



HAL
open science

Optimal growth when consumption takes time

Thai Ha-Huy, Cuong Le Van, Thi-Do-Hanh Nguyen

► **To cite this version:**

Thai Ha-Huy, Cuong Le Van, Thi-Do-Hanh Nguyen. Optimal growth when consumption takes time. 2023. halshs-04310371

HAL Id: halshs-04310371

<https://shs.hal.science/halshs-04310371>

Preprint submitted on 27 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

OPTIMAL GROWTH WHEN CONSUMPTION TAKES TIME *

Thai Ha-Huy[†], Cuong Le Van[‡], Thi-Do-Hanh Nguyen[§]

Juillet 28, 2020

*We are grateful to Jean-Pierre Drugeon, two Referees and one of the Editors for their valuable comments and suggestions. This article is finished during the period Ha-Huy worked as *délégué* in the Theoretical Economics Team at CNRS, Paris School of Economics (campus Jourdan). He acknowledges the members of PSE and CNRS for the warm welcome and excellent working conditions. He would like also to thank the Labex MME-DII (ANR-11-LBX-0023-01) for supports during the writing of this article.

[†]EPEE, Univ of Evry, University Paris-Saclay; TIMAS. E-mail: thai.hahuy@univ-evry.fr

[‡]IPAG Business School, Paris School of Economics, CNRS, TIMAS. Email: levan@univ-parisl.fr

[§]Vietnam Maritime University, 484 Lach Tray - Hai Phong - Vietnam. Email: nguyen.dohanh@vimaru.edu.vn

ABSTRACT

This article analyses a one-sector growth model where the consumption takes time. When the consumption takes time, the consumption set is compact and we meet satiety. However, we prove that dynamic constraints are binding. This result is crucial to prove that, under well-known assumptions in macroeconomic dynamic programming, the optimal path is monotonic and always converges to a unique non-trivial steady state as in the case where consumption is timeless.

Keywords: elastic labour supply, time consuming, dynamic programming, super-modularity.

1 INTRODUCTION

Many well-known economic models are based on the assumptions that consumption is instantaneous, i.e. it does not take time. In the reality, this assumption is not plausible. In the past or even today, it took people time to wash clothes or to prepare the meals. Even with the invention of new machines and technologies, there are activities which cannot be reduced in time, for example watching a match of soccer or listening to a Beethoven's symphony.

Economic agents must always make the trade-off between the time devoted for the production, and the time for leisure. The consumption time constraint was first formally introduced by Gossen [6] and much later generalized by Becker [2] in his famous contribution to the theory of time allocation. Recently, Tran-Nam & Pham [14] consider the question with a general equilibrium approach. Le Van et al. [13] extend this model to a general equilibrium model with many goods and many heterogeneous economic agents. In their article, consumption itself takes time. A typical household is subject to a financial constraint and a time constraint as well. They show that the economy possesses at least one autarkic Walrasian equilibrium.

In our knowledge, though the problem is well analysed in static economy, there is no theoretical research in a dynamic framework. To give a response for this gap, in this paper, we consider a one sector growth model, in which consumption is itself time consuming. At each period, the agent must share her/his available amount of time between consuming and working.

In this case, the consumption set is compact, i.e. there is a satiation point while in the usual models where consuming is timeless the consumption is the positive cone and we have no satiation point. That is the main difference. However, we prove that all the dynamic constraints are binding in our model. This result is crucial to prove that, under well-known assumptions in macroeconomic dynamic programming, such as concavity of the production and utility functions, and complementarity between capital and labour, the optimal path is monotonic and converges to a unique non-trivial steady state as in the case where consumption is timeless. That is our main result¹. We mention that the steady state capital stock is lower than the one obtained in economies where consumption is timeless.

The effects of incorporating an explicit time for consumption in other classes of dynamic models would be worthwhile to investigate in future work, including for models of pollution and growth (e.g., Ghosh et al.,[5]), dynamic allocation of government expenditures (e.g., Le Van et al., [11] and Fan et al., [4]), or impact

¹The results of this model are similar to those obtained from a one sector growth model with elastic labour, see e.g. [12].

of social capital on growth [10], and other dynamic models.

The article is organized as follows. Section 2 describes the model. Section 3.2 presents the indirect utility function and a modified optimization problem which is equivalent to the initial one. Section 3.4 studies the main properties of the optimal paths. The proofs are given in Appendix.

2 FUNDAMENTALS

Time is discrete. The production requires physical capital and labour. The utility of the agent depends only on the consumption which takes time. We assume that in each period, the total amount of time of the agent is inelastic, and is divided in two parts: the first part is devoted for the production, and the second part for the consumption.

At period t , given capital k_t and labour which is measured by the labour time l_t , the production is

$$Y_t = F(k_t, l_t).$$

Let $0 \leq \delta \leq 1$ be the depreciation rate of the physical capital. The dynamics of the accumulation of the capital stocks is

$$k_{t+1} = (1 - \delta)k_t + I_t,$$

where I_t is the investment at time t . It is worth noting that we do not impose that δ is strictly positive.

Given the maximal supply of labour time L and the working time l_t , the available time for consumption is $L - l_t$. Hence, there are two constraints for the consumers, the resource constraint and the time constraint:

$$\begin{aligned} c_t + I_t &\leq Y_t, \\ ac_t &\leq L - l_t, \end{aligned}$$

where $a > 0$ is the required time to consume a unit of good.

With the discount factor $\beta \in (0, 1)$, the utility function u , for given capital stock $k_0 > 0$, the economic agent solves the following maximization problem (P):

$$v(k_0) = \max \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right],$$

$$\text{s.t. } c_t + k_{t+1} \leq (1 - \delta)k_t + F(k_t, l_t), \tag{2.1}$$

$$ac_t + l_t \leq L, \tag{2.2}$$

$$c_t, l_t, k_t \geq 0 \text{ for any } t. \tag{2.3}$$

The function v is called the value function of problem (P) .

