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**On human capital accumulation in times of epidemic**

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**JEL Codes: C61, I18, O41.**

**Keywords: SIS model, human capital.**



# On human capital accumulation in times of epidemic\*

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## Abstract

Can a policy-independent epidemic dynamics change agents' optimal consumption-investment decisions and by so much so that an economy based on human capital and labor à la Lucas (1988) would reach a steady state and stagnate instead of thriving and reaching a Balanced Growth Path?

We prove here that, under strictly decreasing returns on human capital, the economy will optimally reach a steady state in levels. Conversely, under constant returns, the economy will experience a balanced growth either from the beginning if preferences are logarithmic, or asymptotically otherwise.

**JEL codes:** C61, I18, O41.

**Keywords:** SIS model, human capital.

## 1 Introduction

The covid-19 pandemic has popularized the epidemiological models in economics.<sup>1</sup> Among the compartment models to represent disease spread, the SIS is presumably the simplest one. Population splits in two groups: susceptibles ( $S$ ) and infectives ( $I$ ). At any date, some susceptibles meet infectives, catch the disease depending on the prevailing contamination rate and become infectives themselves. Moreover, at any date, a share of infectives recovers from the disease and go back to the susceptibles group. In the long run, two equilibria

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<sup>1</sup>The interested reader is referred to Acemoglu et al. (2021), Alvarez et al. (2021) or Bosi et al. (2021) among others.

emerge: the epidemic converges either to the disease-free or the endemic steady state.

To the best of our knowledge, the first attempt to introduce a SIS dynamics into an economic growth model<sup>2</sup> is Goenka and Liu (2012). In their discrete-time framework, infectives are too ill to work and the labour force consists only of susceptibles. They found that periodic cycles and chaos can emerge around the endemic steady state when the disease is highly infective, and that both vaccination and isolation policies can prevent from chaos implementing a monotonic convergence towards the endemic steady state. In a continuous-time framework, Goenka et al. (2014) have introduced health expenditures. Health capital is assumed to reduce the contamination rate and increase the recovery rate. Differently from Goenka and Liu (2012), where the epidemic affects the economic dynamics but the converse does not hold, in Goenka et al. (2014), economic variables affect in turn the disease spread. After checking that the first-order conditions are sufficient, they show that optimal health expenditures become positive when the disease is highly infectious. In the same spirit, Bosi and Desmarchelier (2018) endogenize both the contamination rate and the recovery rate as functions of a pollution externality coming from production activities. Focusing on the market solution, they point out the emergence of a limit cycle through a Hopf bifurcation around the endemic steady state.

More recently, Goenka and Liu (2020) study the interplay between the spread of infectious disease, human capital accumulation and economic growth in the long run. They consider an endogenous growth model à la Lucas (1988) with both physical and human capital, where an infectious disease evolves according to an SIS dynamics. As in Goenka et al. (2014) or in Bosi and Desmarchelier (2018), they endogenize the contamination rate assuming a positive impact of physical capital and a negative impact of human capital.

Interestingly, Goenka and Liu (2020) assume that ill households are not only unable to work but also unable to accumulate human capital. The representative household optimally chooses how to split her time between working and the accumulation of human capital. In both the market and the centralized solutions, there exists a poverty trap when a large prevalence of disease forces the economy to converge to a steady state instead of to a Balanced Growth Path (BGP hereafter). Along the BGP, the endemic disease always lowers the economic growth rate. In a centralized economy, the planner internalizes the positive externality of human capital and this capital turns out to be higher than it is in a market economy. As a consequence, a poverty trap is more likely to occur in a decentralized economy.

The existence of an endemic disease affects the trade-off between consumption and human capital accumulation. It is known that the BGP is optimal in a Lucas world with no epidemics. The very question is whether it remains optimal when epidemic spreads. Indeed, when the disease is highly infective, the will to smooth consumption over time can lead to slow down human capital accumula-

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<sup>2</sup>Earlier contributions have considered epidemiological models to study optimal disease control but in a partial equilibrium framework (Geoffard and Philipson (1996), Gersovitz and Hammer (2004), Barrett and Hoel (2007)).

tion to preserve current consumption. In this case, it should be optimal to pick the initial consumption such that the corresponding path converges to a steady state instead of to a BGP. In this paper, we tackle and answer positively the question of BGP optimality.

To know how the endemic disease affects the BGP and the long run, we develop a simple model of endogenous growth with human capital à la Lucas (1988). To keep things as simple as possible, we ignore the physical capital. In this respect, our model is close to Ben-Porath (1967). As in Goenka and Liu (2020), the infectious dynamics follow a SIS compartmental model where labour supply only consists of susceptibles. Differently from them, we assume that ill households can accumulate human capital (even if they are not able to work) and preferences are represented by a CIES utility function. The production sector involves both labour and human capital to produce a final good which can be consumed or invested in human capital. There exists a policy maker who chooses the optimal trajectories for consumption and human capital to maximize an overall measure of social welfare.

To have a clear picture of the epidemic impact on the BGP, we assume that the planner knows the negative effect of illness on labour supply, but she is unable to control the disease (infectivity parameters are not affected by economic variables).

In the first part of the paper, we consider the case of exogenous growth, where the production function exhibits decreasing marginal returns on human capital, and we prove the monotonic convergence to an endemic steady state. As expected, the more infective is the endemic disease, the lower are the levels of consumption and human capital at the steady state.

In the second part, we focus on the case of endogenous growth, where the production function is linear in human capital. We prove that, when preferences are represented by a logarithmic utility, the BGP is unique and optimal: it is rational to pick the initial consumption (control variable) to jump to the BGP from the very beginning. When the utility function is more general (CIES), the optimal path remains unique, but converges asymptotically to the BGP.

One may question whether the negative impact of disease prevalence on growth highlighted by Goenka and Liu (2020) depends on their logarithmic utility and whether the converse holds under a larger elasticity of intertemporal substitution, that is the lower income level results in higher investment in human capital today and consumption tomorrow because of the intertemporal smoothing. For instance, during the covid-19 pandemics, many countries have massively invested in hardware and software to maintain schooling activities, boosting in fact the growth rate.

We propose a simple model, but general enough in terms of intertemporal substitution, to fully appreciate the effect of this elasticity on the BGP. However, we find that a more infectious disease always lowers the growth rate in the long run, whatever the value of the elasticity of intertemporal substitution in consumption. In this respect, the result found by Goenka and Liu (2020) with logarithmic preferences is robust to our generalization.

The paper is organized as follows. Section 2 and 3 present the compartmental

(SIS) and the economic models respectively. Section 4 studies the exogenous growth case while the endogenous growth one is considered in Section 5. Section 6 concludes. All proofs are gathered in the Appendix.

## 2 SIS model

Let us consider the Susceptible-Infective-Susceptible (SIS) model to study the spread of a transmissible endemic disease in which infection does not confer immunity

In this model, the population ( $N(t)$ ) is composed of infectives ( $I(t)$ ) and susceptibles ( $S(t)$ ) with

$$I(t) + S(t) = N(t)$$

For simplicity, let us assume that population remains constant over time, that is  $N'(t) = 0$  implying that

$$I'(t) + S'(t) = 0$$

Let us define the shares of susceptibles and infectives in total population as

$$i(t) = \frac{I(t)}{N(t)} \text{ and } s(t) = \frac{S(t)}{N(t)}$$

with  $i(t) + s(t) = 1$ .

As mentioned in the Introduction, the spread of a given disease depends on the rate of transmissibility  $\tau$ , which captures the virulence and pathogenicity of the disease's microbes. We define  $\tau$  as the average number of contacts per unit of time that an infective needs in order to transmit the disease.  $s(t)$  is the probability to face a susceptible during a contact. Thus,  $\tau s(t)$  is the average number of adequate contacts between an infective and the set of susceptibles. The number of new infectives per unit of time is given by  $\tau s(t) I(t)$ . An infective is seek during a given interval of time which also depends on the disease under study. After the convalescence time, she recovers and becomes susceptible anew.

Let us assume that infected recover at a constant rate  $\rho = -I'(t)/I(t)$  in absence of new contamination. The main feature of the SIS model is that the infection does not confer immunity.

The ratio between the rates of transmissibility and recovery is the well-known basic reproduction number:

$$R_0 \equiv \frac{\tau}{\rho}$$

We can solve for the share of infectives and obtain its entire trajectory as a function of the initial share of infected and  $R_0$ .

