



HAL
open science

Dialogical Logic and Constructive Type Theory New Explorations

Shahid Rahman, Lily Gallois

► **To cite this version:**

Shahid Rahman, Lily Gallois. Dialogical Logic and Constructive Type Theory New Explorations. Springer. The Architecture and Archeology of Modern Logic, Springer, In press. halshs-04144017

HAL Id: halshs-04144017

<https://shs.hal.science/halshs-04144017v1>

Submitted on 28 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Dialogical Logic and Constructive Type Theory

New Explorations

Shahid Rahman and Lily Gallois

Université de Lille UMR-CNRS 8163: STL, France.

Contributing authors: shahid.rahman@univ-lille.fr;
zermelomath@gmail.com;

Abstract

The present contribution delves into a recent development of the dialogical rules for proof-theory penned by Ansten Klev that arose from Per Martin-Löf's take on assertoric knowledge as involved in the correctness criterion for assertion. The main aim of our paper is to set out the central steps towards a new Dialogical Framework called Immanent Reasoning II, which on the one hand integrates some of the CTT-rules developed by Klev, but on the other, develops further the general tenets of the dialogical perspective on meaning and logic. In contrast to Klev's approach, IR-II contains rules that indicate how to develop plays and winning strategies for a thesis.

Keywords: dialogical logic, constructive type theory, dialogical meaning explanations, local meaning, local reasons, pragmatism, ethics and logic.

1 Introduction

The present contribution delves into a recent development of the dialogical rules for proof-theory penned by Ansten Klev (202x) that arose from Per Martin-Löf (2017)'s take on assertoric knowledge. According to this view, the more fundamental application of the notion of assertoric knowledge is in providing the correctness criterion for assertion: J is correct iff the assertor assertorically knows J . An inference is valid iff it preserves correctness from premisses to conclusion – so this definition depends on the prior definition of correctness, which in turn makes use of the notion of assertoric knowledge. The main aim of our paper is to set out the central steps towards a new Dialogical Framework called *Immanent Reasoning* II (IR-II), which on the one hand

integrates some of the CTT-rules developed by Klev (202x), but on the other it studies these rules in the larger context of the general tenets of the dialogical perspective on meaning and logic. In contrast to Klev's approach, which has not been designed as a system for developing demonstrations within it, IR-II contains rules that indicate how to develop plays and winning strategies for a thesis.

Basics in Dialogical Logic

Dialogical logic is a dialogue-based framework for meaning, knowledge and logic rooted in a research tradition on argumentation that goes back to dialectics in Greek Antiquity¹, when semantic, epistemic and ethical issues were approached through debates in which opposing parties discussed a thesis through questions and answers.

Contemporary "Dialogic logic" was conceived by Paul Lorenzen in 1958, the year of his talk *Logik und Agon* held in Rome at the Twelfth World Congress of Philosophy and was further developed by Kuno Lorenz. Initially *Dialogical logic* aimed at overcoming some limitations of Lorenzen's own *Operative Logik*, Lorenzen (1955); Lorenz (2001); Schroeder-Heister (2008).

The work of Lorenzen and Lorenz and the *Constructivist School of Erlangen*, took later on the shape of an overall pragmatist and game-theoretical *Dialogical Framework* for philosophy of language, logic and sciences. This framework was the outcome of the intertwining of constructive mathematics and logic Lorenzen (1978), a dialectical reading of the emergence of philosophy of language and logic in the Ancient Greek tradition Ebbinghaus and Aristotle (1964); Lorenz and Mittelstrass (1966, 1967); Lorenz (2011), and Wittgenstein's notion of language games Lorenz (1970); Kamlah and Lorenzen (1967); Lorenz (2008, 2011), with a special accent on the ethical undergirding of conceptual thinking Lorenzen (1969); Lorenzen and Schwemmer (1973); Lorenz (2021). Lorenz added three important methodological perspectives to the project, namely (i) the reconstruction of ancient theories of meaning and knowledge including not only Plato but also Indian traditions, (ii) a semiotic, Peircean, understanding of language games, and (iii) the explicit use of mathematical game theory in logic Lorenz (1961, 1998, 2008, 2010, 2015, 2021).

Dialogical logic studies dialogues; but it also takes the form of dialogues. In a dialogue, two parties (players) argue on a thesis (a certain statement that is the subject of the whole argument) and follow certain fixed rules in their argument.

The player who states the thesis is the *Proponent*, called **P**, and his interlocutor, the player who challenges the thesis, is the *Opponent*, called **O**. In challenging the Proponent's thesis, the Opponent is requiring the Proponent that he defends his statement by a sequence of moves, shaped by the adopted rules for the development of a dialogue, involving declarative utterances, called *statements*, and interrogative utterances called *requests*.

The rules that shape a dialogue are divided into two kinds: *particle rules* and *structural rules*, also called development rules. The particle rules set the local dialogical

¹Though there have been parallel, often independent approaches in other traditions and periods, particularly in the Indian Subcontinent, Lorenz (1998), and also in the context of Medieval Islamic Juridical Argumentation Theories, that were developed long before the reception of Aristotelian Logic, Young (2016, 2022).

meaning explanation of an expression, the structural rules **set** its *global dialogical meaning explanation*.

More precisely, the rules for local meaning, namely particle rules – i.e., rules for connectives and quantifiers, explain the meaning of an expression independently of the rules setting the development of a dialogue (also called play) and independently of who of the interlocutors is the one setting the thesis, i.e., these rules prescribe how to challenge a statement and how to defend it, but they do not fix who is the challenger and who is the defender.

The dialogical meaning explanations for the standard first order logical constants, in the original setting of Lorenzen and Lorenz, are the following Rückert (2011); Clerbout (2014c,b); Clerbout and McConaughey (2022) :

\vee	Request	Answer	\wedge	Request	Answer
$\mathbf{X}! A \vee B$	$\mathbf{Y}?\vee$	$\mathbf{X}! A$ or $\mathbf{X}! B$	$\mathbf{X}! A \wedge B$	$\mathbf{Y}?\wedge_L$ or $\mathbf{Y}?\wedge_R$	$\mathbf{X}! A$ $\mathbf{X}! B$
\mathbf{X} : I state the following disjunction. \mathbf{Y} : Which side are you prepared to defend? \mathbf{X} : I choose the left side. or \mathbf{X} : I choose the right side.			\mathbf{X} : I state the following conjunction. \mathbf{Y} : I choose you to state the left side. or \mathbf{Y} : I choose you to state the right side. \mathbf{X} : I state the requested side.		

\supset	Request	Answer	\neg	Request
$\mathbf{X}! A \supset B$	$\mathbf{Y}! A$	$\mathbf{X}! B$	$\mathbf{X}! \neg A$	$\mathbf{Y}! A$
\mathbf{X} : I state the following implication. \mathbf{Y} : I grant you the antecedent, you must state the consequent \mathbf{X} : Yes, this is what I now state.			\mathbf{X} : I state the following negation. \mathbf{Y} : I rather state the contrary to what you just stated.	

\exists	Request	Answer	\forall	Request	Answer
$\mathbf{X}! \exists x A(x)$	$\mathbf{Y}?\exists$	$\mathbf{X}! A[a/x]$	$\mathbf{X}! \forall x A(x)$	$\mathbf{Y}?\forall a$	$\mathbf{X}! A[a/x]$
\mathbf{X} : I state that there is at least one x which is an A . \mathbf{Y} : Who is it? \mathbf{X} : a is one of those x that are an A .			\mathbf{X} : I state that every x is an A . \mathbf{Y} : This should also hold for a , right? \mathbf{X} : Indeed, a is an A .		

Structural rules set the local dialogical meaning explanations in the context of the development of a play, development rules set global dialogical meaning explanations. Several development rules are possible. Each determines which patterns of reasoning count as meaningful and which not, e.g., the choice between the rules **SR1i** and **SR1c** determines if the reasoning is shaped by either an intuitionistic or classical theory of meaning. The present rules are basic in the sense that different logics and meaning approaches might require extending them by either adding new rules or adding further conditions to the existing rules.

SR0 (Starting rule)

A play starts with a player stating a proposition called the thesis; that player becomes the Proponent (**P**) and the move is labeled move 0².

The other player, the Opponent (**O**), chooses a repetition rank determining how many times she is allowed to challenge or defend any move in a play.

It is usually enough for **O** to choose a repetition rank of 1 ($m := 1$), labelled move 1 and for **P** to choose a repetition rank of 2 ($n := 2$), labelled move 2.

Some remarks on SR0 — This rule should assure that plays are finite (though there might be an infinite number of them). Indeed, one important feature of Lorenzen–Lorenz conception of proposition and statement is its finiteness: for an expression to count as a proposition A there must exist an individual play about the statement A , that reaches a final position ~~must be reached~~ after a **finite number** of moves according to definite particle and structural rules Lorenz (2001). If language-games are to be conceived as mediators of meaning carried out by social interaction, these language-games must be games that we **can actually perform**. This is what Lorenz’s notion of dialogue-definiteness is about. Clerbout (2014c,b,a) delved into the metalogical consequences of the repetition rank and showed how it is related to the semi undecidability of first-order logic. One important general philosophical result of Clerbout’s work is that though every play must be finite, this does not prevent the existence of an infinite number of them.

SR1i (Intuitionistic game-playing rule)

After the repetition ranks have been chosen, each move is a challenge or a defense in reaction to a previous move, in accordance with the particle rules shown previously.

Each player can challenge the same previous move at most n times, where n is the player’s repetition rank, or defend against the adversary’s last unanswered challenge³.

SR1c (Classical game-playing rule)

After the repetition ranks have been chosen, each move is a challenge or a defense in reaction to a previous move, in accordance with the particle rules shown previously.

Each player can challenge *the same previous move, or defend against the same previous challenge*, at most n times, where n is that player’s repetition rank, even if it’s not the last unanswered challenge (contrary to the Intuitionistic rule).

Some remarks on SR1i and SR1c — The perspicuity and explanatory power won by a dialogical setting, become a salient issue when it aims at motivating the choice of some structural rules over others, such as the choice between **SR1i** and **SR1c**. Classical and intuitionistic connectives share the same local meaning, but differ only on one rule on global.

²This labeling has the effect that whereas even numbers indicate moves by P, odd numbers indicate moves by O.

³Notice that any challenge on a negation will always remain unanswered, since there is no defense against it. Thus, such a challenge will always count as the last challenge. This rule establishes that the duty of defending the last challenge first has priority over the repetition rank for defending.

Thus, in a dialogical setting, if there is some common ground between classical and intuitionistic approaches to meaning, this common ground is to be found in the *local dialogical meaning explanations*, given by player independent rules. By 1996 this stance on plural approaches to meaning and knowledge was generalized and developed into the project of Dialogical Pluralism launched by S. Rahman, Keiff (2007); Rückert (2011).

