}$$

with the convention that if $\sum_{k\notin A} \beta_k = +\infty$, then $v(A) = \sum_{k\in A} \alpha_k$. Note that $v(A_n) = \max\{1 - \sum_{k=1}^{n-1} \beta_k, \sum_{k=n}^{\infty} \alpha_k\}$ where $A_n = \{m \in \mathbb{N} \mid m \geq n\} = [\![n, \infty [\![$.

We prove that for all $x \in K$, for all $A \in \mathcal{A}$, $x(A) = \sum_{k\in A} x_k \geq v(A)$, that is, x belongs to the core of v[2]. Let $x \in K$, then $x(A) \geq \sum_{k\in A} \alpha_k$ since $x_k \geq \alpha_k$ for all $k \in \mathbb{N}^*$. Furthermore $x(A) + x(A^c) = 1$ implies $x(A) \geq 1 - \sum_{k\notin A} \beta_k$ if $\sum_{k\notin A} \beta_k$ is finite, so $x(A) \geq v(A)$ for all $A \in \mathcal{A}$. If $\sum_{k\notin A} \beta_k$ is infinite, then $v(A) = \sum_{k\in A} \alpha_k$ and $x(A) \geq v(A)$. Finally, $\sum_{k\in\mathbb{N}^*} x_k = 1 = v(\mathbb{N}^*)$.

Let $x \in K$. Then $-\mu(x) = \sum_{n=1}^{\infty} -\mu_n x_n$, hence from $\sum_{n=1}^{\infty} x_n = 1$, one deduces that

$$-\mu(x) = -\mu_1 + (\mu_1 - \mu_2)\left(\sum_{k=2}^{\infty} x_k\right) + (\mu_2 - \mu_3)\left(\sum_{k=3}^{\infty} x_k\right) + \\ \ldots + (\mu_n - \mu_{n+1})\left(\sum_{k=n+1}^{\infty} x_k\right) + \ldots$$

Hence since $\mu_n > \mu_{n+1}$ for all n, one gets that for all $x \in K$,

$$-\mu(x) \geq -\mu_1 + \sum_{n=1}^{\infty} (\mu_n - \mu_{n+1}) v(A_{n+1})$$

[1]For example, we can build $(\tilde{\beta}_n)$ as follows: from our assumption on (β_n) there exists $\bar{m}$ such that $\sum_{n=1}^{\bar{m}} \beta_n > 1$. Then, we let $\tilde{\beta}_n = \beta_n$ is $n \leq \bar{m}$ and $\beta_n = \min\{\beta_n, 2\alpha_n + (1/n^2)\}$ if $n > \bar{m}$.

[2]We can prove the converse inclusion showing that the core of v is exactly K.

or else

$$-\mu(x) \geq -\sum_{n=1}^{\infty} \mu_n(v(A_{n+1}) - v(A_n))$$

So, it turns out that if $(\hat{x}_n)$ defined by $\hat{x}_n = v(A_n) - v(A_{n+1})$ belongs to K, then it is the unique solution of the problem $(\mathcal{P})$. Note that if $n \geq 2$,

$$v(A_n) - v(A_{n+1}) = \max\{1 - \sum_{k=1}^{n-1} \beta_k, \sum_{k=n}^{\infty} \alpha_k\} - \max\{1 - \sum_{k=1}^{n} \beta_k, \sum_{k=n+1}^{\infty} \alpha_k\}$$

and $v(A_1) - v(A_2) = 1 - \max\{1 - \beta_1, \sum_{k=2}^{\infty} \alpha_k\}$.

Lemma 1 will complete the proof by providing an explicit formula of $\hat{x}_n$ and, as a dividend, shows that $\hat{x}$ belongs to K. $\square$

Lemma 1 *(An explicit formula.) Let* $\mathcal{S}_1 = \sum_{k=1}^{\infty} \alpha_k$*,* $\mathcal{S}_n = \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^{\infty} \alpha_k$ *for* $n \geq 2$ *and* $n_0 = \sup\{n \in \mathbb{N}^* \mid \mathcal{S}_n \leq 1\}$*.*

Consider the optimal solution $\hat{x}$ *as given by Theorem 1. Then,* n_0 *is finite and*

Case 1: if $n_0 > 1$*, then* $\hat{x}$ *is defined by:*

(1.1) $\hat{x}_n = \beta_n$ *for* $1 \leq n < n_0$*,* $\hat{x}_{n_0} = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{\infty} \alpha_k \in [\alpha_{n_0}, \beta_{n_0}[$

(1.2) $\hat{x}_n = \alpha_n$ *for* $n > n_0$*.*

Case 2: if $n_0 = 1$*, then* $\hat{x}$ *is defined by:*

(2.1) $\hat{x}_1 = 1 - \sum_{k=2}^{\infty} \alpha_k \in [\alpha_1, \beta_1[$

(2.2) $\hat{x}_n = \alpha_n$ *if* $n \geq 2$*.*

Remark 2 *Note that, in Case 1, if* $\mathcal{S}_{n_0} = 1$*, then* $\hat{x}_{n_0} = \alpha_{n_0}$ *and similarly, in Case 2, if* $\mathcal{S}_1 = 1$*, then* $\hat{x}_1 = \alpha_1$*.*

So, Lemma 1 states that the optimal extraction is obtained by choosing first the upper bound of the feasible interval at each date as long as it is possible without compelling the extraction to be below the lower bound afterwards. Then the lower bound is chosen after a date (n_0) *where the extraction level is* *chosen in between the lower and the upper bound to adjust the total extraction to* 1.

In all cases, the extraction is at the maximal level for the first periods and then at the minimal one, showing that independently from the sequence of decreasing factors (μ_n)*, the intertemporal maximisation of the present value leads the future generations to benefit from a better ecological environment.*

Remark 3 *The pivotal period* n_0 *can be interpreted as the first period where alternative clean technologies should be available allowing to stop using the scarce polluting resource at its upper acceptable regime and therefore to switch to its lower admissible regime.*

The pivotal period n_0 *is the first one if* $\beta_1 + \sum_{k=2}^{\infty} \alpha_k > 1$*, meaning that the lower bounds are quite large. Then, the only room to increase the extraction above the lower bound is at the first period without having the possibility to extract a quantity equals to the upper bound* β_1*. Keeping the upper bounds fixed, if the lower bounds decrease, then the switching period* n_0 *increases but it never exceeds a finite upper bound* $\bar{n}$*, which is* $\max\{n \in \mathbb{N} \mid \sum_{\nu=1}^{n} \beta_\nu \leq 1\}$.