**Proposition 1 (disease dynamics)** *The share of infectives at any time  $t \geq 0$  is given by*

$$i(t) = \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0 - 1)t}} \quad (1)$$

where  $i_0 \equiv i(0)$  and

$$i_1 \equiv \frac{R_0 - 1}{R_0} < 1 \quad (2)$$

**Proof.** See the Appendix. ■

We observe that the asymptotic share of infectives increases in the reproduction number:

$$\frac{di_1}{dR_0} = \frac{1}{R_0^2} > 0 \quad (3)$$

**Corollary 2 (stability)** (1) If  $R_0 < 1$ , there is one steady state:  $i(t) = 0$  (disease-free), and the share of infectives decreases from  $i_0$  to zero.

(2) If  $R_0 > 1$ , there are two steady states:  $i(t) = 0$  (disease-free) and  $i(t) = i_1$  (endemic).

(2.1) If  $i_0 < i_1$ , then  $i(t)$  increases from  $i_0$  to  $i_1$ .

(2.2) If  $i_0 > i_1$ , then  $i(t)$  decreases from  $i_0$  to  $i_1$ .

**Proof.** See the Appendix. ■

We can infer from Corollary 2 that the disease-free steady state is globally stable when  $R_0 < 1$  and that it becomes unstable when  $R_0 > 1$ . In contrast, when  $R_0 > 1$ , the endemic steady state  $i_1$  is globally stable.

In the rest of the paper, we will assume the existence of an endemic steady state.

**Assumption 1**  $R_0 > 1$ .

### 3 Economic model

We will study here the problem of a policy maker who knows the epidemiological dynamics and decides how to invest in education and health in order to increase human capital and labor productivity over time so as to implement the optimal trajectories that will maximize overall welfare.

In this economy there exists a unique final good, which is all purpose and is produced by a single firm.

Following Goenka and Liu (2012), only healthy agents ( $[1 - i(t)] N(t)$ ) can work, while sick people ( $i(t) N(t)$ ) do not. Each healthy agent provides with labour services that depend on her human capital, that we can measure by  $h(t)^\alpha$  with  $\alpha > 0$ . Labor supply is the only input, and its productivity is constant and equal to  $A$ . Thus, the production is given by:

$$Y(t) = Ah(t)^\alpha [1 - i(t)] N(t) \quad (4)$$

Since the unique final good is all purpose, it will be used both for consumption and investment in human capital. As a result, we can write that a fraction  $l(t)$  of the final good (4) will be devoted to consumption, that is  $C(t) = l(t) Y(t)$ . The remaining final good,  $[1 - l(t)] Y(t)$  will be used to increase human capital through schooling and medical services without distinction between healthy and sick individuals.

Agents are identical in terms of their initial human capital and preferences, and, for simplicity, we also assume that public investment in education and health equally increases individuals' human capital independently of their physical condition. Hence, individual human capital equals the average.

Let  $\tilde{B}$  represent a linear technology transforming a fraction  $1 - l(t)$  of the whole output  $Y(t)$  in human capital, and  $B \equiv A\tilde{B}$  be the product of both productivities.

**Lemma 3 (consumption and human capital)** *Individual consumption and human capital accumulation are driven by the following equations:*

$$c(t) = A[1 - i(t)]l(t)h(t)^\alpha \quad (5)$$

$$h'(t) = B[1 - i(t)][1 - l(t)]h(t)^\alpha \quad (6)$$

**Proof.** See the Appendix. ■

As said, agents are identical and public investment in human capital benefits healthy and sick individuals equally. The policy maker maximizes an overall social welfare function  $\int_0^\infty e^{-\theta t} u(c(t)) dt$ , which depends solely on consumption, taking into account the law of human capital accumulation. More precisely, she solves the following program:

$$\max_l \int_0^\infty e^{-\theta t} u(A[1 - i(t)]l(t)h(t)^\alpha) dt \quad (7)$$

subject to the law of capital accumulation (6).

Hence, the policy maker decides the fraction of output  $l(t)$  to consume at any period that will maximize welfare. Here,  $l$  and  $h$  are the control and the state variables, respectively, while the epidemiological variable  $i$  is taken as given and it remains outside her control.

Next, let

$$\varepsilon(c) \equiv -\frac{u'(c)}{cu''(c)}$$

be the elasticity of intertemporal substitution. In the following, that this elasticity is constant, that is  $\varepsilon(c) = \varepsilon$ .

Under this assumption, the felicity function becomes

$$u(c) = \frac{c^{1-\frac{1}{\varepsilon}}}{1-\frac{1}{\varepsilon}}$$

**Proposition 4 (dynamic system)** *Economic dynamics are driven by a three-dimensional dynamic system*

$$i(t) = \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0 - 1)t}} \quad (8)$$

$$h'(t) = B[1 - i(t)]h(t)^\alpha - \frac{B}{A}c(t) \geq 0 \quad (9)$$

$$\frac{c'(t)}{c(t)} = \varepsilon \left( \alpha B[1 - i(t)]h(t)^{\alpha-1} - \theta \right) \quad (10)$$



together with the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) \in \mathbb{R}_+ \quad (11)$$

**Proof.** See the Appendix. ■

**Proposition 5 (Arrow-Mangasarian)** *Furthermore, conditions (8)-(10) and the transversality condition  $\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) = 0$  are not only necessary but also sufficient to solve the planner's program (7) if  $\alpha \leq 1$ .*

**Proof.** See the Appendix. ■

At the beginning of any disease spread, there are only few cases. For this reason, we will assume in the following that the initial number of infectives is relatively small.

**Assumption 2**  $i_0 < i_1$ .

In the following sections, we will dig further into the optimal solution provided in Proposition 4 focusing on the role of human capital in production. Section 4 analyzes the long-term optimal solution when  $\alpha < 1$ . To complete our analysis, Section 5 studies instead the optimal solution when human capital has a more preeminent role in production, in particular, we assume that  $\alpha = 1$ .

## 4 Exogenous growth

Let us assume in this section that  $\alpha < 1$ . That is, we will focus first on the case of strictly decreasing returns on human capital.

By 6,  $h'(t) > 0$  so that human capital is increasing. The following lemma proves that human capital is bounded from above.

**Lemma 6** *Human capital is increasing and bounded from above:*

$$0 < \lim_{t \rightarrow \infty} h(t) = \bar{h} < \infty$$

**Proof.** See the Appendix. ■

**Proposition 7**  *$h(t)$  increases monotonically and converges asymptotically to the steady state*

$$\bar{h} = \left[ \frac{\alpha B (1 - i_1)}{\theta} \right]^{\frac{1}{1-\alpha}} \quad (12)$$

*Consumption converges as well with*

$$\lim_{t \rightarrow \infty} c(t) = A (1 - i_1)^{\frac{1}{1-\alpha}} \left( \frac{\alpha B}{\theta} \right)^{\frac{\alpha}{1-\alpha}}$$

**Proof.** See the Appendix. ■

It is easy to see that a larger disease prevalence, that is a higher reproduction number  $R_0$ , has a negative impact on human capital in the long run. Indeed, according to (3), the asymptotic share of infectives  $i_1$  increases in  $R_0$ , and, according to (12),  $\bar{h}$  decreases in  $R_0$ .

## 5 Endogenous growth

Let us assume next that  $\alpha = 1$ , that is that human capital exhibits constant returns. In the case  $\alpha = 1$ , the optimal dynamics of human capital and consumption (9) and (10) become

$$\frac{h'(t)}{h(t)} = B[1 - i(t)] - \frac{B}{A}x(t) \quad (13)$$

$$\frac{c'(t)}{c(t)} = \varepsilon B[1 - i(t)] - \varepsilon\theta \quad (14)$$

where we define  $x(t)$  as the ratio of consumption to human capital:

$$x(t) \equiv \frac{c(t)}{h(t)} \quad (15)$$

Since human capital can not be consumed and does not depreciate, we have  $h'(t) \geq 0$  for any  $t$ . Equivalently, from (13),

$$x(t) \leq A[1 - i(t)] \quad (16)$$

which can be reinterpreted as an additional constraint. Clearly, from (15),

$$\frac{x'(t)}{x(t)} = \frac{c'(t)}{c(t)} - \frac{h'(t)}{h(t)}$$

According to (1), (13), (14) and (16), we obtain a two-dimensional dynamic system

$$i(t) = \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0 - 1)t}} \quad (17)$$

$$\frac{x'(t)}{x(t)} = \frac{B}{A}x(t) + (\varepsilon - 1)B[1 - i(t)] - \varepsilon\theta \quad (18)$$

$$x(t) \leq A[1 - i(t)] \quad (19)$$

Under Assumption 2, according to (17),  $i(t)$  grows monotonically from  $i_0$  to  $i_1$ , which implies that the upper bound  $A[1 - i(t)]$  for  $x(t)$  decreases monotonically from  $A(1 - i_0)$  to  $A(1 - i_1)$ .