For $k_0 \geq 0$, let $\Pi(k_0)$ denote the set of feasible paths of the capital stocks $\{k_t\}_{t=0}^\infty$. That is the set of capitals for which there exist a consumption sequence $\{c_t\}_{t=0}^\infty$, and a labour sequence $\{l_t\}_{t=0}^\infty$ such that for any $t \geq 0$, the constraints (2.1) to (2.3) are satisfied.

Before analysing the properties of this economy, we set assumptions on its fundamentals. These assumptions are usual in economic dynamic programming literature.

Assumption A1. i) *The production function F is strictly concave, strictly increasing, twice differentiable. It satisfies $F(0, l) = 0, F(k, 0) = 0$ for any $(k, l) \geq 0$. Moreover, for any $k, l > 0$, we have $F_1(0, l) = \infty$ and $F_2(k, 0) = \infty$ where F_1, F_2 are respectively the partial derivatives with respect to k and to l (Inada conditions).*

ii) *The utility function u is strictly concave, strictly increasing, twice differentiable. It satisfies the Inada condition: $\lim_{c \rightarrow 0} u'(c) = \infty$.*

iii) *No triviality: For any $k_0 > 0$, there exists $\{k_t\}_{t=0}^\infty \in \Pi(k_0)$ such that there exist a consumption sequence $\{c_t\}_{t=0}^\infty$ and labour sequence $\{l_t\}_{t=0}^\infty$ satisfying (2.1) to (2.3) and*

$$\sum_{t=0}^{\infty} \beta^t u(c_t) > -\infty.$$

iv) *Tail-insensitivity: For any $k_0 > 0$, any $\epsilon > 0$, there exist T_0 , a neighbourhood \mathcal{V} of k_0 such that for any $k'_0 \in \mathcal{V}$, any feasible path $\{k'_t\}_{t=0}^\infty$, any consumption sequence $\{c'_t\}_{t=0}^\infty$, and labour sequence $\{l'_t\}_{t=0}^\infty$ satisfying (2.1) to (2.3), we have, for any $T \geq T_0$, $\sum_{t=T}^{\infty} \beta^t u(c'_t) < \epsilon$.*

Conditions (i) and (ii) are well-known. Condition (iii) ensures that even in the case where the utility function u is unbounded from below, the problem is not trivial. Under condition (iv), *tail-insensitivity*, the inter-temporal sum is upper semi-continuous with respect to the product topology. Moreover, the value function v is upper semi-continuous. By Weierstrass theorem, this continuity property on a compact set implies the existence of an optimal path. The strict concavity of the production function F and of the utility function u ensures the concavity of the value function and hence, the uniqueness of the solution. For details and discussions about *tail-insensitivity* condition, see Le Van & Morhaim [9].

3 THE DYNAMICS

3.1 THE OPTIMAL PATH

Under condition A1, there exists a unique solution. By the Inada conditions, the optimal consumption and the capital stock are strictly positive. Naturally, it raises the question whether the constraints in resource and time are binding. Lemma 3.1 provides an answer for this question.

LEMMA 3.1. *Assume A1. Let $k_0 > 0$. Assume that the sequence $\{(c_t^*, k_{t+1}^*, l_t^*)\}_{t=0}^\infty$ is the solution to problem (P). Then for any $t \geq 0$:*

$$\begin{aligned} k_t^*, c_t^* &> 0, \\ 0 &< l_t^* < L, \\ k_{t+1}^* &= F(k_t^*, l_t^*) + (1 - \delta)k_t^* + \frac{l_t^* - L}{a}, \\ ac_t^* + l_t^* &= L. \end{aligned}$$

To understand more deeply the global dynamics of the economy and the role of the time devoted for consumption, we define and study the indirect utility function and a modified optimization problem. Section 3.2 is devoted to this question.

3.2 THE INDIRECT UTILITY FUNCTION

Consider the correspondence Γ defined on \mathbb{R}_+ :

$$\Gamma(k) = \{y \geq 0 : \exists c \geq 0, l \geq 0, c + y \leq (1 - \delta)k + F(k, l) \text{ and } ac + l \leq L\}$$

We verify easily that the correspondence Γ is non empty, convex, compact valued. We will characterize the interior of $\Gamma(k)$ for $k > 0$ and also the interior of graph Γ .

LEMMA 3.2. i) *For any $k > 0$*

$$\text{int } \Gamma(k) = \{y : y > 0, \exists c \geq 0, \exists l \geq 0, \text{ such that } c + y < F(k, l) + (1 - \delta)k \text{ and } ac + l \leq L\}.$$

ii) *The couple $(k, y) \in \text{int}(\text{graph } \Gamma)$ if and only if*

$$\exists c \geq 0, \exists l \geq 0, \text{ s.t. } c + y < F(k, l) + (1 - \delta)k \text{ and } ac + l \leq L.$$

We define the indirect utility function $V : \text{graph } \Gamma \rightarrow \mathbb{R}$ as:

$$V(k, y) = \max u(c), \tag{3.1}$$

$$\text{s.t. } c + y \leq (1 - \delta)k + F(k, l), \tag{3.2}$$

$$ac + l \leq L, \tag{3.3}$$

$$c \geq 0 \text{ and } l \geq 0. \tag{3.4}$$

As in Stokey, Lucas & Prescott [15], we rewrite the optimization problem as follows. For $k_0 > 0$, the agent solves the problem (P'):

$$\max \left[\sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}) \right],$$

$k_{t+1} \in \Gamma(k_t)$ for all $t \geq 0$.

The two optimization problems (P) and (P') are equivalent and have the same optimal paths of capital stocks.

LEMMA 3.3. i) *If a sequence of consumptions, labour times, and capital stocks $\{(c_t^*, l_t^*, k_t^*)\}_{t=0}^{\infty}$ is solution to problem (P), then the sequence $\{k_t^*\}_{t=0}^{\infty}$ is solution to problem (P').*

ii) *A sequence of capital stocks $\{k_t^*\}_{t=0}^{\infty}$ is solution to problem (P') if and only if there exist a sequence of consumptions $\{c_t^*\}_{t=0}^{\infty}$ and a sequence of labour times $\{l_t^*\}_{t=0}^{\infty}$ such that $\{(c_t^*, l_t^*, k_t^*)\}_{t=0}^{\infty}$ is a solution to problem (P).*

The proof of Lemma 3.3 uses usual arguments in dynamic programming literature. For the detail, see Stokey & Lucas (with Prescott) [15].