SR2.1 (Formal rule)

P may not state an elementary proposition unless **O** stated it first. Elementary propositions cannot be challenged. This rule shapes *formal plays or purely assertoric plays, whereby no thesis can state an elementary proposition.*

SR2.2 (Socratic rule)

P may not state an elementary proposition unless **O** stated it first. Elementary propositions stated by **P** can be challenged, but not those of **O**. This rule shapes *material plays or material assertoric plays*. In the case that the thesis states an elementary proposition such as $!A$, **O** can launch the challenge $?_A$, iff she (**O**) did not state the same elementary proposition before. The elementary propositions of material plays have content⁴.

This requires a special rule for each of the elementary propositions involved in a play, which stipulate that **O** can state some primitives when requested. For example, in the case of statements involving natural numbers, the Socratic Rule stipulates that **P** can always ask **O** to state that 0 is a natural number. Once **O** has responded to such a request, **P** can overtake it. This is combined with a rule for local meaning establishing that for any statement by **X** of the form the successor of n is a natural number, the questioner can request **X** to further state that n is also a natural number. In the case of empirical elementary propositions some verification process determined by epistemological context at work will determine which elementary propositions can be stated by **O** without challenge.

Some remarks on SR2 — These rules constitute one of the main features of the Dialogical Framework. In fact, while working out a new concept of judgmental content, Martin-Löf (2015, 2017, 2019a,b) highlighted the contribution of the Dialogical Framework to the notion of assertoric knowledge and to Göran Sundholm (1997, 2012, 2019)’s germane concept of *epistemic assumption*, which turns out to have an essential role in an explanation of validity of inference that does not fall into the very well-known justification circle. The *Socratic Rule* owes its name to Marion and Rückert (2015), who indicate some crucial texts in Plato’s and Aristotle’s *Dialectics* as its historical source – see too Crubellier et al. (2019); McConaughey (2021). It extends Lorenzen (1978) and Lorenz’s *Formal Rule* to material dialogues. Usually, the statements governed by this rule are elementary statements and definitions, Clerbout and McConaughey (2022). These rules are absent in Jaakko Hintikka’s *Game Theoretical*

⁴In fact, material assertoric plays, rather than formal plays, are the ones involved in the CTT approach to assertoric knowledge.

Semantics (GTS) and also in the further developments of it such as the ones by van Benthem and collaborators – cf. Hintikka (1973) pp.77-82.

SR3 (Winning rule)

The play ends when it is a player’s turn to make a move but (s)he has no available move left. That player loses and the other player wins.

Example of play

O			P		
				$! A \vee B \supset B \wedge A$	0
1	$m := 1$			$n := 2$	2
3	$! A \vee B$	0		$! B \wedge A$	4
5	$?_L \wedge$	4		$! B$	8
7	$! B$		3	$?_V$	6

- **P** states $A \vee B \supset B \wedge A$ (move 0)
- **O** and **P** choose their repetition rank (moves 1 and 2)
- **O** challenges \supset by granting the antecedent (move 3) and **P** answers with the consequent (move 4)
- **O** challenges \wedge from move 4 by asking **P** to defend the left side (move 5). **P** can’t answer immediately since **SR2**.
- **P** challenges \vee from move 3 by asking **O** to chose one side (move 6). **O** chooses the left side, and concedes B (move 7).
- **P** answers the challenge at move 5 with B (move 8) and finally wins !

Winning strategy and logical validity

The notion of winning a play is not enough to render the notion of inference or of logical validity: winning a play by **P** does not amount to the (logical) validity of the thesis. In the previous example, $A \vee B \supset B \wedge A$ is of course not valid, but **P** wins. However, **P** wins because **O** made the wrong choice: when **O** defends the disjunction, she chooses to state precisely what **P** needs to state, and the chosen repetition rank does not allow her to challenge twice the same move. There is obviously another play where **O** wins, namely, if she chooses the left side of the conjunction. Dually, a valid thesis can be lost because **P** this time, makes the wrong choice; e.g. **P** can lose a play with the thesis $A \wedge B \supset B$, if he chooses repetition rank 1 and asks **O** to state the left side of the conjunction.

Another telling example has been mentioned by Piecha (2015), namely the case of the thesis $\neg A \vee (A \supset A)$, which can be lost by **P** when playing with **SR1i**, though it is intuitionistically valid, namely when **P** chooses to answer the challenge on the disjunction with the left side.

There are several ways to define a winning strategy within a Dialogical Framework, Krabbe (1985); Clerbout (2014c,b,a); Rahman et al. (2018); Lion (2023). For the sake

of a simple presentation we will offer a variation of Felscher (1985), which can be also found in Redmond and Fontaine’s dialogical tables with branches Redmond and Fontaine (2011), and in Piecha (2015) – Rahman et al. (2018), pp. 90-109 describe a method for finding a winning strategy out of a succession of plays. A player **X** has a winning strategy if for every move made by the other player **Y**, player **X** can make another move, such that each resulting play is eventually won by **X**.

Definition. A *strategy* for **P** stating a thesis *A* is a tree *T* defined by:

- The root (node of depth 0) of *T* is the move **P** ! *A*
- If the node is an **O**-move (i.e., if the number of the node is odd), then it has exactly one successor node (which is a **P**-move)⁵.
- If the node is a **P**-move (i.e., if the number of a node is even), then it has as many successor nodes as there are possible moves for **O** at this position⁶.

A strategy for **P** is a **winning strategy** iff all the branches end by an elementary statement by **P**.

Proposition. A proposition is valid if **P** has a winning strategy for a thesis stating this proposition.

Proof. There are proofs in both case of classical or intuitionistic first-order logic in Felscher (1985); Clerbout (2014c,b,a). □

Some remarks on winning strategy — Winning strategies for quantifier-free theses are always finite trees. Winning strategies for first-order theses can, in general, be trees of countably infinitely many finite branches (each branch is a play). Infinite winning strategies for **P** can be avoided by introducing some provisos grounded on the following rationale led by the commitments prescribed by the Formal (Socratic) Rule: when **O** has the chance to choose how to challenge a universal or defend an existential, her optimal move is to always choose an *a* which is new with regard to the development of the play. On the contrary **P**, who need that **O** states an elementary proposition; challenges a universal of **O** or defends an existential by copying **O**’s own choices when she challenged a universal or defend an existential when is possible.

Dialogical meaning explanations and the invariant core

Central to the dialogical perspective is that *preserving local meaning* plays the role that *admissibility* had in Lorenzen’s earlier *Operative Logik*. Indeed, as pointed out by Schroeder-Heister (2008), admissibility was Lorenzen’s proof-theoretical venue to meaning and what distinguished his approach from formalism: admissibility allows us to identify a core of meaning invariant in relation to a set of rules. Admissibility continues to be one of the main tenets of contemporary proof-theoretical approaches

⁵Branches are introduced by **O**’s choices such as when she challenges a conjunction or when she defends a disjunction.

⁶If there are many options for **P** only one will be retained. Indeed if there is a winning strategy for **P** it is built out of one of **P**’s choices.

after Prawitz's developed attributions of admissibility into a system of normalization procedures, Lecomte and Quatrini (2011a,b).

After the dialogical turn, the identification of the invariant core of meaning came down to identifying the local meaning at work in both structural rules and strategic rules. Moreover, attributions of admissibility to an inference are now replaced by attributions of dialogue-definiteness: an expression enjoys the property of *dialogue-definiteness*, if it can be stated as thesis in an individual finite Zero-Sum-play, that ends with loss or win and is shaped by structural rules that preserve local dialogical meaning explanations. Actually dialogue-definiteness is the condition for being counted as a proposition Lorenz (2001). However, win or loss amounts neither to truth-value definiteness nor to proof-theoretical definiteness. Indeed, according to the novel approach, truth-value indefinite propositions such as the Law of Excluded Middle, can give place to a decidable play on the thesis stating that Law, where \mathbf{P} loses the game, but this of course does not mean that there is a proof for its refutability \Rightarrow , i.e., that \mathbf{P} has a winning strategy for its negation.

Furthermore, in the dialogical setting, the justification of inference rules amounts to showing that they preserve local meaning, that is, that they preserve a core of dialogical meaning explanation invariant in relation to players, and to structural rules and strategic rules⁷.

- In short, from the Dialogician's perspective, the root of inference rules as expressed by rules for building a winning strategy, is the player independent local dialogical meaning explanations put at work by structural rules that preserve them. Dialogical rules are not a dialogical interpretation of inference rules but they justify them.
- The former point is crucial for any form of dialogical pluralism, since choosing between different patterns of reasoning assumes an invariant core in relation to which the norms for determining both the development of a play and of a winning strategy are to be constituted.

Actually, as pointed out by Lion (2020) pp. 125-159; 265-310, the invariant core itself is the result of a prior dynamic semiotic process whereby this core meaning has been fixed \Rightarrow though the process can later on be launched again. According to this view, the choices inherent in dialogical pluralism start before the discussion on structural rules come to the fore.

As pointed out by Clerbout and McConaughey (2022), the development of the Dialogical Framework experiences currently a thriving interest especially in the field of argumentation theory, Gethmann (1982); Barth and Krabbe (1982); Walton (1984); Johnson (1999); Woods and Walton (1989); Woods et al. (2002); van Eemeren and Grootendorst (2004); Prakken (2005); Vaidya (2013); Novaes (2015, 2020); Dutilh Novaes and French (2018); French (2019), history and philosophy of logic

⁷This is why tonk-like operators can be dismissed in the dialogical framework, Rahman and Redmond (2016). Moreover, on one hand Crubellier et al. (2019), further developed by McConaughey (2021), interpret Aristotelian proofs by ecthesis, as laying down the local meaning of the quantifiers given the Dictum de Omni, on the other we know since the work of Ebbinghaus and Aristotle (1964), that Aristotelian reductions to the perfect syllogism, inspired Lorenzen for the invention of admissibility, who by the way was the first on proposing a dialogical reconstruction of Aristotle's proof by ecthesis. Thus, according to Lorenzen's interpretation, reductions to perfect syllogism, if successful, show that the reduced rules preserve the local meaning layed down by the dialectical rules for quantifiers as encoded by the rules for ecthesis.

and mathematics, Ebbinghaus and Aristotle (1964); Lorenz and Mittelstrass (1966, 1967); Miller (2020); Keffer (2001); Yrjönsuuri (2001); Hintikka (2006); Novaes (2007); Castelnérac and Marion (2009); Crubellier (2011); Clerbout et al. (2011); Marion and Rückert (2015); Gorisse (2017, 2018); Crubellier et al. (2019); Uckelman (2013); Young (2016, 2022); McConaughey (2021); Iqbal (2022), non-classical logics, Rahman and Rückert (1999); Keiff (2007, 2010); Rückert (2011), computer science, Blass (1992); Lecomte and Quatrini (2011a,b); Fermüller (2003), applied linguistics, legal reasoning, artificial intelligence and game theory, Ranta (1988, 1994); Hintikka (1996); Hintikka and Sandu (1997); Prakken (2005); Ginzburg (2012). In the present paper we will focus on the deployment of the Dialogical Framework within Per Martin-Löf’s Constructive Type Theory (CTT), Martin-Löf (2015, 2017, 2019a,b); Klev (2022, 202x); Rahman et al. (2018), which takes anew one of the initial motivations of the dialogical logic, namely the constructivist foundations of mathematics and logic, Beth (1955); Coquand (1995); Felscher (1985); Sørensen and Urzyczyn (2006); Alama et al. (2011); Uckelman (2013); Sterling and Angiuli (2021).

