



HAL
open science

The Radical Dialogician

Shahid Rahman

► **To cite this version:**

| Shahid Rahman. The Radical Dialogician. 2023. halshs-04144109

HAL Id: halshs-04144109

<https://shs.hal.science/halshs-04144109>

Preprint submitted on 28 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The Radical Dialogician

(draft 27/06/2023)

Erst wenn Wissenschaft auch noch weiß, was sie tut und warum sie es tut, kehrt sie den zu ihr gehörigen philosophischen Aspekt heraus. Philosophie ist nur in der Tätigkeit des Nachdenkens darüber, was man sagt und tut und warum – in der Selbstreflexion –, und im Dieses-selbstsagen können wirklich. Weil aber ein solches Sagen-können das Darüber-sich-verständigen-können einschließt – woran sollte sich das Sagen-können sonst bewähren –, ist Philosophie eine Einheit von Selbstreflexion und Kommunikation.

Kuno Lorenz, “Dialogischer Konstruktivismus”, 2009, p. 5.

1 Introduction

Abstract

Contemporary "Dialogic logic" was conceived by Paul Lorenzen by 1958. Dialogical logic studies dialogues; but it also takes the form of dialogues. In a dialogue, two parties (players) argue on a thesis (a certain statement that is the subject of the whole argument) and follow certain fixed rules in their argument. In fact, the rules that shape a dialogue are divided into two kinds of rules: *particle rules* and *structural rules*. Whereas the first determine *local meaning* of an expression the second determines its *global meaning*. More precisely, the rules for local meaning explain the meaning of an expression independently of the rules setting the development of a dialogue (play) and independently of who of the interlocutors is the one setting the thesis. Norms for *winning-strategies*, the dialogical analogue to inference rules, select those plays shaped by the structural rules in which both players make their optimal moves towards win. Validity, is defined via the notion of winning strategy. In the dialogical setting, the justification of inference rules, amounts to showing that they preserve local meaning, that is that they preserve a core of dialogical meaning explanation invariant in relation to players, structural and strategic rules

1.1 Basics in Dialogical Logic

Dialogical logic is a dialogue-based framework for meaning, knowledge and logic rooted in a research tradition on argumentation that goes back to dialectics in Greek Antiquity,¹ when semantic, epistemic and ethical issues were approached through debates in which opposing parties discussed a thesis through questions and answers.

Contemporary "Dialogic logic" was conceived by Paul Lorenzen by 1958, the year of his talk *Logik und Agon* held in Rome at the XIIth World Congress of Philosophy and was further developed by Kuno Lorenz. Initially Dialogical logic aimed at overcoming some limitations of Lorenzen's (1955) own *Operative Logik*.²

The work of Lorenzen and Lorenz and the *Constructivist School of Erlangen*, took later on the shape of an overall pragmatist and game-theoretical *Dialogical Framework* to philosophy of language, logic and sciences. This framework was the outcome of the intertwining of constructive mathematics and logic,³ a dialectical reading of emergence of philosophy of language and logic in the Ancient Greek tradition,⁴ and Wittgenstein's notion of language games,⁵ with a special accent on the ethical undergirding of conceptual

¹ Though there have been parallel, often independent approaches in other traditions and periods, particularly in the Indian Subcontinent – see Lorenz (1998), and also in the context of Medieval Islamic Juridical Argumentation Theories, that were developed long before the reception of Aristotelian Logic – see Young (2017, 2021).

² See Lorenz (2001), Schroeder-Heister (2008).

³ See Lorenzen & Lorenz (1978).

⁴ See Ebbinghaus (1964, 2016), Lorenz & Mittelstrass (1966, 1967), Lorenz (2011).

⁵ See Lorenz (1970), Kamlah & Lorenzen (1972), Lorenz (2009, 2011).

thinking.⁶ Lorenz added three important methodological perspectives to the project, namely (i) the reconstruction of ancient theories of meaning and knowledge including not only Plato but also Indian traditions, (ii) a semiotic, Peircean, understanding of language games (iii) the explicit use of mathematical game theory in logic.⁷

Dialogical logic studies dialogues; but it also takes the form of dialogues. In a dialogue, two parties (players) argue on a thesis (a certain statement that is the subject of the whole argument) and follow certain fixed rules in their argument.

The player who states the thesis is the Proponent, called "P", and his interlocutor, the player who challenges the thesis, is the Opponent, called "O". In challenging the Proponent's thesis, the Opponent is requiring of the Proponent that he defends his statement by a sequent moves, shaped by the adopted rules for the development of a dialogue, involving declarative utterances, called *statements*, and interrogative utterances called *requests* (in the original setting of Lorenzen and Lorenz, some requests, may adopt the form of a statement, namely, when stating the antecedent of an implication (or negation), in order to challenge a consequent).

In fact, the rules that shape a dialogue are divided into two kinds of rules: *particle rules* and *structural rules*. Whereas the first the determine *local meaning* of an expression the second determines its *global meaning*. More precisely, the rules for local meaning explain the meaning of an expression independently of the rules setting the development of a dialogue (play) and independently of who of the interlocutors is the one setting the thesis – i.e. these rules prescribe how to challenge a statement and to defend it but it does not fix who the challenger and who the defender is. Norms for *winning-strategies*, the dialogical analogue to inference rules, select those plays shaped by the structural rules in which both players make their optimal moves towards win. In fact, validity, as discussed below, is defined via the notion of winning strategy.

Thus, there is a play perspective on plays and a strategic perspective on plays, whereby the latter presupposes the former. The very point of just playing, can be seen as mechanism for checking if the thesis brought forward, does or not enjoy the property of dialogue definiteness: it does not matter who wins, so long as one does. The strategic perspective then, chooses among the plays on a dialogue definite thesis, those where the moves are optimal with regard to justifying the thesis.

Nota Bene: Notice that since dialogue definiteness is defined on **individual** plays, the play perspective on plays does not preclude the coexistence of two (or more) plays on the same thesis but where different sequences of moves lead to different outcomes. This allows shaping plays where *revisions* take place by choosing one rather than another sequence of moves (for example by the explicit collaboration of the interlocutors), and/or plays aiming at conciliation, whereby the different sequences merge into a new play on an expanded, modified, new thesis.

The local meaning of an expression sets the core of its interactive meaning, which we call its *local dialogical meaning explanation* is *invariant*⁸.

⁶ See Lorenzen (1969), Lorenzen & Schwemmer (1975); Lorenz (2021, chapter 2).

⁷ See Lorenz (1961, 1998, 2009, 2010, 2014, 2021).

⁸ The expression *invariant core* has been borrowed from *Linear Logic* and *Ludics*. Ludics proposes a kind of theory of meaning as interaction focused only on what we just called local dialogical meaning explanations. Indeed, within Ludics, structural rules are reduced to local rules by introducing unary connectives into the object language such as an operator for repetition ranks that allows to convert an intuitionistic negation into a classical one – cf. Lecomte & Quatrini (2011a,b). Ludics proposes a dynamic interaction process involving only the rules preserving the invariant meaning core, whereby structural and strategic rules are constituted. The invariant core amounts to the bare interaction of questions and answers

in relation to the players,
to the development of a play (including winning and losing it),
to the constitution of winning strategy.

Global meaning set the meaning of an expression in the context of some specific form of developing a dialogue given by rules called *structural rules* (since they generate at the strategy level the known structural rules of Gentzen systems). Accordingly, we say that the structural rules provide *global dialogical meaning explanations*. Global Meaning explanations are only *partially* invariant in relation to the players, since, as we will discuss below, there is a rule which allows one player, namely **O**, to state elementary statements, that cannot be challenged. Still, *global dialogical meaning explanations are invariant*

in relation to the players, when they state non-elementary statements,
to the development of a play (including winning and losing it),
to the constitution of winning strategy.

Central to the dialogical perspective is that *preserving local meaning*, plays the role that *admissibility* had in Lorenzen's earlier *Operative Logik*. Indeed, as pointed out by Schröder-Heister (2008), admissibility was Lorenzen's proof-theoretical venue to meaning and what distinguished his approach from formalism: admissibility allows, to identify a core of meaning invariant in relation to a set of rules. Admissibility continues to be one of the main tenets of contemporary proof-theoretical approaches as the one of Substructural Logics, Linear Logic and Ludics, after Prawitz's developed attributions of admissibility into a system of normalization procedures.⁹

After the dialogical turn, the identification of the invariant core of meaning came down to identifying the local meaning at work in both structural rules and strategic rules. Moreover, attributions of admissibility to an inference are now replaced by attributions of *dialogue-definiteness*: an expression enjoys the property of dialogue-definiteness, if it **can** be stated as thesis in an individual finite Zero-Sum-play, that ends with loss or win and is shaped by structural rules that preserve local dialogical meaning explanations. Actually dialogue-definiteness is the condition for being counted as a proposition – see Lorenz (2001, p. 258). However, win or loss amounts neither to truth-value definiteness nor to proof-theoretical definiteness. Indeed according to the novel approach, truth-value indefinite propositions such as the Law of Excluded Middle, can give place to a decidable play on the thesis stating that Law, where **P** loses the game, but this of course does not mean that it false or that there is a proof for its refutability – i.e. that **P** has a winning strategy for its negation.

Furthermore, in the dialogical setting, the justification of inference rules, amounts to showing that they preserve local meaning, that is that they preserve a core of dialogical meaning explanation invariant in relation to players, structural and strategic rules.¹⁰

- In short, from the Dialogician's perspective, the root of inference rules as

or *expectations and intentions*, to use Lecomte & Quatrini's (2011a) words.

⁹ See Lecomte & Quatrini (2011a,b).

¹⁰ This is why tonk-like operators can be dismissed in the dialogical framework – see Rahman & Redmond (2016). Moreover, on one hand Crubellier et al. (2019) – further developed by McConaughey (2021), interpret Aristotelian proofs by *ecthesis*, as laying down the local meaning of the quantifiers given the *Dictum de Omni*, on the other we know since the work of Ebbinghaus (1964), that Aristotelian reductions to the perfect syllogism, inspired Lorenzen for the invention of admissibility, who by the way was the first on proposing a dialogical reconstruction of Aristotle's proof by *ecthesis*. Thus, according to Lorenzen's interpretation, reductions to perfect syllogism, if successful, show that the reduced rules preserve the local meaning layed down by the dialectical rules for quantifiers as encoded by the rules for *ecthesis*.

expressed by rules for a winning strategy, is the player independent local dialogical meaning explanations put at work by structural rules that preserve them. Dialogical rules are not a dialogical interpretation of inference rules but they justify them.

- The former point is crucial for any form of dialogical pluralism since choosing between different patterns of reasoning assumes an invariant core in relation to which the norms for determining both the development of a play and of a winning strategy are to be constituted.

Actually, as pointed out by Lion (2021, pp. 125-159; 265-310), the invariant core itself is the result of a prior dynamic semiotic process whereby this core meaning has been fixed – though the process can later on be launched again. According to this view, the choices inherent to dialogical pluralism, start before the discussion on structural rules come to the fore.

Let us now put this into work in the context of the logical constants for first-order logic. This represents indeed quite of a restricted context but it has the advantage of providing a simple language game for the study of the different levels of meaning shaping the Dialogical Framework.

Particle Rules and Local Dialogical Meaning Explanations

The dialogical meaning explanations for the standard first order logical constants, in the original setting of Lorenzen and Lorenz, are the following: ¹¹

	Request	Answer
$X ! A \vee B$ (player X states a disjunction)	$Y ?_{\vee}$ (Questioner asks X to choose which side of the disjunction s(h)e is willing to defend)	$X ! A$ -----OR----- $X ! B$ (X chooses the sides to be defended)
$X ! A \wedge B$ (player X states a conjunction)	$Y ?_{L\wedge}$ -----OR----- $Y ?_{R\wedge}$ (Questioner chooses to ask either for the Left or the Right)	$X ! A$ ----- $X ! B$ (X defends the side requested by Y)
$X ! A \supset B$ (player X states an implication)	$Y ! A$ (Y: I grant you the antecedent. Are you able to state the consequent given this grant of mine?)	$X ! B$ (X: yes, indeed)
$X ! \neg A$ (player X states a negation)	$Y ! A$ (Y: On the contrary, A is the case)	——— (No defence is available. X might try a counterattack on Y ! A)
$X ! \exists x A(x)$ (player X states an existential quantifier)	$Y ?_{\exists}$ (Y: Who/Which is the x you said to be an A?)	$X ! A(a/x)$ (X: a is one of those that is an A)
$X ! \forall x A(x)$	$Y ?_{\forall} a$	$X ! A(a/x)$

¹¹ See Rückert (2011), Clerbout (2014a,b), Clerbout & McConaughey (2022).

(player X states a universal quantifier)	(Y : Since you stated that every x is an A , this should also hold for a . <u>Right?</u>)	(X : Indeed, a is an A)
---	--	--------------------------------------

Structural Rules and Global Dialogical Meaning Explanations

As already mentioned, structural rules set the local dialogical meaning explanations in the context of the development of a play. This renders their global dialogical meaning explanations. Several development rules are possible. Each determine what pattern of reasoning counts as meaningful and which not, e.g., the choice between the rules **SR1i** and **SR1c**. below, determines if the reasoning is shaped by either an intuitionistic or classical theory of meaning. The present rules are *basic* in the sense that different logics and meaning approaches might require extending them by either adding new rules or adding further conditions to the existing rules. Some of these extensions will be mentioned in the commentaries of the appendix.

SR0 (starting rule): A play starts with a player stating a proposition called the thesis; that player becomes the Proponent (**P**) and the move is labelled move 0.¹²

The other player—the Opponent (**O**)—chooses a repetition rank determining how many times she is allowed to challenge or defend any move in a play. It is usually enough for **O** to choose a repetition rank of 1 ($m := 1$), it is move 1. **P** then chooses a repetition rank: 2 is usually enough ($n := 2$); it is move 2.¹³

SR1i (intuitionistic game-playing rule): After the repetition ranks have been chosen, each move is a challenge or a defence in reaction to a previous move, in accordance with the rules for **local** dialogical meaning explanations. Each player can challenge the same previous move at most n times, where n is the player's repetition rank, or defend against the adversary's *last*

¹² This labelling has the effect that whereas even numbers indicate moves by **P**, odd numbers indicate moves by **O**.

¹³ This rule should assure that plays are finite (though there might be an infinite number of them). Indeed, one important feature of Lorenzen-Lorenz conception of proposition and statement is its finiteness: for an expression to count as a proposition A there must exist an individual play about the statement $!A$, in a play which must reach a final position after a **finite number** of moves according to definite particle and structural rules – cf. Lorenz (2001, p. 258). If language-games are to be conceived as mediators of meaning carried out by social interaction, these language-games must be games that we **can actually perform**. This is what Lorenz's notion of **dialogue-definiteness** is about. Technically speaking, finiteness is implemented by the explicit introduction of a repetition rank which bounds **the number of times a move can be challenged and/or defended** - for discussion on this point See Rahman et al. (2018, pp. 278-282). As pointed out by Lorenz (2001, p. 260) the bounds fixed by **the repetition rank is the play-level source of the strategy level rule for contraction**. Blass (1992) related the repetition rank with *Linear Logic*. Clerbout (2014a,b,c) delved into the metalogical consequences of the repetition rank and showed how it is related to the semi undecidability of first-order logic. One important general philosophical result of Clerbout work is that though every play must be finite, it does not prevent the existence of an infinite number of them. This also answers to a usual misunderstanding of Lorenzen-Lorenz dialogues: **most of them, are in principle, zero-sum plays, but winning or losing a play does not preclude strategic Turing-Church-semi undecidability of first-order logic: we cannot in general fix a uniform repetition rank for every thesis stating a first order proposition**. Here again, we must recall the distinction between play perspective on plays and strategic perspective on plays : the **zero-sum** feature of plays is part of the mechanism for checking dialogue definiteness, but it **does not in general transmit to the strategy level**. Clerbout also showed that repetition rank $m := n$ for **O** and $n := n+1$ for **P**, are the optimal choices for developing a winning strategy for intuitionistic and classical logic.