For simplicity, let us further assume also that agents are not too impatient with respect to the speed of capital accumulation.

**Assumption 3**  $\theta < B(1 - i_1)$ .

**Proposition 8** *Under Assumptions 1, 2 and 3, if  $x(T) = A[1 - i(T)]$  for some  $T \geq 0$ , then  $x(t) = A[1 - i(t)]$  for any  $t \geq T$ . Moreover,  $h(t) = h(T) \equiv h_T$  and  $c(t) = A[1 - i(t)]h_T$  for any  $t \geq T$ .*

**Proof.** See the Appendix. ■

According (13) and (14), we have

$$\begin{aligned} [\ln h(t)]' &= \frac{h'(t)}{h(t)} = B - Bi(t) - \frac{B}{A}x(t) \\ [\ln c(t)]' &= \frac{c'(t)}{c(t)} = \varepsilon(B - \theta) - \varepsilon Bi(t) \end{aligned}$$

and, therefore,

$$\begin{aligned} \ln \frac{h(t)}{h_0} &= Bt - B \int_0^t i(s) ds - \frac{B}{A} \int_0^t x(s) ds \\ \ln \frac{c(t)}{c_0} &= \varepsilon(B - \theta)t - \varepsilon B \int_0^t i(s) ds \end{aligned} \quad (20)$$

Using (17), we compute the integral of the share of infected from 0 to  $t$  that appears in (20):

$$\int_0^t i(s) ds = \frac{i_1}{\rho(R_0 - 1)} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0 - 1)t} - 1 \right] \right) \quad (21)$$

Replacing (21) in (20), we obtain the optimal trajectory for consumption:

$$c(t) = c_0 e^{\varepsilon(B - \theta)t} \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0 - 1)t} - 1 \right] \right)^{-\frac{\varepsilon B i_1}{\rho(R_0 - 1)}} \quad (22)$$

Observe that

$$\lim_{t \rightarrow \infty} c(t) = c_0 \left( \frac{i_1}{i_0} \right)^{\frac{\varepsilon B i_1}{\rho(R_0 - 1)}} \lim_{t \rightarrow \infty} e^{\varepsilon[B(1 - i_1) - \theta]t} \quad (23)$$

provided that  $x(t) < A[1 - i(t)]$ . Thus,

$$\lim_{t \rightarrow \infty} c(t) = +\infty \Leftrightarrow \theta < B(1 - i_1)$$

Hence, under Assumption 3, we have  $\lim_{t \rightarrow \infty} c(t) = +\infty$  if  $x(t) < A[1 - i(t)]$ .

Using the previous asymptotic results, we can delve into the crucial question of the transversality condition. Recall that according to (11), the discounted product  $u'(c(t))h(t)$  optimally converges to a positive real number. However, contrary to the well spread belief, nothing ensures that this constant is zero. We do prove next that if the ratio  $x(t)$  converges towards 0, then the limit in transversality condition is 1. Otherwise, if  $x(t) = A[1 - i(t)]$  for any  $t \geq T$ , then the limit in the transversality condition will be equal to 0.

## 5.1 Transversality condition

According to (13) and (14), if  $x(t) \rightarrow 0$ ,

$$\begin{aligned}\lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} &= B(1 - i_1) \\ \lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} &= \varepsilon [B(1 - i_1) - \theta]\end{aligned}$$

and the transversality condition becomes

$$\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) = \lim_{t \rightarrow \infty} e^{-\theta t} \frac{h(t)}{c(t)^{\frac{1}{\varepsilon}}} = \lim_{t \rightarrow \infty} e^{-\theta t} \frac{e^{B(1-i_1)t}}{e^{\frac{1}{\varepsilon}[B(1-i_1)-\theta]t}} = 1 \in \mathbb{R}_+$$

If, instead,  $x(t) = A[1 - i(t)]$  for any  $t \geq T$ , the transversality condition becomes

$$\begin{aligned}\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) &= \lim_{t \rightarrow \infty} e^{-\theta t} \frac{h(t)}{c(t)^{\frac{1}{\varepsilon}}} = \lim_{t \rightarrow \infty} e^{-\theta t} x(t)^{-\frac{1}{\varepsilon}} h(t)^{1-\frac{1}{\varepsilon}} \\ &= [A(1 - i_1)]^{-\frac{1}{\varepsilon}} h_T^{1-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-\theta t} = 0\end{aligned}$$

## 5.2 Balanced growth path

Let us focus next on the Asymptotic Balanced Growth Path (ABGP). There exists an ABGP for  $h(t)$  and  $c(t)$  if and only if human capital and consumption grow at the same constant rate. As a consequence, there exists an ABGP if and only if  $\lim_{t \rightarrow 0} [x'(t)/x(t)] = 0$ . According to (18), we require

$$\lim_{t \rightarrow 0} \frac{x'(t)}{x(t)} = \frac{B}{A} \lim_{t \rightarrow 0} x(t) + (\varepsilon - 1) B \left[ 1 - \lim_{t \rightarrow 0} i(t) \right] - \varepsilon \theta = 0$$

with

$$x(t) \leq A[1 - i(t)]$$

that is

$$x_\infty = (1 - \varepsilon) A(1 - i_1) + \varepsilon \theta \frac{A}{B} \quad (24)$$

where

$$x_\infty = \lim_{t \rightarrow \infty} x(t)$$

## 5.3 The case of logarithmic utility

We close this section with an in-depth analysis of the special case of the logarithmic utility, that is, the case  $\varepsilon = 1$ .

In this case, the Euler equation (18) writes

$$\frac{x'(t)}{x(t)} = \frac{B}{A} x(t) - \varepsilon \theta$$

with explicit solution

$$x(t) = \frac{A\theta x_0}{Bx_0 + (A\theta - Bx_0)e^{\theta t}} \quad (25)$$

(1) If  $x_0 < \theta A/B$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

(2) If  $x_0 > \theta A/B$ , then  $x(t)$  increases until the constraint (19) becomes binding and, because of Proposition 8, will remain binding forever. Let us denote by  $T$  the date at which (19) becomes first binding. That is,  $T$  is defined as the solution to  $x(T) = A[1 - i(T)]$ . Thus, the critical date  $T$  is solution to the following equation:

$$\frac{A\theta x_0}{Bx_0 + (A\theta - Bx_0)e^{\theta t}} = A[1 - i(t)] \quad (26)$$

where  $i(t)$  is given by (1).

We observe that the left-hand side is increasing from  $x_0$  to  $\infty$  for any

$$t < T_\infty \equiv \frac{1}{\theta} \ln \frac{Bx_0}{Bx_0 - A\theta}$$

while the right-hand side is decreasing from  $A(1 - i_0)$  to  $A(1 - i_1)$ . Thus, if  $x_0 < A(1 - i_0)$ , a solution  $T < T_\infty$  exists and this solution is unique. In this case, dynamics are given by

$$\begin{aligned} x(t) &= \frac{A\theta x_0}{Bx_0 + (A\theta - Bx_0)e^{\theta t}} \text{ for } t \in [0, T] \\ x(t) &= A \left[ 1 - \frac{i_0 i_1}{i_0 + (i_1 - i_0)e^{-\rho(R_0 - 1)t}} \right] \text{ for } t > T \end{aligned}$$

The solution  $x(t)$  in (25) allows us to compute the remaining unknown integral in (20):

$$\int_0^t x(s) ds = -\frac{A}{B} \ln \left[ 1 + \frac{Bx_0}{A\theta} (e^{-\theta t} - 1) \right]$$

According to (20) and (21), we can write  $h(t)$  as

$$h(t) = h_0 e^{Bt} \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0 - 1)t} - 1 \right] \right)^{-\frac{B i_1}{\rho(R_0 - 1)}} \left[ 1 + \frac{Bx_0}{A\theta} (e^{-\theta t} - 1) \right] \quad (27)$$