2 Immanent Reasoning

2.1 Enriching the Expressivity of the Dialogical Framework

Dialogues are games of giving and asking for reasons; yet, in the standard dialogical framework, the reasons for each statement are left implicit. The *Immanent Reasoning* (IR) framework imports a form of judgement from Martin-Löf’s Constructive Type Theory (CTT) in Martin-Löf and Sambin (1984) in order to make these reasons explicit in Rahman et al. (2018). Fully developed, statements thus have the following form:

$$\mathbf{X}! a \varepsilon B^8$$

where B is a proposition and a its local reason, i.e., the particular, circumstantial reason that entitles one to state B , which is read as \mathbf{X} states that a provides evidence for B . In this fashion, the reasons interlocutors have for making a statement are specified at the object-language level. As expected, IR distinguishes local reasons and strategic reasons. Local reasons are brought forth in particular plays, whereas strategic reasons are a recapitulation of all the relevant plays for constituting a winning strategy.

Moreover, IR allows us to enrich the standard dialogical framework within an interactive theory of meaning that makes use of the expressivity of the fully interpreted languages at work in CTT. This expansion of the dialogical framework does not only allow to integrate the CTT analyses of the notorious donkey-sentences, branching quantifiers, and of generalized quantifiers (such as Most, At-Most, At-Least, and so on) Sundholm (1986, 1989); Ranta (1994), but it also provided a venue for studies in the history and philosophy of logic that brings to the light the rationale behind ancient and medieval logical analyses Crubellier et al. (2019); Iqbal (2022), the dialogical stance on contemporary and historical legal reasoning Rahman and Zarepour (2019).

⁸Whereby “ ε ” stands for type-token relation or element of a set. The most recent notation in CTT is “ $;$ ”, however, we use “ ε ” which was used by Lorenz and Mittelstrass (1966, 1967) for the instantiation-relation within their dialogical analysis of Plato’s theory of meaning. In *Immanent Reasoning* “ $a \varepsilon \alpha$ ” stands for a local reason for α .

2.2 Immanent Reasoning revisited

While working out a new concept of judgmental content, Martin-Löf (2015, 2017, 2019a,b) highlighted the contribution of the Dialogical Framework to the notion of *assertoric knowledge* and to Göran Sundholm (1997, 2012, 2019)’s germane concept of *epistemic assumption*, which turns out to have an essential role in an explanation of validity of inference that does not fall into a well-known justification-circle⁹. According to this view, when justifying an inference it is enough to assume that *someone else* has *asserted* the premises and under that assumption, show that *you may then assert* the conclusion¹⁰ Klev (2022, 202x).

In those lectures and texts, which discuss the deontic undergirding of logic, Martin-Löf points out that the main philosophical tenet of the dialogical approach is that judgmental content amounts to the pragmatic and normative notion of a task to be solved. On this account, *a judgment, or an assertion, is correct if and only if the agent making it knows how to (is able to, can) perform the task that constitutes its content* Klev (202x).

This takes us back to the inception of Dialogical Logic and to the work of Lorenz (2010) and to his formulation of *meaning as the mastery of an action schema*. Moreover, the main philosophical tenet of dialogicians, Lorenz observes, is the rejection of the usual reductionist ways to understand the interface between semantics and pragmatics.

Lorenz’s general dialogical principle is that the interface between semantics and pragmatics should be understood neither as the result of the *semantization of pragmatics* at work in different varieties of formal semantics — where deontic, epistemic, ontological, and temporal constraints become truth-functional operators; nor as the result of the *pragmatization of semantics* as promoted by speech act theories — where a propositional kernel, when put into use, is complemented by moods yielding assertions, questions, commands and so on Lorenz (2011).

In short, Lorenz (2010)’s view is that the differentiation of semantic and pragmatic layers is the result of the articulation within one and the same utterance: each utterance displays in principle both features, as it signifies (semantic layer) and communicates (pragmatic layer).

In contrast to the proof-theoretical and inferentialist focus on assertions regarded as the minimal unit of conceptual content (Brandom (2000)), the Dialogical Framework, bestows moves, called *statements*, the role of basic units of meaning and knowledge rather than assertions. Whereas statements constitute some specific language games called “Plays”, and assertions constitute the basic units of strategies.

⁹The usual conception in mathematics of judgment known as demonstrated judgment yields a circle when it is used to justify inference. Indeed, “In mathematics, it is natural to count a judgment as known if it has been demonstrated. Under this account of knowledge, the explanation of the validity of inference becomes: the conclusion J can be demonstrated under the assumption that the premises J_1, \dots, J_n have been demonstrated. That a judgment has been demonstrated means, however, that it is the final conclusion in a chain of valid inferences (we count an axiom as the conclusion of a valid inference with no premises). At least for mathematical discourse, the explanation of inferential validity is therefore circular: it appeals to the notion of demonstrated judgment, which in turn is explained in terms of validity.” Klev (202x)

¹⁰The point is that *assertoric knowledge* is weaker than *apodictic knowledge*: the latter but not the former requires demonstration. The knowledge obtained by an epistemic assumption is of the assertoric kind. When discussing the notion of epistemic assertion Sundholm (2019) produced the following formulation (already present in some of Sundholm’s previous talks): “When I say therefore, I give others my authority for asserting the conclusion, given theirs for asserting the premisses.”

So, statements are, in the most general form they take in rules for local meaning, player independent; however, assertions encode a monological can-must **P**-perspective. Assertions encode the tasks that both players require to solve in order to build a winning strategy.

From the dialogical point of view, strategies are constituted by plays, such that the strategic objects won by a process of “recapitulation” on those plays encode the sequence of moves which need to be carried out in order to accomplish the task associated, now indeed, to the assertion at stake. Thus, though the dialogical perspective shares the view with CTT that propositions are neither assertorically nor apodictically known – what is known is that the proposition is true (at the strategy level) – it departs from the latter view by grounding knowledge – or acknowledged truth – expressed by assertions, on the more basic notion of statement.

The interaction of statements, governed by the rules setting the dialogical meaning explanations, and in particular by the *Socratic Rule*, constitutes the root of what at the strategic level yields assertoric knowledge¹¹.

The new approach to Dialogical Logic called *Immanent Reasoning* (IR), which imports some of the insights of the fully interpreted language of CTT Rahman et al. (2018), puts the play perspective in foreground. This seems to open a path for casting Martin-Löf’s dialogical rules in a setting that puts these rules into plays.

Perhaps the main general differences ensue from the fact that whereas CTT focuses on mathematics and demonstration, the main perspective of the Dialogical Framework, particularly after the developments of Kuno Lorenz (1970, 2010, 2011, 2021), is natural language.

The aim of the present chapter is to accomplish a double task, namely:

1. It aims at **setting** Martin-Löf’s dialogue rules **for CTT (DCTT)** as presented by Klev (2022) **in a framework** for dialogical plays regulated by the Formal (Socratic) Rule, the point of which is to break down a given content into simpler and simpler components, reaching eventually those elementary statements upon which assertoric knowledge is grounded. We restrict our study to the rules for logical constants.
2. It aims at offering an alternative formulation of IR, that we call IR-II, which should on one hand simplify the notation of the original IR by adopting some ideas of DCTT, and on the other, build bridges for crossing from the Dialogician’s perspective to the Inferentialist’s perspective. However, IR-II is not conceived as a Dialogue setting plus a ready-made theory of Inference, but a Dialogical grounding of Inference.

¹¹A usual misunderstanding is to assume that the *Formal Rule* occurs in both, Hintikka’s GTS and Lorenzen and Lorenz’s dialogical logic. This is a mistake which Ranta (1988) is not guilty of but it is ubiquitous in the literature. Though Hintikka adopted the dialogical interpretation of the logical constants of Lorenzen, he explicitly rejected the *Formal Rule*. According to Hintikka (1973) the Formal Rule does only provide *indoor games*, and so he proposes to appeal to a model theoretical perspective once the analysis procedure arrives at the level of elementary propositions; to this matter see too the further developments of GTS-approach published in the collection *Texts in Logic and Games*, launched by Johan van Benthem in 2007. Dutilh Novaes Novaes (2020) passes in silence over this difference (and its bearing on Sundholm’s notion of *epistemic assumption*) when she compares Lorenzen and Hintikka’s game theoretical stances on logic. In short, no assertoric knowledge or epistemic assumptions can be found in game theoretical approaches other than the dialogical one.

Of course, these objectives are on one side quite ambitious and on the other they do not include finite sets in general. This will require a larger development: the present paper should nevertheless take some first steps towards such a study.

3 The Dialogicians Stance: Towards Immanent Reasoning II

Martin-Löf's DCCT constitute important progress towards linking CTT and dialogical logic, since it avoids the proliferation of notation used in IR-I. Moreover, Martin-Löf's remarks on the contribution of dialogical logic to assertoric knowledge and valid inference is an important point towards motivating such an interactive stance.

From the dialogician's point of view, the contribution of the dialogical framework amounts to furnishing a purely interactive notion of meaning invariant to introduction and elimination rules and to the underlying structural rules. Justifying an inference rule eventually comes down to showing that it emerges from the interaction of local meaning explanations of the expressions involved. Moreover, instead of the type-checking mechanism assumed by the CTT framework, the dialogical framework is endowed with the decidable notion of dialogue-definiteness. In other words, the two new main innovations of dialogical logic are:

- The introduction of a level of meaning explanations invariant in relation to players, plays and strategies
- The introduction of a level of plays upon which a mechanism for checking dialogue definiteness is built and winning strategies are generated.

3.1 The Dialogical Meaning Explanations for IR-II

Disjunction, Conjunction, Existential Quantifier

The dialogical meaning explanations for disjunction, conjunction and existential are almost the same as those in type theory Klev (2022, 202x). The main difference is that the rules are formulated as player independent. The double arrow, \Rightarrow , occurring in the tables below, is to be understood as the play level analogue of the evaluation arrow used in Klev (2022, 202x). As discussed further on in our text, the task associated with a statement of the form $d \Rightarrow c \in C$ is solved by certain form of calculation which shows, when successful, how to yield c from d . The calculation itself, essentially a process of what Klev calls definition unfolding, can also be shaped by means of dialogical interaction rules.