An additional interesting proof-theoretical feature of arguments with restricted contraction or explicit repetition rank, is that they can show the degree of dependence of a proof upon an axiom. Indeed, assume that for some proof the axiom of choice is used twice, it makes perfect sense to ask if it one can develop a proof with only one use of the axiom of choice – this is a paraphrase of Mares&Paoli (2014, p. 452f). So, it can be very useful, to make the repetition rank at use explicit – to mention just another example, recall that in order to prove the validity of the double negation of third excluded in a dialogical setting for constructive logic, the Proponent needs to challenge twice Opponent's statement $\neg(A \vee \neg A)$: if the repetition rank were one, then Proponent would not be able to win the thesis $\neg\neg(A \vee \neg A)$.

unanswered challenge.¹⁴

Nota bene: Notice that any challenge on a negation will always remain unanswered, since since there is no defence against them. Thus such a challenge will always count as the last challenge.

SR1c (classical game-playing Rule): After repetition ranks have been chosen, each move is a challenge or a defence in reaction to a previous move, in accordance with the particle rules shown previously. Each player can challenge *the same previous move, or defend against the same previous challenge, at most n times*, where *n* is that player's repetition rank.

SR2 (Formal Rule and Socratic Rule).¹⁵

¹⁴ **Choosing between SR1i and SR1c: Dialogical Pluralism.** Perspicuity and explanatory power won by a dialogical setting, become a salient issue when it aims at motivating the choice of some structural rules over others, such as the choice between **SR1i and SR1c**. Indeed, Lorenzen and Lorenz dialogical logic offered a *substructural* response (to use the terminology launched by Peter Schröder-Heister and Kosta Došen (1993) years after the birth of dialogical logic) to W. v. Quine's (1986, second edition, p. 81) dictum, that when non-classical logics speak of a *deviant* connective they change the subject of conversation. The point of Quine can be put as positing the question on how to establish a meaningful dialogue on the standards of inferential rationality associated to an expression if those standards differ. The answer of Lorenzen and Lorenz, was to distinguish *local meaning* and *global meaning*. Thus, classical and intuitionistic connectives share the same local meaning, but differ only on one rule on global, namely whereas in classical logic the global ruling allows the Proponent to, so to say, *redo* his response to a challenge (or ignore the last challenge launched by the Opponent and answer to a previous one) such kind of backwards moves, are not allowed when in an intuitionistic setting – which, as mentioned above allows **P** to bring forward two theses and generates the rejection of weakening to the right at the proof-theoretical level. Thus, if there is some common ground between classical and intuitionist approaches to meaning, this common ground, is in a dialogical setting the *local dialogical meaning explanations*, given by player independent rules. By 1996 this stance on plural approaches to meaning and knowledge was generalized and developed into the project of *Dialogical Pluralism* launched by S. Rahman, H. Rückert and L. Keiff – See Keiff (2007), Rückert (2011). The rationale behind Dialogical Pluralism is that if deductive rationality is declined as dialogical interaction and the latter is governed by the general properties of the inference-relation, disagreeing with such standards of rationality amounts to disagreeing with the structural rules that shape inference. One new brand of Dialogical Pluralism, called the *Built-in-Opponent Pluralism* (BIO-Pluralism) has been recently developed by Dutilh Novaes & French (2018) and Dutilh Novaes (2020, pp. 78-84). Whereas the Saarbrücken-Lille-Pluralism is led by the idea that different logics are the result of different conceptual architectures which lay down what **counts as a meaningful expression** by setting the rules that shape the development of plays of giving and asking for reasons, BIO-Pluralism focuses on how cognition is gathered by **classifying the psychological and sociological responsive mechanisms at work** in the use of different logics.

¹⁵ **Formal and Material Plays.** These rules constitute one of the main features of the Dialogical Framework. In fact, while working out a new concept of judgmental content, Per Martin-Löf (2015, 2017a,b, 2019, 2020) highlighted the contribution of the Dialogical Framework to the notion of *assertoric* knowledge and to Göran Sundholm's (1997, 2012, 2019) germane concept of *epistemic assumption*, which turns out to have an essential role in an explanation of validity of inference that does not fall into the very well-known justification-circle. The interaction of statements, governed by these rules constitutes the root of what at the strategic level yields assertoric knowledge. The *Socratic Rule*, which owes its name to Marion & Rückert (2016), who indicate some crucial texts in Plato's and Aristotle's *Dialectics* as its historical source – See too Crubellier et al. (2019) and McConaughy (2022), extends Lorenzen & Lorenz's (1978) *Formal Rule* to material dialogues. The main idea is that allows the Proponent to state some specific statements on the sole ground that the Opponent stated them before. Moreover, the rationale behind is that if the thesis is to be challenged under the toughest conditions, the primitive (or declared as primitive) components of the Opponent's challenge shall be considered as holding. If such a stipulation is left out, the Opponent's challenge could be brought to a breakdown **without the thesis having undergone the** strongest possible scrutiny. Usually, the statements governed by this rule are elementary statements and definitions – See Clerbout & McConaughy (2022). These rules are absent in Jaakko Hintikka's *Game Theoretical Semantics* (GTS) and also in the further developments of it such as the ones by van Benthem and collaborators – cf. Hintikka (1973, pp.77-82) ; to this matter See too the further developments of GTS-approach published in the collection *Texts in Logic and Games*, launched by Johan van Benthem in 2007.

The Formal (Socratic) Rule has been the source of quite an amount of misunderstandings. It has been claimed, for examples, that it does not preserve the player independent meaning explanation of implication, since, if **O** states an implication such that the antecedent is an elementary proposition and **O** did not yet assert that antecedent, then **P** cannot challenge it -see e.g. Tulenheimo (2011). However, this objection misconstrues the prescriptions at work in local dialogical meaning explanations. In the case of implication, the local meaning rules prescribe that a challenge requires the challenger **being able** to state the antecedent. The Formal (Socratic) Rule restricts the space of possibilities for that challenge : **P can** challenge the

SR2.1 (Formal Rule): **P** may not state an elementary proposition unless **O** stated it first. Elementary propositions cannot be challenged.

This rule shapes *formal plays or purely assertoric plays*, whereby no thesis can state an elementary proposition.

Modal and temporal logic require some additional conditions for **P** overtaking an elementary proposition stated by **O**, namely: **P** may not state an elementary proposition in (modal and/or temporal context) unless **O** stated it first in the same context. Elementary propositions cannot be challenged

SR2.2 (Socratic Rule): **P** may not state an elementary proposition unless **O** stated it first. Elementary propositions stated by **P** can be challenged, but not those of **O**.

This rule shapes *material plays or material assertoric plays*. In the case that the thesis states an elementary proposition such as $\neg A$, **O** can launch the challenge $?_A$, iff she (**O**) did not state the same elementary proposition before. The elementary propositions of material plays have content.¹⁶

This requires a special rule for each of the elementary propositions involved in a play, which stipulate that **O** can state some primitives when requested.

For example, in the case of statements involving natural numbers, the Socratic Rule stipulates that **P** can always ask **O** to state that 0 is a natural number. Once **O** responded to such a request **P** can overtake it. This is combined with a rule for local meaning establishing that for any statement by **X** of the form the successor of n is a natural number, the questioner can request **X** to further state that n is also a natural number. In the case of empirical elementary propositions some verification process determined by epistemological context at work will determine which elementary propositions can be stated by **O** without challenge.

SR3 (winning rule):¹⁷ The play ends when it is a player's turn to make a move but (s)he has no available move left. That player loses and the other player wins.

In the dialogical approach validity is defined via the notion of *winning strategy*, where winning strategy for **X** means that for any choice of moves by **Y**, **X** has at least one possible move at his disposal such that (s)he (**X**) wins:

Validity (definition): A proposition is valid in a certain dialogical system iff **P** has a formal winning strategy for this proposition set as thesis of the dialogue.

implication if he can state the antecedent, however this does not contravene its local meaning explanation. Notice that for example, the repetition rank, might also restrict the number of challenges on a conjunction, but this does not defeat the dialogical meaning explanation of conjunction which prescribes that a conjunction can be challenged if the challenger can request for the left and the right component of the conjunction.

¹⁶ In fact, material assertoric plays, rather than formal plays, are the ones involved in the CTT approach to assertoric knowledge.

¹⁷ **Winning a play by P does not amount to the validity of the thesis.** The notion of a winning a play is not enough to render the notion of inference or of logical validity. In the following example, the thesis is of course not valid, but **P** wins. However, **P** wins because **O** made the wrong choice. In fact, **O** loses the play since when she defends the disjunction she chooses to state precisely what she asking Proponent to state, and the chosen repetition rank does not allow her to challenge twice the same move.

Opponent		Proponent	
		$\neg A \vee B \supset B \wedge A$	0
1	$m := 1$	$n := 2$	2
3	$\neg A \vee B$	$\neg B \wedge A$	4
5	$?_L^{\wedge}$	$\neg B$	8
3	B	$?_{\vee}$	6

There is obviously another play, where **O** wins, namely, asking for the left side of the conjunction. Dually a valid thesis can be lost because **P** this time, makes the wrong choice; e.g. **P** can lose a play with the thesis $\neg A \wedge B \supset B$, if he chooses repetition rank 1 and asks **O** to state the left side of the conjunction. Another telling example has been mentioned by Piecha (2015), namely the case of the thesis $\neg A \vee (A \supset A)$, which can be lost by **P** when playing with **SR1i**, though it is intuitionistically valid, namely when **P** chooses to answer the challenge on the disjunction with the left side. All these considerations show that we need to introduce the notion of winning strategy.

Thus,

- α is intuitionistically valid if there is a winning strategy for **P** for plays governed by structural rules shaped by **SR1i**.
- α is classically valid if there is a winning strategy for **P** for plays governed by structural rules shaped by **SR1c**.

Strategies

A *strategy* for player **X** in a dialogical game is a complete conditional plan of action. It is conditional in the sense that the strategy informs how **X** plays depending on the moves of the adversary. It is complete in the sense that it must inform how **X** plays for every possible choice of move by the adversary.

There are several ways to define a winning strategy within a Dialogical Framework – see e.g. Krabbe (1985), Clerbout (2014a,b,c), Rahman et al. (2018), Lion (2023a,b). For the sake of a simple presentation we will yield a variation of Felscher (1985) which can be also found in Redmond & Fontaine's (2011) dialogical tables with branches, and in Piecha (2015) – Rahman et al. (2018, pp. 90-109) describe a method for **finding a winning strategy** out of a succession of plays.

Winning strategy

- A player **X** has a winning strategy if for every move made by the other player **Y**, player **X** can make another move, such that each resulting play is eventually won by **X**.

As already mentioned, in dialogical logic validity is defined in relation to winning strategies for the proponent **P**.

- A proposition is valid if **P** has a winning strategy for a thesis stating this proposition
- A *winning strategy for P* for a thesis *A* is a tree *T* the branches of which end by an elementary statement by **P** (there is no other move available for **O** in that in branch), and where the nodes are those moves, such that
 1. *T* has the move **P A** as root node (depth of the tree 0),¹⁸
 2. if the node is an **O**-move (i.e. if the number of the node is odd), then it has exactly one successor node (which is a **P**-move),¹⁹
 3. if the node is a **P**-move (i.e. if the number of a node is even), then it has as many successor nodes as there are possible moves for **O** at this position.

Branches are introduced by **O**'s choices such as when she challenges a conjunction or when she defends a disjunction.

Finite winning strategies

Winning strategies for quantifier-free theses are always finite trees, whereas winning strategies for first-order theses can, in general, be trees of countably infinitely many finite branches (each branch is a play).

For example, if **P** states some universal quantifier, then each choice of the adversary triggers a different play and will yield a tree with infinite number of branches: one for each possible option. Since temporal (and modal) operators can also quantify over an infinite number of histories or moments (or modal contexts), they can also trigger infinite winning strategies. Infinite winning strategies for **P** can be avoided by introducing some provisos grounded on the following rationale:

- Because of the Formal (Socratic) Rule, when **O** has the chance to choose at some move *n* on how to challenge a universal or defend an existential, her optimal move is to always choose an *a* which is new with regard to the development of the play at that move *n*.
- On the contrary **P**, who will do his best to force **O** to state the elementary proposition she asked **P** for, when he challenges a universal of **O** or defends an existential, will copy **O**'s own choices (when **she** challenged an universal or an universal (if **O** did already make such a choice)).

¹⁸ Recall that the depth of a node in the tree is *the length of the path from the root to the node*.

¹⁹ In this case **P** can either challenge that move or answer (if the **O** move is a challenge). If there are many options for **P** only one will be retained. Indeed if there is a winning strategy for **P** it is built out of one of **P**'s choices.

These take us to the following restrictions:

1. If the depth of a node n is even such that \mathbf{P} stated a universal at n , and if among the possible choices for \mathbf{O} she can choose an a which is new in the branch, then this move counts as the only immediate successor node of n .
2. If the depth of a node n is odd such that \mathbf{O} stated an existential at n , and if among the possible choices for \mathbf{O} she can choose an a which is new in the branch, then this move counts as the only immediate successor node of m , i.e. the node where \mathbf{P} launched the attack on n .^[1]
3. If it is \mathbf{P} who has the choice, then only one of the plays triggered by the choice will be kept.

Exercises

Commented solutions in the Appendix

Check classical and/or intuitionistic validity by running the relevant plays for

1 $\neg\neg A \supset A$, 1a) using **SR1c**, 1b) using **SR1i**.

2 $A \supset \neg\neg A$, using **SR1i**.

3 $\neg(A \vee B) \supset \neg B \wedge \neg A$, 3a) using **SR1i**, 3b) by building the **strategy tree** using **SR1i**.

4 $\neg(A \wedge B) \supset \neg B \vee \neg A$, 4a) using **SR1i**, 4b) **SR1c**, 4c) by building the **strategy tree** using **SR1c**.

5 $A \vee \neg A$, 5a), using **SR1c**, 5b) using **SR1i**

6 $\neg\neg(A \vee \neg A)$, using **SR1i**

7 $!((A \supset B) \supset A) \supset A$, 7a) using **SR1c**, 7b) using **SR1i**, 7c) by building the **strategy tree** using **SR1i**.

8) $\forall x (A(x) \wedge B(x)) \supset (\forall x A(x) \wedge \forall x B(x))$, using **SR1i**.

9) $(\forall x A(x) \vee \forall x B(x)) \supset \forall x (A(x) \vee B(x))$ using **SR1i**.

10) $\forall x (A(x) \vee B(x)) \supset (\forall x A(x) \vee \forall x B(x))$ using **SR1c**.

11) $\exists x (A(x) \vee B(x)) \supset (\exists x A(x) \vee \exists x B(x))$ using **SR1i**.

12) $(\exists x A(x) \vee \exists x B(x)) \supset \exists x (A(x) \vee B(x))$ using **SR1i**.

13) $\exists x (A(x) \supset \forall y (A(y)))$, 14a) using **SR1c**, 14b) using **SR1i**.