It is also well-known that the value function v of the two problems (P) and (P') is the solution to the Bellman functional equation²:

$$v(k_0) = \max_{k_1 \in \Gamma(k_0)} [V(k_0, k_1) + \beta v(k_1)].$$

A feasible path $\{k_t^*\}_{t=0}^{\infty}$ is optimal if and only if

$$v(k_t^*) = V(k_t^*, k_{t+1}^*) + \beta v(k_{t+1}^*),$$

for any $t \geq 0$.

By the strict concavity of v , there exists a unique k_1 which is solution to this equation. Let φ denote the optimal policy function which is defined as follows:

$$\varphi(k) = \operatorname{argmax}_{k_1 \in \Gamma(k)} [V(k_0, k_1) + \beta v(k_1)].$$

The function φ is continuous. If $\{k_t^*\}_{t=0}^{\infty}$ is an optimal path, then for any t , $k_{t+1}^* = \varphi(k_t^*)$. A non trivial steady state is a capital level $k^s > 0$ such that $k^s = \varphi(k^s)$.

²If the utility function u is bounded from below, then v is the unique solution.

3.3 THE SUPER-MODULARITY

This section is devoted to study some important features of the indirect utility function V . The two partial derivatives will play an important role in the analysis the monotonicity of optimal path of accumulation of capital.

LEMMA 3.4. *Assume **A1**. Consider $k > 0$ and $y \in \text{int } \Gamma(k)$. Then*

- i) *The function V is strictly concave, increasing with respect to the first argument and decreasing with respect to the second one.*
- ii) *There exists a unique (c^*, l^*) which satisfies $u(c^*) = V(k, y)$ and constraints (3.2) to (3.4). The couple (c^*, l^*) is the solution to the optimization problem yielding the indirect utility $V(k, y)$.*
- iii) *V is differentiable at (k, y) and*

$$\begin{aligned}\frac{\partial V(k, y)}{\partial k} &= \frac{u'(c^*)}{1 + aF_2(k, l^*)} \times [F_1(k, l^*) + (1 - \delta)], \\ \frac{\partial V(k, y)}{\partial y} &= -\frac{u'(c^*)}{1 + aF_2(k, l^*)}.\end{aligned}$$

The concavity of V and the uniqueness of (c^*, l^*) are direct consequences of conditions (i) and (ii) in **A1**. Under a suitable assumption that requires the complementarity between capital and labour, the function V has positive cross derivatives. Hence, V has strictly increasing differences as showed in Amir [1].

Assumption A2. $F_{12}(k, l) \geq 0$ for all $k > 0$ and $0 < l < L$.

Under **A1** and **A2**, the indirect utility function V is twice differentiable and its cross derivative is strictly positive.

LEMMA 3.5. *Assume **A1** and **A2**. For $k > 0$ and $y \in \text{int}(\Gamma(k))$, the function V is twice differentiable at (k, y) with positive cross derivative:*

$$\frac{\partial^2 V(k, y)}{\partial k \partial y} > 0.$$

Since the function V has increasing differences, it is supermodular.³ Then, by Amir [1], the optimal policy function φ is an increasing mapping. Any optimal path of capitals is therefore monotonic. This monotonicity allows us to a better understanding of the economy.

³The (strict) super-modularity is defined as follows. For every (x, x') and every (y, y') that belong to $\text{Graph}(\Gamma)$, we have $V(x, y) + V(x', y')(>) \geq V(x', y) + V(x, y')$ whenever $(x', y')(>) \geq (x, y)$. When V is twice differentiable, (strict) super modularity is equivalent to positive cross derivative: $V_{12}(x, y)(>) \geq 0$ for any x, y .

3.4 THE GLOBAL DYNAMICS

This section analyses the global behaviour of the economy. The optimal path of capitals always converges monotonically to a unique steady state, even in the case where unbounded feasible paths are possible.

PROPOSITION 3.1. *Assume **A1** and **A2**. For any $k_0 > 0$,*

i) *The optimal capitals sequence $\{k_t^*\}_{t=0}^\infty$ is monotonic.*

ii) *For any $t \geq 0$,*

$$(k_t^*, k_{t+1}^*) \in \text{int}(\text{graph}\Gamma).$$

iii) *The optimal path $\{(k_t^*, c_t^*, l_t^*)\}_{t=0}^\infty$ satisfies the Euler Equation⁴:*

$$\frac{u'(c_t^*)}{1 + aF_2(k_t^*, l_t^*)} = \beta \frac{u'(c_{t+1}^*)}{1 + aF_2(k_{t+1}^*, l_{t+1}^*)} \times [F_1(k_{t+1}^*, l_{t+1}^*) + (1 - \delta)].$$

We may wonder whether extinction of the economy is possible. In a one-sector economy, following the results in Kamihigashi & Roy [7], if the productivity at zero is greater than $\frac{1}{\beta}$, the economy does not converge to zero. The next proposition, Proposition 3.2, provides a similar answer for this question, thanks to the Inada condition in Assumption **A1**.

PROPOSITION 3.2. *Assume **A1** and **A2**. For any $k_0 > 0$, the optimal capitals path beginning from k_0 does not converges to zero.*

As in Amir [1], the super-modularity ensures the monotonicity of the optimal capitals path. Given an initial level $k_0 > 0$, can we know whether the optimal capitals path beginning from k_0 is increasing or decreasing? Moreover, is it possible that it converges to infinity?

First, we characterize the non trivial steady states. The proof of Proposition 3.3 merely uses usual arguments in dynamic programming.

PROPOSITION 3.3. *The list (c^s, l^s, k^s) is an interior steady state if it satisfies*

i) *(c^s, l^s) solves (3.1) to (3.4) for $k = y = k^s$.*

ii) *$F_1(k^s, l^s) = \frac{1}{\beta} - 1 + \delta$.*

There exists a unique non trivial steady state. Any optimal capitals path converges monotonically to this value.