More generally n_0 *is weakly decreasing with respect to* β *and* α*. Note that if we change the upper bounds after period* $n_0 - 1$ *keeping them above the lower bounds, then the optimal solution remains the same as well as if we change the lower bounds before period* $n_0 - 1$ *keeping them below the upper bounds. So, as mentioned previously, the asymptotic behaviour of the upper bounds does not modify the result.*

Finally, a planner can design suitable lower and upper bounds given a switching period $n_0 > 1$ *and positive quantities* (q_n) *such that* $\sum_{n=1}^{\infty} q_n = 1$*. Indeed, it suffices to choose* $\beta_n = q_n$ *and* $\alpha_n = (1/2)q_n$ *for* $n < n_0$ *and* $\beta_n = 2q_n$ *and* $\alpha_n = q_n$ *for* $n \geq n_0$.

Proof of Lemma 1.

Note that $(\mathcal{S}_n)$ is strictly increasing. Actually,

$$\mathcal{S}_2 - \mathcal{S}_1 = \beta_1 + \sum_{k=2}^{\infty} \alpha_k - \sum_{k=1}^{\infty} \alpha_k = \beta_1 - \alpha_1 > 0$$

and for all $n \geq 2$,

$$\mathcal{S}_{n+1} - \mathcal{S}_n = \sum_{k=1}^{n} \beta_k + \sum_{k=n+1}^{\infty} \alpha_k - \sum_{k=1}^{n-1} \beta_k - \sum_{k=n}^{\infty} \alpha_k = \beta_n - \alpha_n > 0$$

Note that $\mathcal{S}_1 = \sum_{k=1}^{\infty} \alpha_k \leq 1$ by assumption. Since $\sup\{\sum_{n=1}^{m} \beta_n \mid m \geq 1\} > 1$, there exists m_0 such that $\sum_{n=1}^{m_0} \beta_n > 1$. Therefore, $E = \{n \in \mathbb{N}^* \mid \mathcal{S}_n \leq 1\}$ is nonempty since 1 belongs to E and it is bounded above by m_0. Consequently, there exists $n_0 = \sup E$ and $n_0 \geq 1$.

Several cases have to be examined.

Case 1: $n_0 > 1$.

Let us first show that $\hat{x}_1 = \beta_1$. From the formula above,

$$\hat{x}_1 = 1 - \max\left\{1 - \beta_1, \sum_{k=2}^{\infty} \alpha_k\right\}$$

so, it suffices to prove that $1 - \beta_1 \geq \sum_{k=2}^{\infty} \alpha_k$, which is equivalent to $\mathcal{S}_2 \leq 1$, which is true since $\mathcal{S}_2 \leq \mathcal{S}_{n_0} \leq 1$.

Let us now show that $\hat{x}_n = \beta_n$ for any n such that $1 < n < n_0$. From the definition of n_0 and the monotony of $\mathcal{S}_n$, if $1 < n < n_0$, we get $\mathcal{S}_n \leq \mathcal{S}_{n+1} \leq 1$, so $1 - \sum_{k=1}^{n-1} \beta_k \geq \sum_{k=n}^{\infty} \alpha_k$ and $1 - \sum_{k=1}^{n} \beta_k \geq \sum_{k=n+1}^{\infty} \alpha_k$. Hence, since

$$\hat{x}_n = \max\{1 - \sum_{k=1}^{n-1} \beta_k, \sum_{k=n}^{\infty} \alpha_k\} - \max\{1 - \sum_{k=1}^{n} \beta_k, \sum_{k=n+1}^{\infty} \alpha_k\}$$

we get that $\hat{x}_n = 1 - \sum_{k=1}^{n-1} \beta_k - (1 - \sum_{k=1}^{n} \beta_k) = \beta_n$.

At n_0, $\mathcal{S}_{n_0} \leq 1$ and $\mathcal{S}_{n_0+1} > 1$. Using the same argument as above, it comes $\hat{x}_{n_0} = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{\infty} \alpha_k$.

We remark that $\hat{x}_{n_0} - \alpha_{n_0} = 1 - \mathcal{S}_{n_0}$ and $\hat{x}_{n_0} - \beta_{n_0} = 1 - \mathcal{S}_{n_0+1}$. So $\mathcal{S}_{n_0} \leq 1$ and $\mathcal{S}_{n_0+1} > 1$ imply $\hat{x}_{n_0} \in [\alpha_{n_0}, \beta_{n_0}[$.

Let us finally show that $\hat{x}_n = \alpha_n$ for any $n > n_0$. From the definition of n_0, if $n > n_0$, we get $\mathcal{S}_n > 1$ and $\mathcal{S}_{n+1} > 1$, so $1 - \sum_{k=1}^{n-1} \beta_k < \sum_{k=n}^{\infty} \alpha_k$ and $1 - \sum_{k=1}^{n} \beta_k < \sum_{k=n+1}^{\infty} \alpha_k$. Hence, since

$$\hat{x}_n = \max\left\{1 - \sum_{k=1}^{n-1} \beta_k, \sum_{k=n}^{\infty} \alpha_k\right\} - \max\left\{1 - \sum_{k=1}^{n} \beta_k, \sum_{k=n+1}^{\infty} \alpha_k\right\}$$

we get that $\hat{x}_n = \sum_{k=n}^{\infty} \alpha_k - \sum_{k=n+1}^{\infty} \alpha_k = \alpha_n$.

Case 2: $n_0 = 1$.

Since $n_0 = 1$, $\mathcal{S}_2 > 1$, so $1 - \beta_1 < \sum_{k=2}^{\infty} \alpha_k$. Consequently, $\hat{x}_1 = 1 - \max\{1-\beta_1, \sum_{k=2}^{\infty} \alpha_k\} = 1-\sum_{k=2}^{\infty} \alpha_k$. Note that the above inequality implies that $\hat{x}_1 < \beta_1$ and, since $\sum_{k=1}^{\infty} \alpha_k \leq 1$, one gets that $\alpha_1 \leq \hat{x}_1$.