14) $[(\forall x \forall y \forall z ((R(x,y) \wedge R(y,z)) \supset R(x,z))) \wedge (\forall x \forall y (R(x,y) \supset R(y,x))) \wedge \forall x \exists y (R(x,y))] \supset \forall x R(x,x)$, using **SR1i**.

1.2 Main Research Programs on Philosophy and History of Logic Related to the Dialogical Framework

Currently, the philosophical development of the Dialogical Framework experiences a thriving interest especially in the field of argumentation theory, history and philosophy of logic and mathematics, computer science, applied linguistics, legal reasoning, and artificial intelligence. In the present paper we will focus on the deployment of the Dialogical Framework within Per Martin-Löf's Constructive Type Theory (CTT), but let us first mention succinctly those main research game theoretical approaches that relate to the Dialogical framework.

1. Dialogical Logic and Constructivist Approaches to Meaning and Knowledge.

This is the main tradition of dialogical logic and in the next section we will focus on new developments in that direction. Felscher (1985) re-writes winning strategies using the semantic tableaux of Beth (1955), and his formalizations are strongly present in nowadays literature on the dialogical framework or some sequent-calculus variant of it – see Fermüller (2003), Sørensen and Urzyczyn (2007); Alama et al. (2011); Uckelman et al. (2014); Researchers at the U. Saarland and Lille, have made dialogical logic a framework for expressing various logics other than the initial intuitionistic logic and this launched *Dialogical Pluralism*. The rationale behind Dialogical Pluralism is that if deductive rationality is declined as dialogical interaction and the latter is governed by the general properties of the inference-relation, disagreeing with such standards of rationality amounts to disagreeing with the structural rules that shape inference. – see Rahman & Rückert (2001), Rahman and Keiff (2005), Keiff (2011), Rückert (2011). The

most recent version of Dialogical Pluralism is the one of Dutilh Novaes (2015; 2020). We will come back to it in our discussion on Argumentation theory.

Ranta (1988) launched a dialogical interpretation of Martin-Löf's Constructive Type Theory (CTT), whereby winning strategies are identified with proof-objects. This interpretation opened the way to develop is Type Theoretical Grammar for natural languages based on language games – see Ranta (1994), Ginzburg (2012). By these times Coquand (1995) applied dialogical logic for the foundations of mathematics and logic – see Sterling (2021).

More recently, while working out a new concept of judgmental content, Martin-Löf (2015, 2017a,b, 2019, 2020) highlighted the contribution of the dialogical framework to the notion of *assertoric* knowledge, which turns out to have an essential role in an explanation of validity of inference that does not fall into the very well-known justification-circle. This produced different research projects linking CTT and the dialogical framework such as Klev's (2022, 2023) dialogical rules for assertoric knowledge within CTT and Rahman et al. (2018) *Immanent Reasoning* that shall be discussed in the last section of our paper.

- 2. Game Theoretical Semantics.** This approach developed by Hintikka (1968) shares the game-theoretical tenets of dialogical logic for logical constants, but switches to standard truth-functional model theory at the level of elementary statements. This approach gave rise to large and successful programmes on dynamic epistemic logics launched by van Benthem in Amsterdam – see the collection *Texts in Logic and Games*. Lorenz's *semantization of pragmatics*, just mentioned, targets these kind of game theoretical approaches.

A different direction of GTS, closer to the constructivist tenet of the dialogical framework is Hintikka & Sandu's Independence Friendly Logic (IF-Logic), which deals with games with incomplete information where third-excluded is not generally valid. It is well known that cases of the so-called Henkin branching quantifiers such as $\forall x \forall y \exists z \exists u A(x, y, z, u)$ present an important limitation of first-order logic. In Hintikka's IF-notation, this is written: $(\forall x)(\forall y)(\exists z \ / \forall y)(\exists u \ / \forall x) \exists u A(x, y, z, u)$. The dialogical way to deal with it is to define the meaning of $(\exists z \ / \forall y)A$ by requiring that when **X** defends a challenge on \exists and has therefore to choose a value for z , s(h)e that player *does not know* the choice for y made by his **Y** – see Hintikka (1996), Hintikka & Sandu (1997).

- 3. Argumentation theory and Dialogical Logic.** This framework initiated by Barth and Krabbe (1982; see also Gethmann 1979) links dialogical logic with informal logic, which originated in the work of Perelman and Olbrechts-Tyteca (1958) and Toulmin's (1958) argumentation theory. It also includes the work of Walton (1984), Johnson (1999), Woods' argumentation theory – see Woods (1989) and Woods et al. (2000) –, and critical thinking – see Vaidya 2013, among others). Here we must also mention the work of Dutilh Novaes (2015; 2020); Dutilh Novaes and French (2018); French (2021), who combine informal logic with the formal setting as formalized by Sørensen and Urzyczyn (2007). Indeed, Dutilh Novaes approach, called the *Built-In-Opponent* (BIO), amounts to a dialogical interpretation of inferential steps carried out in an already pre-determined inferential system. Thus, according to the BIO approach, the dialogical perspective has as main purpose to lead to convincing and/or persuading interlocutors concerning the reasoning behind the inferential steps employed in a deduction carried out by means of a sequent calculus. This tenet also grounds her *Dialogical Pluralism*, which focus on a dialogical interpretation – in the sense just mentioned – of the structural rules at work in Gentzen-style systems. More generally, in order to analyse the convincing power of an argument a *three-tiered model of epistemic exchange* is deployed, consisting in (i) identifying possible sources for epistemic

exchange, (ii) choosing among those possible sources, and (iii) engaging with the content. This approach that amounts to assume a given inferential system and add a dialogical interpretative layer on it, is of course very different to Lorenzen and Lorenz proposal who aimed at providing dialogical foundations of meaning and logic. According to Lorenz's characterization such kind approaches fall under the category of *pragmatization of semantics*.

Another project towards linking the dialogical framework with informal argumentation is the one of Prakken (2005) who deploys the dialogical framework in order to shape non-monotonic reasoning particularly in the context of legal reasoning. Another important strand in the literature closer to the work of Krabbe than to work of Lorenzen and Lorenz is Eemeren & Grootendorst (2004) pragma-dialectics whereby consensus is seen as the main goal of an argumentative process.

- 4. Ludics and Linear Logic.** This tradition provides an overall theory of proof-theoretical meaning based on interactive computation. Blass (1992) proposed a dialogical semantics (in the sense of tradition 1) for Girard's (2001) linear logic, to which *Ludics* is often associated - see Lecomte & Quatrini (2011a,b).

Ludics proposes a kind of theory of meaning as interaction focused only on what we called above the invariant meaning core at work within local dialogical meaning explanations. Within Ludics, structural rules are reduced to local rules by introducing special operators ; e.g. in order to introduce contraction an operator for repetition ranks is explicitly introduced in the object language as a unary connective, which allows turning an intuitionistic negation into a classical negation – see Lecomte & Quatrini (2011a,b). Moreover, Ludics proposes a dynamic interaction process, involving only those rules that preserve the invariant meaning core, whereby structural and strategic rules are constituted. The invariant core amounts to the bare interaction of questions and answers or *expectations* and *intentions*, to use Lecomte & Quatrini's (2011a) words.

- 5. Dialogical Perspectives in the History of Logic and Argumentation.** Research in the history of logic with particular accent on the dialectical genealogy of Logic is experiencing nowadays a huge revival that includes studies not only in the ancient Greek and Medieval (Arabic, Byzantine, Hebrew and Latin traditions), but also in Indian, Chinese and Japanese traditions. The literature on the subject is immense and is not possible in this paper to even attempt to provide a short overview. Notice that, very much to our regret, with few exceptions, scholars working in contemporary Argumentation theory do not know or neglect the work of their colleagues in the vast historical sources on the subject or have quite of a superficial understanding of it. So, allow us to mention just a few that have been directly influenced by the work of Lorenzen and Lorenz and Game Theoretical Approaches. Before doing so, let us stress one methodological, hermeneutical, point that might elucidate the rationale behind the use of the Dialogical Framework for reconstructing historical patterns for logical reasoning. According to the dialogical stance, meaning is dialectical interaction including that of logical constants. Thus, in order to explain the meaning of some expression within a dialectical context we need bring to the fore the rules for request and answer involving such an expression. To put it bluntly, dialogical meaning explanations, serve as a bridge between informal and formal analyses for the study of logical constants and the underlying inference theory. Most of the handful of scholars mentioned below, explicitly or implicitly adopted this methodological principle. In ancient Greek philosophy, let us mention the work a Dialectical reading of Aristotelian Syllogism of McConaughy (2021), who on one hand develops further the work, already mentioned, of Ebbinghaus (1964), Lorenz & Mittlestrass (1966, 1967), and on the other she deploys the formal means of recent studies on

dialogical logic and syllogism – see Castelnérac & Marion (2009), Crubellier (2011), Marion & Rückert (2016), Crubellier et al. (2019). Hintikka’s (2004) includes reprints of papers discussing Aristotle’s dialectics under the background the GTS. The GTS reading of traditional logic and Keffer’s (2001) dialogical reconstruction of *Obligationes* underlies Dutilh Novaes (2007) and Uckelman’s (2013) work on Medieval debate theory and Meaning. One landmark in this field is the work of Yrjönsuuri (2001) on Obligations and Insolubles. In relation to the Arabic tradition, so far we know, Miller (1984, repr. 2020) was the first to explicitly deploy Lorenzen and Lorenz Dialogues for the analysis of Arabic Debate Theory. The most impressive work in this direction is the one of Young (2017, 2021), who influenced also Iqbal’s (2022) reconstruction of the reception of Arabic Juridical Debate Theory in Indonesia. Notice that the texts in Arabic Disputation theory represent most probably the richest source of manuscripts on the subject in Medieval Times – by large yet unexplored and neglected.²⁰ Let us finish by mentioning Gorisse’s (2017, 2018) dialogical reconstruction of Jaina dialectics – see too Clerbout et al. (2011).

1 Immanent Reasoning

1.1 Enriching the Expressivity of the Dialogical Framework

Dialogues are games of giving and asking for reasons; yet, in the standard dialogical framework, the reasons for each statement are left implicit. The *Immanent Reasoning* (IR) framework imports a form of judgement from Martin-Löf’s Constructive Type Theory (CTT) in Martin-Löf and Sambin (1984) in order to make these reasons explicit in Rahman et al. (2018). Fully developed, statements thus have the following form:

$$\mathbf{X} ! a \varepsilon B^8$$

where B is a proposition and a its local reason, i.e., the particular, circumstantial reason that entitles one to state B , which is read as \mathbf{X} states that a provides evidence for B . In this fashion, the reasons interlocutors have for making a statement are specified at the object-language level. As expected, IR distinguishes local reasons and strategic reasons. Local reasons are brought forth in particular plays, whereas strategic reasons are a recapitulation of all the relevant plays for constituting a winning strategy. Moreover, IR allows us to enrich the standard dialogical framework within an interactive theory of meaning that makes use of the expressivity of the fully interpreted languages at work in CTT. This expansion of the dialogical framework does not only allow to integrate the CTT analyses of the notorious donkey-sentences, branching quantifiers, and of generalized quantifiers (such as Most, At-Most, At-Least, and so on) Sundholm (1986, 1989); Ranta (1994), but it also provided a venue for studies in the history and philosophy of logic that brings to the light the rationale behind ancient and medieval logical analyses Crubellier et al. (2019); Iqbal (2022), the dialogical stance on contemporary and historical legal reasoning Rahman and Zarepour (2019)

⁸ Whereby “ ε ” stands for type-token relation or element of a set. The most recent notation in CTT is “:”, however, we use “ ε ” which was used by Lorenz and Mittelstrass (1966, 1967) for the instantiation-relation within their dialogical analysis of Plato’s theory of meaning. In *Immanent Reasoning* “ $a \varepsilon a$ ” stands for a local reason for a .

²⁰ Dutilh Novaes (2020), in her book on the *Dialogical Roots of Deduction*, astonishingly, not only does she almost not acknowledge the existence of Islamicate Disputation theory – letting by side her rather thin overview and interpretation of Arabic Logic, but also Seems to neglect the studies of scholars in the history of the dialogical perspective on Logic in general, when she develops her own theory of dialogues and argumentation.

1.2 Immanent Reasoning revisited

While working out a new concept of judgmental content, Martin-Löf (2015, 2017, 2019a,b) highlighted the contribution of the Dialogical Framework to the notion of *assertoric knowledge* and to Göran Sundholm (1997, 2012, 2019)'s germane concept of *epistemic assumption*, which turns out to have an essential role in an explanation of validity of inference that does not fall into a well-known justification-circle⁹. According to this view, when justifying an inference it is enough to assume that *someone else* has *asserted* the premises and under that assumption, show that *you may then assert* the conclusion¹⁰ Klev (2022, 202x).

In those lectures and texts, which discuss the deontic undergirding of logic, Martin-Löf points out that the main philosophical tenet of the dialogical approach is that judgmental content amounts to the pragmatic and normative notion of a task to be solved. On this account, *a judgment, or an assertion, is correct if and only if the agent making it knows how to (is able to, can) perform the task that constitutes its content* Klev (202x).

This takes us back to the inception of Dialogical Logic and to the work of Lorenz (2010) and to his formulation of *meaning as the mastery of an action schema*. Moreover, the main philosophical tenet of dialogicians, Lorenz observes, is the rejection of the usual reductionist ways to understand the interface between semantics and pragmatics.

Lorenz's general dialogical principle is that the interface between semantics and pragmatics should be understood neither as the result of the *semantization of pragmatics* at work in different varieties of formal semantics — where deontic, epistemic, ontological, and temporal constraints become truth-functional operators; nor as the result of the *pragmatization of semantics* as promoted by speech act theories — where a propositional kernel, when put into use, is complemented by moods yielding assertions, questions, commands and so on Lorenz (2011).

In short, Lorenz (2010)'s view is that the differentiation of semantic and pragmatic layers is the result of the articulation within one and the same utterance: each utterance displays in principle both features, as it signifies (semantic layer) and communicates (pragmatic layer).

In contrast to the proof-theoretical and inferentialist focus on assertions regarded as the minimal unit of conceptual content (Brandom (2000)), the Dialogical Framework, bestows moves, called *statements*, the role of basic units of meaning and knowledge rather than assertions. Whereas statements constitute some specific language games called “Plays”, and assertions constitute the basic units of strategies.

⁹ The usual conception in mathematics of judgment known as demonstrated judgment yields a circle when it is used to justify inference. Indeed, “In mathematics, it is natural to count a judgment as known if it has been demonstrated. Under this account of knowledge, the explanation of the validity of inference becomes: the conclusion J can be demonstrated under the assumption that the premises J_1, \dots, J_n have been demonstrated. That a judgment has been demonstrated means, however, that it is the final conclusion in a chain of valid inferences (we count an axiom as the conclusion of a valid inference with no premises). At least for mathematical discourse, the explanation of inferential validity is therefore circular: it appeals to the notion of demonstrated judgment, which in turn is explained in terms of validity.” Klev (202x)

¹⁰ The point is that *assertoric knowledge* is weaker than *apodictic knowledge*: the latter but not the former requires demonstration. The knowledge obtained by an epistemic assumption is of the assertoric kind. When discussing the notion of epistemic assertion Sundholm (2019) produced the following formulation (already present in some of Sundholm's previous talks): “When I say therefore, I give others my authority for asserting the conclusion, given theirs for asserting the premisses.”

So, statements are, in the most general form they take in rules for local meaning, player independent; however, assertions encode a monological can-must **P**-perspective. Assertions encode the tasks that both players require to solve in order to build a winning strategy.