$h(t) > 0$  requires the last term to be positive, that is we need

$$x_0 < \theta \frac{A}{B} \text{ or } t < \frac{1}{\theta} \ln \frac{Bx_0}{Bx_0 - A\theta}$$

If  $x_0 < \theta A/B$ , taking the limit, we have

$$\lim_{t \rightarrow \infty} h(t) = h_0 \left( 1 - \frac{Bx_0}{A\theta} \right) \left( \frac{i_1}{i_0} \right)^{\frac{B i_1}{\rho(R_0 - 1)}} \lim_{t \rightarrow \infty} e^{B(1 - i_1)t}$$

and, using (23), we obtain

$$\lim_{t \rightarrow \infty} x(t) = \frac{\lim_{t \rightarrow \infty} c(t)}{\lim_{t \rightarrow \infty} h(t)} = \frac{c_0}{h_0} \frac{A\theta}{A\theta - Bx_0} \frac{\lim_{t \rightarrow \infty} e^{[B(1-i_1)-\theta]t}}{\lim_{t \rightarrow \infty} e^{B(1-i_1)t}} = 0$$

That is, if the initial ratio of human capital to consumption is too low, then it will decrease with time eventually reaching zero.

Let us focus now on the existence of a Balanced Growth Path (BGP).

According to (24), in the case  $\varepsilon = 1$ , we require

$$\theta \frac{A}{B} = x_\infty \equiv \lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \frac{A\theta x_0}{Bx_0 + (A\theta - Bx_0)e^{\theta t}}$$

Hence, the only initial value compatible with the BGP is

$$x_0 = \theta \frac{A}{B} \quad (28)$$

The transversality condition in the form  $\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) \in \mathbb{R}_+$  is always verified.

**Proposition 9 (BGP)** *The Balanced Growth Path is the unique optimal solution, and it is attainable only when  $x_0 = \theta A/B$ .*

**Proof.** See the Appendix. ■

**Proposition 10 (BGR)** *Under Assumptions 1, 2 and 3, the optimal growth path is balanced all along the transition with a positive Balanced Growth Rate:*

$$g(t) \equiv \frac{h'(t)}{h(t)} = \frac{c'(t)}{c(t)} = B[1 - i(t)] - \theta > 0 \quad (29)$$

The BGR decreases over time and converges to

$$g^* \equiv \lim_{t \rightarrow \infty} g(t) = \frac{B}{R_0} - \theta > 0 \quad (30)$$

Therefore, the more infective the disease (and the higher the reproduction number  $R_0$ ), the lower the asymptotic BGR:

$$\frac{\partial g^*}{\partial R_0} = -\frac{B}{R_0^2} < 0$$

**Proof.** See the Appendix. ■

According to (30), the sensitivity of the BGR with respect to the  $R_0$  depends on  $B$ :

$$\frac{\partial^2 g^*}{\partial B \partial R_0} = -\frac{1}{R_0^2} < 0$$

The more efficient the learning process (the higher  $B$ ), the lower the sensitivity of the BGR with respect to the reproduction number. In the limit, and always according to (30), if the time preference  $\theta$  and the efficiency of the learning process  $B$  are close to zero, then the asymptotic BGR  $g^*$  becomes insensitive to the reproduction number  $R_0$ .

## 5.4 General case

The previous section has proved that the BGP is the optimal trajectory when preferences are logarithmic ( $\varepsilon = 1$ ). What happens when preferences are more general ( $\varepsilon \neq 1$ )?

The optimal growth path is no longer balanced, but we can show that the optimal growth path is actually asymptotically balanced.

Let

$$g^* \equiv \varepsilon [B(1 - i_1) - \theta] = \varepsilon \left( \frac{B}{R_0} - \theta \right)$$

be the Asymptotically Balanced Growth Rate (ABGR).

**Proposition 11** *Optimal growth is asymptotically balanced, that is the optimal growth path converges to the following balanced growth path:*

$$\begin{aligned} c(t) &\sim C e^{g^* t} \\ h(t) &\sim H e^{g^* t} \end{aligned}$$

where  $C$  and  $H$  are positive constants.

**Proof.** See the Appendix. ■

As above, it is easy to see that the ABGR is decreasing with the disease prevalence, that is with  $R_0$ :

$$\frac{\partial g^*}{\partial R_0} = -\frac{\varepsilon B}{R_0^2} < 0$$

Thus, a more infectious disease always lowers the growth rate in the long run, whatever the value of the elasticity of intertemporal substitution in consumption. In this respect, the result found by Goenka and Liu (2020) with logarithmic preferences is robust.

## 6 Appendix

**Proof of Proposition 1.**

The evolution of the numbers of infectives and susceptibles across time is given by:

$$\begin{aligned} I'(t) &= \tau s(t) I(t) - \rho I(t) \\ S'(t) &= -\tau s(t) I(t) + \rho I(t) \end{aligned} \tag{31}$$

Dividing (31) by  $I(t)$  and noting that  $s(t) = 1 - i(t)$ , we obtain

$$\frac{i'(t)}{i(t)} = \tau [1 - i(t)] - \rho \tag{32}$$

that is a simple one-dimensional dynamics.

Equation (32) equivalently writes:

$$\frac{i'(t)}{i(t)} = \rho(R_0[1 - i(t)] - 1) \quad (33)$$

with explicit solution (1). ■

**Proof of Corollary 2.**

According to (1), if  $R_0 < 1$ ,

$$\lim_{t \rightarrow \infty} i(t) = \frac{i_0 i_1}{i_0 + (i_1 - i_0) \lim_{t \rightarrow \infty} e^{-\rho(R_0 - 1)t}} = 0$$

Conversely, if  $R_0 > 1$ ,

$$\lim_{t \rightarrow \infty} i(t) = \frac{i_0 i_1}{i_0 + (i_1 - i_0) * 0} = i_1$$

■

**Proof of Lemma 3.**

At the aggregate level,

$$H'(t) = \tilde{B} [Y(t) - C(t)]$$

with  $C(t) = l(t)Y(t)$ .

Therefore, using (4), we get

$$C(t) = l(t) Ah(t)^\alpha [1 - i(t)] N(t) \quad (34)$$

$$H'(t) = \tilde{B} [1 - l(t)] Ah(t)^\alpha [1 - i(t)] N(t) \quad (35)$$

Since  $H(t) = h(t)N(t)$  and  $N(t) = N$  is constant, we get  $H'(t)/H(t) = h'(t)/h(t)$ . Dividing equation (34) by  $N(t)$  and (35) by  $H(t)$ , we obtain (5) and (6). ■

**Proof of Proposition 4.**

The Hamiltonian associated to the welfare maximization problem (7) subject to the constraints (5) and (6) is given by:

$$H(h, l, \lambda, t) = e^{-\theta t} u(A[1 - i(t)]lh^\alpha) + \lambda B[1 - i(t)](1 - l)h^\alpha \quad (36)$$

Taking the first-order conditions  $\partial H/\partial l = 0$ ,  $\partial H/\partial h = -\lambda'(t)$  and  $\partial H/\partial \lambda = h'(t)$ , we obtain

$$\frac{\partial H}{\partial l} = e^{-\theta t} u'(c(t)) A[1 - i(t)] h(t)^\alpha - \lambda(t) B[1 - i(t)] h(t)^\alpha = 0 \quad (37)$$

$$\frac{\partial H}{\partial h} = \alpha e^{-\theta t} u'(c(t)) A[1 - i(t)] l(t) h(t)^{\alpha-1} + \alpha \lambda(t) B[1 - i(t)] [1 - l(t)] h(t)^{\alpha-1} = -\lambda'(t) \quad (38)$$

$$\frac{\partial H}{\partial \lambda} = B[1 - i(t)] [1 - l(t)] h(t)^\alpha = h'(t)$$



jointly with the transversality condition  $\lim_{t \rightarrow \infty} \lambda(t) h(t) \in \mathbb{R}^+$ . (37) yields the multiplier

$$B\lambda(t) = Ae^{-\theta t} u'(c(t)) \quad (39)$$

Setting  $\mu(t) \equiv e^{\theta t} \lambda(t)$ , we obtain

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{\mu'(t)}{\mu(t)} - \theta \quad (40)$$

Replacing (39) and (40) in (38), we obtain the Euler equation

$$\frac{\dot{\mu}(t)}{\mu(t)} = \theta - \alpha B [1 - i(t)] h(t)^{\alpha-1} \quad (41)$$

with  $\mu(t) = u'(c(t)) A/B$ . Taking the derivatives of  $\ln \mu(t) = \ln u'(c(t)) + \ln(A/B)$ , we find

$$\frac{\dot{c}(t)}{c(t)} = -\varepsilon(c(t)) \frac{\mu'(t)}{\mu(t)} \quad (42)$$

Putting together (41) and (42), we obtain

$$\frac{c'(t)}{c(t)} = \varepsilon(c(t)) \left( \alpha B [1 - i(t)] h(t)^{\alpha-1} - \theta \right)$$

Under a constant elasticity of intertemporal substitution, we find (10). According to (5), we have

$$l(t) = \frac{c(t)}{A[1 - i(t)] h(t)^\alpha}$$

Replacing it in (6), we get (9) and, reconsidering (1), we obtain a three-dimensional dynamic system. According to (6),  $l(t) \in [0, 1]$  implies the inequality in (9).