\vee	Request	Answer	...
$\mathbf{X}! d \in A \vee B$	$\mathbf{Y} ?_{\vee d}$	$\mathbf{X}! d \Rightarrow L^{\vee}(a) \in A \vee B$ or $\mathbf{X}! d \Rightarrow R^{\vee}(b) \in A \vee B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\vee}$	$\mathbf{X}! L^{\vee}(a) \in A \vee B$ $\mathbf{X}! R^{\vee}(b) \in A \vee B$	$\mathbf{Y} ?_{L^{\vee}(a)}$ $\mathbf{Y} ?_{R^{\vee}(b)}$	$\mathbf{X}! a \in A$ $\mathbf{X}! b \in B$

\wedge	Request	Answer	...
$\mathbf{X}! d \in A \wedge B$	$\mathbf{Y} ?_{\wedge d}$	$\mathbf{X}! d \Rightarrow \langle a, b \rangle \in A \wedge B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\wedge}$	$\mathbf{X}! \langle a, b \rangle \in A \wedge B$	$\mathbf{Y} ?_{L^{\wedge}}$ or $\mathbf{Y} ?_{R^{\wedge}}$	$\mathbf{X}! a \in A$ $\mathbf{X}! b \in B$

\exists	Request	Answer	...
$\mathbf{X}! d \in (\exists x \in A) B$	$\mathbf{Y} ?_{\exists d}$	$\mathbf{X}! d \Rightarrow \langle a, b \rangle \in (\exists x \in A) B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\exists}$	$\mathbf{X}! \langle a, b \rangle \in (\exists x \in A) B$	$\mathbf{Y} ?_{L^{\exists}}$ or $\mathbf{Y} ?_{R^{\exists}}$	$\mathbf{X}! a \in A$ $\mathbf{X}! b \in B[a]$

Implication, Universal Quantifier

It is the implication where the difference with the type-theoretical dialogue rules is manifest. Indeed the dialogical meaning explanation of the implication, in a setting for dialogues where the Formal Rule is at work, requires distinguishing:

1. The player independent rules for local meaning
2. The effects of the global rules, which put at work the rules for local meaning in the context of the development of a play
3. The strategic meaning explanation that results from abstracting the function that sets the meaning dependence of the consequent upon the antecedent shaped by the play level global rules.

Concerning 1., let us recall that the local meaning explanations are not only player independent: they also set what counts as a legitimate move independently of optimality considerations. In the case of implication, the rules simply determine that challenging it amounts to the challenger's providing a local reason for the antecedent and the defender's providing a local reason for the consequent.

The notation $(L_{\mathbf{Y}}^{\supset}) R_{\mathbf{X}}^{\supset} \varepsilon A \supset B$ for the local reason does not yet read as a function; it simply indicates that the defense of the implication requires the defender, \mathbf{X} , to defend the consequent by committing to $b \varepsilon B$ after the challenger \mathbf{Y} commits to the antecedent by stating $a \varepsilon A$. In order to simplify the notation, we will drop in the tables below the indexes \mathbf{X} and \mathbf{Y} occurring in $(L_{\mathbf{Y}}^{\supset}) R_{\mathbf{X}}^{\supset}$ and in $(L_{\mathbf{Y}}^{\forall}) R_{\mathbf{X}}^{\forall}$. Who states the antecedent, and who states the consequent, is anyway clear when the implication is challenged.

\supset	Request	Answer	...	
$\mathbf{X}! d \varepsilon A \supset B$	$\mathbf{Y}?\supset d$	$\mathbf{X}! d \Rightarrow (L^{\supset}) R^{\supset} \varepsilon A \supset B$...	
...	Request	Answer	Request	Answer
...	$\mathbf{Y}?\supset$	$\mathbf{X}! (L^{\supset}) R^{\supset} \varepsilon A \supset B$	$\mathbf{Y}! a \varepsilon A$	$\mathbf{X}! b \varepsilon B$

\forall	Request	Answer	...	
$\mathbf{X}! d \varepsilon (\forall x \varepsilon A) B$	$\mathbf{Y}?\forall d$	$\mathbf{X}! d \Rightarrow (L^{\forall}) R^{\forall} \varepsilon (\forall x \varepsilon A) B$...	
...	Request	Answer	Request	Answer
...	$\mathbf{Y}?\forall$	$\mathbf{X}! (L^{\forall}) R^{\forall} \varepsilon (\forall x \varepsilon A) B$	$\mathbf{Y}! a \varepsilon A$	$\mathbf{X}! b \varepsilon B(a/x)$

Abstracting the function

The local reason for the implication indicates, that the answerer's commitment to the consequent is dependent upon the challenger providing a local reason for the antecedent. From these moves a function can be abstracted, whereby the challenge provides an argument for the function. But this abstraction takes place at the strategy level.

Notice that if an implication such as $d \varepsilon A \vee B \supset B \vee A$ constitutes the thesis of a dialogue, then there will be two plays for it which can be won by \mathbf{P} , dependent upon \mathbf{O} 's choice : If \mathbf{O} chooses the left side for defending the disjunction in the antecedent, \mathbf{P} can win by choosing the right side for defending the disjunction in the consequent. If \mathbf{O} chooses the right side for defending the disjunction in the antecedent, \mathbf{P} can win by choosing the left side for defending the disjunction in the consequent.

Now, from the carrying out of those plays, we abstract the function $f \varepsilon A \vee B \supset B \vee A$ – provided it has been verified that identical arguments yield identical values, and that a criterion of identity for functions is given, by producing the strategic object $\lambda(f) \varepsilon A \vee B \supset B \vee A$ ¹². The strategic object $\lambda(f)$ encodes the information that whatever results by breaking down what \mathbf{O} brings as an argument for the antecedent, namely $a \varepsilon A$ (or $b \varepsilon B$), \mathbf{P} can produce $f(a)$ (resp. $g(b)$) for the defense of the analyzed consequent. Notice that this abstraction procedure is similar to the one described by

¹²Verifying that the condition of functionality and that the criterion of identity for functions have been fulfilled, can be implemented by suitable dialogical rules set at the strategy level.

Sundholm (2013) for generalizing a demonstration with concrete content to *logical validity*.

3.2 Demonstrations and Winning Strategies

In his talk for the award of Rolf Schock prize, Martin-Löf (2022) summarized the dialogical view on inference with the following remarks:

The concluder receives the premises from the premissers, and in turn gives away, or passes on, the conclusion. As a result of receiving the premisses from the premissers, the concluder gets the right, or permission, to request the premissers to perform their respective tasks C_1, \dots, C_n . The validity of the rule is tantamount to the concluder’s ability to perform C when given this help from the premissers. Thus, the effect is that the premissers together with the concluder can perform the conclusion task C . You see the novelty that is not present in the usual explanations of inference and rules of inference: the novelty is that in an inference, the concluder gets the right to ask the premissers to perform their respective tasks C_1, \dots, C_n , and that means that the concluder gets helped by the premissers to perform these tasks. The validity of the rule is tantamount to the concluder’s being able to do C given this help from the premissers to do C_1, \dots, C_n . This is a less elegant formulation, but it has the advantage of bringing out the interactive character of inference more clearly than Sundholm’s formulation.

These words indeed stress the main feature of the interactive stance, which we mentioned while discussing the dialogue rules for implication. Now, the term “helping”, triggered an unfruitful discussion to decide if dialogical logic is collaborative or adversarial: whereas Hodges (2001) accuses the dialogical rules for implication and inference to involve collaboration rather than challenge, Novaes (2015, 2020) accuses dialogical logic for being only or mainly adversarial¹³. The truth is that, when the focus is on assertoric knowledge, the dialogical setting is both, and this is what the Formal (Socratic) Rule accomplishes.

As mentioned above, stipulating the Opponent’s (elementary) statements as unchallengeable, helps to put the thesis under the toughest possible scrutiny. Be that as it may, let us run a dialogue for proving the commutation of disjunction using IR’s local dialogue rules in the context of the **structural rules** for the development of a play and the definition of strategy.

¹³Let us recall once more the example of $A \vee B \supset B \wedge A$, but let us consider now a concrete material dialogue such as, say, $(1 = 1 \vee 1 = 0) \supset (1 = 0 \wedge 1 = 1)$, with repetition rank 1. Let us further assume that **O** is in a “good” mood and that, after conceding the antecedent and **P** is stating the consequent, **O** decides to ask for the right. Certainly, these are legitimate choices, but is **O** being collaborative? Moreover, let us switch again to the more general case of formal dialogue for $A \vee B \supset B \wedge A$, where the content is not apparent, and where **O**, after asking for the right, chooses herself A , if asked to choose for one of the sides of the disjunction. On one side **O** collaborates in stressing the meaningful components of the implication, on the other she is not by not submitting the thesis to the toughest possible scrutiny.

Let us recall that in dialogical logic a proposition is valid iff **P** has a winning strategy for a thesis constituted by a succession of relevant plays stating this proposition.

3.2.1 Example

Prove $e \varepsilon B \vee A$, given the premise $d \varepsilon A \vee B$

The idea is clear: whatever **O** chooses for solving the task attached to the premise, **P** can use it to solve his own task, depending upon **O**'s choice. So, the Opponent is indeed helping the Proponent to solve his own task. This is the way a dialogicians understands a dialogical demonstration. For the sake of simplicity, we silent the choice of repetition ranks.

Notational convention : In the context of IR the notation $\mathbf{P}! e \varepsilon B \vee A \llbracket d \varepsilon A \vee B \rrbracket$ indicates that **P** is committed to building a winning strategy for $e \varepsilon B \vee A$, provided **O** grants $d \varepsilon A \vee B$. Thus, a challenge on $\mathbf{P}! e \varepsilon B \vee A \llbracket d \varepsilon A \vee B \rrbracket$, amounts to **O** stating $d \varepsilon A \vee B$. In fact, this notation expresses the dialogical counterpart to the proof-theoretically inference stroke, whereby the double square brackets enclose the premises and the expression to the left of those brackets is the conclusion.