From the dialogical point of view, strategies are constituted by plays, such that the strategic objects won by a process of “recapitulation” on those plays encode the sequence of moves which need to be carried out in order to accomplish the task associated, now indeed, to the assertion at stake. Thus, though the dialogical perspective shares the view with CTT that propositions are neither assertorically nor apodictically known – what is known is that the proposition is true (at the strategy level) – it departs from the latter view by grounding knowledge – or acknowledged truth – expressed by assertions, on the more basic notion of statement.

The interaction of statements, governed by the rules setting the dialogical meaning explanations, and in particular by the *Socratic Rule*, constitutes the root of what at the strategic level yields assertoric knowledge¹¹.

The new approach to Dialogical Logic called *Immanent Reasoning* (IR), which imports some of the insights of the fully interpreted language of CTT Rahman et al. (2018), puts the play perspective in foreground. This seems to open a path for casting Martin-Löf’s dialogical rules in a setting that puts these rules into plays.

Perhaps the main general differences ensue from the fact that whereas CTT focuses on mathematics and demonstration, the main perspective of the Dialogical Framework, particularly after the developments of Kuno Lorenz (1970, 2010, 2011, 2021), is natural language.

The aim of the present chapter is to accomplish a double task, namely:

1. It aims at setting Martin-Löf’s dialogue rules for CTT (DCTT) as presented by Klev (2022) in a framework for dialogical plays regulated by the Formal (Socratic) Rule, the point of which is to break down a given content into simpler and simpler components, reaching eventually those elementary statements upon which assertoric knowledge is grounded. We restrict our study to the rules for logical constants.
2. It aims at offering an alternative formulation of IR, that we call IR-II, which should on one hand simplify the notation of the original IR by adopting some ideas of DCTT, and on the other, build bridges for crossing from the Dialogician’s perspective to the Inferentialist’s perspective. However, IR-II is not conceived as a Dialogue setting plus a ready-made theory of Inference, but a Dialogical grounding of Inference.

¹¹ A usual misunderstanding is to assume that the *Formal Rule* occurs in both, Hintikka’s GTS and Lorenzen and Lorenz’s dialogical logic. This is a mistake which Ranta (1988) is not guilty of but it is ubiquitous in the literature. Though Hintikka adopted the dialogical interpretation of the logical constants of Lorenzen, he explicitly rejected the *Formal Rule*. According to Hintikka (1973) the Formal Rule does only provide *indoor games*, and so he proposes to appeal to a model theoretical perspective once the analysis procedure arrives at the level of elementary propositions; to this matter see too the further developments of GTS-approach published in the collection *Texts in Logic and Games*, launched by Johan van Benthem in 2007. Dutilh Novaes Novaes (2020) passes in silence over this difference (and its bearing on Sundholm’s notion of *epistemic assumption*) when she compares Lorenzen and Hintikka’s game theoretical stances on logic. In short, no assertoric knowledge or epistemic assumptions can be found in game theoretical approaches other than the dialogical one.

Of course, these objectives are on one side quite ambitious and on the other they do not include finite sets in general. This will require a larger development: the present paper should nevertheless take some first steps towards such a study.

2 The Dialogicians Stance: Towards Immanent Reasoning II

The DCCT rules constitutes progress towards linking CTT and dialogical logic, since it avoids the proliferation of notation used in IR-I. Moreover, Martin-Löf's remarks on the contribution of dialogical logic to assertoric knowledge and valid inference is an important point towards motivating such an interactive stance.

From the dialogician's point of view, the contribution of the dialogical framework amounts to furnishing a purely interactive notion of meaning invariant to introduction and elimination rules and to the underlying structural rules. Justifying an inference rule eventually comes down to showing that it emerges from the interaction of local meaning explanations of the expressions involved. Moreover, instead of the type-checking mechanism assumed by the CTT framework, the dialogical framework is endowed with the decidable notion of dialogue-definiteness. In other words, the two new main innovations of dialogical logic are:

- The introduction of a level of meaning explanations invariant in relation to players, plays and strategies
- The introduction of a level of plays upon which a mechanism for checking dialogue definiteness is built and winning strategies are generated.

2.1 The Dialogical Meaning Explanations for IR-II

1.1.1 Disjunction, Conjunction, Existential Quantifier

The dialogical meaning explanations for disjunction, conjunction and existential are almost the same as those in type theory Klev (2022, 202x). The main difference is that the rules are formulated as player independent. The double arrow, \Rightarrow , occurring in the tables below, is to be understood as the play level analogue of the evaluation arrow used in Klev (2022, 202x). As discussed further on in our text, the task associated with a statement of the form $d \Rightarrow c \in C$ is solved by certain form of calculation which shows, when successful, how to yield c from d . The calculation itself, essentially a process of what Klev calls definition unfolding, can also be shaped by means of dialogical interaction rules.

\vee	Request	Answer	...
$\mathbf{X}! d \varepsilon A \vee B$	$\mathbf{Y} ?_{\vee d}$	$\mathbf{X}! d \Rightarrow L^{\vee}(a) \varepsilon A \vee B$ or $\mathbf{X}! d \Rightarrow R^{\vee}(b) \varepsilon A \vee B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\vee}$	$\mathbf{X}! L^{\vee}(a) \varepsilon A \vee B$ $\mathbf{X}! R^{\vee}(b) \varepsilon A \vee B$	$\mathbf{Y} ?_{L^{\vee}(a)}$ $\mathbf{Y} ?_{R^{\vee}(b)}$	$\mathbf{X}! a \varepsilon A$ $\mathbf{X}! b \varepsilon B$

\wedge	Request	Answer	...
$\mathbf{X}! d \varepsilon A \wedge B$	$\mathbf{Y} ?_{\wedge d}$	$\mathbf{X}! d \Rightarrow \langle a, b \rangle \varepsilon A \wedge B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\wedge}$	$\mathbf{X}! \langle a, b \rangle \varepsilon A \wedge B$	$\mathbf{Y} ?_{L\wedge}$ or $\mathbf{Y} ?_{R\wedge}$	$\mathbf{X}! a \varepsilon A$ $\mathbf{X}! b \varepsilon B$

\exists	Request	Answer	...
$\mathbf{X}! d \varepsilon (\exists x \varepsilon A) B$	$\mathbf{Y} ?_{\exists d}$	$\mathbf{X}! d \Rightarrow \langle a, b \rangle \varepsilon (\exists x \varepsilon A) B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\exists}$	$\mathbf{X}! \langle a, b \rangle \varepsilon (\exists x \varepsilon A) B$	$\mathbf{Y} ?_{L\exists}$ or $\mathbf{Y} ?_{R\exists}$	$\mathbf{X}! a \varepsilon A$ $\mathbf{X}! b \varepsilon B[a]$

1.1.2 Implication, Universal Quantifier

It is the implication where the difference with the type-theoretical dialogue rules is manifest. Indeed, the dialogical meaning explanation of the implication, in a setting for dialogues where the Formal Rule is at work, requires distinguishing:

1. The player independent rules for local meaning
2. The effects of the global rules, which put at work the rules for local meaning in the context of the development of a play
3. The strategic meaning explanation that results from abstracting the function that sets the meaning dependence of the consequent upon the antecedent shaped by the play level global rules.

Concerning 1., let us recall that the local meaning explanations are not only player independent: they also set what counts as a legitimate move independently of optimality considerations. In the case of implication, the rules simply determine that challenging it amounts to the challenger's providing a local reason for the antecedent and the defender's providing a local reason for the consequent.

The notation $(L_{\mathbf{Y}}^{\supset}) R_{\mathbf{X}}^{\supset} \varepsilon \mathcal{A} \quad B$ for the local reason does not yet read as a function; it simply indicates that the defense of the implication requires the defender, \mathbf{X} , to defend the consequent by committing to $b \varepsilon B$ after the challenger \mathbf{Y} commits to the antecedent by stating $a \varepsilon A$. In order to simplify the notation, we will drop in the tables below the indexes \mathbf{X} and \mathbf{Y} occurring in $(L_{\mathbf{Y}}^{\supset}) R_{\mathbf{X}}^{\supset}$ and in $L_{\mathbf{Y}}^{\supset} R_{\mathbf{X}}^{\supset}$. Who states the antecedent, and who states the consequent, is anyway clear when the implication is challenged.

\supset	Request	Answer	...
$\mathbf{X}! d \varepsilon A \supset B$	$\mathbf{Y} ?_{\supset d}$	$\mathbf{X}! d \Rightarrow (L^{\supset}) R^{\supset} \varepsilon A \supset B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\supset}$	$\mathbf{X}! (L^{\supset}) R^{\supset} \varepsilon A \supset B$	$\mathbf{Y}! a \varepsilon A$	$\mathbf{X}! b \varepsilon B$

\forall	Request	Answer	...
$\mathbf{X}! d \varepsilon (\forall x \varepsilon A) B$	$\mathbf{Y} ?_{\forall d}$	$\mathbf{X}! d \Rightarrow L^{\forall} R^{\forall} \varepsilon (\forall x \varepsilon A) B$...

...	Request	Answer	Request	Answer
...	$\mathbf{Y} ?_{\forall}$	$\mathbf{X}! L^{\forall} R^{\forall} \varepsilon (\forall x \varepsilon A) B$	$\mathbf{Y}! a \varepsilon A$	$\mathbf{X}! b \varepsilon B(a/x)$

1.1.3 Abstracting the function

The local reason for the implication indicates, that the answerer's commitment to the consequent is dependent upon the challenger providing a local reason for the antecedent. From these moves a function can be abstracted, whereby the challenge provides an argument for the function. But this abstraction takes place at the strategy level. Notice that if an implication such as $d \varepsilon A \vee B \supset B \vee A$ constitutes the thesis of a dialogue, then there will be two plays for it which can be won by P, dependent upon O's choice : If O chooses the left side for defending the disjunction in the antecedent, P can win by choosing the right side for defending the disjunction in the consequent. If O chooses the right side for defending the disjunction in the antecedent, P can win by choosing the left side for defending the disjunction in the consequent. Now, from the carrying out of those plays, we abstract the function $f \varepsilon A \vee B \supset B \vee A$ – provided it has been verified that identical arguments yield identical values, and that a criterion of identity for functions is given, by producing the strategic object $\lambda(f) \varepsilon A \vee B \supset B \vee A_{12}$. The strategic object $\lambda(f)$ encodes the information that whatever results by breaking down what O brings as an argument for the antecedent, namely $a \varepsilon A$ (or $b \varepsilon B$), P can produce $f(a)$ (resp. $g(b)$) for the defense of the analysed consequent. Notice that this abstraction procedure is similar to the one described by

¹² Verifying that the condition of functionality and that the criterion of identity for functions have been fulfilled, can be implemented by suitable dialogical rules set at the strategy level.

Sundholm (2013) for generalizing a demonstration with concrete content to *logical validity*.

2.2 Demonstrations and Winning Strategies

In his talk for the award of Rolf Schock prize, Martin-Löf (2022) summarized the dialogical view on inference with the following remarks:

The concluder receives the premises from the premissers, and in turn gives away, or passes on, the conclusion. As a result of receiving the premisses from the premissers, the concluder gets the right, or permission, to request the premissers to perform their respective tasks C_1, \dots, C_n . The validity of the rule is tantamount to the concluder's ability to perform C when given this help from the premissers. Thus, the effect is that the premissers together with the concluder can perform the conclusion task C . You see the novelty that is not present in the usual explanations of inference and rules of inference: the novelty is that in an inference, the concluder gets the right to ask the premissers to perform their respective tasks C_1, \dots, C_n , and that means that the concluder gets helped by the premissers to perform these tasks. The validity of the rule is tantamount to the concluder's being able to do C given this help from the premissers to do C_1, \dots, C_n . This is a less elegant formulation, but it has the advantage of bringing out the interactive character of inference more clearly than Sundholm's formulation.

These words indeed stress the main feature of the interactive stance, which we mentioned while discussing the dialogue rules for implication. Now, the term "helping", triggered an unfruitful discussion to decide if dialogical logic is collaborative or adversarial: whereas Hodges (2001) accuses the dialogical rules for implication and inference to involve collaboration rather than challenge, Novaes (2015, 2020) accuses dialogical logic for being only or mainly adversarial¹³. The truth is that, when the focus is on assertoric knowledge, the dialogical setting is both, and this is what the Formal (Socratic) Rule accomplishes.

As mentioned above, stipulating the Opponent's (elementary) statements as unchallengeable, helps to put the thesis under the toughest possible scrutiny. Be that as it may, let us run a dialogue for proving the commutation of disjunction using IR's local dialogue rules in the context of the **structural rules** for the development of a play and the definition of strategy.

¹³ Let us recall once more the example of $A \vee B \supset B \wedge A$, but let us consider now a concrete material dialogue such as, say, $(1 = 1 \vee 1 = 0) \supset (1 = 0 \wedge 1 = 1)$, with repetition rank 1. Let us further assume that O is in a "good" mood and that, after conceding the antecedent and P is stating the consequent, O decides to ask for the right. Certainly, these are legitimate choices, but is O being collaborative? Moreover, let us switch again to the more general case of formal dialogue for $A \vee B \supset B \wedge A$, where the content is not apparent, and where O, after asking for the right, chooses herself A, if asked to choose for one of the sides of the disjunction. On one side O collaborates in stressing the meaningful components of the implication, on the other she is not by not submitting the thesis to the toughest possible scrutiny.

Let us recall that in dialogical logic a proposition is valid iff **P** has a winning strategy for a thesis constituted by a succession of relevant plays stating this proposition.

2.2.1 Example

Prove $e \varepsilon B \vee A$, given the premise $d \varepsilon A \vee B$

The idea is clear: whatever **O** chooses for solving the task attached to the premise, **P** can use it to solve his own task, depending upon **O**'s choice. So, the Opponent is indeed helping the Proponent to solve his own task. This is the way a dialogicians understands a dialogical demonstration. For the sake of simplicity, we silent the choice of repetition ranks.

Notational convention: In the context of IR the notation $\mathbf{P} !! e \varepsilon B \vee A \llbracket d \varepsilon A \vee B \rrbracket$ indicates that **P** is committed to building a winning strategy for $e \varepsilon B \vee A$, provided **O** grants $d \varepsilon A \vee B$. Thus, a challenge on $\mathbf{P} ! e \varepsilon B \vee A \llbracket d \varepsilon A \vee B \rrbracket$, amounts to **O** stating $d \varepsilon A \vee B$. In fact, this notation expresses the dialogical counterpart to the proof-theoretically inference stroke, whereby the double square brackets enclose the premises and the expression to the left of those brackets is the conclusion.

O		P			
				$! e \varepsilon B \vee A \llbracket d \varepsilon A \vee B \rrbracket$	0. 0
0.1	$! d \varepsilon A \vee B$			$! e \varepsilon B \vee A$	0.2
1	$?_{ve}$	0.2		$! e \Rightarrow R^v(a) \varepsilon B \vee A$	8
3	$! d \Rightarrow L^v(a) \varepsilon A \vee B$		0.1	$?_{vd}$	2
5	$! L^v(a) \varepsilon A \vee B$		3	$?_v$	4
7	$! a \varepsilon A$		5	$?_{L^v(a)}$	6
9	$?_v$	8		$! R^v(a) \varepsilon B \vee A$	10
11	$?_{R^v(a)}$	10		$! a \varepsilon A$	12

- 0.0. **P** I can prove $e \varepsilon B \vee A$, provided $d \varepsilon A \vee B$
- 0.1. **O** Fine, let me grant $d \varepsilon A \vee B$, show me how you defend $e \varepsilon B \vee A$
- 0.2. **P** indeed, $e \varepsilon B \vee A$
1. **O** What side are you going to choose in order to solve the task associated with the disjunction?
 2. **P** Well I will not answer yet. Tell me first which side are you going to solve your own task?
 3. **O** d evaluates as a local reason for the left.
 4. **P** So, let us play on the side of the disjunction of your own choice.
 5. **O** The left side of $A \vee B$ is my choice.
 6. **P** How do you solve the task $L^v(a) \varepsilon A \vee B$.
 7. **O** $a \varepsilon A$.
 8. **P** I will now answer your question posed at 1, by stating $e \Rightarrow R^v(a) \varepsilon B \vee A$.
 9. **O** Fine, let us continue the play, according to this choice of yours.
 10. **P** Yes, let us play on $R^v(a) \varepsilon B \vee A$.
 11. **O** How do you solve the task $R^v(a) \varepsilon B \vee A$.
 12. **P** $a \varepsilon A$. In fact, when you stated $a \varepsilon A$ in move 7 you gave me the authority of stating it myself.