Finally, the transversality condition is given by  $\lim_{t \rightarrow \infty} \lambda(t) h(t) \in \mathbb{R}_+$ . Using (39), we obtain (11). ■

**Proof of Proposition 5.**

The Arrow-Mangasarian condition is given by Arrow and Kurz (1970). Define  $H^*(h, \lambda, t)$  to be the maximum of  $H(h, l, \lambda, t)$  with respect to  $l$ , given  $h$ ,  $\lambda$ , and  $t$  (see (36)). The Arrow-Kurz theorem says that, if  $H^*(h, \lambda, t)$  is concave in  $h$ , for given  $\lambda$  and  $t$ , then the necessary (first-order) conditions are also sufficient. More precisely, according to Seierstad and Sydsaeter (1987, Page 107, Theorem 5), if  $H^*(h, \lambda, t)$  is strictly concave for any  $t$  the state function  $h^*(t)$  is the unique solution, while if  $H^*(h, \lambda, t)$  is concave for any  $t$ ,  $h^*(t)$  is solution (maybe not unique). In both the case, the policy function  $l^*(t)$  may be not unique.

Let  $l^*$  be the optimal  $l$ , solution to

$$B\mu = Au'(A[1 - i(t)]lh^\alpha) \quad (43)$$

Thus  $l^* = l^*(h, \mu, t)$ . Given  $\mu$  and  $t$ , we compute the partial elasticity of  $l^*$  with respect to  $h$  by differentiating (43):

$$\frac{h}{l^*} \frac{\partial l^*}{\partial h} = -\alpha \quad (44)$$

Consider equivalently the current-value Hamiltonian:

$$\tilde{H}(h, l, \mu, t) \equiv e^{\theta t} H(h, l, \lambda, t) = u(A[1 - i(t)]lh^\alpha) + \mu B[1 - i(t)](1 - l)h^\alpha$$

and its maximal value:

$$\begin{aligned} \tilde{H}^*(h, \mu, t) &\equiv \tilde{H}(h, l^*(h, \mu, t), \mu, t) \\ &= u(A[1 - i(t)]l^*(h, \mu, t)h^\alpha) + \mu B[1 - i(t)][1 - l^*(h, \mu, t)]h^\alpha \end{aligned}$$

Deriving with respect to  $h$  and using (44), we obtain:

$$\frac{\partial \tilde{H}^*}{\partial h} = \alpha B[1 - i(t)]\mu h^{\alpha-1}$$

Eventually,

$$\frac{\partial^2 \tilde{H}^*}{\partial h^2} = \alpha(\alpha - 1)B[1 - i(t)]\mu h^{\alpha-2} \leq 0$$

if and only if  $\alpha \leq 1$ . ■

**Proof of Lemma 6**

Suppose the contrary, that is, that human capital is not bounded.

Since  $h'(t) = B[1 - i(t)][1 - l(t)]h(t)^\alpha > 0$  and  $h(t)$  is unbounded, we have  $\lim_{t \rightarrow \infty} h(t) = \infty$ .

Since  $\alpha < 1$ , according to (10), it must be

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon \left[ \alpha B(1 - i_1) \lim_{t \rightarrow \infty} h(t)^{\alpha-1} - \theta \right] = -\varepsilon \theta$$

Hence,  $c(t) \sim \tilde{c}e^{-\varepsilon \theta t}$  asymptotically, where  $\tilde{c}$  is a positive constant.

Noticing that  $u'(c) = c^{-1/\varepsilon}$ , the transversality condition (11) becomes:

$$\begin{aligned} \infty &> \lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) = \lim_{t \rightarrow \infty} e^{-\theta t} c(t)^{-\frac{1}{\varepsilon}} h(t) = \lim_{t \rightarrow \infty} e^{-\theta t} (\tilde{c}e^{-\varepsilon \theta t})^{-\frac{1}{\varepsilon}} h(t) \\ &= \tilde{c}^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-\theta t} e^{\theta t} h(t) = \tilde{c}^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} h(t) = \infty \end{aligned}$$

which is a contradiction. Therefore, human capital is bounded from above:  $\lim_{t \rightarrow \infty} h(t) = \bar{h} < \infty$ . ■

**Proof of Proposition 7**

From Lemma 6, we know that human capital increases and converges asymptotically to  $\bar{h} < \infty$ . Therefore, according to (9),

$$0 = \lim_{t \rightarrow \infty} h'(t) = B \lim_{t \rightarrow \infty} [1 - i(t)]h(t)^\alpha - \frac{B}{A} \lim_{t \rightarrow \infty} c(t) = B(1 - i_1)\bar{h}^\alpha - \frac{B}{A} \lim_{t \rightarrow \infty} c(t)$$

that is

$$\lim_{t \rightarrow \infty} c(t) = A(1 - i_1) \bar{h}^\alpha$$

Noticing that  $c'(t) = \varepsilon \left( \alpha B [1 - i(t)] h(t)^{\alpha-1} - \theta \right) c(t)$ , we obtain

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} c'(t) = \varepsilon \lim_{t \rightarrow \infty} \left( \alpha B [1 - i(t)] h(t)^{\alpha-1} - \theta \right) \lim_{t \rightarrow \infty} c(t) \\ &= \varepsilon \left[ \alpha B (1 - i_1) \bar{h}^{\alpha-1} - \theta \right] A(1 - i_1) \bar{h}^\alpha \end{aligned}$$

entailing

$$\alpha B (1 - i_1) \bar{h}^{\alpha-1} - \theta = 0$$

that is

$$\bar{h} = \left[ \frac{\alpha B (1 - i_1)}{\theta} \right]^{\frac{1}{1-\alpha}}$$

and

$$\lim_{t \rightarrow \infty} c(t) = A(1 - i_1) \bar{h}^\alpha = A(1 - i_1)^{\frac{1}{1-\alpha}} \left( \frac{\alpha B}{\theta} \right)^{\frac{\alpha}{1-\alpha}}$$

■

**Proof of Proposition 8.**

Since  $i(t)$  increases and Assumption 2 holds, we have

$$B[1 - i(T)] > B(1 - i_1) \geq \theta$$

that is

$$A[1 - i(T)] > (1 - \varepsilon) A[1 - i(T)] + \varepsilon \theta \frac{A}{B}$$

and, since  $x(T) = A[1 - i(T)]$ ,

$$(\varepsilon - 1) B[1 - i(T)] - \varepsilon \theta + \frac{B}{A} x(T) > 0$$

Thus, according to (18),  $x'(T) > 0$  but, since  $A[1 - i(t)]$  decreases in time, constraint (19) remains binding at  $T + dt$ :

$$x(T + dt) = A[1 - i(T + dt)]$$

Repeating the same argument, we get that constraint (19) remains binding forever. ■

**Proof of Proposition 9.**

We have to evaluate the functional (7), that is  $\int_0^\infty e^{-\theta t} \ln c(t) dt$  with  $x(t) \leq A[1 - i(t)]$  in the following three cases: (1)  $x_0 < \theta A/B$ , (2)  $x_0 = \theta A/B$ , (3)  $x_0 > \theta A/B$ .