For $n > 1$, we apply the same argument as the final one for Case 1, to conclude that $\hat{x}_n = \alpha_n$. $\square$

2.2 Solution for a minimisation problem

We now consider the problem of minimizing the present value cost of the reimbursement of a debt. The underlying economic motive can be understood as stemming from a debt incurred to finance alternative clean technologies. q_0 is the amount of the debt, (α_n) and (β_n) are the lower and upper bounds of the reimbursement per period and (λ_n) is the sequence of discounting factors.

$$(\mathcal{P}_2) \left\{ \begin{array}{l} \text{Minimise } \sum_{n=1}^{\infty} \lambda_n x_n \\ \forall n \geq 1,\ \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{\infty} x_n = q_0 \end{array} \right.$$

We posit the following assumption on the parameters:

Assumption 2

a) $q_0 = 1$;

b) $(\alpha_n) \in \ell_1^+$ and $(\beta_n) \in \ell_\infty^+$, $\alpha_n < \beta_n$ for all n and $\sup\{\sum_{n=1}^{m} \beta_n \mid m \geq 1\} > 1 > \sum_{n=1}^{\infty} \alpha_n$;

c) $\lambda = (\lambda_n) \in \ell_\infty^+$ is a strictly increasing sequence.

Theorem 2 *The problem $(\mathcal{P}_2)$ has a unique solution $(\bar{x}_n)$ which is the same as the one of the previous problem, namely $\hat{x}_1 = 1 - \max\{1 - \beta_1, \sum_{k\geq 2} \alpha_k\}$ and*

$$\hat{x}_n = \max\left\{1 - \sum_{k=1}^{n-1} \beta_k, \sum_{k=n}^{\infty} \alpha_k\right\} - \max\left\{1 - \sum_{k=1}^{n} \beta_k, \sum_{k=n+1}^{\infty} \alpha_k\right\} \ \textit{for } n \geq 2$$

With the notation of Lemma 1,

Case 1: if $n_0 > 1$, then $\hat{x}$ is defined by:

(1.1) $\hat{x}_n = \beta_n$ for $1 \leq n < n_0$, $\hat{x}_{n_0} = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{\infty} \alpha_k \in [\alpha_{n_0}, \beta_{n_0}[$

(1.2) $\hat{x}_n = \alpha_n$ for $n > n_0$.

Case 2: if $n_0 = 1$, then $\hat{x}$ is defined by:

(2.1) $\hat{x}_1 = 1 - \sum_{k=2}^{\infty} \alpha_k \in [\alpha_1, \beta_1[$

(2.2) $\hat{x}_n = \alpha_n$ if $n \geq 2$.

Remark 4 *Therefore, one gets an optimal allocation satisfying some kind of intergenerational fairness since the highest effort is supported by the first generations, who are mainly responsible of the pollution, and then it decreases for the remaining ones. Note that in actual fact, the first generations need to reimburse at the upper level while the next ones need to reimburse only at the lower level.*

Proof of the theorem Note that $\min \sum_{n=1}^{\infty} \lambda_n x_n$ exists and is equal to $-\max \sum_{n=1}^{\infty} \mu_n x_n$ where $\mu_n = -\lambda_n$ is strictly decreasing. The same proof as the one of Theorem 1 applies since the sign of μ_n does not matter, the key hypothesis being μ_n strictly decreasing, which gives the result. $\square$.

Remark 5 *Let us assume now that $\mu \in \ell_\infty^+$ is strictly increasing and we would like to solve the following maximisation problem:*

$$(\mathcal{Q}) \left\{ \begin{array}{l} \textit{Maximise } \sum_n \mu_n x_n \\ \forall n \geq 1,\ \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{\infty} x_n = 1 \end{array} \right.$$

Then we can apply the same methodology as above but considering the dual capacity $\bar{v}$ instead of v. $\bar{v}$ is defined by $\bar{v}(A) = 1 - v(\bar{A})$ or equivalently:

$$\bar{v}(A) = \min\{1 - \sum_{k \notin A} \alpha_k, \sum_{k \in A} \beta_k\}$$

We then prove that for all $x \in K$, for all $A \in \mathcal{A}$, $x(A) = \sum_{k \in A} x_k \leq \bar{v}(A)$. With similar computations as the previous ones, one gets that the optimal solution of the problem $(\mathcal{Q})$ is given by $(x_n^* = \bar{v}(A_n) - \bar{v}(A_{n+1}))$. So, (x_n^*) has a reverse structure compare to the previous solution $(\hat{x}_n)$, namely, $x_n^* = \alpha_n$ for the periods before a turning period and then β_n afterwards with an intermediate value for the turning period.

3 Some facets of the solution

3.1 The finite horizon case

If we consider the same problem as the one in Section 2 with a finite horizon N, then we get exactly the optimal solution with the same functional forms. Indeed, all the proofs work identically replacing the infinite sums by finite ones. The only difference in the statement of Lemma 1 appears when n_0 is equal to N, so that the lower bound α_n is never reach at the optimal solution.

Let (α_n) and (β_n) in $\mathbb{R}_+^N$ such that $\alpha_n < \beta_n$ for all n and $\sum_{n=1}^{N} \beta_n > 1 > \sum_{n=1}^{N} \alpha_n$.

Let $\mu = (\mu_n)$ be a strictly decreasing element of $\mathbb{R}_+^N$. Let us consider the following maximisation problem of the present value of an extraction plan $(x_n) \in \mathbb{R}_+^N$:

$$(\mathcal{P}_N) \begin{cases} \text{Maximise } \sum_{n=1}^{N} \mu_n x_n \\ \forall n \in \{1, \ldots, N\},\, \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{N} x_n = 1 \end{cases}$$

Corollary 1 *Let $\mathcal{S}_1^N = \sum_{k=1}^{N} \alpha_k$, $\mathcal{S}_n^N = \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^{N} \alpha_k$ for $n = 2, \ldots, N$, and $n_0 = \sup\{n \in \{1, \ldots, N\} \mid \mathcal{S}_n^N \leq 1\}$. The problem $(\mathcal{P}_N)$ has a unique solution $(\hat{x}_n)$ given by the explicit formula:*

Case 1: $N > n_0 > 1$, then $\hat{x}$ satisfies:

(1.1) $\hat{x}_n = \beta_n$ for $1 \leq n < n_0$, $\hat{x}_{n_0} = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{N} \alpha_k \in [\alpha_{n_0}, \beta_{n_0}[$

(1.2) $\hat{x}_n = \alpha_n$ for $N \geq n > n_0$.

