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## *On the “reduction to identicals” in Leibniz*

David Rabouin

SPHERE, UMR 7219, CNRS-Univ. Paris Cité & ERC Philiumm (Adg n°101020985)

### **Introduction**

In his second letter to Clarke, Leibniz famously declared that the principle of identity  $A$  is  $A$  is “the great foundation of mathematics”. Moreover, he continued, this principle “is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles”. This intriguing stance has given rise to the belief that Leibniz was a kind of “logician”. According to this view, the philosopher was convinced that all of mathematics could be reduced to some logical principles, the basis of all of them being the principle of identity (or the law of non-contradiction, since he conflates the two in the letter to Clarke). This interpretation, however, was objected to very early, as it is apparently not consistent with Leibniz’s mathematical practice and in particular, the way in which the philosopher put great emphasis on the inventive role of various systems of symbols (such as those he devised for his differential calculus and his *analysis situs*). In his mathematics, one does not seem to find what the logicist would expect, that is to say a practice relying on purely logical derivations. Quite to the contrary, one finds symbolic practices which do not always seem to pay much attention to “rigor”. As Michel Serfati puts it in the title of one of his papers, “symbolic inventiveness” often seems to go hand in hand in Leibniz with “irrationalist practices”. This opposition between two views on Leibniz’s mathematics, the one Dietrich Mahnke called “conceptual realism” (beginning with Russell and Couturat) and the one he called “rational idealism” (beginning with Cassirer’s – and one might add here: Brunschvicg) has had a deep influence on Leibnizian scholarship until today. It is my aim in this paper to confront this tension and to sketch a resolution. In the first section of the paper, I will recall some of the features of the “logician” picture to underline its plausibility, before indicating, in the second section, some of its limitations. In the third and fourth sections, I will consider one aspect of Leibniz’s mathematical practice, which has not yet received enough attention and which is based on logical analysis (in the sense of the “analysis of notions and truths”). My main claim will be that the missing link between the logicist picture and the picture based on mathematical practice is given by a form of logical analysis seen *as* a mathematical practice and based on the “reduction to identicals”. To put it more briefly, logical analysis was not seen by Leibniz as a foundation, but as an important ingredient of a new mathematical *ars inveniendi*.

### **I. The logicist picture**

Let me first recall the whole passage from the letter to Clarke which gives the most prominent example of Leibniz “logician” stance:

“The great foundation of mathematics is the *principle of contradiction or identity*, that is, that a proposition cannot be true and false at the same time, and that therefore  $A$  is  $A$  and cannot be not  $A$ . This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles. But in order to proceed from mathematics to natural philosophy, another principle is required, as I have observed in my Theodicy; I mean the *principle of sufficient reason*, namely, that nothing happens without a reason why it should be so rather than otherwise.” (To Clarke, GP VII, 355)<sup>1</sup>

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<sup>1</sup> Transl. from D. Garber and R. Ariew (ed.): *G.W. Leibniz. Philosophical Essays*, Indianapolis 1989, p. 321.

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Although this passage is quite particular, in its very condensed form, and although one can find other places in which Leibniz puts to the fore the role of other principles in mathematics, it certainly concurs with the exposition of the kind of truths given in the *Monadologie*:

“31. Our reasonings are based on *two great principles, that of contradiction*, in virtue of which we judge that which involves a contradiction to be false, and that which is opposed or contradictory to the false to be true.  
32. And *that of sufficient reason* [...].  
33. There are also two kinds of *truths*, those of *reasoning* and those of *fact*. The truths of reasoning are necessary and their opposite is impossible; the truths of fact are contingent, and their opposite is possible. When a truth is necessary, its reason can be found by analysis, resolving it into simpler ideas and simpler truths until we reach the primitives.  
34. This is how the speculative *theorems* and practical *canons* of mathematicians are reduced by analysis to *definitions*, *axioms* and *postulates*.  
35. And there are, finally, *simple ideas*, whose definition cannot be given. There are also axioms and postulates, in brief, *primitive principles*, which cannot be proved and which need no proof. And these are *identical propositions*, whose opposite contains an explicit contradiction.” (*Monadology* § 33-34)<sup>2</sup>

Moreover, the *Nouveaux Essais* presents us with what seems to be a beautiful example of such a logical analysis of truths by resolution into primitives:

“It is not a fully immediate truth that two and two are four. Assuming that *four* signifies three and one. We can thus demonstrate it, and here is how.