⁴Using the indirect utility function V , the Euler equation can be written as: $V_2(k_t^*, k_{t+1}^*) + \beta V_1(k_{t+1}^*, k_{t+2}^*) = 0$.

PROPOSITION 3.4. i) *There exists a unique interior steady state (c^s, l^s, k^s) .*

ii) *If $0 < k_0 < k^s$, then the optimal capitals path is increasing and converges to k^s .*

iii) *If $k_0 > k^s$, then the optimal capitals path is decreasing and converges to k^s .*

It is well known that for $0 < \delta \leq 1$, the economy is bounded, in the sense that every feasible capitals path is bounded⁵. For the case $\delta = 0$, we verify easily that there exist feasible capitals paths that converge to infinity. Proposition 3.4 states that, even in this case, every optimal capitals path is bounded from above and it converges to the unique steady state.

REMARK 3.1. *Observe that the steady state capital stock is lower than the one we obtain in economies where consumptions does not take time. Indeed, let k'^s denote the steady state capital stock corresponding to the case where consumption is timeless. It satisfies $F_1(k'^s, L) = \frac{1}{\beta} - 1 + \delta$. In our model we have $F_1(k^s, l^s) = \frac{1}{\beta} - 1 + \delta$, with $l^s < L$. Since $F_{12} > 0$, we have*

$$F_1(k^s, l^s) = F_1(k'^s, L) > F_1(k'^s, l^s)$$

and since $F_{11} < 0$, we have $k'^s > k^s$.

4 CONCLUSION

This article considers a growth model in which agent needs time to consume. Under usual conditions, a detailed description of the dynamics is given. We obtain that the optimal capitals path still is monotonic and converges to a unique steady state as in the case where consumption is timeless. However, the steady state capital stock is lower than the one we obtain in economies where consumptions does not take time.

Many economic problems may be reconsidered in future work with time consuming consumption, including for models of pollution and growth (e.g., Ghosh et al.[5]), dynamic allocation of government expenditures (e.g., Le Van et al. [11], and Fan et al. [4]), or impact of social capital on growth [10], and other dynamic models.

Finally, it would be interesting to understand what happens in the case where the agent invests in technology, or in infrastructure in order to diminish the time for consumption, (for example the use of machines may reduce the washing time, or the Internet helps us to shop online), and frees people for other activities. For us, this problem, though interesting, represents a challenge and should be subject of a future research.

⁵The proof for this statement is given in the Appendix.

5 APPENDIX

5.1 PROOF OF LEMMA 3.1

Consider a sequence $\{(c_t^*, k_{t+1}^*, l_t^*)\}_{t=0}^{\infty}$ which is the optimal solution to problem (P).

We prove that $ac_t^* + l_t^* = L$ for any t . Assume the contrary, for some t we have $ac_t^* + l_t^* < L$. Then there exists $\epsilon_c, \epsilon_l > 0$ such that

$$\begin{aligned} a(c_t^* + \epsilon_c) + (l_t^* + \epsilon_l) &\leq L, \\ (c_t^* + \epsilon_c) + k_{t+1}^* - (1 - \delta)k_t^* &\leq F(k_t^*, l_t^* + \epsilon_l). \end{aligned}$$

This implies $u(c_t^* + \epsilon_c) > u(c_t^*)$ and we obtain a feasible sequence which provides a higher utility level: a contradiction.

We prove the existence of some $t \geq 0$ such that $l_t^* > 0$. Indeed, assume the contrary, $l_t^* = 0$ for all $t \geq 0$. This implies $c_t^* = \frac{L}{a}$ and $F(k_t^*, l_t^*) = 0$ for any t . We have

$$\begin{aligned} k_{t+1}^* &\leq (1 - \delta)k_t^* - \frac{L}{a} \\ &< k_t^*. \end{aligned}$$

Thus, the sequence $\{k_t^*\}_{t=0}^{\infty}$ is decreasing and converges to some k^* satisfying $k^* \leq (1 - \delta)k^* - \frac{L}{a}$: a contradiction.

Now, we prove that if $l_t^* > 0$, then

$$k_{t+1}^* = F(k_t^*, l_t^*) + (1 - \delta)k_t^* + \frac{l_t^* - L}{a}.$$

Assume the contrary, for some T , $l_T^* > 0$ and

$$k_{T+1}^* < F(k_T^*, l_T^*) + (1 - \delta)k_T^* + \frac{l_T^* - L}{a}.$$

Take some $\epsilon > 0$ small enough such that $0 < \epsilon < l_T^*$ and

$$k_{T+1}^* \leq F(k_T^*, l_T^* - \epsilon) + (1 - \delta)k_T^* + \frac{l_T^* - \epsilon - L}{a}.$$

Define the sequence $\{(\hat{c}_t, \hat{l}_t, \hat{k}_t)\}_{t=0}^{\infty}$ as follows:

$$\begin{aligned} \hat{c}_t &= c_t^* \text{ for any } t \neq T, \\ \hat{l}_t &= l_t^* \text{ for any } t \neq T, \\ \hat{c}_T &= c_T^* + \frac{\epsilon}{a}, \\ \hat{l}_T &= l_T^* - \epsilon, \\ \hat{k}_t &= k_t^* \text{ for any } t. \end{aligned}$$

We verify easily that the sequence $\{(\hat{c}_t, \hat{l}_t, \hat{k}_t)\}_{t=0}^{\infty}$ is feasible. Moreover,

$$\sum_{t=0}^{\infty} \beta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*) = \beta^T u(\hat{c}_T) - \beta^T u(c_T^*) > 0,$$

a contradiction.

Now, we prove that $l_t^* > 0$ for any $t \geq 0$.