O			P		
				$\mathbf{P}! e \varepsilon B \vee A \llbracket d \varepsilon A \vee B \rrbracket$	0.0
0.1	$! d \varepsilon A \vee B$			$! e \varepsilon B \vee A$	0.2
1	$?_{\vee e}$	0.2		$! e \Rightarrow R^{\vee}(a) \varepsilon B \vee A$	8
3	$! d \Rightarrow L^{\vee}(a) \varepsilon A \vee B$		0.1	$?_{\vee d}$	2
5	$! L^{\vee}(a) \varepsilon A \vee B$		3	$?_{\vee}$	4
7	$! a \varepsilon A$		5	$?_{L^{\vee}(a)}$	6
9	$?_{\vee}$	8		$! R^{\vee}(a) \varepsilon B \vee A$	10
11	$?_{R^{\vee}(a)}$	10		$! a \varepsilon A$	12

- 0.0. **P** I can prove $e \varepsilon B \vee A$, provided $d \varepsilon A \vee B$.
- 0.1. **O** Fine, let me grant $d \varepsilon A \vee B$, show me how you defend $e \varepsilon B \vee A$.
- 0.2. **P** Indeed, $e \varepsilon B \vee A$.
1. **O** What side are you going to choose in order to solve the task associated with the disjunction?
 2. **P** Well I will not answer yet. Tell me first which side are you going to solve your own task ?
 3. **O** d evaluates as a local reason for the left.
 4. **P** So, let us play on the side of the disjunction of your own choice.
 5. **O** The left side of $A \vee B$ is my choice.
 6. **P** How do you solve the task $L^{\vee}(a) \varepsilon A \vee B$?
 7. **O** $a \varepsilon A$.
 8. **P** I will now answer your question posed at 1, by stating $e \Rightarrow R^{\vee}(a) \varepsilon B \vee A$.
 9. **O** Fine, let us continue the play, according to this choice of yours.
 10. **P** Yes, let us play on $R^{\vee}(a) \varepsilon B \vee A$
 11. **O** How do you solve the task $R^{\vee}(a) \varepsilon B \vee A$?

12. **P** $a \in A$. In fact, when you stated $a \in A$ in move 7 you gave me the authority of stating it myself.

Obviously, the sequence of moves of this play above does not yet constitute a winning strategy for **P**'s thesis. A winning strategy is one according to which **P** wins whatever **O** chooses. So now we have to run a play, where **O** chooses the right with his move 3. A winning strategy for such a thesis results by composing two plays, such that whatever way **O** chooses to defend the disjunction $d \in A \vee B$, it will provide a play for $e \in B \vee A$ won by **P**¹⁴.

Following the definition of strategy described in the introduction, a winning strategy generated by a succession of relevant plays yields the following tree, with the thesis at the root of the tree (depth 0), and where we assume that the repetition ranks have been already fixed as $m := 1$ for **O** and $n := 2$ for **P**:

$$\begin{array}{c}
\mathbf{P}! e \in B \vee A \llbracket d \in A \vee B \rrbracket \\
\mathbf{O}! d \in A \vee B \\
\mathbf{P}! e \in B \vee A \\
\mathbf{O} ?_{\vee e} \\
\mathbf{P} ?_{\vee d} \\
\mathbf{O}! d \Rightarrow L^\vee(a) \in A \vee B \quad \mathbf{O}! d \Rightarrow R^\vee(b) \in A \vee B \\
\mathbf{P} ?_{\vee} \quad \mathbf{P} ?_{\vee} \\
\mathbf{O}! L^\vee(a) \in A \vee B \quad \mathbf{O}! R^\vee(b) \in A \vee B \\
\mathbf{P} ?_{L^\vee(a)} \quad \mathbf{P} ?_{R^\vee(b)} \\
\mathbf{O}! a \in A \quad \mathbf{O}! b \in B \\
\mathbf{P}! e \Rightarrow R^\vee(a) \in B \vee A \quad \mathbf{P}! e \Rightarrow L^\vee(b) \in B \vee A \\
\mathbf{O} ?_{\vee} \quad \mathbf{O} ?_{\vee} \\
\mathbf{P}! R^\vee(a) \in B \vee A \quad \mathbf{P}! L^\vee(b) \in B \vee A \\
\mathbf{O} ?_{R^\vee(a)} \quad \mathbf{O} ?_{L^\vee(b)} \\
\mathbf{P}! a \in A \quad \mathbf{P}! b \in B
\end{array}$$

3.2.2 C-Strategic Objects

Notice in the example above, the plays that shape the winning strategy, do not make use of other rules than the dialogical meaning explanation for the disjunction. Moreover, the operator $\mathbf{D}(c, x.d, y.e)$ used in the CTT framework for the elimination rules does not occur in any of those **plays**.

From a dialogical point of view, strategic rules, corresponding to elimination rules, apply the local dialogical meaning explanations to those **O**-statements out of which winning strategies for **P** will be built. As pointed out in Rahman et al. (2018) (chapter 7), strategies are conceived as a record or *recapitulation* of the interactions shaping relevant plays, and thus their strategic objects, in their most general abstract form, encode the most salient moves constituting those plays.

¹⁴Notice that all this very close to *Immanent Reasoning*. Indeed the arrow is very similar to instructions, and evaluating the arrow is very close to our solving instructions.

More precisely, C -strategic objects for the standard logical constants, the dialogical counterpart to CTT-proof-objects occurring in elimination rules for those constants, are operators composed by player indexed dialogical meaning explanations.

The equality rules for these strategic objects indicate how to reconstruct the plays building a \mathbf{P} -winning strategy for a thesis C , a family of sets, defined over disjunction, implication/universal quantifier, or conjunction/universal quantifier.

Let us thus, discuss briefly the strategic objects for disjunction, conjunction and implication.

The C -Strategic Object for Disjunction

Let us assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the disjunction $A \vee B$, and let us further assume that

- \mathbf{O} stated $d \in A \vee B$,
- for any $x \in A$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[L^\vee(x)]$ won by himself,
- for any $y \in B$ chosen by \mathbf{O} , \mathbf{P} has a method m' for developing a play on $C[R^\vee(y)]$ won by himself.

Then, the C -strategic object for the thesis $C[d]$ has the following form, which of course, follows the CTT-notation for the selector for disjunction:

$$\mathbf{P} ! \mathbf{D}(d, x.m, y.m') \in C[d]$$

How to **evaluate** the strategic object in order to obtain the relevant plays encoded by this C -strategic object is determined by the equality rules.

Assume that

- \mathbf{O} stated $a \in A$,
- for any $x \in A$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[L^\vee(x)]$ won by himself,
- for any $y \in B$ chosen by \mathbf{O} , \mathbf{P} has a method m' for developing a play on $C[R^\vee(y)]$ won by himself.

Then the strategic object for $C[L^\vee(x)]$ is:

$$\mathbf{P} ! \mathbf{D}(L^\vee(a), x.m, y.m') = m[a] \in C[L^\vee(a)]$$

Similarly, assume this time that

- \mathbf{O} stated $b \in B$,
- for any $x \in A$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[L^\vee(x)]$ won by himself,
- for any $y \in B$ chosen by \mathbf{O} , \mathbf{P} has a method m' for developing a play on $C[R^\vee(y)]$ won by himself.

Then the strategic object for $C[R^\vee(y)]$ is:

$$\mathbf{P}! \mathbf{D}(R^\vee(b), x.m, y.m') = m'[b] \in C[R^\vee(b)]$$

Given these rules, it is straightforward to show that the strategic object \mathbf{D} preserves the local dialogical meaning explanation of disjunction. Indeed, assume that \mathbf{P} stated $\mathbf{D}(d, x.m, y.m') \in C[d]$ and that \mathbf{O} granted $d \in A \vee B$. \mathbf{P} will respond to \mathbf{O} 's challenge on \mathbf{D} by asking her to choose which side of the disjunction she is willing to defend, before he commits himself to defend against the challenge on \mathbf{D} . If \mathbf{O} 's choice is $L^\vee(a) \in \underline{A}$ and the play on this move by \mathbf{O} develops in such a way that she ends up by stating $a \in A$; then, if \mathbf{P} has a winning strategy, he will win with the move $\mathbf{P}! m[a] \in C[L^\vee(a)]$. This last move is \mathbf{P} 's defence of the challenge on \mathbf{D} . Same applies if the play on $\mathbf{O}! R^\vee(b) \in \underline{B}$ develops in such a way that \mathbf{O} ends up by stating $b \in B$. Notice that what m and m' are depend on the local dialogical meaning explanation for C . Thus, the plays that build the winning strategy encoded by \mathbf{D} develop the local dialogical meaning explanations for $A \vee B$ and C . Similar dialogical “justification” can be given for all of the C -strategic objects discussed below.

Remark

Concerning equality Rahman et al. (2018) pointed out that, from a dialogical perspective, equality between local reasons expresses the result, at the level of objects, of the act of \mathbf{P} stating a local reason on behalf of \mathbf{O} 's authority. In other words, within IR-I every use of the Socratic Rule at the level of local reasons gives rise to a judgmental equality. In contrast to IR-I, here we do not systematically “objectivize” at the play level the result of applying the Socratic Rule. Moreover, in IR-II the fundamental role of judgmental equalities is to indicate how to obtain those plays (where the Socratic Rules is at work) that build a winning strategy out of the resulting strategic object.

The C -Strategic Object for Conjunction

Let us assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the disjunction $A \wedge B$, and let us further assume that

- \mathbf{O} stated $d \in A \wedge B$,
- for any $x \in A$ and any $y \in B$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[\langle a, b \rangle]$ won by himself,

Then, the C -strategic object for the thesis $C[d]$ has the following form:

$$\mathbf{P}! \mathbf{E}(d, xy.m) \in C[d]$$

How to compute the strategic object in order to obtain the relevant plays encoded by this C -strategic object is determined by the equality rules.

Assume that

- \mathbf{O} stated both $a \in A$ and $b \in B[a]$,

- for any $x \in A$ and any $y \in B$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[\langle a, b \rangle]$ won by himself,

Then, the strategic object for $C[\langle a, b \rangle]$ is:

$$\mathbf{P}! \mathbf{E}(\langle a, b \rangle, xy.m) = m[a, b] \in C[\langle a, b \rangle]$$

In order to make explicit those plays that build the winning strategy it is useful to have at hand the operators $L^\wedge(d)$, and $R^\wedge(d)$. These operators, which are the dialogical counterpart of the left and right projections within CTT, can be defined in terms of \mathbf{E} as follows:

- $L^\wedge(d) = \mathbf{E}(d, xy.x) \in A$ for $d \in A \wedge B$
- $R^\wedge(d) = \mathbf{E}(d, xy.y) \in B$ for $d \in A \wedge B$

Notice that if \mathbf{O} stated $\langle a, b \rangle \in A \wedge B$, the further statements $L^\wedge(\langle a, b \rangle) \in A$ and $L^\wedge(\langle a, b \rangle) = a \in A$, can be introduced as answers specific to \mathbf{P} 's request $?_{L^\wedge}$ and \mathbf{P} 's request $?_{L^\wedge}$ to $\mathbf{O}! \langle a, b \rangle \in A \wedge B$. Similar holds for $R^\wedge(\langle a, b \rangle) \in B$ and $R^\wedge(\langle a, b \rangle) = b \in B$.