Case 2: $n_0 = 1$, then $\hat{x}$ satisfies:

(2.1) $\hat{x}_1 = 1 - \sum_{k=2}^{N} \alpha_k \in [\alpha_1, \beta_1[$

(2.2) $\hat{x}_n = \alpha_n$ if $n \geq 2$.

Case 3: $n_0 = N$, then $\hat{x}$ satisfies:

$\hat{x}_n = \beta_n$ for $1 \leq n < n_0$, $\hat{x}_{n_0} = 1 - \sum_{k=1}^{n_0-1} \beta_k \in [\alpha_{n_0}, \beta_{n_0}[$

Truncation of the horizon

The above result shows that considering a finite or an infinite horizon model does not change the optimal solution for the first periods if the bounds are the same for these first periods . More precisely, let (α_n) and (β_n) in ℓ_1^+ satisfying the assumptions of Theorem 1. Let us choose a finite horizon N and let us assume that the lower and upper bounds are the same for the periods 1 to $N-1$ and, at the period N, $\hat{\alpha}_N = \sum_{n=N}^{\infty} \alpha_n$ and $\hat{\beta}_N = \sum_{n=N}^{\infty} \beta_n$. Then one easily check that for the problem $(\mathcal{P})$ and for the problem $(\mathcal{P}_N)$, the solutions are the same for the periods up to $N-1$. Indeed, we remark that $\mathcal{S}_n = \mathcal{S}_n^N$ for $n \leq N$ and, if $n_0 < N$, $\hat{x}_{n_0} = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{\infty} \alpha_k = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{N-1} \alpha_k - \hat{\alpha}_N$.

The same result works when we start with a finite horizon N and we push it forward to $\tilde{N} > N$ or $+\infty$. If the lower and upper bounds α_n and β_n are not modified for the periods 1 to $N-1$ and the final bounds α_N and β_N are shared over the remaining periods, that is, $\alpha_N = \sum_{n=N}^{\tilde{N}} \tilde{\alpha}_n$ and $\beta_N = \sum_{n=N}^{\tilde{N}} \tilde{\beta}_n$ or $\alpha_N = \sum_{n=N}^{\infty} \tilde{\alpha}_n$ and $\beta_N = \sum_{n=N}^{\infty} \tilde{\beta}_n$, then the optimal plans for the problem $(\mathcal{P}_N)$ and for the extended horizon problem $(\mathcal{P}_{\tilde{N}})$ or $(\mathcal{P})$ coincide up to the period $N-1$.

So the initial extraction plan until a period N does not depend on the lower and upper bounds for each period after $N+1$ but just on the remaining aggregate bounds at $N+1$ $\sum_{n=N+1}^{\infty} \alpha_n$ and $\sum_{n=N+1}^{\infty} \beta_n$. So, the choice of the lower and upper bounds, which are the key parameters of the model, is really important for the first periods together with the lower and the upper bounds for the total extraction defined as $\sum_{n=1}^{\infty} \alpha_n$ and $\sum_{n=1}^{\infty} \beta_n$. Hence, if we have in mind that these bounds are determined by experts using forecasts over the evolution of productivity of the current technologies and the one of alternative technologies, we just need a finite (even short) horizon expectations to take the right decisions for the first periods.

3.2 Intertemporal consistency

Let us now consider that the initial decision be revised at a given period N. So, starting from Problem $(\mathcal{P})$ and the optimal solution $(\hat{x}_n)$ given by Theorem 1, let $N > 1$ and $q_N = 1 - \sum_{n=1}^{N} \hat{x}_n$. q_N is the remaining stock of the non renewable resource at the end of Period N if the optimal extraction plan is realised until Period N. Keeping the lower and upper bounds unchanged, for the forthcoming periods, we get a new maximisation problem, which is:

$$(\mathcal{Q}^N) \begin{cases} \text{Maximise } \sum_{n=N+1}^{\infty} \mu_n x_n \\ \forall n \geq N+1,\ \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=N+1}^{\infty} x_n = q_N \end{cases}$$

We remark that the set defined by the constraints is nonempty since the truncation of the optimal plan $(\hat{x}_n)_{n=N+1}^{\infty}$ satisfies the constraints. The following proposition shows that the initial optimal solution is time consistent since there is no change in the optimal extraction plan.

Proposition 1 *The optimal solution $(\tilde{x}_n)_{n=N+1}^{\infty}$ of Problem $(\mathcal{Q}^N)$ is just the truncation of the optimal solution $(\hat{x}_n)$ that is $\tilde{x}_n = \hat{x}_n$ for $n \geq N+1$.*

Proof. Let us first start with the case $N \geq n_0$. Then, we remark that $\hat{x}_n = \alpha_n$ for all $n > N_0$, so $q_N = \sum_{n=N+1}^{\infty} \alpha_n$. In this case, there is a unique $(x_n)_{n=N+1}^{\infty}$ satisfying the constraints of Problem $(\mathcal{Q}^N)$, namely $x_n = \alpha_n = \hat{x}_n$. So, this unique point is the solution of the problem and we get the result.