Suppose the contrary. Without loss of generality, we can suppose that $l_0^* = 0$ and $l_1^* > 0$. We have successively

$$c_0^* = \frac{L}{a}, \tag{5.1}$$

$$\frac{L}{a} + k_1^* \leq F(k_0, 0) + (1 - \delta)k_0 \equiv G(k_0, 0), \tag{5.2}$$

$$c_1^* + k_1^* = F(k_1^*, l_1^*) + (1 - \delta)k_1^* \equiv G(k_1^*, l_1^*), \tag{5.3}$$

$$c_1^* = \frac{L - l_1^*}{a}.$$

With some $\epsilon > 0$, define k_1, c_0, c_1, l_1 as follows:

$$c_0 + k_1 = F(k_0, \epsilon) + (1 - \delta)k_0 \equiv G(k_0, \epsilon), \tag{5.4}$$

$$c_0 = \frac{L - \epsilon}{a}, \tag{5.5}$$

$$c_1 + k_2^* = F(k_1, l_1) + (1 - \delta)k_1 \equiv G(k_1, l_1), \tag{5.6}$$

$$ac_1 + l_1 = L. \tag{5.7}$$

(5.6) and (5.7) imply

$$c_1 + k_2^* = G(k_1, L - ac_1). \tag{5.8}$$

Combining (5.8) and (5.3), one gets

$$\begin{aligned} c_1 - c_1^* &= G(k_1, L - ac_1) - G(k_1^*, L - ac_1^*), \\ &\geq G_k(k_1, L - ac_1)(k_1 - k_1^*) - aG_l(k_1, L - ac_1)(c_1 - c_1^*). \end{aligned} \tag{5.9}$$

This implies

$$(1 + aG_l(k_1, L - ac_1))(c_1 - c_1^*) \geq G_k(k_1, L - ac_1)(k_1 - k_1^*) \tag{5.10}$$

(5.4) and (5.5) yield

$$k_1 = G(k_0, \epsilon) - \frac{L_0 - \epsilon}{a}. \tag{5.11}$$

Combine with (5.2), we obtain

$$k_1 - k_1^* \geq G_l(k_0, \epsilon)\epsilon + \frac{\epsilon}{a}. \tag{5.12}$$

Let $\Delta = u(c_0) + \beta u(c_1) - u(c_0^*) - \beta u(c_1^*)$. We have

$$\Delta \geq u'(c_0)(c_0 - c_0^*) + \beta u'(c_1)(c_1 - c_1^*).$$

Using (5.1), (5.5), (5.10) and (5.12), it comes

$$\begin{aligned} \Delta &\geq u'(c_0) \left(-\frac{\epsilon}{a} \right) + \beta u'(c_1) \frac{G_k(k_1, L - ac_1)}{1 + aG_l(k_1, L - ac_1)} \left[G_l(k_0, \epsilon)\epsilon + \frac{\epsilon}{a} \right], \\ \frac{\Delta}{\epsilon} &\geq -\frac{u'(c_0)}{a} + \beta u'(c_1) \frac{G_k(k_1, L - ac_1)}{1 + aG_l(k_1, L - ac_1)} \left[G_l(k_0, \epsilon) + \frac{1}{a} \right]. \end{aligned}$$

Let $\epsilon \rightarrow 0$. Then

$$\begin{aligned} u'(c_0) &\rightarrow u'(c_0^*) < \infty, \\ u'(c_0) &\rightarrow u'(c_0^*) < \infty, \\ u'(c_1) &\rightarrow u'(c_1^*) > 0, \\ G_k(k_1, L - ac_1) &\rightarrow G_k(k_1^*, l_1^*) > 0, \\ G_l(k_1, L - ac_1) &\rightarrow G_l(k_1^*, l_1^*) \in (0, \infty), \\ G_l(k_0, \epsilon) &\rightarrow G_l(k_0, 0) = \infty. \end{aligned}$$

Thus $\frac{\Delta}{\epsilon}$ converges to infinity when ϵ converges to 0. This implies implying $\Delta > 0$ when ϵ is small enough, a contradiction.

□

5.2 PROOF OF LEMMA 3.2

i) Consider $k, y > 0$ such that there exist $c \geq 0, l \geq 0$ satisfying

$$\begin{aligned} c + y &< F(k, l) + (1 - \delta)k, \\ ac + l &\leq L. \end{aligned}$$

For any $\epsilon > 0$ small enough, we have,

$$\begin{aligned} c + (y + \epsilon) &< F(k, l) + (1 - \delta)k, \\ c + (y - \epsilon) &< F(k, l) + (1 - \delta)k, \\ ac + l &\leq L. \end{aligned}$$

that means $y \in \text{int}\Gamma(k)$.

Now, consider $y \in \text{int}(\Gamma(k))$. Then there exists $\epsilon > 0$ such that the interval $(y - \epsilon, y + \epsilon) \subset \Gamma(k)$. For some $c \geq 0, l \geq 0$

$$\begin{aligned} c + y + \epsilon &\leq F(k, l) + (1 - \delta)k, \\ ac + l &\leq L. \end{aligned}$$

Therefore

$$\begin{aligned} c + y &< F(k, l) + (1 - \delta)k, \\ ac + l &\leq L. \end{aligned}$$

ii) Consider $(k, y) \in \text{int}(\Gamma)$. Then for any $\epsilon > 0$ small enough, $(k, y + \epsilon), (k, y - \epsilon), (k - \epsilon, y) \in \Gamma(k)$. This implies $y > 0$ and $k > 0$.

There exist $c \geq 0, l \geq 0$ such that

$$\begin{aligned} c + (y + \epsilon) &\leq F(k, l) + (1 - \delta)k, \\ ac + l &\leq L. \end{aligned}$$

Hence, $c + y < F(k, l) + (1 - \delta)k, ac + l \leq L$.