Recall that from a dialogical point of view, the dialogical counterpart to elimination rules, are rules that apply the local dialogical meaning explanations to those \mathbf{O} -statements out of which winning strategies for \mathbf{P} will be built.

The C -Strategic Object for Implication

Let us assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the implication $A \supset B$, and \mathbf{O} stated $d \in A \supset B$. This requires the winning strategy for C to be built from the function abstracted from the local reason $(L^\supset)R^\supset \in A \supset B$, provided a criterion of identity for functions is given, and the local reason fulfils the condition of functionality¹⁵. More precisely \mathbf{O} 's d is then the abstract $\lambda(f)$, whereby the function is an element of the function type $(A)B$ ¹⁶. Moreover, \mathbf{P} has a method m for developing a play on $C[\lambda(f)]$ won by himself, for every f in $(A)B$.

In other words, assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the implication $A \supset B$, and further assume that

- \mathbf{O} stated $d \in A \supset B$,
- Whenever \mathbf{O} produces a f , such that $f \in (A)B$, \mathbf{P} has a method $m(f)$ for developing a play on $C[\lambda(f)]$, such that $\mathbf{P}! m(f) \in C[\lambda(f)]$ is won by \mathbf{P} himself,

Then the strategic object for $C[d]$ is:

$$\mathbf{P}! \mathbf{F}(d, m) \in C[d]$$

¹⁵As pointed out in Klev (2022, 202x), dialogically speaking, functionality requires the defender of $f \in (\alpha)\beta$ to produce $f(a) = f(a') \in \beta$ whenever the challenger provides $a = a' \in \alpha$. In the dialogical setting, the criterion of identity for functions amounts to the following 1) the defender of $f \in (\alpha)\beta$ must produce $f \in (\alpha')\beta'$ whenever the challenger provides $(\alpha)\beta = (\alpha')\beta' \in \text{type}$; 2) the defender of $f = f' \in (\alpha)\beta$ must produce $f = f' \in (\alpha')\beta'$ whenever the challenger provides $(\alpha)\beta = (\alpha')\beta' \in \text{type}$

¹⁶The dialogical notation $(L^\supset)R^\supset$ for the local reason of an implication, where “ L^\supset ” stands for the antecedent and “ R^\supset ” of an implication already suggests that such a local reason, in its most general.

In higher-order notation the CTT-operator \mathbf{Ap} can be defined as follows:

$$\mathbf{Ap}(d, a) := \mathbf{F}(d, (y)y(a))$$

where y is a bound function variable. Then we obtain

$$\mathbf{P}! \mathbf{Ap}(d, a) \varepsilon B[a]$$

The point is that dialogical speaking, an application $\mathbf{Ap}(d, a)$ amounts to P producing a winning strategy for $B[a]$ when d is a local reason for $A \supset B$ and a is a local reason for A .

3.3 The Dialogician's view on Evaluations

According to the DCTT-rules as presented in Klev (202x), evaluation assertions are solved by acts of calculation¹⁷. However, in a lecture Klev (2022) has developed rules for challenging and answering evaluations. Let us discuss both approaches separately.

3.3.1 Evaluation as Calculation

From a dialogical point of view, the answers $c \varepsilon C$ (where c stands for a canonical element) and $d \Rightarrow c \varepsilon C$ to the request on solving the task expressed by some statement of the form $d \varepsilon C$, are understood as commitments, namely,

- To make it explicit how the task expressed by the challenged statement is solved. This commitment amounts to choosing a canonical element.
- The commitment that d evaluates to the previously chosen canonical element¹⁸.

If we wish to integrate calculations within a dialogue we can place those acts in sub-plays, such that their outcome influences the development of the main play:

1. the evaluation is challenged,
2. the evaluation is then carried out outside the main play,
3. the result is then brought into the main play, if the result is positive the play continues, if the result is negative the play stops and the antagonist wins.

With regard to the challenge, we adopt the following form of request¹⁹:

$$\mathbf{P} \boxed{!} d \Rightarrow c \varepsilon C \text{ or } \mathbf{P} \boxed{!} \not\Rightarrow \text{-give-up}$$

Whereby the box with the exclamation mark indicates that the statement is associated to an act of calculation that takes place at a sub-play. Notice that from the dialogical point of view calculations can also be integrated to the dialogical framework, as a sequence of requests and answers determined by the dialogical meaning explanations for the expressions involved.

¹⁷If an assertion involves a purely mathematical quantity, its value is determined by calculation. However, if an assertion involves an empirical quantity its value is determined by experiment – Martin-Löf (2014) p. 5.

¹⁸Thus, both parts for solving the task are to be accomplished in interdependence. So, assume that it is asserted that $2 + 2 \varepsilon \mathbb{N}$ and as answer to the request, the assertions $2 : \mathbb{N}$ and $2 + 2 \Rightarrow 2 \varepsilon \mathbb{N}$ are produced. Clearly the first response is fine and the second response, applies blindly the rule, but the second answer asserts that an impossible calculation is doable.

¹⁹This is an adaptation of Ranta (1994)'s formulation of questions.

Let us put this together in a table

\Rightarrow	Request	Answer
$\mathbf{P}! d \Rightarrow c \in C$	$\mathbf{O}?\Rightarrow \not\Rightarrow$	$\mathbf{P} \boxed{!} d \Rightarrow c \in C$ or $\mathbf{P} \boxed{!} \not\Rightarrow$ -give-up

3.3.2 Evaluation as Definitional Unfolding

Klev’s evaluation rules are inspired by the inductive definition of the evaluation relation given by Martin-löf (2021) p. 106. In type theory, evaluation is understood as definitional unfolding, that is, repeated replacement of definiendum by the corresponding definiens until a primitive object is reached. Each such replacement can be understood as an evaluation step. If challenged, an evaluation judgement, $d \Rightarrow c \in C$, is defended by shortening the evaluation by one step: replace d by its definiens, say d' , and assert $d' \Rightarrow c \in C$. If d' is c , that is, if there is a definitional equation with d as definiendum and c as definiens, the defence is complete, and there is no assertion of the form $d \Rightarrow c \in C$ to be made. Otherwise, this assertion must be made, and it may in turn be challenged.

The following are the DCTT-rules for challenging an evaluation assertion:

$$\frac{\vdash d \Rightarrow c : C}{\left\{ \begin{array}{l} \vdash d' : C \\ \vdash d' \Rightarrow c : C \\ \vdash d = d' : C \end{array} \right. ?}$$

or $\frac{\vdash d = c : C}{\vdash d = c : C}$

Whereby “ \vdash ”, stands for stipulation. The dotted line indicates that defender has the choice between making the three assertions displayed above the dotted line and asserting the identity displayed below that line.

In IR-I, as already mentioned, we used *local reasons* for statements. Local reasons were resolved by rewriting, in a sense close to the rules for challenging and answering evaluations mentioned above. However, the evaluation procedures, which in IR-I were called *resolutions*, required some redundant repetitions of identities. Klev (2022)’s new evaluation rules, which are simpler, can be adapted to the present dialogical setting. The main idea is to link the answer to the evaluation to the answer to a request on the canonical object involved.

In fact, from the dialogical point of view, the key notion with regard to the identities evoked by an evaluation is *play-invariance*: when stating $d \Rightarrow c \in C$ the player stipulates that a play on $d \in C$ will yield c , and thus, if this is the case, the further plays are indistinguishable. For example, given $d \Rightarrow i(a) \in A \vee B$, if the play on $d \in A \vee B$, yields $a \in A$, then the identity $d = i(a) \in A \vee B$ is said to have been verified modulo play invariance.

In *formal* dialogues, when the content of the elementary proposition(s) constituting C are not known, the identity between d and c is established by either a one-step rewriting modulo (formal) play invariance, or by a chain of one-step rewritings rules such that the last yields the identity to c :

\Rightarrow	Request	Answer
$\mathbf{X}! d \Rightarrow c \in C$	$\mathbf{Y}?\Rightarrow$	$\mathbf{X}! d = c \in C$ or $\mathbf{X}! d = d' \in C$
$\mathbf{X}! d = d' \in C$	$\mathbf{Y}?\Rightarrow d'$	$\mathbf{X}! d' \Rightarrow c \in C$
$\mathbf{X}! d' \Rightarrow c \in C$	$\mathbf{Y}?\Rightarrow$	same alternatives as before

However, in practice, in formal dialogues, the first option is the one that will be systematically applied. Notice that the play invariance condition will prevent statements such as $2 + 2 = 1 \in \mathbb{N}$. In material dialogues, play invariance amounts to having explicit dialogical meaning explanations for all of the expressions occurring in an evaluation statement, including the elementary proposition(s) occurring the statement. This yields, dialogical *calculations*, within material plays. Moreover, given such meaning explanations, the identity statement as a whole can be verified at once.

Summing up, within the IR-II setting, since identities are verified by play invariance, the distinction between calculation and definitional unfolding procedures is less rigid, unless empirical procedures are involved. However, even in the latter case, the result can be integrated into the dialogical setting.

- In relation to the structural Rules for IR-II, they are basically the same as the ones for basic Dialogical Logic, with one exception. A challenge on an evaluation is restricted by a repetition rank specific to this operator. More precisely, since the arrow can be challenged twice – namely, once for “introducing” the arrow, and once for developing the evaluation itself, the repetition rank on the evaluation arrow is fixed to 2^{20} .

4 The True Dialogician

From the perspective of Per Martin-Löf’s Constructive Type Theory (CTT) dialogical rules have as main purpose to serve as formalized meaning explanations for the forms of assertoric content employed in type theory.

From the perspective of Catarina Dutillh Novaes (2020)’s Built-In-Opponent (BIO), the dialogical interpretation of inferences has as main purpose to lead to convince and/or persuade a (sometimes plural) addressee concerning the reasoning behind the inferential steps employed in a deduction carried out by means of a sequent calculus.

Both approaches stress important insights of the Dialogical Framework: the intertwining of deontic and epistemic features of meaning, in the case of CTT, and the role of psychological and sociological aspects, in the case of the BIO-approach. However,

²⁰Notice that all this very close to Immanent Reasoning, the arrow, is very similar to instructions, and evaluating the arrow is very close to our solving instructions.

in contrast to the CTT and the BIO understanding of the Dialogical Framework, Dialogicians do not conceive their approach as the result of combining a Dialectical framework with a ready-made theory of Inference, but as a setting that provides a Dialogical Theory of Meaning, within which winning strategies accomplish the inferential task.