If $N < n_0$, then, for $n \leq N$, $\hat{x}_n = \beta_n$ and $q_N = 1 - \sum_{n=1}^{N} \beta_n$. For $n \geq N+1$, let $\tilde{\mathcal{S}}_n = \sum_{k=N+1}^{n-1} \beta_k + \sum_{k=n}^{\infty} \alpha_k$. We note that $\mathcal{S}_n = 1 - q_N + \tilde{\mathcal{S}}_n$ where $\mathcal{S}_n$ is defined before Lemma 1. So, since $\mathcal{S}_n$ is non decreasing, $n_0 = \sup\{n \in \mathbb{N}^* \mid \mathcal{S}_n \leq 1\} = \sup\{n \geq N+1 \mid \tilde{\mathcal{S}}_n \leq q_N\}$ Mimicking the proof of Theorem 1 and Lemma 1, we get that the optimal solution of Problem $(\mathcal{Q}^N)$ is

- if $N+1 < n_0$, $\tilde{x}_n = \beta_n = \hat{x}_n$ for $N+1 \leq n < n_0$, $\tilde{x}_{n_0} = q_N - \sum_{k=N+1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{\infty} \alpha_k = 1 - \sum_{k=1}^{n_0-1} \beta_k - \sum_{k=n_0+1}^{\infty} \alpha_k = \hat{x}_n$ and $\tilde{x}_n = \alpha_n = \hat{x}_n$ for $n > n_0$;

- if $N+1 = n_0$, $\tilde{x}_{N+1} = q_N - \sum_{k=N+2}^{\infty} \alpha_n = 1 - \sum_{k=1}^{N} \beta_k - \sum_{k=N+2}^{\infty} \alpha_k = \hat{x}_{N+1}$ since $N+1 = n_0$ and $\tilde{x}_n = \alpha_n = \hat{x}_n$ for $n > N+1$.

Then the optimal solutions coincide in all cases for the periods after $N+1$. □

3.3 No regret after a revision of the bounds

We now assume that the lower and upper bounds after the Period N are revised thanks to a new analysis of the experts or a better knowledge of the technology transition. We still assume that they are coherent with the available stock $q_N = 1 - \sum_{n=1}^{N} \hat{x}_n$. Let us denote $(\tilde{\alpha}_n)_{n=N+1}^{\infty}$ and $(\tilde{\beta}_n)_{n=N+1}^{\infty}$ the new bounds satisfying $\tilde{\alpha}_n < \tilde{\beta}_n$ for all $n \geq N+1$ and $\sum_{n=N+1}^{\infty} \tilde{\alpha}_n < q_N < \sum_{n=N+1}^{\infty} \tilde{\beta}_n$. Then, the question is to know if this change could have modified the optimal path before Period N if they were anticipated at Period 0. The answer is clearly yes but the next proposition shows that the new optimal plan leads to a larger extraction level with the new bounds before period N. So, at Period N, there is no regret about the previous extractions levels since it remains a greater quantity of the ressources for the next generations.

Let us denote by $(\mathring{\alpha}_n)_{n\geq 1}$ and $(\mathring{\beta}_n)_{n\geq 1}$ the sequences defined by: $\mathring{\alpha}_n = \alpha_n$ and $\mathring{\beta}_n = \beta_n$ for $n \leq N$ and $\mathring{\alpha}_n = \tilde{\alpha}_n$ and $\mathring{\beta}_n = \tilde{\beta}_n$ for $n > N$. We remark that the modified lower and upper bounds satisfy the feasibility condition. Indeed:

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathring{\beta}_n &= \sum_{n=1}^{N} \beta_n + \sum_{n=N+1}^{\infty} \tilde{\beta}_n \\
&> \sum_{n=1}^{N} \beta_n + q_N \\
&= \sum_{n=1}^{N} \beta_n + 1 - \sum_{n=1}^{N} \hat{x}_n \\
&= \sum_{n=1}^{N} (\beta_n - \hat{x}_n) + 1 \\
&\geq 1
\end{aligned}$$

and the same type of computation shows that $\sum_{n=1}^{\infty} \mathring{\alpha}_n < 1$.

Let $(\mathring{x}_n)$ the optimal solution associated to $(\mathring{\alpha}_n)_{n\geq 1}$ and $(\mathring{\beta}_n)_{n\geq 1}$.

Proposition 2 *Let $(\mathring{x}_n)$ the optimal solution associated to $(\mathring{\alpha}_n)_{n\geq 1}$ and $(\mathring{\beta}_n)_{n\geq 1}$. Then, $\sum_{n=1}^{N} \hat{x}_n \leq \sum_{n=1}^{N} \mathring{x}_n$.*

Proof. Let us first start with the case $N \geq n_0$. Then $\hat{x}_n = \alpha_n$ for all $n > N_0$, so $q_N = \sum_{n=N+1}^{\infty} \alpha_n$. Consequently, for all $n \leq N$,

$$
\begin{aligned}
\mathcal{S}_n &= \sum_{k=1}^{n-1} \beta_n + \sum_{k=n}^{\infty} \alpha_n \\
&= \sum_{k=1}^{n-1} \beta_n + \sum_{k=n}^{N} \alpha_n + \sum_{n=N+1}^{\infty} \alpha_n \\
&= \sum_{k=1}^{n-1} \beta_n + \sum_{k=n}^{N} \alpha_n + q_N \\
&> \sum_{k=1}^{n-1} \beta_n + \sum_{k=n}^{N} \alpha_n + \sum_{n=N+1}^{\infty} \tilde{\alpha}_n \\
&= \mathring{\mathcal{S}}_n
\end{aligned}
$$

One deduces that $\mathring{\mathcal{S}}_{n_0} \leq \mathcal{S}_{n_0} \leq 1$, which implies that the new pivotal period $\mathring{n}_0$ is greater or equal to the initial one n_0. If $N < \mathring{n}_0$, then for all for all $1 \leq n \leq N$, $\mathring{x}_n = \mathring{\beta}_n = \beta_n \geq \hat{x}_n$.

If $\mathring{n}_0 \leq N$, we distinguish two cases. If $n_0 < \mathring{n}_0$, then for all $1 \leq n < \mathring{n}_0$, $\mathring{x}_n = \mathring{\beta}_n = \beta_n \geq \hat{x}_n$, and for all $\mathring{n}_0 \leq n \leq N$, $\mathring{x}_n \geq \mathring{\alpha}_n = \alpha_n = \hat{x}_n$.