**Definitions:**

- (1) *Two* is one and one
- (2) *Three* is two and one
- (3) *Four* is three and one

**Axiom:** Putting equals in place of equals, equality remains.

**Demonstration:**

2 and 2 is 2 and 1 and 1 (per def. 1)	2 + 2
2 and 1 and 1 is 3 and 1 (def. 2)	(2 + 1) + 1
3 and 1 is 4 (def. 3)	3 + 1

Therefore (by the axiom)

2 and 2 is 4, which is what was to be demonstrated.”<sup>3</sup>

Unsurprisingly, it is this very proof which Frege recalled at the beginning of his *Grundlagen der Arithmetick* (1884) in order to criticize the view of those who, following Kant, wanted to base mathematics on intuition (rather than on logical principles and derivations)<sup>4</sup>. As expressed by Frege, not only do we have here an example of the way in which a mathematical truth can be derived from definitions and an “identical axiom”, but this axiom too can be “transformed into a definition” by relying on the logical definition of identity: “things are the same as each other, of which one can be substituted for the other without loss of truth”.<sup>5</sup>

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<sup>2</sup> Transl. from Ariew and Garber (ed.): *G.W. Leibniz. Philosophical Essays*, p. 217. See also: “Necessary truths are those that can be demonstrated through an analysis of terms, so that in the end they become identities (*identicas*), just as in Algebra an equation expressing an identity ultimately results from the substitution of values. That is, necessary truths depend upon the principle of contradiction.” (*On Contingency*, transl. Ariew and Garber, p. 28)

<sup>3</sup> NEEH IV, chap. 7, § 10; A VI, 6, 413-414. Unless otherwise stated, the translations are mine. Leibniz indicates in the calculation the various ways of regrouping terms by the use of curly braces. For typographical reasons, I was not able to reproduce them although they are of tremendous importance, see Michel Fichant: “Les axiomes de l’identité et la démonstration des formules arithmétiques :  $2+2=4$ ”, in: *Revue Internationale de Philosophie* 48.188 (1994), p. 85-119.

<sup>4</sup> I quote the English translation by J. L. Austin, G. Frege: *The Foundations of Arithmetic: a logico-mathematical enquiry into the concept of number* (1884), London 1950.

<sup>5</sup> *Eadem sunt quorum unum potest substitui alteri salva veritate*. Frege quotes this definition from the *Non inelegans specimen demonstrandi in abstractis* (1687), which he knows through Erdmann’s edition and which is now edited in A VI, 4, 845-855.

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In a subsequent chapter in which he deals with what would later be called “definitions by abstraction” (with the famous example of the identity of direction defined through the equivalence relation of parallelism), Frege remarks that such a definition rests on a form of logical equality (*Gleichheit*) and that the laws governing this equality, as analytic truths, should be derivable “from the concept alone”. He recalls at this occasion the definition given in the *Non inelegans specimen demonstrandi in abstractis*. This, however, raises a natural question: is it a matter of defining a relation of an equality (*Gleich*) or, as Leibniz phrased it, an absolute sameness (*Dasselbe*)? To this question, Frege answers that it does not matter because an absolute identity and an equality would differ only in the sense that the first is an agreement “in all respects”. In this sense, as soon as one has chosen a given point of view, it makes no difference to say that the two entities are “the same” or “equal”. A logical equality “in all respects” holds *a fortiori* according to this or that respect (for example from a quantitative point of view). In fact, Leibniz had already made a similar comment just after the proof given in the *Nouveaux Essais*:

“Instead of saying that 2 and 2, is 2 and 1 and 1, I could have put that 2 and 2 is equal to 2 and 1 and 1, and the same with the others. But it can be understood everywhere, since we did it already, on the basis of another axiom which holds that a thing is equal to itself, or that what is the same is equal.” (NEEH IV, chap. 7, § 10)

Frege expresses this general situation by saying that all the laws of equality are *comprised* in the general law of substitution (meaning substitution *salva veritate*). This gave rise to an idea, explicitly put forward by Russell and Carnap, according to which absolute identity could be defined by substitution “in all respects”, that is to say as agreement according to every possible predicate – a definition which they thought was expressed by Leibniz as the “principle of identity of indiscernibles”.<sup>6</sup>

One thing which is quite typical of the logicist picture is to see Logic as coming before mathematics, as its condition and its foundation<sup>7</sup>. Another crucial stance is that through this foundation, all of mathematics could be derived from logical truths by specifying various types of relations between entities, provided one can engender in this way the basic mathematical objects (typically natural numbers) through equivalence relations. This last move was considered to be what was behind the proof of “ $2 + 2 = 4$ ” given in the *Nouveau Essais* – although Frege thought that Leibniz did not take things far enough since he still considered “unity” as a primitive notion and did not see that it was itself definable from the concept of zero (itself definable from the pure concept of logical equality).<sup>8</sup>

Relying on the various passages mentioned above, it was quite natural to consider that Leibniz was the grandfather of the logicist position or, as Russell puts it in the clearest way in the very first pages of his *Principles of mathematics* that:

“The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself.

The general doctrine that all mathematics is deduction by logical principles from logical principles was strongly advocated by Leibniz, who urged constantly that axioms ought to be proved and that all except a few fundamental notions ought to be defined”. (Bertrand Russell: *Principles of Mathematics*, Cambridge 1903, *Preface*)

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<sup>6</sup> See R. Carnap *Abriss der Logistik*, Berlin 1929, § 7, which contains a historical note referring to the use of this principle in Russell, Hilbert, Ramsey and criticism from Wittgenstein.

<sup>7</sup> This point was put forward very clearly by Russell against Couturat, who conceived of logic in the tradition of Universal Algebra and described logical calculus as a particular type of mathematical structures (“Recent Work on The Philosophy of Leibniz “, In: *Mind* 12 (46) (1903), p. 186-187).

<sup>8</sup> G. Frege: *The Foundations of Arithmetic*, §55, p. 67.

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Since then, at least in this tradition, “Gottfried Leibniz is today almost universally credited as being the first logicist”.<sup>9</sup>

My aim, in this paper, is not to scrutinize this picture in every detail. However, I wanted to bring to mind the way in which it relies on Leibniz explicit declarations to emphasize that it has a lot of plausibility to it. In fact, the objections that were raised by authors such as Cassirer and Brunschvicg were not that Leibniz did not endorse such a view, but that it was for him an ideal, a dream he never succeeded to achieve.<sup>10</sup> Still today, the most common objection to the logicist picture is that it does not fit with what we find in Leibniz’s mathematical practice. What we have is not the completed project of a *characteristica universalis*, but a non-unified plurality of symbolic systems testifying to Leibniz’s great inventiveness in mathematics. However, this answer remains weak from a conceptual point of view. All that it says, and which should not surprise us much, is that Leibniz’s practice was not in perfect accordance with his ideal. This does not undermine in any way the nature of the mathematical truths attached to this ideal. One could add that mathematical truths are certainly not limited, in Leibniz’s eyes, to the way in which our finite human minds struggle to find them. They are supposed to be already there in God’s mind as “eternal truths”. In this “region of eternal truths”, it is difficult to see how they could be ordered, except by logical relations or to claim that they are contaminated by any kind of “irrationalism”.<sup>11</sup> The logicist could thus easily reply that the appeal to Leibniz’s practice, although very interesting from a historical point of view, is of no interest when it comes to understanding the way in which Leibniz, as a philosopher, saw the architecture of mathematical truths.