Now, consider (k, y) such that $k, y > 0$ and there exist $c \geq 0, l \geq 0$ satisfying

$$\begin{aligned} c + y &< F(k, l) + (1 - \delta)k, \\ ac + l &\leq L. \end{aligned}$$

There exists $\epsilon > 0$ such that for any $y' \in (y - \epsilon, y + \epsilon)$, any $k' \in (k - \epsilon, k + \epsilon)$ we have

$$c + y' < F(k', l) + (1 - \delta)k', \quad ac + l \leq L.$$

□

5.3 PROOF OF LEMMA 3.4

1) Consider the Lagrangian

$$\mathcal{L} = u(c) + \lambda[F(k, l) + (1 - \delta)k - c - y] + \mu(L - ac - l) + \xi_1 c + \xi_2 l.$$

Consider the Kuhn-Tucker first-order conditions:

$$\begin{aligned} u'(c^*) - \lambda - a\mu + \xi_1 &= 0, \\ \lambda F_2(k, l^*) - \mu + \xi_2 &= 0, \\ \lambda \geq 0, \lambda[F(k, l^*) + (1 - \delta)k - c^* - y] &= 0, \\ \mu \geq 0, \mu[L - ac^* - l] &= 0, \\ \xi_1 \geq 0, \xi_1 c^* &= 0, \\ \xi_2 \geq 0, \xi_2 l^* &= 0. \end{aligned}$$

The Inada condition of the utility function implies that $c^* > 0$. Hence, $\xi_1 = 0$. The Inada condition on labour's marginal productivity implies that $l^* > 0$, hence $\xi_2 = 0$. The strict increasingness of F implies that $\lambda > 0$ and $\mu > 0$. This implies $0 < l^* < L$. The uniqueness of the solution comes from the strict concavity of u .

2) We have

$$u'(c^*) - \lambda - a\mu = 0, \quad (5.13)$$

$$\lambda F_2(k, l^*) - \mu = 0, \quad (5.14)$$

$$c^* + y = F(k, l^*) + (1 - \delta)k,$$

$$ac^* + l^* = L.$$

Differentiating the above equations with respect to k and y gives:

$$u''dc^* - d\lambda - ad\mu = 0,$$

$$F_2d\lambda + \lambda[F_{12}dk + F_{22}dl^*] - d\mu = 0,$$

$$dc^* + dy - [F_1dk + F_2dl^*] - (1 - \delta)dk = 0,$$

$$adc^* + dl^* = 0.$$

Writing these equations in a matrix form we get:

$$\underbrace{\begin{pmatrix} u'' & 0 & -1 & -a \\ 0 & \lambda F_{22} & F_2 & -1 \\ 1 & -F_2 & 0 & 0 \\ a & 1 & 0 & 0 \end{pmatrix}}_A \begin{pmatrix} dc^* \\ dl^* \\ d\lambda \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\lambda F_{12} & 0 \\ F_1 + 1 - \delta & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dk \\ dy \end{pmatrix}.$$

The determinant of matrix A is:

$$\det(A) = (1 + aF_2)^2.$$

Since the production function F is strictly increasing then $\det(A) > 0$ and A is invertible. It implies that $c^*(k, y), l^*(k, y), \lambda(k, y), \mu(k, y)$ are continuously differentiable in a neighborhood of (k, y) .

By the Envelope Theorem we have:

$$\frac{\partial V(k, y)}{\partial k} = \frac{\partial \mathcal{L}}{\partial k}(c^*, l^*, \lambda, \mu) = \lambda[F_1(k, l^*) + 1 - \delta],$$

$$\frac{\partial V(k, y)}{\partial y} = \frac{\partial \mathcal{L}}{\partial y}(c^*, l^*, \lambda, \mu) = -\lambda.$$

and $v'(k) = V_1(k, \varphi(k)) = \lambda[F_1(k, l^*) + 1 - \delta]$

From (5.13) and (5.14) we have

$$\lambda = \frac{u'(c^*)}{1 + aF_2(k, l^*)}; \quad \mu = \frac{u'(c^*)F_2(k, l^*)}{1 + aF_2(k, l^*)}.$$

3) This is a direct consequence of (i) and (ii).

□

5.4 PROOF OF LEMMA 3.5

In the following, λ denotes the same notation as in the proof of Lemma 3.4. Using the same arguments as in the proof of Lemma 3.4, by tedious computations, we get:

$$\frac{\partial \lambda}{\partial k} = \frac{1}{(1 + aF_2)^2} [-a\lambda(1 + aF_2)F_{12} + (a^2\lambda F_{22} + u'')(F_1 + 1 - \delta)].$$

Consider the expression in the parentheses. Observe that the concavity of F and strict concavity of u imply $(a^2\lambda F_{22} + u'') < 0$. Then the second term is strictly negative. By Assumption A2, the first term is strictly negative. Hence

$$\frac{\partial \lambda}{\partial k} < 0.$$

It follows that

$$\frac{\partial^2 V(k, y)}{\partial y \partial k} = -\frac{\partial \lambda}{\partial k} > 0.$$

The optimal policy φ is therefore increasing (see Amir [1]). Obviously, if $\{k_t^*\}_{t=0}^\infty$ is the optimal path, then for any t , $k_{t+1}^* = \varphi(k_t^*)$.

□

5.5 PROOF OF PROPOSITION 3.1

i) We have that for all $t \geq 0$,

$$\begin{aligned} k_t^* &> 0, c_t^* > 0, 0 < l_t^* < L, \\ c_t^* + k_{t+1}^* &= F(k_t^*, l_t^*) + (1 - \delta)k_t^*, \\ ac_t^* + l_t^* &= L. \end{aligned}$$

Consider $0 < c < c_t^*$. Then

$$\begin{aligned} c + k_{t+1}^* &< F(k_t^*, l_t^*) + (1 - \delta)k_t^*, \\ ac + l_t^* &< L. \end{aligned}$$

From Lemma 3.2, $(k_t^*, k_{t+1}^*) \in \text{int graph } \Gamma$.

ii) Since $(k_t^*, k_{t+1}^*) \in \text{int}(\text{graph } \Gamma)$, By Lemma 3.4, the indirect utility function is differentiable at (k_t^*, k_{t+1}^*) . Therefore the Euler equation holds.

□

5.6 PROOF OF PROPOSITION 3.2

Fix $k_0 > 0$. Assume that the optimal path $\{k_t^*\}_{t=0}^\infty$ converges to zero. By the monotonicity of the optimal capitals path, this is possible only if this path is strictly decreasing.

First, we prove that for T large enough, we obtain an interior solution. This allows us to work with *Euler equations*.