1. **P** $a \in A$. In fact, when you stated $a \in A$ in move 7 you gave me the authority of stating it myself.

Obviously, the sequence of moves of this play above does not yet constitute a winning strategy for **P**'s thesis. A winning strategy is one according to which **P** wins whatever **O** chooses. So now we have to run a play, where **O** chooses the right with his move 3. A winning strategy for such a thesis results by composing two plays, such that whatever way **O** chooses to defend the disjunction $d \in A \vee B$, it will provide a play for $e \in B \vee A$ won by **P**¹⁴.

Following the definition of strategy described in the introduction, a winning strategy generated by a succession of relevant plays yields the following tree, with the thesis at the root of the tree (depth 0), and where we assume that the repetition ranks have been already fixed as $m := 1$ for **O** and $n := 2$ for **P**:

P ! $e \in B \vee A$ [[$d \in A \vee B$]]	
O ! $d \in A \vee B$	
P ! $e \in B \vee A$	
O ? _{$\vee e$}	
P ? _{$\vee d$}	
O ! $d \Rightarrow L^{\vee}(a) \in A \vee B$	O ! $d \Rightarrow R^{\vee}(b) \in A \vee B$
P ? _{\vee}	P ? _{\vee}
O ! $L^{\vee}(a) \in A \vee B$	O ! $R^{\vee}(b) \in A \vee B$
P ? _{$L^{\vee}(a)$}	P ? _{$R^{\vee}(b)$}
O ! $a \in A$	O ! $b \in B$
P ! $e \Rightarrow R^{\vee}(a) \in B \vee A$	P ! $e \Rightarrow L^{\vee}(b) \in B \vee A$
O ? _{\vee}	O ? _{\vee}
P ! $R^{\vee}(a) \in B \vee A$	P ! $L^{\vee}(b) \in B \vee A$
O ? _{$R^{\vee}(a)$}	O ? _{$L^{\vee}(b)$}
P ! $a \in A$	P ! $b \in B$

2.2.2 C-Strategic Objects

Notice in the example above, the plays that shape the winning strategy, do not make use of other rules than the dialogical meaning explanation for the disjunction. Moreover, the operator $\mathbf{D}(c, x, d, y, e)$ used in the CTT framework for the elimination rules does not occur in any of those **plays**.

From a dialogical point of view, strategic rules, corresponding to elimination rules, apply the local dialogical meaning explanations to those **O**-statements out of which winning strategies for **P** will be built. As pointed out in Rahman et al. (2018) (chapter 7), strategies are conceived as a record or *recapitulation* of the interactions shaping relevant plays, and thus their strategic objects, in their most general abstract form, encode the most salient moves constituting those plays.

¹⁴ Notice that all this very close to *Immanent Reasoning*. Indeed the arrow is very similar to instructions, and evaluating the arrow is very close to our solving instructions.

More precisely, C -strategic objects for the standard logical constants, the dialogical counterpart to CTT-proof-objects occurring in elimination rules for those constants, are operators composed by player indexed dialogical meaning explanations.

The equality rules for these strategic objects indicate how to reconstruct the plays building a \mathbf{P} -winning strategy for a thesis C , a family of sets, defined over disjunction, implication/universal quantifier, or conjunction/universal quantifier.

Let us thus, discuss briefly the strategic objects for disjunction, conjunction and implication.

1.1.4 The C -Strategic Object for Disjunction

Let us assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the disjunction $A \vee B$, and let us further assume that

- \mathbf{O} stated $d \varepsilon A \vee B$,
- for any $x \varepsilon A$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[L^\vee(x)]$ won by himself,
- for any $y \varepsilon B$ chosen by \mathbf{O} , \mathbf{P} has a method m' for developing a play on $C[R^\vee(y)]$ won by himself.

Then, the C -strategic object for the thesis $C[d]$ has the following form, which of course, follows the CTT-notation for the selector for disjunction:

$$\mathbf{P} ! \mathbf{D}(d, x.m, y.m') \varepsilon C[d]$$

How to evaluate the strategic object in order to obtain the relevant plays encoded by this C -strategic object is determined by the equality rules.

Assume that

- \mathbf{O} stated $a \varepsilon A$,
- for any $x \varepsilon A$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[L^\vee(x)]$ won by himself,
- for any $y \varepsilon B$ chosen by \mathbf{O} , \mathbf{P} has a method m' for developing a play on $C[R^\vee(y)]$ won by himself.

Then the strategic object for $C[L^\vee(x)]$ is:

$$\mathbf{P} ! \mathbf{D}(L^\vee(a), x.m, y.m') = m[a] \varepsilon C[L^\vee(a)]$$

Similarly, assume this time that

- \mathbf{O} stated $b \varepsilon B$,
- for any $x \varepsilon A$ chosen by \mathbf{O} , \mathbf{P} has a method m for developing a play on $C[L^\vee(x)]$ won by himself,
- for any $y \varepsilon B$ chosen by \mathbf{O} , \mathbf{P} has a method m' for developing a play on $C[R^\vee(y)]$ won by himself.

Then the strategic object for $C[R^\vee(y)]$ is:

$$\mathbf{P} ! \mathbf{D}(R^\vee(b), x.m, y.m') = m'[b] \varepsilon C[R^\vee(b)]$$

Given these rules, it is straightforward to show that the strategic object \mathbf{D} preserves the local dialogical meaning explanation of disjunction. Indeed, assume that \mathbf{P} stated $\mathbf{D}(d, x.m, y.m') \varepsilon C[d]$ and that \mathbf{O} granted $d \varepsilon A \vee B$. \mathbf{P} will respond to \mathbf{O} 's challenge on \mathbf{D} by asking her to choose which side of the disjunction she is willing to defend, before he commits himself to defend against the challenge on \mathbf{D} . If \mathbf{O} 's choice is $L^\vee(a) \varepsilon A \vee B$. and the play on this move by \mathbf{O} develops in such a way that she ends up by stating $a \varepsilon A$; then, if \mathbf{P} has a winning strategy, he will win with the move $\mathbf{P} ! m(a) \varepsilon C[L^\vee(a)]$. This latter move is \mathbf{P} 's defence against the challenge on \mathbf{D} . Same applies if the play on $\mathbf{O} ! R^\vee(b) \varepsilon A \vee B$. develops in such a way that \mathbf{O} ends up by stating $b \varepsilon B$. **Notice that what m and m' are depend on the local dialogical meaning explanations for C . Thus, the plays that build the winning strategy encoded by \mathbf{D} develop the local dialogical meaning explanations for $A \vee B$ and C .** Similar dialogical "justification" can be given

for all of the \mathbf{C} -strategic objects discussed below.

1.1.4.1.1 Remark

Concerning equality Rahman et al. (2018) pointed out that, from a dialogical perspective, equality between local reasons expresses the result, at the level of objects, of the act of **P** stating a local reason on behalf of **O**'s authority. In other words, within IR-I every use of the Socratic Rule at the level of local reasons gives rise to a judgmental equality. In contrast to IR-I, here we do not systematically "objectivize" at the play level the result of applying the Socratic Rule. Moreover, in IR-II the fundamental role of judgmental equalities is to indicate how to obtain those plays (where the Socratic Rules is at work) that build a winning strategy out of the resulting strategic object.

1.1.5 The C-Strategic Object for Conjunction

Let us assume that **P** has a winning strategy for the thesis C , which is a family of sets defined over the disjunction $A \wedge B$, and let us further assume that

- **O** stated $d \varepsilon A \wedge B$,
- for any $x \varepsilon A$ and any $y \varepsilon B$ chosen by **O**, **P** has a method m for developing a play on $C[(a, b)]$ won by himself,

Then, the C -strategic object for the thesis $C[d]$ has the following form:

$$\mathbf{P} ! \mathbf{E}(d, xy.m) \varepsilon C[d]$$

How to compute the strategic object in order to obtain the relevant plays encoded by this C -strategic object is determined by the equality rules.

Assume that

- **O** stated both $a \varepsilon A$ and $b \varepsilon B[a]$,
- for any $x \varepsilon A$ and any $y \varepsilon B$ chosen by **O**, **P** has a method m for developing a play on $C[(a, b)]$ won by himself,

Then, the strategic object for $C[(a, b)]$ is:

$$\mathbf{P} ! \mathbf{E}((a, b), xy.m) = m[a, b] \varepsilon C[(a, b)]$$

In order to make explicit those plays that build the winning strategy it is useful to have at hand the operators $L^\wedge(d)$, and $R^\wedge(d)$. These operators, which are the dialogical counterpart of the left and right projections within CTT, can be defined in terms of **E** as follows:

- $L^\wedge(d) = \mathbf{E}(d, xy.x) \varepsilon A$ for $d \varepsilon A \wedge B$
- $R^\wedge(d) = \mathbf{E}(d, xy.y) \varepsilon B$ for $d \varepsilon A \wedge B$

Notice that if **O** stated $(a, b) \varepsilon A \wedge B$, the further statements $L^\wedge((a, b)) \varepsilon A$ and $L^\wedge((a, b)) = a \varepsilon A$, can be introduced as answers specific to **P**'s request $?_{L^\wedge}$

and \mathbf{P} 's request $?_{L^2}$ to $\mathbf{O} ! (a, b)$. Similar holds for $R^A((a, b)) \varepsilon A$ and $R^A((a, b)) = b \varepsilon A$

Recall that from a dialogical point of view, the dialogical counterpart to elimination rules, are rules that apply the local dialogical meaning explanations to those \mathbf{O} -statements out of which winning strategies for \mathbf{P} will be built.

1.1.6 The C-Strategic Object for Implication

Let us assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the implication $A \supset B$, and \mathbf{O} stated $d \vDash A \supset B$. This requires the winning strategy for C to be built from the function abstracted from the local reason $\neg(L^2)R^2 \varepsilon A \supset B$, provided a criterion of identity for functions is given, and the local reason fulfils the condition of functionality¹⁵. More precisely \mathbf{O} 's d is then the abstract $\lambda(f)$, whereby the function is an element of the function type $(A)B$ ¹⁶. Moreover, \mathbf{P} has a method m for developing a play on $C[\lambda(f)]$ won by himself, for every f in $(A)B$.

In other words, assume that \mathbf{P} has a winning strategy for the thesis C , which is a family of sets defined over the implication $A \supset B$, and further assume that

- \mathbf{O} stated $d \vDash A \supset B$,
- Whenever \mathbf{O} produces a f , such that $f \vDash (A)B$, \mathbf{P} has a method $m(f)$ for developing a play on $C[\lambda(f)]$, such that $\mathbf{P} ! m(f) \vDash C[\lambda(f)]$ is won by \mathbf{P} himself,

Then the strategic object for $C[d]$ is:

$$\mathbf{P} ! \mathbf{F}(d, m) \vDash C[d]$$

In higher-order notation the CTT-operator \mathbf{Ap} can be defined as follows:

$$\mathbf{Ap}(d, a) := \mathbf{F}(d, (y)y(a))$$

where y is a bound function variable. Then we obtain

$$\mathbf{P} ! \mathbf{Ap}(d, a) \vDash B[a]$$

The point is that dialogical speaking, an application $\mathbf{Ap}(d, a)$ amounts to producing a winning strategy for $B[a]$ when d is a local reason for $A \supset B$ and a is a local reason for A .

¹⁵ As pointed out in Klev (2022, 202x), dialogically speaking, functionality requires the defender of $f \vDash (a)\beta$ to produce $f(a) = f(a') \vDash \beta$ whenever the challenger provides $a = a' \vDash a$. In the dialogical setting, the criterion of identity for functions amounts to the following 1) the defender of $f \vDash (a)\beta$ must produce $f \vDash (a')\beta$ whenever the challenger provides $(a)\beta = (a')\beta \vDash \varepsilon$ type; 2) the defender of $f = f' \vDash (a)\beta$ must produce $f = f' \vDash (a')\beta$ whenever the challenger provides $(a)\beta = (a')\beta \vDash \varepsilon$ type

¹⁶ The dialogical notation $(L^2)R^2$ for the local reason of an implication, where “ L^2 ” stands for the antecedent and “ R^2 ” of an implication already suggests that such a local reason, in its most general.

2.3 The Dialogician's view on Evaluations

According to the DCTT-rules as presented in Klev (202x), evaluation assertions are solved by acts of calculation¹⁷. However, in a lecture Klev (2022) has developed rules for challenging and answering evaluations. Let us discuss both approaches separately.

2.3.1 Evaluation as Calculation

From a dialogical point of view, the answers $c \in C$ (where c stands for a canonical element) and $\nexists c \in C$ to the request on solving the task expressed by some statement of the form $d \in C$, are understood as commitments, namely,

- To make it explicit how the task expressed by the challenged statement is solved. This commitment amounts to choosing a canonical element.
- The commitment that d evaluates to the previously chosen canonical element¹⁸.

If we wish to integrate calculations within a dialogue we can place those acts in sub-plays, such that their outcome influences the development of the main play:

1. the evaluation is challenged,
2. the evaluation is then carried out outside the main play,
3. the result is then brought into the main play, if the result is positive the play continues, if the result is negative the play stops and the antagonist wins.

With regard to the challenge, we adopt the following form of request¹⁹:

$$\mathbf{P} \boxed{!} d \in C \text{ or } \mathbf{P} \boxed{!} \text{-give-up}$$

Whereby the box with the exclamation mark indicates that the statement is associated to an act of calculation that takes place at a sub-play. Notice that from the dialogical point of view calculations can also be integrated to the dialogical framework, as a sequence of requests and answers determined by the dialogical meaning explanations for the expressions involved.

Let us put this together in a table

\Rightarrow	Request	Answer
$\mathbf{P} ! d \Rightarrow c \in C$	$\mathbf{O} ? \Rightarrow \neq$	$\mathbf{P} \boxed{!} d \Rightarrow c \in C$ C or $\mathbf{P} \boxed{!} \neq \text{-give-up}$

2.3.2 Evaluation as Definitional Unfolding

Klev's evaluation rules are inspired by the inductive definition of the evaluation relation given by Martin-Löf (2021) p. 106. In type theory, evaluation is understood

¹⁷If an assertion involves a purely mathematical quantity, its value is determined by calculation. However, if an assertion involves an empirical quantity its value is determined by experiment – Martin-Löf (2014) p. 5. ¹⁸ Thus, both parts for solving the task are to be accomplished in interdependence. So, assume that it is asserted that $2 + 2 \in \mathbb{N}$ and as answer to the request, the assertions $2 : \mathbb{N}$ and $2 + 2$

Clearly the first response is fine and the second response, applies blindly the rule, but the second answer asserts that an impossible calculation is doable.