This is why I think it is important to take the “logicist” picture seriously and to bring to the fore other kinds of limitations to which it is subject. For lack of space, I will just quickly mention two of them: the first is related to the use of the “principle of the identity of indiscernibles”, the second to the chronology of Leibniz’s various achievements in mathematics and in logic.

## II. Some limitations of the logicist picture

As I just mentioned, the logicist picture is based on the idea that basic mathematical relations can be related to an absolute identity given in logic, which is characterized on the one hand by what is now called “Leibniz’s law” (two terms are logically equal if and only if they can be substituted one for the other by preserving the truth) and on the other hand by the “principle of identity of indiscernibles” (two things are absolutely identical or “indiscernible” if and only if they share all of their properties/predicates).<sup>12</sup> A very straightforward difficulty, which was raised right away by some authors, is that these two principles are not on the same level: the first is a definition on which logical calculi are based, the second is a metaphysical principle.<sup>13</sup> The failure to make this distinction has had tremendous (and unfortunate) consequences on the discussion surrounding the identity of indiscernibles in Leibniz and more generally in the philosophical discussions deriving from Russell and Carnap.

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<sup>9</sup> Martin Godwyn and Andrew Irvine: “Russell’s Logicism”, in: N. Griffin (ed.), *The Cambridge Companion to Bertrand Russell*, Cambridge 2003, p. 173.

<sup>10</sup> See Léon Brunschvicg: *Les Étapes de la philosophie mathématique*, Paris 1912, p. 204.

<sup>11</sup> To borrow an expression from Michel Serfati: “Symbolic Inventiveness and ‘Irrationalist’ Practices in Leibniz’s Mathematics”, in: Marcelo Dascal (ed.): *What Kind of Rationalist ?*, Dordrecht 2008.

<sup>12</sup> There are variations in the contemporary literature on what is called the “principle of identity of indiscernibles”. Some authors require it to be a logical equivalence, whereas others just require an implication (the converse being sometimes called the “indiscernibility of identicals”). I followed Carnap’s initial formulation, but these variations are of no importance for what I will say about it.

<sup>13</sup> See already K. Grelling: “Identitas indiscernibilium”, in: *Erkenntnis* 6-1 (1936), p. 252-259. More recently, Ignacio Angelelli: “On identity and interchangeability in Leibniz and Frege”, in: *Notre Dame Journal of Formal Logic* 8.1-2 (1967).

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First, let me emphasize that this principle is in Leibniz a *negative* thesis.<sup>14</sup> It claims that there cannot be two strictly identical individuals in nature (or two individuals differing only *solo numero*). To my knowledge, it is never used as a positive characterization of identity. Moreover, it is never mentioned in a context in which abstract entities would enter, but deals with substances and their appearances. By contrast, the definition of logical equality through substitution *salva veritate* is introduced *only* in the context of abstract objects (as recalled by the very title of the piece quoted by Frege which is a *specimen demonstrandi in abstractis*). Accordingly, it is not a matter of simply confusing two formulations, but of ignoring their range of application.

This distinction is expressed in the clearest way by Leibniz:

“It follows also that *there cannot be two individual things in nature which differ only numerically*. For surely it must be possible to give a reason why they are different, and this must be sought in some differences within themselves [...]. Never are two eggs, two leaves, or two blades of grass in a garden to be found exactly similar to each other. So perfect similarity occurs only in incomplete and abstract concepts, where matters are conceived, not in their totality but according to a certain single viewpoint, as when we consider only figures and neglect the figured matter.” (*Principia logico-metaphysica* 1689, A VI, 4, 1645)<sup>15</sup>

So perfect identity certainly makes sense in mathematics (and in physics), but it makes sense *by difference* from what happens with real individuals. Moreover, it makes sense precisely because things are not considered in mathematics under *all* of their properties. They are “incomplete” concepts. The post-Fregean way cannot grasp this distinction precisely because it assumes that abstract predicates directly qualify any kind of objects, including real individuals<sup>16</sup> – a view which seems to go directly against what Leibniz said about predicates “taken in abstraction”:

“Thus, taken in abstraction from the subject, the quality of being a king which belongs to Alexander the Great is not determinate enough to constitute an individual and does not include the other qualities of the same subject, nor does it include everything that the notion of this prince includes. On the other hand, God, seeing Alexander's individual notion or haecceity, sees in it at the same time the basis and reason for all the predicates which can be said truly of him, for example, that he vanquished Darius and Porus.” (*Discourse of Metaphysics* §7)<sup>17</sup>

Here one should not believe that what we are lacking is the complete list of *abstract* predicates. What Leibniz claims is that the real predicate, the one which applies to Alexander, has to include specificities related to the presence of other properties in the same subject.<sup>18</sup> The idea of seeing “all the predicates” makes perfect sense. However, it makes sense precisely because God does not grasp abstract concepts, but “the basis and reason for all the predicates” (in one and the same subject), that is

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<sup>14</sup> For a very useful list of the occurrences of the principle in Leibniz, see Rodriguez-Pereyra: *Leibniz's Principle of Identity of Indiscernibles*, Oxford 2014, p. 15-20.

<sup>15</sup> English translation from L. Loemker: *G. W. Leibniz. Philosophical Papers and Letters*, Dordrecht 1989 (2<sup>nd</sup> edition), p. 268.

<sup>16</sup> There is a difficulty in the way in which Rodriguez-Pereyra conducts his discussion on the principle of the indiscernibles. In fact, he mentions one text in which Leibniz identifies this principle with the sharing of all properties (p. 23). But the fact that one finds only one text (in the entire corpus) going in this direction is already quite significant. Moreover, the passage mentioned (A VI, 4, 306) deals with another principle, that of individuation, and makes a different claim (that there are as many singular substances as they are combinations of compatible attributes). The analysis preceding this statement in the text makes it clear that Leibniz is talking of complete notions and that in this context abstract properties are just “ways of talking” of singular things. I'll come back to this issue in [note 18](#) below.

<sup>17</sup> Transl. in Garber and Ariew 41.

<sup>18</sup> This is why the passage quoted by Rodriguez-Pereyra should not be read too quickly as characterizing the principle of identity of indiscernibles. In fact, it presupposes it. *Provided the principle is true*, one can say something of one thing which cannot be said of another thing and this holds already at the level of abstract predicates (this argument is made explicitly by Leibniz in A VI, 4, 553). The fiction of abstract predicates is a good instrument for *testing* individuation, which, by no means, entails that one can transform this test into a positive characterization for identity.

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to say the complete notion. As Leibniz explains in another text, the wisdom of Socrates cannot be exactly the same as the wisdom of Seneca, because the latter was rich, contrary to the former and that to be wise when one is rich is not the same as to be wise when one is not rich. But it is not a matter of simply adding abstract properties one with the other (precisely because we are lacking an identity of predicates to begin with). The way in which Seneca was rich was also different from other ways of being rich (for instance, Cresus's), and so forth and so on, with any other predicate. What we do by using abstract properties is simply to use a fiction, similar to what mathematicians do when introducing infinitesimals.<sup>19</sup> Accordingly, there is no abstract predicate "applying" to individuals and acquiring in this mysterious application some specificities. There are only predicates attached to complete notion. However, they can be taken in abstraction outside of these subjects, exactly in the same way that there are natural bodies, which can be taken in abstraction in mathematics and in the natural sciences (which certainly does not entail that there are abstract bodies). The important fact to keep in mind is that even when Leibniz associates his definition of coincidence with the sharing of properties/predicates, as in his logical calculi, it occurs precisely in the context of *abstract* entities. Indeed, it is only in this context that the notion of identity of predicates, which grounds the identity of terms in logical calculi, makes sense.

Now, since we are interested in mathematics, in which one deals with incomplete notions, the preceding remark does not yet exclude a picture in which the logical calculi, based on the coincidence *salva veritate*, would be the general framework in which Leibniz conceived of his idea of a reduction of all of mathematics to "identicals". I will now show that this cannot be the case with a very simple argument.