(1) Assumption 3 implies  $x_0 < \theta A/B < A(1 - i_1)$  and, thus,  $x_0 < A[1 - i(t)]$  for any  $t$ . Since  $x(t)$  decreases over time  $x(t) < A[1 - i(t)]$  and the functional

(7), according to (22) becomes

$$\begin{aligned}
& \int_0^\infty e^{-\theta t} \ln c(t) dt \\
&= \int_0^\infty e^{-\theta t} \ln \left[ c_0 e^{(B-\theta)t} \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right)^{-\frac{Bi_1}{\rho(R_0-1)}} \right] dt \\
&= \ln c_0 \int_0^\infty e^{-\theta t} dt + (B-\theta) \int_0^\infty t e^{-\theta t} dt \\
&\quad - \frac{Bi_1}{\rho(R_0-1)} \int_0^\infty e^{-\theta t} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right) dt
\end{aligned}$$

We observe that

$$\begin{aligned}
\int_0^\infty e^{-\theta t} dt &= \frac{1}{\theta} \\
\int_0^\infty t e^{-\theta t} dt &= \frac{1}{\theta^2}
\end{aligned}$$

and, under Assumptions 1 and 2,

$$\begin{aligned}
& \int_0^\infty e^{-\theta t} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right) dt \\
&< \int_0^\infty e^{-\theta t} \ln e^{\rho(R_0-1)t} dt \\
&= \rho(R_0-1) \int_0^\infty t e^{-\theta t} dt = \frac{\rho(R_0-1)}{\theta^2}
\end{aligned}$$

Thus, the integral  $\int_0^\infty e^{-\theta t} \ln c(t) dt$  converges (the transversality condition  $\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) \in \mathbb{R}_+$  is satisfied) and

$$\begin{aligned}
& \int_0^\infty e^{-\theta t} \ln c(t) dt \\
&= \frac{\ln c_0}{\theta} + \frac{B-\theta}{\theta^2} - \frac{Bi_1}{\rho(R_0-1)} \int_0^\infty e^{-\theta t} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right) dt \quad (45) \\
&< \frac{\ln c_0}{\theta} + \frac{B(1-i_1) - \theta}{\theta^2}
\end{aligned}$$

This integral is increasing in  $c_0$ . Since  $x_0 < \theta A/B$ , we have  $c_0 < \theta A h_0/B$ . Therefore,  $c_0 < \theta A h_0/B$  does not maximize the utility functional.

(2) Assumption 3 implies  $x_0 = \theta A/B < A(1-i_1)$  and, thus, under Assumption 2,  $x_0 < A[1-i(t)]$  for any  $t$ . Thus, solution (45) still holds and, comparing cases (1) and (2) with  $x_0 \in [0, \theta A/B]$ , that is  $c_0 \in [0, h_0 \theta A/B]$ , we obtain

$$h_0 \theta \frac{A}{B} = \arg \max_{0 < c_0 < h_0 \theta \frac{A}{B}} \int_0^\infty e^{-\theta t} \ln c(t) dt$$

In other words, the BGP dominates all the trajectories  $x(t)$  starting with  $x_0 < \theta A/B$ .

(3) If  $x_0 > \theta A/B$ , then  $x(t)$  increases until the constraint (19) becomes binding and, because of Proposition 8 remains binding forever:  $x(T) = A[1 - i(T)]$ . Thus the critical date  $T$  is solution to equation (26). Clearly,  $T$  depends on  $x_0$ , that is on  $c_0$  with  $T'(c_0) < 0$ .

We have to maximize with respect to  $c_0$  the following functional:

$$\begin{aligned}
& \int_0^\infty e^{-\theta t} \ln c(t) dt \\
&= \int_0^{T(c_0)} e^{-\theta t} \ln c(t) dt + \int_{T(c_0)}^\infty e^{-\theta t} \ln c(t) dt \\
&= \int_0^{T(c_0)} e^{-\theta t} \ln \left[ c_0 e^{(B-\theta)t} \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right)^{-\frac{Bi_1}{\rho(R_0-1)}} \right] dt \\
&\quad + \int_{T(c_0)}^\infty e^{-\theta t} \ln \left( A \left[ 1 - \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0-1)t}} \right] \right) dt \\
&\quad + \int_{T(c_0)}^\infty e^{-\theta t} \ln h_{T(c_0)} dt
\end{aligned}$$

where  $h_{T(c_0)}$  is given by (27).

We observe that the impact of  $c_0$  on the first integral is ambiguous because  $c_0$  increases the argument of the logarithm but it decreases the extremum  $T(c_0)$ . The impact of  $c_0$  on the second integral is positive if  $\ln[A(1 - i_1)] > 0$  and negative if  $\ln[A(1 - i_0)] < 0$ . The impact on the third integral is ambiguous because  $c_0$  decreases  $T(c_0)$  and, so, *ceteris paribus*, increases the integral, and decreases  $h_{T(c_0)}$  and, so, decreases the integral.

$$\begin{aligned}
& \int_0^\infty e^{-\theta t} \ln c(t) dt \\
&= \frac{1 - e^{-\theta T(c_0)}}{\theta} \ln c_0 + \frac{e^{-\theta T(c_0)}}{\theta} \ln A h_{T(c_0)} + (B - \theta) \frac{1 - [1 + \theta T(c_0)] e^{-\theta T(c_0)}}{\theta^2} \\
&\quad - \frac{Bi_1}{\rho(R_0 - 1)} \int_0^{T(c_0)} e^{-\theta t} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right) dt \\
&\quad + \int_{T(c_0)}^\infty e^{-\theta t} \ln \left[ 1 - \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0-1)t}} \right] dt
\end{aligned}$$

since

$$\begin{aligned}
\int_0^{T(c_0)} e^{-\theta t} dt &= \frac{1 - e^{-\theta T(c_0)}}{\theta} \\
\int_{T(c_0)}^\infty e^{-\theta t} dt &= \frac{e^{-\theta T(c_0)}}{\theta} \\
\int_0^{T(c_0)} e^{-\theta t} t dt &= \frac{1 - [1 + \theta T(c_0)] e^{-\theta T(c_0)}}{\theta^2}
\end{aligned}$$

Then,

$$\begin{aligned} \int_0^\infty e^{-\theta t} \ln c(t) dt &= \frac{B-\theta}{\theta^2} + \frac{\ln c_0}{\theta} - \frac{Bi_1}{\rho(R_0-1)} \varphi(c_0) + \psi(c_0) \\ &\quad + \frac{e^{-\theta T(c_0)}}{\theta} \left( \ln Ah_{T(c_0)} - \ln c_0 - [1 + \theta T(c_0)] \frac{B-\theta}{\theta} \right) \end{aligned}$$

where

$$\begin{aligned} \varphi(c_0) &\equiv \int_0^{T(c_0)} e^{-\theta t} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)t} - 1 \right] \right) dt \\ \psi(c_0) &\equiv \int_{T(c_0)}^\infty e^{-\theta t} \ln \left[ 1 - \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0-1)t}} \right] dt \end{aligned}$$

We apply the Leibniz' rule:

$$\varphi'(c_0) \equiv T'(c_0) e^{-\theta T(c_0)} \ln \left( 1 + \frac{i_0}{i_1} \left[ e^{\rho(R_0-1)T(c_0)} - 1 \right] \right) \quad (46)$$

$$\begin{aligned} \psi'(c_0) &\equiv -T'(c_0) e^{-\theta T(c_0)} \ln \left[ 1 - \frac{i_0 i_1}{i_0 + (i_1 - i_0) e^{-\rho(R_0-1)T(c_0)}} \right] \\ &= -T'(c_0) e^{-\theta T(c_0)} \ln [1 - i_{T(c_0)}] \end{aligned} \quad (47)$$

Moreover, according to (27), we have

$$\begin{aligned} \frac{d}{dc_0} \ln h_{T(c_0)} &= BT'(c_0) \left[ 1 - i_{T(c_0)} - \frac{\theta x_0}{Bx_0 + (A\theta - Bx_0) e^{\theta T(c_0)}} \right] \\ &\quad + \frac{1}{c_0} \frac{Bx_0 [1 - e^{\theta T(c_0)}]}{Bx_0 + (A\theta - Bx_0) e^{\theta T(c_0)}} \end{aligned}$$

and, observing that  $T(c_0)$  is determined by (26):

$$\frac{\theta x_0}{Bx_0 + (A\theta - Bx_0) e^{\theta T(c_0)}} = 1 - i_{T(c_0)} \quad (48)$$