¹⁹ This is an adaptation of Ranta (1994)'s formulation of questions.

as definitional unfolding, that is, repeated replacement of definiendum by the corresponding definiens until a primitive object is reached. Each such replacement can be understood as an evaluation step. If challenged, an evaluation judgement, $d = c \in C$, is defended by shortening the evaluation by one step: replaced by its definiens, say d' , and assert $d' = c \in C$. If d' is c , that is, if there is a definitional equation with d as definiendum and c as definiens, the defence is complete, and there is no assertion of the form $d = c \in C$ to be made. Otherwise, this assertion must be made, and it may in turn be challenged.

The following are the DCTT-rules for challenging an evaluation assertion:

$$\frac{\frac{\frac{\vdash a \Rightarrow c : \mathcal{C}}{\vdash b : \mathcal{C}} ?}{\vdash b \Rightarrow c : \mathcal{C}}}{\vdash a = b : \mathcal{C}}}{\vdash a = c : \mathcal{C}}$$

Whereby “ $\frac{\vdash}{\vdash}$ ”, stands for stipulation. The dotted line indicates that defender has the choice between making the three assertions displayed above the dotted line and asserting the identity displayed below that line.

In IR-I, as already mentioned, we used *local reasons* for statements. Local reasons were resolved by rewriting, in a sense close to the rules for challenging and answering evaluations mentioned above. However, the evaluation procedures, which in IR-I were called *resolutions*, required some redundant repetitions of identities. Klev (2022)’s new evaluation rules, which are simpler, can be adapted to the present dialogical setting. The main idea is to link the answer to the evaluation to the answer to a request on the canonical object involved.

In fact, from the dialogical point of view, the key notion with regard to the identities evoked by an evaluation is *play-invariance*: when stating $d = c \in C$ player stipulates that a play on $d \in C$ will yield c , and thus, if this is the case, the further plays are indistinguishable. For example, given $d \Rightarrow L^v(a) : A \vee B$, if the play on $d \in A \vee B$, yields $a \in A$, then the identity $d = L^v(a) \in A \vee B$ is said to have been verified modulo play-invariance.

In *formal* dialogues, when the content of the elementary proposition(s) constituting C are not known, the identity between d and c is established by either a one-step rewriting modulo (formal) play invariance, or by a chain of one-step rewritings rules such that the last yields the identity to c :

\Rightarrow	Request	Answer
$\mathbf{X} ! d \Rightarrow c \in C$	$\mathbf{Y} ? \Rightarrow$	$\mathbf{X} ! d = c \in C$ or $\mathbf{X} ! d = d' \in C$
$\mathbf{X} ! d = d' \in C$	$\mathbf{Y} ? \Rightarrow d'$	$\mathbf{X} ! d' \Rightarrow c \in C$
$\mathbf{X} ! d' \Rightarrow c \in C$	$\mathbf{Y} ? \Rightarrow$	same alternatives as before

However, in practice, in formal dialogues, the first option is the one that will be systematically applied. Notice that the play invariance condition will prevent state-ments such as $2 + 2 = 1 \in \mathbb{N}$. In material dialogues, play invariance amounts to having explicit dialogical meaning explanations for all of the expressions occurring in an evaluation statement, including the elementary proposition(s) occurring the state-ment. This yields, dialogical *calculations*, within material plays. Moreover, given such meaning explanations, the identity statement as a whole can be verified at once.

Summing up, within the IR-II setting, since identities are verified by play invari-ance, the distinction between calculation and definitional unfolding procedures is less rigid, unless empirical procedures are involved. However, even

in the latter case, the result can be integrated into the dialogical setting.

- In relation to the structural Rules for IR-II, they are basically the same as the ones for basic Dialogical Logic, with one exception. A challenge on an evaluation is restricted by a repetition rank specific to this operator. More precisely, since the arrow can be challenged twice – namely, once for “introducing” the arrow, and once for developing the evaluation itself, the repetition rank on the evaluation arrow is fixed to 2.²⁰

6 Topics in History and Philosophy of Logic

6.1 Aristotelian Logic is Dialogic I: Assertoric Logic

6.2 Aristotelian Logic is Dialogic II: Modalities

6.3 Dialectics and Legal Reasoning in Medieval Islam

6.4 Temporal Logic within Post-Classical Islam and its Contemporary Bearings

6.5 Contemporary Legal Reasoning

7 The True Dialogician

From the perspective of Per Martin-Löf’s Constructive Type Theory (CTT) dialogical rules have as main purpose to serve as formalized meaning explanations for the forms of assertoric content employed in type theory.

From the perspective of Catarina Dutilh Novaes (2020)’s Built-In-Opponent (BIO), the dialogical interpretation of inferences has as main purpose to lead to convince and/or persuade a (sometimes plural) addressee concerning the reasoning behind the inferential steps employed in a deduction carried out by means of a sequent calculus. Both approaches stress important insights of the Dialogical Framework: the inter- twining of deontic and epistemic features of meaning, in the case of CTT, and the role of psychological and sociological aspects, in the case of the BIO-approach.

However, in contrast to the CTT and the BIO understanding of the Dialogical Framework, Dialogicians do not conceive their approach as the result of combining a Dialectical framework with a ready-made theory of Inference, but as a setting that provides a Dia- logical Theory of Meaning, within which winning strategies accomplish the inferential task.

According to the *old* approach of what we indulge in calling the perspective of the *True Dialogician*, the play level is the level where meaning is forged. It reduces neither to the (singular) performances that actualize the interaction of the play level, nor to the constitution of the schema that yields a winning strategy. Nevertheless, the possibility of selecting not only those plays that accomplish best the explicative task towards knowledge and endorsement of a targeted (sometimes plural) or more

²⁰ Notice that all this very close to Immanent Reasoning, the arrow, is very similar to instructions, and evaluating the arrow is very close to our solving instructions.

generally putative addressee, but also the norms for the development of these plays, is indeed one of the most salient virtues of the Dialogical Framework.

We call our dialogues *Dialogues for Immanent Reasoning* precisely because *reasons* behind a statement, that are now *explicit* denizens of the object-language of plays, are *internal* to the development of the dialogical interaction itself.

The emergence of concepts, so we claim, are not only games of giving and asking for reasons (games involving *Why* -questions): they are also games that aim at displaying *how is it that the reason brought forward accomplishes an explicative task* on the moves leading to the proposed conclusion. Dialogues for immanent reasoning are dialogical games of *Why* and *How* constituted by actualizations and schematizations.

However, within the Dialogical Framework actualizing and schematizing should not be understood as performing two separate actions: through these actions we acquire the competence that is associated with the meaning of an expression by *learning to play*.

This feature of Dialogical Constructivism stems from Herder's view – Herder (2016)

– that the cultural process is a process of education, in which teaching and learning always occur together: dialogues display this double nature of the cultural process in which concepts emerge from a complex interplay of *why* and *how* questions.

In this sense, as pointed out by Lorenz Lorenz (2010), the dialogical teaching-learning situation is where *competition* – the I-perspective – and *cooperation* – the You-perspective – interact: both intertwine in collective forms of dialogical interaction that take place at the play level.

The insights of Paul Lorenzen and Kuno Lorenz on dialogical logic as re-establishing the historical links of ethics and logic provides a clear answer to Wilfried Hodges's skeptical view Hodges and Väätänen (2019) to what the Dialogical Framework's contribution is. Hodges's criticism seems to target the *mathematical* interest of a dialogical conception of logic, rather than a philosophical interest, which does not to attract much of his interest after all. The deep links concerning meaning between dialogical logic and constructive type theory might give an answer to Hodges's worries on the mathematical or logical interest of the dialogical framework.

In lieu of a general plaidoyer for the Dialogical Framework's philosophical contribution to the foundations of logic and mathematics, which would bring us too far Lion (2023), let us highlight three points which result from the above remarks:

1. The dialogical framework offers a sound and robust venue for the development of inference-based foundations of logic.
2. The interaction of epistemic and deontic notions rooted in *local dialogical meaning explanations* sets new ways of implementing the pragmatics/semantics interface within logic.
3. The dialogical setting is an irreducible feature of meaning and knowledge, whereby knowledge in general and assertoric knowledge in particular is rooted in the interactive stance on elementary statements.

Formal semantics in the Tarski-style is blind to the first point, misunderstands the nature of the interface involved in the second, and ignores the third.

Of course, this goes towards a radical form of dialogicism that we happily profess and promote and can understand that not everyone would like to endorse. However, the present proposal can also be seen as an invitation for reflecting together on some common ground that, at least some of us share.

1.1.6.1.1 Acknowledgements:

Special thanks to Ansten Klev (Academy of Sciences, Prague) whom we harassed with hundreds of questions and that he answered with patience and accuracy.

Klev also suggested a good deal of reformulations of notions central to our paper. Many thanks to Zoe McConaughey (Lille U. STL) with whom we started developing Immanent Reasoning and with whom we discussed many issues relevant to the context of the present paper and who suggested some thoughts in relation to the new formulation for the rules for evaluations. Many thanks to Leone Gazziero (CNRS, STL), Walter E. Young (McGill), Farid Zidani (Alger II) and the PhD students at the laboratory UMR 8163 STL: Louis Blazejewski, Rayane Boussad, Meriem Drissi, Amissah Moukama, Alioune Seck and Vincent Wistrand for important inputs and discussions on dialogical logic.

APPENDIX

Solutions to Exercises²¹

Chapter 1

1a) Playing with the Classical Structural Rule **SR1c**

Opponent		Proponent	
		$! \neg A \supset A$	0
1	$m := 1$	$n := 2$	2
3	$! \neg A$ 0	$! A$	6
	—	3 $! \neg A$	4
5	$! A$ 4	—	

P wins since **SR1c** allows move 6 that defends against move 3 even if it's not the last one (move 5). Moreover **P** can affirm the atomic formula A after **O** conceded it at move 5 since **SR2**.

- Notice that in this case and others below, when there are no relevant options for **O** (given the fixed repetition rank) the strategy tree has only one path so the dialogue and this tree are similar.

1b) Playing with the Intuitionistic Structural Rule **SR1i**

Opponent		Proponent	
		$! \neg A \supset A$	0
1	$m := 1$	$n := 2$	2
3	$! \neg A$ 0	???	
	—	3 $! \neg A$	4
5	$! A$ 4	—	

P loses since **SR1i** does not allow to defend against move 3, it is not the last attack from **O**, **P** can't answer after move 5. Hence, the implication rule $\neg A \supset A$ is not valid in intuitionistic dialogic.

2) Playing with the Intuitionistic Structural Rule **SR1i**

Opponent		Proponent	
		$! A \supset \neg A$	0
1	$m := 1$	$n := 2$	2
3	$! A$ 0	$! \neg A$	4
5	$! \neg A$ 4	—	
	—	5 $! A$	6

P wins since the play does not contravene **SR1i**.

- Notice that each formula valid in intuitionistic dialogic is also valid in classical since intuitionistic structural rules are more restrictive.

3a) Playing with the Intuitionistic Structural Rule **SR1i**

Opponent		Proponent	
		$! \neg(A \vee B) \supset \neg B \wedge \neg A$	0
1	$m := 1$	$n := 2$	2
3	$! \neg(A \vee B)$ 0	$! \neg B \wedge \neg A$	4
5	$?_L \wedge$ 4	$! \neg B$	6
7	$! B$ 6	—	
	—	3 $! A \vee B$	8
9	$?_V$ 8	$! B$	10

²¹ Revised by Lily Gallois.

P wins this play, but in order to prove validity, **P** must win with all the other choices of **O**. Namely, at move 5, requesting the right side of the conjunction instead of the left. But this also leads to **P** winning the play:

Opponent		Proponent	
		$\neg(A \vee B) \supset \neg B \wedge \neg A$	0
1	$m := 1$	$n := 2$	2
3	$\neg(A \vee B)$	$\neg B \wedge \neg A$	4
5	$?_{R\wedge}$	$\neg A$	6
7	$! A$	—	
	—	3 $! A \vee B$	8
9	$?_{\vee}$	$! A$	10

- The converse (establishing the De Morgan's Law) is not true in intuitionistic dialogic (in coherence with exercise 1b).

3b) The Strategy Tree

Since **P** wins in both of the available options for **O**, the thesis is logically valid. Moreover, since the plays used **SR1i**, the thesis is valid in intuitionistic logic and a fortiori also valid in classical logic. The winning strategy can be expressed by means of the following tree – whereby we assume that the repetition ranks have been already fixed as **O** $m := 1$ and **P** $n := 2$. The numbers within the brackets, preceded by an interrogation mark, indicate the move challenged; the ones preceded by an exclamation mark indicate the number of the request answered. When **O** has several options, the move before will get several children, one for each option.

	P $\neg(A \vee B) \supset \neg B \wedge \neg A$	0
	O $\neg(A \vee B)$	I (? ₀)
	P $\neg B \wedge \neg A$	II (! ₁)
O $?_{L\wedge}$	V (? _{II})	O $?_{R\wedge}$
P $\neg B$	VI (! _V)	P $\neg A$
O $! B$	VII (? _{VI})	O $! A$
P $! A \vee B$	VIII (? _I)	P $! A \vee B$
O $?_{\vee}$	IX (? _{VIII})	O $?_{\vee}$
P $! B$	X (! _{IX})	P $! A$
		VIII (? _I)
		IX (? _{VIII})
		X (! _{IX})

The last move in both branches is a **P** move (labelled with an even number). So, the tree displays a winning strategy for **P** and the thesis is logically valid.

4a) Playing with the Intuitionistic Structural Rule **SR1i**

Opponent		Proponent	
		$\neg(A \wedge B) \supset \neg B \vee \neg A$	0
1	$m := 1$	$n := 2$	2
3	$\neg(A \wedge B)$	$\neg B \vee \neg A$	4
5	$?_{\vee}$	$\neg B$	6
7	$! B$	—	
	—	3 $! A \wedge B$	8
9	$?_{L\wedge}$???	

P loses using **SR1i**, since he has to answer with $! A$ against **O**'s move 9 but it's possible for him to answer an atomic formula only if **O** conceded it before. To ensure **O** to do that, **P** should answer against move 5 to state $\neg A$ and later A , but it is not possible in **SR1i** to come back.

4b) Playing with the Classical Structural Rule **SR1c**

Opponent		Proponent	
		$\neg(A \wedge B) \supset \neg B \vee \neg A$	0
1	$m := 1$	$n := 2$	2
3	$\neg(A \wedge B)$	$\neg B \vee \neg A$	4
5	$?_{\vee}$	$\neg B$	6
7	B	—	

—		3	$!A \wedge B$	8
9	$?_L \wedge$	4	$!A$	12
[5	$?_V$	8]	$!\neg A$	10
11	$!A$	10	—	

P wins using **SR1c**, since he is allowed to come back to the defence of the disjunction and state this time $\neg A$. This new defence of **P** does not follow a new challenge of **O**: **P** is making use of his right to defend twice from the **same** challenge – this is why **O**'s challenge move 5 is written within square brackets. **O**'s challenge on the negation allows him to defend the **O**'s request posited at move 9. If instead of the left, **O** decides to ask for the right, a second play will be launched, where **P** can win (by symmetry) if **SR1c** is used again. The reader might wish to develop the tree of the winning strategy.

5a) Playing with the Classical Structural Rule **SR1c**

Opponent		Proponent	
		$!A \vee \neg A$	0
1	$m := 1$	$n := 2$	2
3	$?_V$	0	$!\neg A$
5	$!A$	4	—
[3	$?_V$	0]	$!A$
			6

P wins using **SR1c**, since he is allowed to come back to the defence of the disjunction (move 3) and state A , making by the way the last move of the play. As in the preceding example, the square brackets indicate that the challenge has not been repeated and **P** makes use of his right to defend twice from the **same** challenge.