This comes from our better knowledge of the chronology of Leibniz's achievements – a knowledge which was not accessible to scholars from the beginning of the last century. In particular, we now have a much clearer view on the way in which Leibniz developed his logical calculi. A first phase occurred at the beginning of the Hanover period, and particularly in 1679 with calculi based on the coding of notions by numbers. The overall framework was still at the time that of syllogistic and the logical coincidence was not isolated as a central relation on which operations could be built. Things changed in the middle of the 1680s with the realization of two important enterprises: the *Generales Inquisitiones de analysis notionum et veritatum* on the one side and the various sketches of the "calculus of real addition" on the other. Now, the fact that mathematical truths can be reduced to identities cannot be a consequence of these new calculi for a simple reason: this stance appears long before this period in Leibniz's thought and in a completely different context.

In fact, as Leibniz recalls in the *Historia et Origo calculi differentialis*, "while still a boy, when studying logic he perceived that the ultimate analysis of truths that depended on reasoning reduced to two things, definitions and identical truths, and that these alone of the essentials were primitive and undemonstrable". Or, as he puts it in a variant passage:

"He marveled very much at the power of identical truths, for they were generally considered to be useless and nugatory. But later he considered that the whole of arithmetic and geometry arose from identical truths, and in general that all undemonstrable truths depending on reasoning were identical, and that these combined with definitions yield identical truths. He gave as an elegant example of this analysis a proof of the theorem, The whole is greater than its part."<sup>20</sup>

We know that this late narrative is not a sheer *a posteriori* reconstruction for a very good reason: we possess texts prior to the arrival in Paris in which this idea is clearly put to the fore, and on the very example mentioned in the *Historia et origo*. The main testimony is the text entitled *Demonstratio propositionum primarum* (1671-1672), in which Leibniz undertakes to prove the Euclidean axiom "the

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<sup>19</sup> The comparison is from Leibniz in A VI, 4, 991.

<sup>20</sup> English translation from J.-M. Child, *G. W. Leibniz. The Early Mathematical Manuscripts of Leibniz*, Chicago London 1920, p. 29.

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whole is greater than the part". As can be seen in the proof, its source has nothing to do with a new logical calculus since it is taken from Hobbes and is written in the form of a syllogism:

*“Demonstration*

That which has a part equal to another whole, is greater. By def. of the Greater

The part of the whole *cde*, that is to say *de*, is equal to the whole *de* (that is to say to itself)

Hence *cde* is greater than *de*, the whole than the part

What was to be demonstrated (A VI, 2, 482-483)

Just after the proof, Leibniz comments: “Here is a beautiful example of first propositions, i.e. definitions and propositions close to identicals, demonstrated by a unique syllogism of the first figure, whose major premise is a definition, whose minor premise is an identical proposition and the conclusion the given theorem”.

Another interesting document from this early period is the letter to Gallois from the end of 1672 in which Leibniz intended to present what he would later call his “harmonic triangle”. Here one finds more algebraic formulations of how to prove Euclidean “common notions”, such as *eidem tertio aequalia esse inter se aequalia*. This shows that the position derived from Hobbes was already generalized to other axioms (Leibniz gives several other examples in the letter). But the most important feature of this demonstration is that it is based on the substitution of equals, itself based on the following definition: *aequalia sunt, quorum eadem est quantitas seu quorum alterum alteri substitui potest salva quantitate*.<sup>21</sup> This was seven years before the earliest known occurrence of the definition of logical coincidence through substitution *salva veritate* (A VI, 4, 141 and 154).

### III. First appearances of “identicals” in Leibniz’s thought

The last example provides us with a very interesting entry toward the role played by the substitution of identicals in Leibniz. Indeed, it is clear that this process appears in Leibniz first in mathematics, as a mechanism regulating algebraic proofs, *before* being used in logic. Over time, as I will now show, Leibniz realized that this mechanism was at the heart of any reasoning and that this could give a very useful way to reorganize the conceptual architecture of mathematics. Then, and only then, did he have the idea of transferring it to logic in order to build efficient logical calculi (on the model of algebraic calculus).