If $n_0 = \mathring{n}_0$, then for all $1 \leq n < \mathring{n}_0$, $\mathring{x}_n = \mathring{\beta}_n = \beta_n = \hat{x}_n$, for all $\mathring{n}_0 < n \leq N$, $\mathring{x}_n = \mathring{\alpha}_n = \alpha_n = \hat{x}_n$, and

$$
\begin{aligned}
\mathring{x}_{\mathring{n}_0} &= 1 - \sum_{k=1}^{\mathring{n}_0 - 1} \mathring{\beta}_k - \sum_{k=\mathring{n}_0+1}^{\infty} \mathring{\alpha}_k \\
&= 1 - \sum_{k=1}^{\mathring{n}_0 - 1} \beta_k - \sum_{k=\mathring{n}_0+1}^{\infty} \mathring{\alpha}_k \\
&> 1 - \sum_{k=1}^{\mathring{n}_0 - 1} \beta_k - \sum_{k=\mathring{n}_0+1}^{\infty} \alpha_k \\
&= \hat{x}_{\mathring{n}_0}
\end{aligned}
$$

The last inequality comes from the fact that $\mathring{n}_0 + 1 \leq N + 1$ and $q_N = \sum_{n=N+1}^{\infty} \alpha_n > \sum_{n=N+1}^{\infty} \tilde{\alpha}_n = \sum_{n=N+1}^{\infty} \mathring{\alpha}_n$.

Gathering the previous results, in all cases, we get that $\mathring{x}_n \geq \hat{x}_n$ for $1 \leq n \leq N$, hence $\sum_{n=1}^{N} \hat{x}_n \leq \sum_{n=1}^{N} \mathring{x}_n$.

If $n_0 > N$, then, $\hat{x}_n = \beta_n$ for $1 \leq n \leq N$, so $q_N = 1 - \sum_{k=1}^{N} \beta_k$ and $\sum_{k=N+1}^{\infty} \tilde{\alpha}_k < 1 - \sum_{k=1}^{N} \beta_k$. So

$$
\begin{aligned}
\mathring{\mathcal{S}}_{N+1} &= \sum_{k=1}^{N} \mathring{\beta}_k + \sum_{k=N+1}^{\infty} \mathring{\alpha}_k \\
&= \sum_{k=1}^{N} \beta_k + \sum_{k=N+1}^{\infty} \tilde{\alpha}_k \\
&< 1
\end{aligned}
$$

Hence $\mathring{n}_0 \geq N+1$ and $\mathring{x}_n = \mathring{\beta}_n = \beta_n = \hat{x}_n$ for $1 \leq n \leq N$, hence $\sum_{n=1}^{N} \hat{x}_n = \sum_{n=1}^{N} \mathring{x}_n$. □

3.4 On implicit discounting factors

In this last subsection, we analyse the compatibility of the above solution with the standard discounted utilitarian model. Actually, we follow two complementary approaches: in the first case, we take an instantaneous utility function as given and we prove that there is a unique normalized sequence of discounting factors so that the solution of Problem $(\mathcal{P})$ is a maximum of the discounted utility. Then, we take discounting factors as given and we provide the greatest and the smallest concave instantaneous utility functions which leads to the same optimal solution than Problem $(\mathcal{P})$. So, beyond the discounting factors, our approach through lower and upper bounds also avoid the question of what is the right instantaneous utility function. Finally, we show that some discounting factors are not compatible with a solution if we impose that the utility function be bounded from below at 0, which is a measure of the intertemporal substituability of the consumptions.

Let us consider again Problem $(\mathcal{P})$

$$
(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{n=1}^{\infty} \mu_n x_n \\ \forall n \geq 1,\ \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{\infty} x_n = 1 \end{cases}
$$

and the solution $(\hat{x}_n)$. We assume that $\alpha_n > 0$ for all n, so $\hat{x}_n > 0$.

Let u be an instantaneous utility function from $]0, +\infty[$ to $\mathbb{R}$. We posit the following assumption:

Assumption U

a) u is concave and strictly increasing.

b) u is differentiable.

Proposition 3 *If the solution $(\hat{x}_n)$ of Problem $(\mathcal{P})$ is decreasing and the instantaneous utility function u satisfies Assumption U, then $(\gamma_n = \frac{u'(\hat{x}_1)}{u'(\bar{x}_n)})$ is the unique sequence of discounting factors satisfying $\gamma_1 = 1$ for which $(\hat{x}_n)$ is a solution of:*

$$(\mathcal{P}_u) \begin{cases} \textit{Maximise } \sum_{n=1}^{\infty} \gamma_n u(x_n) \\ \sum_{n=1}^{\infty} x_n = 1 \\ x_n > 0 \textit{ for all } n \end{cases}$$

Proof. Let (x_n) satisfying the constraints of the problem $(\mathcal{P}_u)$. Then, since u is concave, for all n, $u(x_n) \leq u'(\hat{x}_n)(x_n - \hat{x}_n) + u(\hat{x}_n)$, so

$$\frac{u'(\hat{x}_1)}{u'(\hat{x}_n)} u(x_n) \leq u'(\hat{x}_1)(x_n - \hat{x}_n) + \frac{u'(\hat{x}_1)}{u'(\hat{x}_n)} u(\hat{x}_n)$$

Hence, if $\gamma_n = \frac{u'(\hat{x}_1)}{u'(\bar{x}_n)}$,

$$\sum_{n=1}^{\infty} \gamma_n u(x_n) \leq u'(\hat{x}_1)(\sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} \hat{x}_n) + \sum_{n=1}^{\infty} \gamma_n u(\hat{x}_n) = \sum_{n=1}^{\infty} \gamma_n u(\hat{x}_n)$$

the last equality coming from the constraint $\sum_{n=1}^{\infty} x_n = 1$. So, $(\hat{x}_n)$ is a solution of $(\mathcal{P}_u)$.