From the Inada condition, for any k small enough we have

$$F_1(k, L) + (1 - \delta) > \frac{1}{\beta}.$$

This implies that for k small enough

$$F(k, L) + (1 - \delta)k > k.$$

Hence, for T large enough, for any $t \geq T$ we have

$$0 < k_{t+1}^* < F(k_t^*, L) + (1 - \delta)k_t^*,$$

which implies that $k_{t+1}^* \in \text{int}\Gamma(k_t^*)$ for any $t \geq T$. Let $\{c_t^*\}_{t=0}^\infty$ and $\{l_t^*\}_{t=0}^\infty$ be the corresponding consumption and labour sequences. Recall that from Lemma 3.1,

$$k_{t+1}^* = F(k_t^*, L) + (1 - \delta)k_t^* + \frac{l_t^* - L}{a}.$$

Since $\{k_t^*\}_{t=0}^\infty$ converges to zero, this implies $\{l_t^*\}_{t=0}^\infty$ converges to L .

Denote by V_1 and V_2 respectively the partial derivatives corresponding to the first and the second arguments. Let $\lambda_t = -V_2(k_t^*, k_{t+1}^*)$. By Lemma 3.4, we have

$$V_1(k_t^*, k_{t+1}^*) = \lambda_t (F_1(k_t^*, l_t^*) + (1 - \delta)).$$

Consider the Euler equations for $t \geq T$:

$$V_2(k_t^*, k_{t+1}^*) + \beta V_1(k_{t+1}^*, k_{t+2}^*) = 0,$$

which is equivalent to

$$-\lambda_t + \beta \lambda_{t+1} [F_1(k_{t+1}^*, l_{t+1}^*) + (1 - \delta)] = 0.$$

Hence, for any $t \geq T$,

$$\beta F_1(k_{t+1}^*, l_{t+1}^*) + (1 - \delta) = \frac{\lambda_t}{\lambda_{t+1}}.$$

By the Inada condition, for t large enough we have

$$\lambda_t > \lambda_{t+1}.$$

On the other hand, by the envelope theorem, for t large enough, we have

$$\begin{aligned} v'(k_{t+1}^*) &= V_1(k_{t+1}^*, k_{t+2}^*) \\ &= -\frac{1}{\beta} V_2(k_t^*, k_{t+1}^*) \\ &= \frac{\lambda_t}{\beta}. \end{aligned}$$

Using the same arguments, we obtain

$$v'(k_{t+2}^*) = \frac{\lambda_{t+1}}{\beta}.$$

Hence $v'(k_{t+1}^*) > v'(k_{t+2}^*)$. By the concavity of the value function v , this implies $k_{t+1}^* < k_{t+2}^*$: a contradiction. The optimal path $\{k_t^*\}_{t=0}^\infty$ does not converge to zero. □

5.7 PROOF OF PROPOSITION 3.4

First, we prove that for any $l > 0$,

$$\lim_{k \rightarrow \infty} F_1(k, l) = 0.$$

Indeed, by the concavity of F , for $k \geq 1$, we have

$$\begin{aligned} \frac{1}{k} F(k, l) + \left(1 - \frac{1}{k}\right) F(0, 0) &\leq F\left(\frac{1}{k} \times k + \left(1 - \frac{1}{k}\right) \times 0, \frac{1}{k} \times l + \left(1 - \frac{1}{k}\right) \times 0\right) \\ &= F\left(1, \frac{l}{k}\right). \end{aligned}$$

This implies

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{F(k, l)}{k} &\leq \lim_{k \rightarrow \infty} F\left(1, \frac{l}{k}\right) \\ &= F(1, 0) \\ &= 0. \end{aligned}$$

By the concavity of F , for any $k \geq 0$, we have $k \times F_1(k, l) \leq F(k, l)$. Hence

$$\lim_{k \rightarrow \infty} F_1(k, l) = 0.$$

Let c^* and l^* be the solutions that solve (3.1) to (3.4). We prove the following claim. The value of labour l^* is the unique solution to

$$\frac{L-l}{a} + k = F(k, l) + (1-\delta)k. \tag{5.15}$$

It is easy to verify that there exists a solution \tilde{l} to equation (5.15). Let

$$\tilde{c} = \frac{L - \tilde{l}}{a}.$$

Assume that $l^* > \tilde{l}$. Since $ac^* + l^* = a\tilde{c} + \tilde{l}$, we have $c^* < \tilde{c}$. Hence, $u(\tilde{c}) > u(c^*)$, a contradiction.

Consider the case $l^* < \tilde{l}$. We have $c^* > \tilde{c}$. Moreover,

$$\begin{aligned} c^* &= F(k, l^*) - \delta k \\ &< F(k, \tilde{l}) - \delta k \\ &= \tilde{c}, \end{aligned}$$

a contradiction.

Hence, $l^* = \tilde{l}$. The claim is proven.

Now, we prove that if a steady state exists, it is unique. For $k > 0$, let $\tilde{l}(k)$ be the solution to

$$F_1(k, l) = \frac{1}{\beta} - 1 + \delta.$$

By the Implicit theorem, the function \tilde{l} is differentiable and

$$\tilde{l}'(k) = -\frac{F_{11}(k, \tilde{l}(k))}{F_{12}(k, \tilde{l}(k))}.$$

Since $F_{11}(k, \tilde{l}(k)) \leq 0$ and $F_{12}(k, \tilde{l}(k)) \geq 0$, the function \tilde{l} is increasing with respect to k .

By the Claim, the capital stock $k > 0$ is a steady state if and only if it is a solution to

$$\frac{L - \tilde{l}(k)}{a} + k = F(k, \tilde{l}(k)) + (1 - \delta)k,$$

that is equivalent to

$$H(k) = 0,$$

where

$$H(k) = F(k, \tilde{l}(k)) + (1 - \delta)k - \frac{L - \tilde{l}(k)}{a} - k.$$

We have

$$\begin{aligned} H'(k) &= F_1(k, \tilde{l}(k)) + F_2(k, \tilde{l}(k))\tilde{l}'(k) - \delta + \tilde{l}'(k) \\ &= \frac{1}{\beta} - 1 + \delta + F_2(k, \tilde{l}(k))\tilde{l}'(k) - \delta + \tilde{l}'(k) \\ &= \frac{1}{\beta} - 1 + F_2(k, \tilde{l}(k))\tilde{l}'(k) + \tilde{l}'(k) \\ &> 0, \end{aligned}$$

since $\frac{1}{\beta} - 1$, $F_2(k, \tilde{l}(k))$, $\tilde{l}'(k)$ are positive.