Thus,

$$\frac{d}{dc_0} \ln h_{T(c_0)} = \frac{B [1 - e^{\theta T(c_0)}] [1 - i_{T(c_0)}]}{\theta c_0} \quad (49)$$

Therefore,

$$\begin{aligned} \frac{d}{dc_0} \int_0^\infty e^{-\theta t} \ln c(t) dt &= \frac{1 - e^{-\theta T(c_0)}}{\theta c_0} - \frac{Bi_1}{\rho(R_0-1)} \varphi'(c_0) + \psi'(c_0) \\ &\quad + T'(c_0) e^{-\theta T(c_0)} \left[ \ln \frac{c_0}{Ah_{T(c_0)}} + (B-\theta) T(c_0) \right] \\ &\quad + \frac{e^{-\theta T(c_0)}}{\theta} \frac{d}{dc_0} \ln h_{T(c_0)} \end{aligned}$$

Replacing (46), (47) and (49), we obtain

$$\begin{aligned} & \frac{d}{dc_0} \int_0^\infty e^{-\theta t} \ln c(t) dt \\ = & \frac{1 - e^{-\theta T(c_0)}}{\theta c_0} \left( 1 - [1 - i_{T(c_0)}] \frac{B}{\theta} \right) \\ & + T'(c_0) e^{-\theta T(c_0)} \left[ (B - \theta) T(c_0) + \ln \frac{c_0 \left( 1 + \frac{i_0}{i_1} [e^{\rho(R_0-1)T(c_0)} - 1] \right)^{-\frac{Bi_1}{\rho(R_0-1)}}}{A [1 - i_{T(c_0)}] h_{T(c_0)}} \right] \end{aligned}$$

According to (27) and (48), we have

$$\ln \frac{c_0 \left( 1 + \frac{i_0}{i_1} [e^{\rho(R_0-1)T(c_0)} - 1] \right)^{-\frac{Bi_1}{\rho(R_0-1)}}}{A [1 - i_{T(c_0)}] h_{T(c_0)}} = -(B - \theta) T(c_0)$$

Therefore,

$$\frac{d}{dc_0} \int_0^\infty e^{-\theta t} \ln c(t) dt = \frac{1 - e^{-\theta T(c_0)}}{\theta c_0} \left( 1 - [1 - i_{T(c_0)}] \frac{B}{\theta} \right)$$

and

$$\frac{d}{dc_0} \int_0^\infty e^{-\theta t} \ln c(t) dt > 0 \Leftrightarrow i_{T(c_0)} > \frac{B - \theta}{B}$$

According to (26), in the critical value,

$$x_0 = \theta \frac{A}{B}$$

we obtain

$$i_{T(c_0)} = \frac{B - \theta}{B}$$

and, thus,

$$\frac{d}{dc_0} \int_0^\infty e^{-\theta t} \ln c(t) dt = 0$$

Moreover, if  $x_0 > \theta A/B$ , that is  $c_0 > h_0 \theta A/B$ , since  $T'(c_0) < 0$  and  $i'(t) > 0$  (Assumption 2),

$$i_{T(c_0)} < \frac{B - \theta}{B}$$

and, hence,

$$\frac{d}{dc_0} \int_0^\infty e^{-\theta t} \ln c(t) dt < 0$$

entailing that the BGP is optimal. ■

**Proof of Proposition 10.**

According to (27), we obtain

$$\frac{h'(t)}{h(t)} = B[1 - i(t)] - \frac{\theta Bx_0}{Bx_0 + (A\theta - Bx_0)e^{\theta t}} \quad (50)$$

Replacing (28) in (50) and reconsidering (22), we obtain (29), that is the BGR (the same for human capital and consumption). Under Assumptions 2 and 3,

$$\theta < B(1 - i_1) < B[1 - i(t)]$$

and, then,  $g(t) > 0$ . Since  $i(t)$  increases, the BGR decreases over time to  $B(1 - i_1) - \theta$ , that is, according to (2), to (30), which remains positive. ■

**Proof of Proposition 11.**

We consider two cases: (1) dominant income effects ( $\varepsilon < 1$ ), (2) dominant substitution effects ( $\varepsilon > 1$ ).

Focus on equation (18), that is on

$$x'(t) = \frac{B}{A} [x(t) - \tilde{x}(t)] x(t) \quad (51)$$

where

$$\tilde{x}(t) \equiv \varepsilon \theta \frac{A}{B} + (1 - \varepsilon) A [1 - i(t)]$$

(1)  $\varepsilon < 1$  (dominant income effects).

In this case, since  $i(t)$  is increasing in  $t$ ,  $\tilde{x}(t)$  is decreasing.

(1.1) If  $x(\tau) > \tilde{x}(\tau)$  for some  $\tau$ , then  $x'(\tau) > 0$  and  $x(t) > \tilde{x}(t)$  for any  $t \geq \tau$ . Then, according to (16),  $x(t)$  attains the upper bound  $A[1 - i(T)]$  in a finite time because

$$x''(t) = \frac{B}{A} [x'(t) - \tilde{x}'(t)] x(t) + \frac{B}{A} [x(t) - \tilde{x}(t)] x'(t) > 0$$

for any  $t \geq \tau$ , where  $T$  is the unique solution to  $x(T) = A[1 - i(T)]$  (the LHS is increasing, the RHS decreasing).

In this case,  $h(t) = h(T)$  for any  $t \geq T$  and, under Assumption 3, according to (14),

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon [B(1 - i_1) - \theta] > 0$$

that is, asymptotically,

$$c(t) \sim C e^{\varepsilon [B(1 - i_1) - \theta] t}$$

where  $C$  is a positive constant.

Since  $u'(c) = c^{-1/\varepsilon}$ , the transversality condition becomes:

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) &= \lim_{t \rightarrow \infty} e^{-\theta t} c(t)^{-\frac{1}{\varepsilon}} h(t) \\ &= \lim_{t \rightarrow \infty} e^{-\theta t} \left[ C e^{\varepsilon [B(1 - i_1) - \theta] t} \right]^{-\frac{1}{\varepsilon}} h(T) \\ &= h(T) C^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-\theta t} e^{-[B(1 - i_1) - \theta] t} \\ &= h(T) C^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-B(1 - i_1) t} = 0 \end{aligned}$$



However, under Assumption 3, condition (9) entails

$$\lim_{t \rightarrow \infty} h'(t) = B(1 - i_1)h(T)^\alpha - \frac{BC}{A} \lim_{t \rightarrow \infty} e^{\varepsilon[B(1-i_1)-\theta]t} = -\infty$$

leading to a contradiction, since  $h(t) = h(T)$  and  $h'(t) = 0$  for any  $t \geq T$ .

(1.2) If  $x(t) \leq \tilde{x}(t)$  for any  $t$ , then  $x'(t) \leq 0$  and

$$\lim_{t \rightarrow \infty} x(t) \leq \lim_{t \rightarrow \infty} \tilde{x}(t) = \varepsilon\theta\frac{A}{B} + (1 - \varepsilon)A(1 - i_1)$$

Since  $x(t)$  is monotonic, either (1.2.1)  $\lim_{t \rightarrow \infty} x(t) = \underline{x} > 0$  or (1.2.2)  $\lim_{t \rightarrow \infty} x(t) = 0$ .

(1.2.1) In the first case, according to (51), we have

$$0 = \lim_{t \rightarrow \infty} x'(t) = \left( \frac{B}{A}\underline{x} + (\varepsilon - 1)B(1 - i_1) - \varepsilon\theta \right) \underline{x}$$

that is

$$\underline{x} = \lim_{t \rightarrow \infty} \tilde{x}(t) = \varepsilon\theta\frac{A}{B} + (1 - \varepsilon)A(1 - i_1)$$

and, according to (13), asymptotically,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} &= B(1 - i_1) - \frac{B}{A}\underline{x} = B(1 - i_1) - \frac{B}{A} \left[ \varepsilon\theta\frac{A}{B} + (1 - \varepsilon)A(1 - i_1) \right] \\ &= \varepsilon[B(1 - i_1) - \theta] > 0 \end{aligned} \quad (52)$$

because of Assumption 3.