Conversely, if $(\hat{x}_n)$ is a solution of Problem $(\mathcal{P}_u)$, then, for all n, for all t in a small enough neighbourhood of 0, $\gamma_1 u(\hat{x}_1 + t) + \gamma_n u(\hat{x}_n - t) \leq \gamma_1 u(\hat{x}_1) + \gamma_n u(\hat{x}_n)$. By a simple derivation, one concludes that $\gamma_1 u'(\hat{x}_1) - \gamma_n u'(\hat{x}_n) = 0$, which together with the normalization $\gamma_1 = 1$ leads to the formula $\gamma_n = \frac{u'(\hat{x}_1)}{u'(x_n^*)}$. $\square$

Remark 6 (Hyperbolic discounting factors) *From Lemma 1, if* $u = \ln$, *then,* $\gamma_n = \frac{\beta_n}{\beta_1}$ *if* $n < n_0$, $\frac{\alpha_{n_0}}{\beta_1} \leq \gamma_{n_0} < \frac{\beta_{n_0}}{\beta_1}$ *and* $\gamma_n = \frac{\alpha_n}{\beta_1}$ *if* $n > n_0$. *Hence, if the lower bound* (α_n) *is hyperbolic,* $\alpha_n = \underline{\alpha}^n$ *for all* n *with* $\underline{\alpha} > 0$, *then the sequence of implicit discounting factors is decreasing and then hyperbolic of parameter* $\underline{\alpha} > 0$ *after the turning period* n_0 *, which is similar to a quasi-hyperbolic discounting factor representation of Phelps & Pollack [17] and Laibson [14]. Further note that if the upper bound* (β_n) *is hyperbolic* $\beta_n = \bar{\beta}^n$, *then strict consistency* ($\sum_{n=1}^{\infty} \beta_n > 1$ *and* $\sum_{n=1}^{\infty} \alpha_n < 1$) *is equivalent to* $1 > \bar{\beta} > 1/2 > \underline{\alpha} > 0$ *and* $n_0 > 1$ *as soon as* $\underline{\alpha}$ *is sufficiently small* ($\underline{\alpha}^2(1-\underline{\alpha}) \leq 1 - \bar{\beta}$), *and, therefore the implicit sequence of discounting factors is strictly decreasing and hyperbolic for* $n < n_0$, $\gamma_n = \bar{\beta}^{n-1}$ *and for* $n > n_0$, $\gamma_n = \underline{\alpha}^n / \bar{\beta}$.

Remark 7 *If* $u = \sqrt{n}$ *we get another discounting factors* $\gamma_n = \sqrt{\frac{\beta_n}{\beta_1}}$ *if* $n < n_0$, $\sqrt{\frac{\alpha_{n_0}}{\beta_1}} \leq \gamma_{n_0} < \sqrt{\frac{\beta_{n_0}}{\beta_1}}$ *and* $\gamma_n = \sqrt{\frac{\alpha_n}{\beta_1}}$ *if* $n > n_0$. *So, if the lower bound* (α_n) *is hyperbolic of parameter* $\underline{\alpha} > 0$, *then the implicit discounting factor is decreasing and then hyperbolic of parameter* $\sqrt{\underline{\alpha}} > 0$ *after the turning period* n_0 *, which is again similar to a quasi hyperbolic discounting factor.*

Let us now consider the problem where the strictly decreasing sequence of discounting factors (γ_n) is given, and we look for a concave strictly increasing instantaneous utility function u such that $(\hat{x}_n)$ is a solution of the problem $(\mathcal{P}_u)$.

Proposition 4 *If the solution* $(\hat{x}_n)$ *is decreasing and the non-negative sequence of discounting factors* (γ_n) *is strictly decreasing, there exists a strictly increasing concave utility function* u *on* $]0, +\infty[$ *such that* $(\hat{x}_n)$ *is a solution of the problem* $(\mathcal{P}_u)$.

Proof. Without any loss of generality, we assume that $\gamma_1 = 1$. We define the function u as a piecewise affine function as follows: $u(\hat{x}_1)$ is an arbitrary real number.

$$u(t) = \begin{cases} u(\hat{x}_1) + (t - \hat{x}_1) & \text{for all } t \geq \hat{x}_1 \\ u(\hat{x}_1) + \frac{1}{\gamma_2}(t - \hat{x}_1) & \text{for all } t \in [\hat{x}_2, \hat{x}_1] \\ \vdots \\ u(\hat{x}_1) + \frac{1}{\gamma_2}(\hat{x}_2 - \hat{x}_1) + \ldots + \frac{1}{\gamma_n}(t - \hat{x}_{n-1}) & \text{for all } t \in [\hat{x}_n, \hat{x}_{n-1}] \\ \vdots \end{cases}$$

Since the sequence $(\hat{x}_n)$ is strictly decreasing and converges to 0, the function u is well defined on $]0, +\infty[$. Since the sequence $(1/\gamma_n)$ is strictly increasing, the function u is concave. From the above formula, one deduces that the super-gradient of u at $\hat{x}_n$ is $[\frac{1}{\gamma_n}, \frac{1}{\gamma_{n+1}}]$ since $\frac{1}{\gamma_n}$ and $\frac{1}{\gamma_{n+1}}$ are respectively the slope of the graph of u on the interval $[\hat{x}_n, \hat{x}_{n-1}]$ and $[\hat{x}_{n+1}, \hat{x}_n]$. So, for all $x \in]0, +\infty[$,

$$u(x) \leq \frac{1}{\gamma_n}(x_n - \hat{x}_n) + u(\hat{x}_n)$$

Consequently, mimicking the proof of the previous proposition, one concludes that $(\hat{x}_n)$ is a solution of the problem $(\mathcal{P}_u)$. $\square$

Remark 8 *Note that u is not unique even among the piecewise affine functions associated to the sequence $(\hat{x}_n)$. Indeed, we obtain the same result with the function $\bar{u}$ defined by:*

$$\bar{u}(t) = \begin{cases} \bar{u}(\hat{x}_1) + (t - \hat{x}_1) \quad \textit{for all } t \geq \hat{x}_2 \\ \bar{u}(\hat{x}_1) + (\hat{x}_2 - \hat{x}_1) + \frac{1}{\gamma_2}(t - \hat{x}_2) \quad \textit{for all } t \in [\hat{x}_3, \hat{x}_2] \\ \vdots \\ \bar{u}(\hat{x}_1) + (\hat{x}_2 - \hat{x}_1) + \frac{1}{\gamma_2}(\hat{x}_3 - \hat{x}_2) + \ldots + \frac{1}{\gamma_{n-1}}(t - \hat{x}_{n-1}) \\ \qquad \textit{for all } t \in [\hat{x}_n, \hat{x}_{n-1}] \\ \vdots \end{cases}$$