Hence, the function H is strictly increasing. This implies that if a steady state exists, it is unique.

To complete the proof, we must prove that a steady states exists, and any non trivial optimal capitals path converges monotonically to this value.

Fix some $k_0 > 0$, let $\{k_t^*\}_{t=0}^\infty$ be the optimal capitals path beginning from k_0 , $\{c_t^*\}_{t=0}^\infty$ and $\{l_t^*\}_{t=0}^\infty$ be the corresponding sequences of consumption and labour.

We prove that $\{k_t^*\}_{t=0}^\infty$ is bounded from above. Assume the contrary, this implies that this sequence is strictly increasing and converges to infinity.

By Lemma 3.4, for any $t \geq 0$, we have

$$\begin{aligned} \frac{u'(c_t^*)}{1 + aF_2(k_t^*, l_t^*)} &= -V_2(k_t^*, k_{t+1}^*) \\ &= \beta V_1(k_{t+1}^*, k_{t+2}^*) \\ &= \beta v'(k_{t+1}^*) \\ &< \beta v'(k_t^*), \end{aligned}$$

since v' is strictly decreasing with respect to k .

We also have

$$\begin{aligned} v'(k_t^*) &= V_1(k_t^*, k_{t+1}^*) \\ &= \frac{u'(c_t^*)}{1 + aF_2(k_t^*, l_t^*)} (F_1(k_t^*, l_t^*) + 1 - \delta). \end{aligned}$$

This implies

$$F_1(k_t^*, l_t^*) + 1 - \delta > \frac{1}{\beta},$$

for any $t \geq 0$.

Let t converges to infinity,

$$\begin{aligned} \lim_{t \rightarrow \infty} F_1(k_t^*, L) &\geq \lim_{t \rightarrow \infty} F_1(k_t^*, l_t^*) \\ &\geq \frac{1}{\beta} - 1 + \delta \\ &> 0, \end{aligned}$$

a contradiction with the property that $\lim_{k \rightarrow \infty} F_1(k, L) = 0$.

Hence, for any $k_0 > 0$, the optimal path beginning from k_0 is bounded from above. Moreover, by Proposition 3.2, this path does not converge to zero. The monotonicity implies that $\{k_t^*\}_{t=0}^\infty$ converges to some k^s that is strictly positive. In other words a steady states exists.

The uniqueness of the steady state ensures that for $0 < k_0 < k^s$, the optimal capitals path beginning from k_0 is increasing and converges to k^s . Otherwise we obtain either the existence of another non trivial steady state, or the extinction of the economy. With a similar argument, for $k_0 > k^s$, the optimal capitals path starting from k_0 is decreasing and converges to k^s .

REFERENCES

- [1] Amir, R. (1996): Sensitivity analysis of multisector optimal of economic dynamics, *Journal of Mathematical Economics* **25**, 123-141.
- [2] Becker, G. S. (1965): A Theory of the Allocation of Time, *The Economic Journal* **75**, no. 299 , 493-517.
- [3] Benhabib, J. & K. Nishimura (1985): Competitive equilibrium cycles, *Journal of Economic Theory*, **35**, 284-306.
- [4] Fan, S, Y. Pang, & P. Pestieau (2020): A model of the optimal allocation of government expenditures, *Journal of Public Economic Theory*, <https://doi.org/10.1111/jpet.12416>.
- [5] Ghosh, S., T. R. Barman & M. R. Gupta (2020): Are short-term effects of pollution important for growth and optimal fiscal policy? *Journal of Public Economic Theory*, <https://doi.org/10.1111/jpet.12438>.
- [6] Gossen, H. H. (1983): *The Laws of Human Relations and the Rules of Human Action Derived Therefrom*, Cambridge MA: MIT Press. English translation by R.C. Bliss of H.H. Gossen, (1854) *Entwicklung der Gesetze des Menschlichen Verkehrs und der daraus fließenden Regeln für menschliches Handeln*, Braunschweig: Vieweg und Sohn.
- [7] Kamihigashi, T. & S. Roy (2007): A nonsmooth, nonconvex model of optimal growth, *Journal of Economic Theory* **132**, 435-460.
- [8] Le Van, C. & R-A. Dana (2003): *Dynamic Programming in Economics*, Kluwer Academic Publishers.
- [9] Le Van, C. & L. Morhaim (2002): Optimal Growth Models with Bounded or Unbounded Returns: A Unifying Approach, *Journal of Economic Theory*, **105**, no 1, 158-187.

- [10] Le Van, C., A.N. Nguyen, N-M. Nguyen & M. Simioni (2018): Growth Strategy with Social Capital, Human Capital and Physical Capital-Theory and Evidence: the Case of Vietnam, *Journal of Public Economic Theory*, **20**, No 5, 768-787.
- [11] Le Van, C., P. Nguyen-Van, A. Barbier-Gauchard, & D.-A. Le (2019): Government expenditure, external and domestic public debt, and economic growth, *Journal of Public Economic Theory*, **21**, 116-134.
- [12] Le Van, C. & Y. Vailakis (2004): Existence of Competitive Equilibrium in a Single-Sector Growth Model with Elastic labour, *Working paper CES*.
- [13] Le Van, C., B. Tran-Nam, N. S. Pham & T. D. H. Nguyen (2018): A General Equilibrium Model in Which Consumption Takes Time, in *Recent Developments in Normative Trade Theory and Welfare Economics*, B. Tran-Nam, Ed., Springer.
- [14] Tran-Nam, B. & N. S. Pham (2014): A Simple General Equilibrium Model Incorporating the Assumption that Consumption Takes Time, *Working paper*.
- [15] Stokey, N. & R. Lucas (with Prescott) (1989): *Recursive methods in economic dynamics*, Harvard University Press.