We obtain an asymptotically balanced growth path since, according to (14), asymptotically, we get also

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon[B(1 - i_1) - \theta] \quad (53)$$

(1.2.2) In the second case, according to (13),

$$\lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} = B(1 - i_1) - \frac{B}{A} \lim_{t \rightarrow \infty} x(t) = B(1 - i_1)$$

Therefore, since, according to (14),

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon[B(1 - i_1) - \theta]$$

we obtain, asymptotically,

$$\begin{aligned} h(t) &\sim He^{B(1-i_1)t} \\ c(t) &\sim Ce^{\varepsilon[B(1-i_1)-\theta]t} \end{aligned}$$

The usual TC is violated

$$\begin{aligned}
\lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) &= \lim_{t \rightarrow \infty} e^{-\theta t} c(t)^{-\frac{1}{\varepsilon}} h(t) \\
&= \lim_{t \rightarrow \infty} e^{-\theta t} \left[ C e^{\varepsilon[B(1-i_1)-\theta]t} \right]^{-\frac{1}{\varepsilon}} H e^{B(1-i_1)t} \\
&= H C^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-\theta t - [B(1-i_1)-\theta]t + B(1-i_1)t} \\
&= H C^{-\frac{1}{\varepsilon}} > 0
\end{aligned}$$

Therefore, the optimal solution is the Asymptotically Balanced Growth Path (ABGP) given by (52) and (53) in the case (1.2.1):

$$\begin{aligned}
h(t) &\sim H e^{\varepsilon[B(1-i_1)-\theta]t} \\
c(t) &\sim C e^{\varepsilon[B(1-i_1)-\theta]t}
\end{aligned}$$

(2)  $\varepsilon > 1$  (dominant substitution effects).

Reconsider equation (51), where now, under Assumption 2, the path

$$\tilde{x}(t) = \varepsilon \theta \frac{A}{B} + (\varepsilon - 1) A [i(t) - 1]$$

is increasing in  $t$ .

(2.1) According to (51), if  $x(\tau) < \tilde{x}(\tau)$  for some  $\tau$ , then  $x'(\tau) < 0$  and  $x(t) < \tilde{x}(t)$  for any  $t \geq \tau$ .

Thus,

$$\lim_{t \rightarrow \infty} x(t) \leq \lim_{t \rightarrow \infty} \tilde{x}(t) < \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A (1 - i_1) < \varepsilon \theta \frac{A}{B}$$

As above, since  $x(t)$  is monotonic for  $t \geq \tau$ , either (2.1.1)  $\lim_{t \rightarrow \infty} x(t) = \underline{x} > 0$  or (2.1.2)  $\lim_{t \rightarrow \infty} x(t) = 0$ .

(2.1.1) In the first case, according to (51), we have

$$0 = \lim_{t \rightarrow \infty} x'(t) = \left[ \frac{B}{A} \underline{x} + (\varepsilon - 1) B (1 - i_1) - \varepsilon \theta \right] \underline{x}$$

that is

$$\underline{x} = \lim_{t \rightarrow \infty} \tilde{x}(t) = \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A (1 - i_1)$$

and, according to (13), asymptotically,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} &= B(1 - i_1) - \frac{B}{A} \underline{x} = B(1 - i_1) - \frac{B}{A} \left[ \varepsilon \theta \frac{A}{B} + (1 - \varepsilon) A (1 - i_1) \right] \\
&= \varepsilon [B(1 - i_1) - \theta] > 0
\end{aligned}$$

We obtain an asymptotically balanced growth path since, according to (14), asymptotically, we have also

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon [B(1 - i_1) - \theta]$$

(2.1.2) In the second case, according to (13), asymptotically,

$$\lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} = B(1 - i_1) - \frac{B}{A} \lim_{t \rightarrow \infty} x(t) = B(1 - i_1)$$

Thus, since, according to (14), asymptotically,

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon [B(1 - i_1) - \theta]$$

we obtain

$$\begin{aligned} h(t) &\sim H e^{B(1-i_1)t} \\ c(t) &\sim C e^{\varepsilon[B(1-i_1)-\theta]t} \end{aligned}$$

The usual transversality condition is violated:

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) &= \lim_{t \rightarrow \infty} e^{-\theta t} c(t)^{-\frac{1}{\varepsilon}} h(t) \\ &= \lim_{t \rightarrow \infty} e^{-\theta t} \left[ C e^{\varepsilon[B(1-i_1)-\theta]t} \right]^{-\frac{1}{\varepsilon}} H e^{B(1-i_1)t} \\ &= H C^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-\theta t - [B(1-i_1)-\theta]t + B(1-i_1)t} \\ &= H C^{-\frac{1}{\varepsilon}} > 0 \end{aligned}$$

(2.2) If  $x(t) \geq \tilde{x}(t)$  for any  $t$ , according to (51), we have  $x'(t) \geq 0$  for any  $t$ . We observe that

$$\lim_{t \rightarrow \infty} \tilde{x}(t) = \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1)$$

Thus, either (2.2.1):

$$\tilde{x}(t) \leq x(t) \leq \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1)$$

for any  $t$  and, then,

$$\varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1) = \lim_{t \rightarrow \infty} \tilde{x}(t) \leq \lim_{t \rightarrow \infty} x(t) \leq \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1)$$

that is

$$\lim_{t \rightarrow \infty} x(t) = \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1)$$

or (2.2.2): there exists  $\tau$  such that

$$x(\tau) > \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1)$$

and, since  $x'(t) \geq 0$  for any  $t$ ,

$$x(t) \geq x(\tau) > \varepsilon \theta \frac{A}{B} - (\varepsilon - 1) A(1 - i_1) = \lim_{t \rightarrow \infty} \tilde{x}(t) > \tilde{x}(t)$$

for any  $t > \tau$  since the path  $\tilde{x}(t)$  is increasing, that is

$$x(t) - \tilde{x}(t) > x(\tau) - \varepsilon\theta\frac{A}{B} + (\varepsilon - 1)A(1 - i_1) \equiv \delta > 0$$

for any  $t > \tau$ .

Therefore, either (2.2.1):

$$\lim_{t \rightarrow \infty} x(t) = \varepsilon\theta\frac{A}{B} - (\varepsilon - 1)A(1 - i_1)$$

or, according to (51), (2.2.2):

$$x'(t) = \frac{B}{A}[x(t) - \tilde{x}(t)]x(t) > \frac{B}{A}\delta x(\tau) > 0$$

for any  $t > \tau$ .

(2.2.1) In the first case, asymptotically,

$$0 = \frac{x'(t)}{x(t)} = \frac{c'(t)}{c(t)} - \frac{h'(t)}{h(t)}$$

that is, according to (14), asymptotically, under Assumption 3,

$$\lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} = \lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon[B(1 - i_1) - \theta] > 0$$

(ABGP).

(2.2.2) In the second case,  $x(t)$  attains the upper bound  $x(T) = A[1 - i(T)]$  with  $h(t) = h(T)$  for any  $t \geq T$ .

Since, according to (14), under Assumption 3,

$$\lim_{t \rightarrow \infty} \frac{c'(t)}{c(t)} = \varepsilon[B(1 - i_1) - \theta] > 0$$

we have

$$c(t) \sim Ce^{\varepsilon[B(1 - i_1) - \theta]t}$$

The usual transversality condition is satisfied:

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-\theta t} u'(c(t)) h(t) &= \lim_{t \rightarrow \infty} e^{-\theta t} c(t)^{-\frac{1}{\varepsilon}} h(t) = \lim_{t \rightarrow \infty} e^{-\theta t} \left[ Ce^{\varepsilon[B(1 - i_1) - \theta]t} \right]^{-\frac{1}{\varepsilon}} h(T) \\ &= h(T) C^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-\theta t - [B(1 - i_1) - \theta]t} = h(T) C^{-\frac{1}{\varepsilon}} \lim_{t \rightarrow \infty} e^{-B(1 - i_1)t} = 0 \end{aligned}$$

However, under Assumption 3, condition (9) entails

$$\lim_{t \rightarrow \infty} h'(t) = B(1 - i_1)h(T)^\alpha - \frac{B}{A} \lim_{t \rightarrow \infty} Ce^{\varepsilon[B(1 - i_1) - \theta]t} = -\infty$$

a contradiction, since  $h'(t) = 0$  for any  $t \geq T$ .

Therefore, only the case (2.2.1) holds and the optimal trajectory converges to the asymptotically balanced growth path

$$\begin{aligned} h(t) &\sim He^{\varepsilon[B(1 - i_1) - \theta]t} \\ c(t) &\sim Ce^{\varepsilon[B(1 - i_1) - \theta]t} \end{aligned}$$

■

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