Remark 9 *The function u (resp. $\bar{u}$) defined above is the smallest (resp. the greatest) one on the interval $]0, \hat{x}_1]$ among the concave utility functions for which $(\hat{x}_n)$ is a solution of $(\mathcal{P}_u)$ for a given value of $u(\hat{x}_1)$.*

We prove the result for u and we leave the other part to the reader. Let v be another suitable concave utility function satisfying $v(t) \leq u(t)$ for all $t \in]0, \hat{x}_1[$ and $v(\hat{x}_1) = u(\hat{x}_1)$. Then, with the same argument as in the proof of the previous proposition, we deduces that $\frac{1}{\gamma_2}$ belongs to the supergradient of u at $\hat{x}_2$. So, on the interval $[\hat{x}_2, \hat{x}_1]$, $v(\hat{x}_1) \leq v(\hat{x}_2) + \frac{1}{\gamma_2}(\hat{x}_1 - \hat{x}_2)$. So, $v(\hat{x}_2) \geq u(\hat{x}_1) + \frac{1}{\gamma_2}(\hat{x}_2 - \hat{x}_1) = u(\hat{x}_2)$. Then, since $v(\hat{x}_1) = u(\hat{x}_1)$, one gets that $v(\hat{x}_2) \geq u(\hat{x}_2)$ so, since $v \leq u$ on $]0, \hat{x}_1[$, $v(\hat{x}_2) = u(\hat{x}_2)$. For all $t \in]\hat{x}_2, \hat{x}_1[$, since v is concave,

$$\begin{aligned} v(t) &= v\left(\frac{\hat{x}_1 - t}{\hat{x}_1 - \hat{x}_2}\hat{x}_2 + \frac{t - \hat{x}_2}{\hat{x}_1 - \hat{x}_2}\hat{x}_1\right) \\ &\geq \frac{\hat{x}_1 - t}{\hat{x}_1 - \hat{x}_2}v(\hat{x}_2) + \frac{t - \hat{x}_2}{\hat{x}_1 - \hat{x}_2}v(\hat{x}_1) \\ &= \frac{\hat{x}_1 - t}{\hat{x}_1 - \hat{x}_2}u(\hat{x}_2) + \frac{t - \hat{x}_2}{\hat{x}_1 - \hat{x}_2}u(\hat{x}_1) \\ &= u(t) \end{aligned}$$

The last equality comes from the fact that u is affine on the interval $[\hat{x}_2, \hat{x}_1]$. So, since $v \leq u$ on $]0, \hat{x}_1[$, $v = u$ on the interval $[\hat{x}_2, \hat{x}_1]$. By a recursive argument, we prove that $v = u$ on the interval $[\hat{x}_{n+1}, \hat{x}_n]$ for all n, so $v = u$ on $]0, \hat{x}_1]$.

At this stage, we have no restriction on the discounting factors compatible with the solution $(\hat{x}_n)$. Nevertheless, this is at the cost of a utility function which could tend to $-\infty$ at 0. If we restrict ourself to consider a continuous utility function on the interval $[0, +\infty[$, then the asymptotic behaviour of the discounting factors is bounded from below by the asymptotic behaviour of the lower bound (α_n).

Proposition 5 *Let us assume that the solution $(\hat{x}_n)$ is decreasing. If there exists a strictly increasing continuous concave utility function u on $[0, +\infty[$ such that $(\hat{x}_n)$ is a solution of the problem $(\mathcal{P}_u)$, then non-negative strictly decreasing sequence of discounting factors (γ_n) are such that the series $\left(\frac{\alpha_{n-1} - \alpha_n}{\gamma_{n-1}}\right)$ is convergent.*

Proof. We have described above the greatest utility function $\bar{u}$ compatible with the sequence of discounting factors (γ_n). So, if $\bar{u}$ is continuous at 0, then for all n, $\bar{u}(\hat{x}_n) \geq \bar{u}(0)$, which implies that

$$(\hat{x}_1 - \hat{x}_2) + \frac{1}{\gamma_2}(\hat{x}_2 - \hat{x}_3) + \ldots + \frac{1}{\gamma_{n-1}}(\hat{x}_{n-1} - \hat{x}_n) \leq \bar{u}(\hat{x}_1) - \bar{u}(0)$$

So, the series $\left(\frac{\hat{x}_{n-1}-\hat{x}_n}{\gamma_{n-1}}\right)$ is convergent. But, we know from Lemma 1, that $\hat{x}_{n-1} - \hat{x}_n = \alpha_{n-1} - \alpha_n$ for $n-1 > n_0$. So, the series $\left(\frac{\alpha_{n-1}-\alpha_n}{\gamma_{n-1}}\right)$ is convergent since it differs from the previous one only by a finite number of entries. $\square$

Remark. To illustrate the previous result, let us assume that the lower bound ($\alpha_n = \underline{\alpha}^n$) and the discounting factor ($\gamma_n = \underline{\gamma}^n$) are both hyperbolic. Then $\frac{\alpha_{n-1}-\alpha_n}{\gamma_{n-1}} = (1-\underline{\alpha})\left(\frac{\underline{\alpha}}{\underline{\gamma}}\right)^{n-1}$. So, the constant discounting factor $\underline{\gamma}$ must be higher than the lower bound actualisation factor $\underline{\alpha}$.

If the lower bound series (α_n) converges "slowly" to 0, like $\alpha_n = \frac{1}{n^\nu}$ for ν given in $]1, +\infty[$, then the solution is not compatible with a bounded below utility function and an hyperbolic discounting factor ($\gamma_n = \underline{\gamma}^n$). Indeed,

$$\frac{\alpha_{n-1} - \alpha_n}{\gamma_{n-1}} = \frac{\frac{1}{n^\nu} - \frac{1}{(n+1)^\nu}}{\underline{\gamma}^{n-1}} \geq \frac{\nu}{n(n+1)^\nu \underline{\gamma}^n}$$

The last series is not convergent since the terms does not converge to 0, so the series $\left(\frac{\alpha_{n-1}-\alpha_n}{\gamma_{n-1}}\right)$ is not convergent.

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