5b) Playing with the Intuitionistic Structural Rule **SR1i**

Opponent		Proponent	
		$!A \vee \neg A$	0
1	$m := 1$	$n := 2$	2
3	$?_V$	0	$!\neg A$
5	$!A$	4	—
		???	

P loses using **SR1i**, since he has to answer to the last request, namely **O**'s move 3, and he is not allowed to come back to the defence of the disjunction and state A even if **O** conceded A before. Hence, **P** can not do such an answer, **O** wins.

- The non-validity of the law of excluded middle ($A \vee \neg A$) illustrates the main philosophical difference between intuitionistic logic and classical propositional logic.

6) Playing with the Intuitionistic Structural Rule **SR1i**

Opponent		Proponent	
		$!\neg(A \vee \neg A)$	0
1	$m := 1$	$n := 2$	2
3	$!\neg(A \vee \neg A)$	0	—
		3	$!A \vee \neg A$
5	$?_V$	4	$!\neg A$
7	$!A$	6	—
		3	$!A \vee \neg A$
9	$?_V$	8	$!A$
			10

P wins using **SR1i**, since repetition rank 2, allows him to challenge twice **O**'s move 3. Since, **P** can launch his second challenge after **O** stated A at move 5, he can use this (forced) help of **O** in order to defend the disjunction newly stated at move 8, and therefore state A himself. As **P** can win with **SR1i** he can also win with **SR1c**.

7a) Playing with the Classical Structural Rule **SR1c**

Opponent		Proponent	
		$!((A \supset B) \supset A) \supset A$	0

1	$m := 1$		$n := 2$	2
3	$!(A \supset B) \supset A$	0	$!A$	6
	—		3 $!A \supset B$	4
5	$!A$	4	—	

P wins this play, but in order to prove validity, **P** has to win with the other options of **O**, namely, answering immediately instead of challenging move 4. But this also leads to **P** winning the play:

Opponent		Proponent	
		$!((A \supset B) \supset A) \supset A$	0
1	$m := 1$	$n := 2$	2
3	$!(A \supset B) \supset A$	$!A$	6
5	$!A$	3 $!A \supset B$	4

7b) Playing with the Classical Structural Rule **SR1i**

P wins the second play using **SR1i**, but not the first. Indeed, in the first play, **P**'s move 6 is not an answer to the last request, namely **O**'s move 5, which challenges the implication. This **P**'s move is an answer to the previous request of **O** at move 3. The rule **SR1i**, does not allow such a move.

Opponent		Proponent	
		$!((A \supset B) \supset A) \supset A$	0
1	$m := 1$	$n := 2$	2
3	$!(A \supset B) \supset A$???	
	—	3 $!A \supset B$	4
5	$!A$	—	

7c) The Strategy Tree with the Intuitionistic Structural Rule **SR1i**

	$\mathbf{P} !((A \supset B) \supset A) \supset A$	0
	$\mathbf{O} ! (A \supset B) \supset A$	I (? ₀)
	$\mathbf{P} ! A \supset B$	II (? ₁)
$\mathbf{O} ! A$	III (? _{II})	III (! _{II})
$\mathbf{P} ???$	$\mathbf{O} ! A$	IV (! _I)
	$\mathbf{P} ! A$	

The right branch ends with a **P** move (even number). This branch displays the play where **P** wins since **O** decides to answer immediately. The left branch ends with a move by **O** (odd number) which is a counterattack on **P**'s move II. This branch displays the play where **O** wins, since **SR1i** prevents **P** to come back to the defence of the main implication and use this move of **O**. Since one of the branches doesn't end with a **P** move, the thesis is not intuitionistically valid.

8) Playing with **SR1i**.

Opponent		Proponent	
		$! \forall x (A(x) \wedge B(x)) \supset (\forall x A(x) \wedge \forall x B(x))$	0
1	$m := 1$	$n := 2$	2
3	$! \forall x (A(x) \wedge B(x))$	$! \forall x A(x) \wedge \forall x B(x)$	4
5	$?_L^\wedge$	$! \forall x A(x)$	6
7	$?_V a$	$! A(a)$	12
9	$! A(a) \wedge B(a)$	3 $?_V a$	8
11	$! A(a)$	9 $?_L^\wedge$	10

P wins this play and has even a winning strategy, since the other options of **O** yields a play where **P** also comes out as the winner.

9) Playing with **SR1i**.

Opponent		Proponent	
		$! (\forall x A(x) \vee \forall x B(x)) \supset \forall x (A(x) \vee B(x))$	0
1	$m := 1$	$n := 2$	2
3	$! \forall x A(x) \vee \forall x B(x)$	$! \forall x (A(x) \vee B(x))$	4

5	$?_{\forall} a$	4	$! A(a) \vee B(a)$	6
7	$?_{\forall}$	6	$! A(a)$	12
9	$! \forall x A(x)$	3	$?_{\forall}$	8
11	$! A(a)$	9	$?_{\forall} a$	10

P wins. There is a winning strategy for **P**, since he also wins the other possible play in which **O** defends the right side of the disjunction. Since the thesis is valid.

10) Playing with SR1c

Opponent		Proponent	
		$! \forall x (A(x) \vee B(x)) \supset (\forall x A(x) \vee \forall x B(x))$	0
1	$m := 1$	$n := 2$	2
3	$! \forall x (A(x) \vee B(x))$	$! \forall x A(x) \vee \forall x B(x)$	4
5	$?_{\forall}$	$! \forall x A(x)$	6
7	$?_{\forall} a$	—	
9	$! A(a) \vee B(a)$	3 $?_{\forall} a$	8
11	$! B(a)$	9 $?_{\forall}$	10
[5	$?_{\forall}$	$! \forall x B(x)$	12
13	$?_{\forall} b$	4]	
15	$! A(b) \vee B(b)$	—	
17	$! A(a)$	3 $?_{\forall} b$	14
		15 $?_{\forall}$	16

O wins. There is no winning strategy for **P**, even if after **O**'s move 11 **P** tries to continue the play by repeating his defence of the disjunction, and then challenge once more the universal (making use of his own choice for the repetition rank).

11) Playing with SR1i

Opponent		Proponent	
		$! \exists x (A(x) \vee B(x)) \supset (\exists x A(x) \vee \exists x B(x))$	0
1	$m := 1$	$n := 2$	2
3	$! \exists x (A(x) \vee B(x))$	$! \exists x A(x) \vee \exists x B(x)$	4
5	$?_{\forall}$	$! \exists x A(x)$	10
7	$! A(a) \vee B(a)$	3 $?_{\exists}$	6
9	$! A(a)$	7 $?_{\forall}$	8
11	$?_{\exists}$	$! A(a)$	12

P wins this play. There is of course another play, where **O**'s answer to the challenge on the disjunction is $B(a)$ instead of $A(a)$. **P** wins this second play too by using an analogue strategy.

12) Playing with SR1i

Opponent		Proponent	
		$! (\exists x A(x) \vee \exists x B(x)) \supset \exists x (A(x) \vee B(x))$	0
1	$m := 1$	$n := 2$	2
3	$! \exists x A(x) \vee \exists x B(x)$	$! \exists x (A(x) \vee B(x))$	4
5	$?_{\exists}$	$! A(a)$	10
7	$! \exists x A(x)$	3 $?_{\forall}$	6
9	$! A(a)$	7 $?_{\exists}$	8

P wins this play, and also the other play, should **O** choose to defend the right side of the disjunction.

13a) Playing with SR1c

Opponent		Proponent	
		$! \exists x (A(x) \supset \forall y A(y))$	0
1	$m := 1$	$n := 2$	2
3	$?_{\exists}$	$! A(a) \supset \forall y A(y)$	4
5	$! A(a)$	$! \forall y A(y)$	6

7	$?_{\forall} b$	6	$! A(b)$	10
[3	$?_{\exists}$	0]	$! A(b) \supset \forall y A(y)$	8
9	$! A(b)$	4	—	

P wins, because **SR1c** allows him to re-do his answer to the challenge on the existential at move 3, and then answer to the challenge on the universal at move 7, despite that the last request on the implication remains unanswered.

13b) Playing with **SR1i**

Opponent		Proponent	
		$! \exists x (A(x) \supset \forall y A(y))$	0
1	$m := 1$	$n := 2$	2
3	$?_{\exists}$	$! A(a) \supset \forall y A(y)$	4
5	$! A(a)$	$! \forall y A(y)$	6
7	$?_{\forall} b$???	

P loses, because **SR1i** does not allow him to re-do his answer to the challenge on the existential at move 3, and so he has to answer to the last challenge on the universal quantification at move 7.

14) Strategy Tree with **SR1i**

In this case, it is faster to directly build the winning strategy instead of going through all the possible plays. Moreover, in order to avoid proliferation of brackets, we assume that the conjunction has been ordered from left to right, in such a way that the challenge $?^n$ is a request for the n-th member of the conjunction. Accordingly, we also assume that **P** has chosen the repetition rank $n = 3$. We will also make use of the notation $?_{\forall} a$, $?_{\forall} b$, or a combination to challenge on a multiple quantification in order to spare some moves. The answer to this challenge consists in implementing the substitutions all at once.

P ! [$(\forall x \forall y \forall z ((R(x,y) \wedge R(y,z)) \supset R(x,z))) \wedge (\forall x \forall y (R(x,y) \supset R(y,x))) \wedge \forall x \exists y (R(x,y))$] $\supset \forall x R(x,x)$	0
O ! $(\forall x \forall y \forall z ((R(x,y) \wedge R(y,z)) \supset R(x,z))) \wedge (\forall x \forall y (R(x,y) \supset R(y,x))) \wedge \forall x \exists y (R(x,y))$	I (? ₀)
P ! $\forall x R(x,x)$	II (! ₁)
O $?_{\forall} a$	III (? _{II})
P $?^{\wedge 3}$	IV (? _I)
O ! $\forall x \exists y R(x,y)$	V (! _{IV})
P $?_{\forall} a$	VI (? _V)
O ! $\exists y R(a,y)$	VII (! _V)
P $?_{\exists}$	VIII (? _{VII})
O ! $R(a,b)$	IX (! _{VIII})
P $?^{\wedge 2}$	X (? _I)
O ! $\forall x \forall y (R(x,y) \supset R(y,x))$	XI (! _X)
P $?_{\forall} a, ?_{\forall} b$	XII (? _{XI})
O ! $R(a,b) \supset R(b,a)$	XIII (! _{XII})
P ! $R(a,b)$	XIV (? _{XIII})
O ! $R(b,a)$	XV (! _{XIV})
P $?^{\wedge 1}$	XVI (? _I)
O ! $\forall x \forall y \forall z ((R(x,y) \wedge R(y,z)) \supset R(x,z))$	XVII (! _{XVI})
P ! $?_{\forall} a, ?_{\forall} b, ?_{\forall} a$	XVIII (? _{XVII})
O ! $(R(a,b) \wedge R(b,a)) \supset R(a,a)$	XIX (! _{XVIII})
P ! $R(a,b) \wedge R(b,a)$	XX (? _{XIX})
O $?_{L^{\wedge}}$ XXI (? _{XX})	XXI (! _{XX})
P ! $R(a,b)$ XXII (! _{XXI})	XXII (! _{XXI})
O ! $R(a,a)$ XXIII (! _{XX})	XXIII (! _{XX})
P ! $R(a,a)$ XXVI (! _{III})	XXVI (! _{III})
O $?_{R^{\wedge}}$ XXI (? _{XX})	XXI (? _{XX})
P ! $R(b,a)$ XXII (! _{XXI})	XXII (! _{XXI})
O ! $R(a,a)$ XXIII (! _{XX})	XXIII (! _{XX})
P ! $R(a,a)$ XXVI (! _{III})	XXVI (! _{III})
O ! $R(a,a)$	XXI (! _{XX})
P ! $R(a,a)$	XXII (! _{III})

O can still challenge the conjunction in XX, but these branches will end by **P** stating $R(a,b)$ if the left is requested and $R(b,a)$ if the right is requested.

After move XX, **O** has the option to either directly answer the challenge or to counterattack before answering. For all of these options **P** wins. So, the thesis is logically valid.

References

- Aristotle (1991).
The Complete Works of Aristotle. The Revised Oxford Translation. Jonathan Barnes (ed.). Oxford: Clarendon Press.
- Blass, A. (1992).
“A Game Semantics for Linear Logic”. *Annals of Pure and Applied Logic*, vol. 56, pp. 183–220.
- Brandon, R. (1994).
Making it Explicit. Cambridge: Harvard University Press.
- Brandon, R. (2000).
Articulating Reasons. Cambridge: Harvard University Press.
- Castelnérac, B. & Marion, M. (2009).
“Arguing for inconsistency: dialectical games in the Academy”. In G. Primiero and S. Rahman, *Acts of Knowledge: History, Philosophy and Logic*, London: College Publication, pp. 37-76.
- Clerbout, N. (2014a).
La Sémantique dialogique. Notions fondamentales et éléments de métathéorie. London: College Publications.
- Clerbout, N. (2014b).
“First-Order Dialogical Games and Tableaux”. *Journal of Philosophical Logic*, vol. 43(4), pp. 785-801.
- Clerbout, N. (2014c).
“Finiteness of plays, and the dialogical problem of decidability”. *IhCoLog Journal of Logics and their Applications*, vol. 1(1), pp. 115-140.
- Clerbout, N. & McConaughey, Z. (2022).
“Dialogical Logic”. ”. In E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy* <https://plato.stanford.edu/entries/logic-dialogical/>.
- Clerbout, N. & Rahman, S. (2019)
“Immanent Reasoning and CTT: Building on Göran Sundholm’s Insight on Dialogical Logic”. Talk at the meeting *Formalisation vs. Meaning in Mathematics: Formal theories as tools for understanding Themes from the work of Göran Sundholm*. June 13th-15th, 2019 - IHPST, Paris. Talk presented by N. Clerbout.
- Clerbout, N. & Gorisse, M.H. & Rahman, S. (2011)
“Context-Sensitivity in Jain Philosophy: A Dialogical Study of Siddharṣiṅaṅi’s *Commentary on the Handbook of Logic*”. *Journal of Philosophical Logic*, vol. 40(5), pp. 633-662.
- Coquand, T. (1995)
“A Semantics of Evidence for Classical Arithmetic”. *Journal of Symbolic Logic*, vol. 60(1), pp. 325–337
- Crubellier, M. (2011).
“Du *sullogismos* au syllogisme ”. *Revue philosophique*, vol. 1/2011, pp. 17–36
- Crubellier, M., Z. McConaughey, M. Marion & S. Rahman (2019).
“Dialectic, The Dictum de Omni and Ecthesis”. *History and Philosophy of Logic.*, vol. 40, No. 3, pp. 207-233.
- Dutilh Novaes, C. (2007).
Formalizing Medieval Logical Theories: Suppositio, Consequentiae and Obligationes. Dordrecht: Springer.
- Dutilh Novaes, C. (2015).
“A Dialogical, Multi-Agent Account of the Normativity of Logic”. *Dialectica*, vol. 69(4), pp. 587–609.
- Dutilh Novaes, C. (2020).
The dialogical roots of deduction : historical, cognitive, and philosophical perspectives on reasoning. Cambridge: Cambridge University Press.
- Dutilh Novaes, C. & French, R. (2018).
“Paradoxes and Structural Rules from a Dialogical Perspective”. *Philosophical Issues*, vol. 28(1), pp. 129–158.
- Ebbinghaus, K. (1964)
Ein formales Modell der Syllogistik des Aristoteles. Göttingen: Hypomnemata, Untersuchungen zur Antike und zu ihrem Nachleben.
- Ebbinghaus, K. (2016)
Un modèle formel de la syllogistique d’Aristote, traduit et commenté par C. Lion. London: College Publications.
- Eemeren, F. H. van & Grootendorst, R. (1984)
Speech Acts in Argumentative Discussions: A Theoretical Model for the Analysis of Discussions Directed towards Solving Conflicts of Opinion, Dordrecht: Foris Publications.
- Felscher, W. (1985).
“Dialogues, Strategies, and Intuitionistic Provability”. *Annals of Pure and Applied Logic*, vol. 28, pp. 217–254.