According to the *old* approach of what we indulge in calling the perspective of the *True Dialogician*, the play level is the level where meaning is forged. It reduces neither to the (singular) performances that actualize the interaction of the play level, nor to the constitution of the schema that yields a winning strategy. Nevertheless, the possibility of selecting not only those plays that accomplish best the explicative task towards knowledge and endorsement of a targeted (sometimes plural) or more generally putative addressee, but also the norms for the development of these plays, is indeed one of the most salient virtues of the Dialogical Framework.

We call our dialogues *Dialogues for Immanent Reasoning* precisely because *reasons* behind a statement, that are now *explicit* denizens of the object-language of plays, are *internal* to the development of the dialogical interaction itself.

The emergence of concepts, so we claim, are not only games of giving and asking for reasons (games involving *Why*-questions): they are also games that aim at displaying *how is it that the reason brought forward accomplishes an explicative task* on the moves leading to the proposed conclusion. Dialogues for immanent reasoning are dialogical games of *Why* and *How* constituted by actualizations and schematizations.

However, within the Dialogical Framework actualizing and schematizing should not be understood as performing two separate actions: through these actions we acquire the competence that is associated with the meaning of an expression by *learning to play*.

This feature of Dialogical Constructivism stems from Herder's view – Herder (2016) – that the cultural process is a process of education, in which teaching and learning always occur together: dialogues display this double nature of the cultural process in which concepts emerge from a complex interplay of *why* and *how* questions.

In this sense, as pointed out by Lorenz (2010), the dialogical teaching-learning situation is where *competition* — the I-perspective — and *cooperation* — the You-perspective — interact: both intertwine in collective forms of dialogical interaction that take place at the play level.

The insights of Paul Lorenzen and Kuno Lorenz on dialogical logic as re-establishing the historical links of ethics and logic provides a clear answer to Wilfried Hodges's skeptical view Hodges and Väänänen (2019) to what the Dialogical Framework's contribution is. Hodges's criticism seems to target the *mathematical* interest of a dialogical conception of logic, rather than a philosophical interest, which does not to attract much of his interest after all. The deep links concerning meaning between dialogical logic and constructive type theory might give an answer to Hodges's worries on the mathematical or logical interest of the dialogical framework.

In lieu of a general plaidoyer for the Dialogical Framework's philosophical contribution to the foundations of logic and mathematics, which would bring us too far Lion (2023), let us highlight three points which result from the above remarks:

1. The dialogical framework offers a sound and robust venue for the development of inference-based foundations of logic.
2. The interaction of epistemic and deontic notions rooted in *local dialogical meaning explanations* sets new ways of implementing the pragmatics/semantics interface within logic.
3. The dialogical setting is an irreducible feature of meaning and knowledge, whereby knowledge in general and assertoric knowledge in particular is rooted in the interactive stance on elementary statements.

Formal semantics in the Tarski-style is blind to the first point, misunderstands the nature of the interface involved in the second, and ignores the third.

Of course, this goes towards a radical form of dialogicism that we happily profess and promote and can understand that not everyone would like to endorse. However, the present proposal can also be seen as an invitation for reflecting together on some common ground that, at least some of us share.

Acknowledgements:

Special thanks to Ansten Klev (Academy of Sciences, Prague) whom we harassed with hundreds of questions and that he answered with patience and accuracy. Klev also suggested a good deal of reformulations of notions central to our paper. Many thanks to Zoe McConaughy (Lille U. STL) with whom we started developing Immanent Reasoning and with whom we discussed many issues relevant to the context of the present paper and who suggested some thoughts in relation to the new formulation for the rules for evaluations. Many thanks to Leone Gazziero (CNRS, STL), Walter E. Young (McGill), Farid Zidani (Alger II) and the PhD students at the laboratory UMR 8163 STL: Louis Blazejewski, Rayane Boussad, Meriem Drissi, Amissah Moukama, Alioune Seck and Vincent Wistrand for important inputs and discussions on dialogical logic.

References

- Alama, J., A. Knoks, and S. Uckelman. 2011. Dialogues games for classical logic, In *Tableaux 2011*, eds. Giese, M. and R. Kuznets, 82–86. Bern: Universiteit Bern.
- Barth, E.M. and E.C. Krabbe. 1982. *From Axiom to Dialogue*. Berlin, New York: De Gruyter.
- Beth, E.W. 1955. Les Fondements Logiques des Mathématiques. *Les Etudes Philosophiques* 10(2): 288–289 .
- Blass, A. 1992. A game semantics for linear logic. *Annals of Pure and Applied Logic* 56(1): 183–220 .
- Brandom, R. 2000. *Articulating Reasons: An Introduction to Inferentialism*. Cambridge, Mass.: Harvard University Press.

- Casteln rac, B. and M. Marion. 2009. Arguing for inconsistency: Dialectical games in the academy, In *Acts of Knowledge: History, Philosophy and Logic*, eds. Primiero, G. and S. Rahman, 37–76. London: College Publications.
- Clerbout, N. 2014a. Finiteness of plays and the dialogical problem of decidability. *IfCoLog Journal of Logics and their Applications* 1: 115–140 .
- Clerbout, N. 2014b. First-Order Dialogical Games and Tableaux. *Journal of Philosophical Logic* 43(4): 785–801 .
- Clerbout, N. 2014c. *La S mantiq  Dialogique. Notions Fondamentales Et Elements de Metatheorie*. Cahiers de logique et d’ pist mologie. Milton Keynes: College Publications.
- Clerbout, N., M.H. Gorisse, and S. Rahman. 2011. Context-Sensitivity in Jain Philosophy: A Dialogical Study of Siddhar igani’s Commentary on the Handbook of Logic. *Journal of Philosophical Logic* 40: 633–662 .
- Clerbout, N. and Z. McConaughy 2022. Dialogical Logic. In E. N. Zalta and U. Nodelman (Eds.), *The Stanford Encyclopedia of Philosophy* (Fall 2022 ed.), Stanford University. Metaphysics Research Lab.
- Coquand, T. 1995. A semantics of evidence for classical arithmetic. *Journal of Symbolic Logic* 60: 325 – 337 .
- Crubellier, M. 2011. Du Sullogismos au Syllogisme. *Revue Philosophique de la France Et de l’Etranger* 136(1): 17–36 .
- Crubellier, M., M. Marion, Z. McConaughy, and S. Rahman. 2019. Dialectic, the Dictum de Omni and Ecthesis. *History and Philosophy of Logic* 40: 1–27 .
- Dutilh Novaes, C. and R. French. 2018. Paradoxes and structural rules from a dialogical perspective. *Philosophical Issues* 28: 129–158 .
- Ebbinghaus, K. and Aristotle. 1964. *Ein Formales Modell der Syllogistik des Aristoteles*. G ttingen: Vandenhoeck & Reprecht.
- Felscher, W. 1985. Dialogues, strategies, and intuitionistic provability. *Annals of Pure and Applied Logic* 28(3): 217–254 .
- Ferm ller, C.G. 2003. Parallel Dialogue Games and Hypersequents for Intermediate Logics, In *Automated Reasoning with Analytic Tableaux and Related Methods*, eds. Cialdea Mayer, M. and F. Pirri, 48–64. Heidelberg: Springer Berlin.
- French, R. 2019. A Dialogical Route to Logical Pluralism. *Synthese* 198(Suppl 20): 4969–4989 .

- Gethmann, C.F. 1982. Protologik. Untersuchungen Zur Formalen Pragmatik von Begründungsdiskursen. *Zeitschrift für Philosophische Forschung* 36(2): 293–296.
- Ginzburg, J. 2012. *The Interactive Stance: Meaning for Conversation*. Oxford, UK: Oxford University Press.
- Gorisse, M.H. 2017. Logic in the Tradition of Prabhācandra, *The Oxford Handbook of Indian Philosophy*. Oxford, UK: Oxford University Press.
- Gorisse, M.H. 2018. Concealing meaning in inferential statements : the practice of patra in Jainism, In *Jaina studies : select Papers presented in the 'Jaina Studies' Section at the 16th World Sanskrit Conference, Bangkok Thailand and the 14th World Sanskrit Conference, Kyoto Japan*, ed. Balbir, Nalini and Flügel, Peter, 111–126. New Delhi: DK Publishers.
- Herder, J.G. 2016. Abhandlung über den Ursprung der Sprache (1772), In *Das Entgegenkommende Denken*, eds. Engel, F. and S. Marienberg. Berlin, Boston: De Gruyter.
- Hintikka, J. 1973. *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic*. Oxford: Clarendon Press.
- Hintikka, J. 1996. *The Principles of Mathematics Revisited*. New York: Cambridge University Press.
- Hintikka, J. 2006. *Analyses of Aristotle*. Dordrecht: Springer.
- Hintikka, J.G. and G. Sandu. 1997. Game-Theoretic Semantics, In *Handbook of Logic and Language*, eds. Benthem and Meulen. Cambridge: MIT Press.
- Hodges, W. 2001. Dialogue Foundations: A Sceptical Look: ~~Wilfrid Hodges~~. *Aristotelian Society Supplementary Volume* 75(1): 17–32 .
- Hodges, W. and J. Väänänen. 2019. Logic and Games, In *The Stanford Encyclopedia of Philosophy* (Fall 2019 ed.), ed. Zalta, E.N. Stanford University: Metaphysics Research Lab.
- Iqbal, M. 2022. *Arsyad al-Banjari's Insights on Parallel Reasoning and Dialectic in Law: The Development of Islamic Argumentation Theory in the 18th Century in Southeast Asia*, Volume 25. Cham: Springer.
- Johnson, R.H. 1999. The Relation Between Formal and Informal Logic. *Argumentation* 13(3): 265–274 .
- Kamlah, W. and P. Lorenzen. 1967. *Logische Propädeutik Oder Vorschule des Vernünftigen Redens*. Stuttgart: J.B. Metzler.

- Keffer, H. 2001. *De Obligationibus : Rekonstruktion Einer Spätmittelalterlichen Disputationstheorie*. Boston: Brill.
- Keiff, L. 2007. *Le Pluralisme Dialogique. Approches dynamiques de l'argumentation formelle*. Ph. D. thesis, Université de Lille.
- Keiff, L. 2010. La Dialectique, Entre Logique et Rhétorique. *Revue de Métaphysique et de Morale* 66(2): 149–178 .
- Klev, A. 2022. Martin-Löf's Dialogue Rules. Talk at the Workshop “Meaning Explanations”, Prague, ~~dec~~ 2022.
- Klev, A. 202x. Martin-Löf on the validity of inference, In *Perspectives on Deduction*, ed. d'Aragona, A. Cham: Springer. Forthcoming.
- Krabbe, E.C.W. 1985. Formal Systems of Dialogue Rules. *Synthese* 63(3): 295–328 .
- Lecomte, A. and M. Quatrini. 2011a. Figures of Dialogue: A View From Ludics. *Synthese* 183(S1): 59–85 .
- Lecomte, A. and M. Quatrini. 2011b. Ludics and Rhetorics. *Ludics, Dialogue and Interaction: PRELUDE Project-2006-2009. Revised Selected Papers* 6505: 32–57 .
- Lion, C. 2020. *L'intuitionnisme dialogique. Une autre lecture du sujet créateur brouwerien*. Ph. D. thesis, Université de Lille, Philosophie. Thèse de doctorat dirigée par Rahman, Shahid et Heinzmann, Gerhard.
- Lion, C. 2023. *L'Intuitionnisme dialogique. Une lecture iconoclaste de l'iconoclasme de Brouwer-Intuitionnisme et constructivisme*. Paris: Classiques Garnier.
- Lorenz, K. 1961. *Arithmetik und Logik als Spiele*. Kiel: Christian - Albrechts - Universität.
- Lorenz, K. 1970. *Elemente der Sprachkritik Eine Alternative Zum Dogmatismus Und Skeptizismus in der Analytischen Philosophie*. Frankfurt: Suhrkamp.
- Lorenz, K. 1998. *Indische Denker*. Beck'sche Reihe. München: C.H. Beck.
- Lorenz, K. 2001. Basic objectives of dialogue logic in historical perspective. *Synthese* 127: 255–263 .
- Lorenz, K. 2008. Dialogischer Konstruktivismus, *Dialogischer Konstruktivismus*. Berlin, New York: De Gruyter.
- Lorenz, K. 2010. *Logic, Language and Method - On Polarities in Human Experience*. Berlin, New York: De Gruyter.

- Lorenz, K. 2011. *Gesammelte Aufsätze unter Einschluss gemeinsam mit Jürgen Mittelstraß geschriebener Arbeiten zu Platon und Leibniz*. Berlin, New York: De Gruyter.
- Lorenz, K. 2015. *Zur Herkunft der Dialogbedingung im Dialogischen Aufbau der Logik*, pp. 55 – 74. Leiden, The Netherlands: Brill — Mentis.
- Lorenz, K. 2021. *Von der dialogischen Logik zum dialogischen Konstruktivismus*. Berlin, Boston: De Gruyter.
- Lorenz, K. and J. Mittelstrass. 1966. Theaitetos fliegt. zur theorie wahrer und falscher sätze bei platon. *Archiv für Geschichte der Philosophie* 48(1-3): 113–152 .
- Lorenz, K. and J. Mittelstrass. 1967. On rational philosophy of language: The programme in plato's cratylus reconsidered. *Mind* 76(301): 1–20 .
- Lorenzen, P & Lorenz, K. 1978. *Dialogische Logik*. Darmstadt: Wissenschaftliche Buchgesellschaft.
- Lorenzen, P. 1955. *Einführung in die operative Logik und Mathematik*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Heidelberg: Springer Berlin.
- Lorenzen, P. 1969. *Normative Logic and Ethics*. Zürich: Bibliographisches Institut.
- Lorenzen, P. and O. Schwemmer. 1973. *Konstruktive Logik, Ethik und Wissenschaftstheorie*, Volume 700. Mannheim: Bibliogr. Institution.
- Marion, M. and H. Rückert. 2015. Aristotle on Universal Quantification: A Study from the Point of View of Game Semantics. *History and Philosophy of Logic* 37: 1–29 .
- Martin-Löf, P. 2014. Truth of Empirical Propositions. **Lecture held at the University of Leiden, February 2014. Transcription by Ansten Klev.**
- Martin-Löf, P. 2015. Is logic part of normative ethics? Transcript by Ansten Klev of a lecture given on 16 April 2015 and in Paris on 15 May 2015.
- Martin-Löf, P. 2017. Assertion and request. Transcript by Ansten Klev of a lecture given in Stockholm on 14 August.
- Martin-Löf, P. 2019a. Epistemic assumptions: are they assumed to be backwards vindicated or forwards vindicable? Transcript of a lecture given in Leiden on 6 September 2019.
- Martin-Löf, P. 2019b. Logic and Ethics, In *Proof-Theoretic Semantics: Assessment and Future Perspectives. Proceedings of the 3rd Tübingen Conference on Proof-Theoretic Semantics*, eds. Piecha, T. and P. Schroeder-Heister, 27–30. Tübingen: Universitätsbibliothek Tübingen.

- Martin-löf, P. 2021. The Sense/Reference Distinction in Constructive Semantics. *Bulletin of Symbolic Logic* 27(4): 501–513 .
- Martin-Löf, P. 2022. Correctness of assertion and validity of inference. Transcript edited by the author and prepared by Ansten Klev, of a lecture given by Per Martin-Löf on 26 October 2022 at the Rolf Schock Symposium in Stockholm in the context of the award of the Schock Price.
- Martin-Löf, P. and G. Sambin. 1984. *Intuitionistic Type Theory. Lecture notes*. Napoli: Bibliopolis.
- McConaughey, Z. 2021. *Aristotle, science and the dialectician's activity : a dialogical approach to Aristotle's logic*. Ph. D. thesis, Université de Lille, Philosophie. Thèse de doctorat dirigée par Rahman, Shahid et Marion, Mathieu.
- Miller, L. 2020. *Islamic Disputation Theory: The Uses & Rules of Argument in Medieval Islam*. Cham: Springer.
- Novaes, C. 2007. *Formalizing Medieval Logical Theories: Suppositio, Consequentiae and Obligationes*. Logic, Epistemology, and the Unity of Science. Dordrecht: Springer.
- Novaes, C. 2015. A Dialogical, Multi-Agent Account of the Normativity of Logic. *Dialectica* 69: 587–609 .
- Novaes, C.D. 2020. *The Dialogical Roots of Deduction: Historical, Cognitive, and Philosophical Perspectives on Reasoning*. New York: Cambridge University Press.
- Piecha, T. 2015. Dialogical Logic. In *The Internet Encyclopedia of Philosophy*.
- Prakken, H. 2005. Coherence and Flexibility in Dialogue Games for Argumentation. *J. Log. Comput.* 15: 1009–1040 .
- Rahman, S., Z. McConaughey, A. Klev, and N. Clerbout. 2018. *Immanent Reasoning or Equality in Action. A Plaidoyer for the Play Level*. Cham: Springer.
- Rahman, S. and J. Redmond. 2016. Armonía Dialógica: Tonk, Teoría Constructiva de Tipos y Reglas Para Jugadores Anónimos. *Theoria: Revista de Teoría, Historia y Fundamentos de la Ciencia* 31(1): 27–53 .
- Rahman, S. and H. Rückert. 1999. Dialogische Modallogik (Für T, B, ~~s4~~, ~~U~~ und S5). *Logique et Analyse* 167(168): 243–282 .
- Rahman, S. and M.S. Zarepour. 2019. On Descriptive Propositions in Ibn Sīnā: Elements for a Logical Analysis, In *Mathematics, Logic, and Their Philosophies: Essays in Honour of Mohammad Ardeshir*, eds. Mojtahedi, M., S. Rahman, and M. Zarepour, 411–431. Springer.

- Ranta, A. 1988. Propositions as Games as Types. *Synthese* 76(3): 377–395 .
- Ranta, A. 1994. *Type-theoretical Grammar*. Indices (Clarendon). New York: Clarendon Press.
- Redmond, J. and M. Fontaine. 2011. *How to play dialogues: An introduction to dialogical logic*. London: College Publications.
- Rückert, H. 2011. *Dialogues as a Dynamic Framework for Logic*. Dialogues and games of logic. London: College Publications.
- Schroeder-Heister, P. 2008. Lorenzen’s operative justification of intuitionistic logic, In *One Hundred Years of Intuitionism (1907–2007): The Cerisy Conference*, eds. van Atten, Markand Boldini, P., M. Bourdeau, and G. Heinzmann, 214–240. Basel: Birkhäuser Basel.
- Sørensen, M. and P. Urzyczyn. 2006. *Lectures on the Curry-Howard Isomorphism*. Number vol. 10 in Lectures on the Curry-Howard isomorphism. New-York: Elsevier.
- Sterling, J. and C. Angiuli. 2021. Normalization for Cubical Type Theory, *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 1–15. New York, USA: Association for Computing Machinery.
- Sundholm, G. 1986. Proof theory and meaning, In *Handbook of Philosophical Logic: Volume III: Alternatives in Classical Logic*, eds. Gabbay, D. and F. Guentner, 471–506. Dordrecht: Springer. https://doi.org/10.1007/978-94-009-5203-4_8.
- Sundholm, G. 1989. Constructive generalized quantifiers. *Synthese* 79(1): 1–12 .
- Sundholm, G. 1997. Implicit epistemic aspects of constructive logic. *Journal of Logic, Language and Information* 6(2): 191–212 .
- Sundholm, G. 2012. “Inference versus consequence” revisited: inference, consequence, conditional, implication. *Synthese* 187: 943–956 .
- Sundholm, G. 2013. Containment and Variation; Two Strands in the Development of Analyticity from Aristotle to Martin-Löf, *Judgement and the Epistemic Foundation of Logic*, 23–35. Heidelberg: Springer.
- Sundholm, G. 2019. The Neglect of Epistemic Considerations in Logic: The Case of Epistemic Assumptions. *Topoi* 38: 1–9 .
- Uckelman, S.L. 2013. Medieval Disputationes de Obligationibus as Formal Dialogue Systems. *Argumentation* 27(2): 143–166 .
- Vaidya, A.J. 2013. Epistemic Responsibility and Critical Thinking. *Metaphilosophy* 44(4): 533–556 .

- van Eemeren, F. and R. Grootendorst. 2004. *A Systematic Theory of Argumentation: The Pragma-dialectical Approach*. Cambridge: Cambridge University Press.
- Walton, D.N. 1984. *Logical Dialogue-Games and Fallacies*. Lanham, Md.: University Press of America.
- Woods, J., A. Irvine, and D. Walton. 2002. Argument: Critical Thinking, Logic and the Fallacies (M. Hogan). *Philosophical Books* 43(1): 43–45 .
- Woods, J.H. and D.N. Walton. 1989. *Fallacies: Selected Papers 1972-1982*. Dordrecht, Netherland: Foris.
- Young, W.E. 2016. *The Dialectical Forge: Juridical Disputation and the Evolution of Islamic Law*. Cham, Switzerland: Springer.
- Young, W.E. 2022. The Formal Evolution of Islamic Juridical Dialectic: A Brief Glimpse, In *New Developments in Legal Reasoning and Logic: From Ancient Law to Modern Legal Systems*, eds. Rahman, S., M. Armgardt, and H.C.N. Kvernenes, 83–113. Cham: Springer International Publishing.
- Yrjönsuuri, M. 2001. *Medieval Formal Logic: Obligations, Insolubles and Consequences*. Dordrecht, Netherland: Springer.