- Heinzmann, G. (2013)
L'intuition épistémique. Paris: Vrin.
- Hodges, W. (2001).
 "Dialogue foundations: A Skeptical look". *Proceedings of the Aristotelian Society, Supp.*, vol. 75, pp 17–32.
- Hodges, W. (2013).
 "Logic and games". In E. Zalta (ed.), *Stanford Encyclopedia of Philosophy*.
<http://plato.stanford.edu/entries/logic-games/>
- Fermüller, C. (2021).
 "Connecting Sequent Calculi with Lorenzen-Style Dialogue Games". In Heinzmann and Wolters 2021, pp. 115–141.
- Fontaine, M. (2013).
Argumentation et engagement ontologique. Être, c'est être choisi, London: College Publications.
- Ginzburg, J. (2012).
The Interactive Stance. Oxford: Oxford University Press.
- Gorisse, M. H. (2017).
 "Logic in the tradition of Prabhācandra". In Ganeri (2017), pp. 486–506.
- Gorisse, M. H. (2018).
 "Concealing meaning in inferential statements: The practice of patra in Jainism". In Balbir and Flügel 2018, pp. 111–126..
- Heinzmann, G. (2013).
 "L'intuition épistémique". *Mathesis*. Paris : Vrin.
- Heinzmann, G. (2017).
 "Dialogue sur le vrai". In *Philosophia Scientiae*, vol? 21(3), pp. 3-10.
- Heinzmann, G. & Lorenzen, P. (1986)
 "Correspondance". Correspondance conservée par G. Heinzmann.
- Hintikka, J. (1973).
Logic, language-games and information: Kantian themes in the philosophy of logic. Oxford, UK: Clarendon Press.
- Hintikka, J. (1996).
The Principles of Mathematics Revisited. Cambridge: Cambridge University Press.
- Hintikka, J. (2004).
Analyses of Aristotle. Cambridge: Cambridge University Press.
- Hintikka, J. & Sandu, G. (1989).
 "Informational Independence as a Semantical Phenomenon,". In J. E. Fenstad, I. T. Frolov, and R. Hilpinen (eds.), *Logic, Methodology and Philosophy of Science*. Amsterdam: Elsevier, vol. 8, pp. 571–589.
- Hodges, W. (2001).
 "Dialogue Foundations: A Sceptical Look", *Proceedings of the Aristotelian Society (Supplement)*, vol. 75, pp. 17–32.
 589.
- Iqbal, M. (2022).
Arsyad al-Banjari's Insights on Parallel Reasoning and Dialectic in Law: The Development of Islamic Argumentation Theory in the 18th Century in Southeast Asia. Cham: Springer
- Kamlah, W. & Lorenzen, P. (1972).
Logische Propädeutik. Suttgart-Weimar: Metzler, Second edition.
- Keffer, H. (2001).
De Obligationibus – Rekonstruktion einer spätmittelalterlichen Disputationstheorie. Leiden : Brill
- Keiff, L. (2011).
Le Pluralisme dialogique : Approches dynamiques de l'argumentation formelle, PhD thesis, Lille: University of Lille Press.
- Klev, A. (2022).
 "Martin-Löf's Dialogue Rules". Talk at the Workshop "Meaning Explanations", Prague, dec 2022
- Klev, A. (2023).
 "Martin-Löf on the validity of inference". In A. P. d'Aragona (Ed.), *Perspectives on Deduction*. Cham: Springer. (Forthcoming).
- Krabbe, E.C. (1985).
 "Formal Systems of Dialogue Rules". *Synthese*, vol. 63, pp. 295-328.
- Lecomte, A., Quatrini, M. (2011a)
 "Figures of dialogue: a view from ludics. *Synthese*, vol. 183, pp. 279–305.
- Lecomte, A., Quatrini, M. (2011b).
 "Ludics and rhetorics". In Lecomte, A., Tronçon, S. (eds.), *Ludics, Dialogue and Interaction. PRELUDE Project 2006–2009: Revised Selected Papers*. Lecture Notes in Artificial Intelligence, vol. 6505, pp. 32–59.
- Lion, C. (2023a).

- L'Intuitionnisme dialogique*. Paris: Classiques Garnier.
- Lion, C. (2023b).
 “Logique et Agon”. Translation and Commentary of Lorenzen (1958). *Minuit – Philosophie*, vol. 157, pp. 3-22.
- Lorenz (1970).
Elemente der SprachKritik. Eine Alternative zum Dogmatismus und Skeptizismus in der Analytischen Philosophie. Frankfurt: Suhrkamp.
- Lorenz (1998).
Indische Denker. München: Beck.
- Lorenz, K. (2001).
 “Basic objectives of dialogue logic in historical perspective”. In S. Rahman and H. Rückert (eds.), *New Perspectives in Dialogical Logic, Synthese*, vol. 127, pp. 255-263.
- Lorenz, K. (2009).
Dialogische Konstruktivismus. Berlin and New York: De Gruyter.
- Lorenz, K. (2010).
Logic, Language and Method: On Polarities in Human Experience, New York/Berlin: Walter deGruyter.
- Lorenz, K. (2011)
Philosophische Variationen: Gesammelte Aufsätze Unter Einschluss Gemeinsam Mit Jürgen Mittelstraß Geschriebener Arbeiten Zu Platon Und Leibniz, New York/Berlin: Walter deGruyter.
- Lorenz, K. (2014)
 “Zur Herkunft der Dialogbedingung im Dialogischen Aufbau der Logik”. In J. Mittelstraß and C. von Bulow, Münster: Mentis, pp. 55-74.
- Lorenz, K. (2021)
Von der dialogischen Logik zum dialogischen Konstruktivismus. New York/Berlin: Walter deGruyter.
- Lorenz, K. & Mittelstrass (1966).
 “Theaitetos fliegt. Zur Theorie wahrer un falscher Sätze in Platons Sophistes”. *Archiv für Geschichte der Philosophie*, vol. 48 (1-3), pp. 113-152.
- Lorenz, K. & Mittelstrass (1967).
 “On Rational Philosophy of Language. The Programme in Plato’s *Cratylus* Reconsidered”. *Mind*, vol. 76, pp- 1-20.
- Lorenzen, P. (1955).
Einführung in die operative Logik und Mathematik, Berlin: Springer.
- Lorenzen, P. (1960).
 “Logik und Agon”. In *Atti del XII Congresso Internazionale di Filosofia*, vol. 4., Florence: Sansoni Editore, pp. 187-194.
- Lorenzen, P. (1968).
Methodisches Denken. Frankfurt am Main : Suhrkamp.
- Lorenzen, P. (1969).
Normative Logic and Ethics. Mannheim/Zürich : Bibliographisches Institut-Hochschultaschenbücher Verlag.
- Lorenzen, P. & Schwemmer, O. (1975).
Konstruktive Logik, Ethik und Wissenschaftstheorie. Mannheim/Wien/Zürich : Bibliographisches Institut-Hochschultaschenbücher Verlag.
- Lorenzen, P. & Lorenz, K. (1978).
Dialogische Logik. Darmstadt: Wissenschaftliche Buchgesellschaft.
- Marion, M. (2010).
 “Between Saying and Doing: From Lorenzen to Brandom and Back”. In Bour, Rebuschi, and Rollet (eds), *Construction: Festschrift for Gerhard Heinzmann*, London: College Publications, pp. 489–497.
- Marion, M. & Rückert, H. (2016).
 “Aristotle on universal quantification: a study from the perspective of game semantics”. *History and Philosophy of Logic*, vol. 37, pp 201–29.
- Martin-Löf, P. (1984).
Intuitionistic Type Theory. Notes by Giovanni Sambin of a Series of Lectures given in Padua, June 1980. Naples: Bibliopolis.
- Martin-Löf, P. (2014).
 “Truth of Empirical Propositions”. *Lecture held at the University of Leiden, February 2014*.
 Transcription by Ansten Klev.
- Martin-Löf, P. (2015).
 “Is logic part of normative ethics?”. Transcript by Ansten Kleve of a lecture given on 16 April 2015 and in Paris on 15 May 2015.
- Martin-Löf, P. (2016).
 “Judgement and inference”. Transcript by Ansten Kleve of a lecture given in Marseille on 1 June 2016.
- Martin-Löf, P. (2017a).
 “Assertion and request”. Transcript by Ansten Klev of a lecture given in Stockholm on 14 August

- 2017.
- Martin-Löf, P. (2017b).
 “Assertion and request”. Transcript by Ansten Kleve of a lecture given in Oslo on 29 August 2017.
- Martin-Löf, P. (2019).
 “Epistemic assumptions: are they assumed to be backwards vindicated or forwards vindicable?”
 Transcript of a lecture given in Leiden on 6 September 2019. Forthcoming in a Festschrift for B. G. Sundholm?
- Martin-Löf, P. (2020).
 “Logic and ethics”. In Sedlar, I. and Blichka, M., editors, *The Logica Yearbook 2019*, pages 83–92. College Publications, London. Also published in the proceedings of the conference Proof-Theoretic Semantics, <http://dx.doi.org/10.15496/publikation-35319>.
- Martin-Löf, P. (2022)
 “Correctness of assertion and validity of inference”. Transcript edited by the author and prepared by Ansten Klev, of a lecture given by Per Martin-Löf on 26 October 2022 at the Rolf Schock Symposium in Stockholm in the context of the award of the Schock Prize.
- McConaughey, Z. (2021).
Aristotle, Science and the Dialectician's Activity. A Dialogical Approach to Aristotle's Logic. PhD, Lille: Université de Lille & Montreal ! UQAM.
- McConaughey, Z. & Rahman, S. (2023, forthcoming).
 “Vindicating the Future. A Dialogical Stance on Belnap’s Tomorrow’s Sea Battle.” In A. Klev (ed.), *The Architecture and Archeology of Modern Logic*. Cham: Springer.
- Miller, Z. (1984, repr. 2020).
Islamic Disputation Theory. The Uses and Rules of Argument in Medieval Islam. PhD, Lille: Université de Lille & Montreal ! UQAM.
- Piecha, T. (2015).
 “Dialogical Logic”. Online entry in the *Internet Encyclopedia of Philosophy*
- Quine, W. V. O (1986).
Philosophy of Logic. Englewood Cliffs: Prentice-Hall. Second Edition.
- Rahman, S. & Rückert, H. (2001).
 “Dialogische Modallogik (für T, B, S4, und S5)”. *Logique et Analyse*, vol. 167-168. pp. 243-282.
- Rahman, S. (2022a).
 “Dialogical Insights on Structural Rules. A Plea for the New Old Ways”. Talk at the Technische Universität Wien, 29-30 June 2002.
- Rahman, S. (2022b).
 “Disagreeing with the Standards of Rationality”. Talk at the *Université Européenne d’Été du réseau OFFRES: La Mécompréhension productive*, Lille 4-12/07/2022.
- Rahman, S. & Redmond, J. (2016).
 “Harmonía Dialógica: tonk, Teoría Constructiva de Tipos y Reglas para Jugadores Anónimos”. *Theoria*, vol. 31/1, , pp. 1-138.
- Rahman, S. & Zarepour, M. S. (2021).
 “On Descriptive Propositions in Ibn Sīnā: Elements for a Logical Analysis”. In M. Mojtahedi et al. (ed.), *Mathematics, Logic and their Philosophies*, Cham: Springer, pp. 411-432.
- Rahman, S., McConaughey, Z., Klev, A & Clerbout, N. (2018).
Immanent Reasoning or Equality in Action: A Plaidoyer for the Play Level. Cham: Springer
- Ranta, A. (1994).
Type Theoretical Grammar. Oxford: Clarendon Press. University Press.
- Recanati, F. (2007).
 “It Is Raining (Somewhere)”. *Linguistics and Philosophy*, vol. 30(1), pp. 123–146.
- Redmond, J. (2011)
Logique Dynamique de la Fiction. Pour une approche dialogique, London: College Publications
- Redmond, J and Fontaine, M. (2011).
How to Play Dialogues: An Introduction to Dialogical Logic. London: College Publications
- Rückert, H. (2011).
Dialogues as a Dynamic Framework for Logic, London: College Publications..
- Shafiei, M. (2018).
Meaning and Intentionality: a Dialogical Approach, London: College Publications.
- Schröder-Heister, P. (2008).
 “Lorenzen’s operative justification of intuitionistic logic”. In M. van Atten, P. Boldini, M. Bourdeau, & G. Heinzmann (Eds.), *One hundred years of intuitionism (1907–2007)*. Basel: Birkhäuser, pp. 214–240.
- Sörensen, M. H. & Urzyczyn, P. (2006).
Lectures on the Curry-Howard Isomorphism. New York: Elsevier Science.
- Sundholm, B. G. (1986).
 “Proof Theory and Meaning”. In D. Gabbay and F. Guenther (eds.), *Handbook of Philosophical Logic*, vol. III, Chapter III:8, Dordrecht : Reidel, pp. 471-506.

- Sundholm, B. G. (1989).
 “Constructive Generalized Quantifiers.” *Synthese*, vol. 79, pp. 1-12.
- Sundholm, B. G. (1997).
 “Implicit epistemic aspects of constructive logic”. *Journal of Logic, Language and Information*, vol. 6(2), pp. 191–212.
- Sundholm, B. G. (2012).
 “*Inference versus consequence* revisited: inference, consequence, conditional, implication”. *Synthese*, vol. 187, pp. 943-956.
- Sundholm, B. G. (2013).
 “Containment and variation: Two strands in the development of analyticity from Aristotle to Martin-Löf. In van der Schaar, M. (ed.) *Judgement and the Epistemic Foundation of Logic*. Springer: Heidelberg, pp. 23-35.
- Sundholm, G. (2019).
 “The Neglect of Epistemic Considerations in Logic: The Case of Epistemic Assumptions. ”. *Topoi*, vol. 38, pp. 551–559.
- Tulenheimo, T. (2011).
 “On some logic games in their philosophical context”. In Lecomte & Tronçon (eds), *Ludics, dialogue and interaction*. Dordrecht: Springer, pp.88–113.
- Uckelman, S. (2013).
 “Medieval Disputationes de obligationibus as Formal Dialogue Systems,”. *Argumentation*, vol. 27(2), pp. 143–166.
- Young, W. E. (2017).
The Dialectical Forge: Juridical Disputation and the Evolution of Islamic Law. Cham: Springer
- Young, W. E. (2021).
 “The Formal Evolution of Islamic Juridical Dialectic: A Brief Glimpse.” In *New Developments in Legal Reasoning and Logic: From Ancient Law to Modern Legal Systems* (eds). S. Rahman & M. Armgardt & H. C. Nordtveit Kvernenes. *Logic, Argumentation & Reasoning 23*. Dordrecht; New York; Cham: Springer, pp. 83-114.
- Yrjönsuuri, M. (2001)
Medieval Formal Logic: Consequences, Obligations and Insolubles. Dordrecht: Kluwer