The first texts we possess on this issue date from the end of the Parisian stay. Since before going to Paris Leibniz had already conceived the idea of the role of identicals as supporting demonstrations for what was taken as axioms in a Euclidean context, one can wonder why at the time he did not develop a program based on this idea. This is strongly related to an important evolution in Leibniz’s views on basic notions, which I shall briefly recall now<sup>22</sup>.

As is well known, Leibniz first thought, already at the time of the *De Arte Combinatoria* (1666), that one could build some kind of a “universal script” based on an alphabet of human notions and ruled by simple combinations. Gradually, however, he came to see the difficulty entailed with his idea of an alphabet of simple notions. This was a significant threat for the initial project. But Leibniz astutely recognized that the mechanism of substitution and reduction to identicals was a way out of a possible collapse of the whole enterprise:

“if it is true that there is a perfect demonstration, i.e. one which leaves nothing unproven, then it necessarily follows that some elements of thought must exist, since the demonstration will be perfect *only when everything is analysed*. But I realize now that this is false and that a demonstration is perfect as soon as one

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<sup>21</sup> A III, 18-20.

<sup>22</sup> For more details, see D. Rabouin: “The Difficulty of Being Simple. Interactions between Mathematics and Philosophy in Leibniz’s Analysis of Notions”, in: Norma B. Goethe, Philip Beeley and David Rabouin (ed.): *G.W. Leibniz, Interrelations between Mathematics and Philosophy*, Dordrecht 2015.p. 49-72.

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can reach identicals, which can happen *even though everything is not analysed*. For even notions which are not absolutely simple (like the parabola, or the ternary) can be stated from one another" (A VI, 3, 504. My emphasis)

What is very striking, and of primary importance for what I want to show, is that this general idea was immediately put *into practice*. In the text that the editors entitled *De magnitudine* (1676), Leibniz states that "after many attempts" (*multa tentando*), he seemed to have found "the most suitable concept of magnitude, which is this: quantity or magnitude is that according to which some thing (which is called 'so big') is said to be capable of being congruent (that is, capable of being brought within the same boundaries)"<sup>23</sup>. This is one of the first definitions by abstraction in his work. The abstract concept of "magnitude" or "quantity" can be introduced as a way to talk about an equivalence relation between objects (since two things would be said to have the same magnitude if and only if they are congruent).<sup>24</sup> In a deleted passage from his *Dissertatio exoterica* from the same year, Leibniz contrasts the branch of mathematics which deals with magnitude to another domain dealing with Forms or similitude.<sup>25</sup> Indeed, in the same way that one can introduce the abstract concept of magnitude through the identity condition between congruent entities, one can introduce the abstract concept of form through the identity condition between similar entities (geometrical shapes or even algebraic formulae).<sup>26</sup> In a text on the *characteristica geometrica* (later to be called *analysis situs*) from January 1677, Leibniz formulates the idea that things are similar when they cannot be discerned *singulatim*, a definition he would repeat again and again until the end of his life. In the autumn of the same year, here is how he presents the importance of this characterization to Gallois (in a letter which was not sent):

"[...] there is no one who has correctly defined what it is to be similar. And yet, before having defined it, one cannot provide natural demonstrations of several important propositions in metaphysics and in mathematics. Having thoroughly inquired, I have found that two things are perfectly similar when they cannot be discerned other than *per compraesentiam*, for example, two unequal circles of the same matter cannot be discerned other than by seeing them together, for in this way we can well see that the one is bigger than the other" (to Gallois, 1677, A II, 1, 568-569; GM I, 180).

This formulation is of primary importance as it shows that Leibniz saw immediately the close connection between equivalence relation and indiscernibility. Moreover, indiscernibility goes hand in hand with the possibility of substitution: each time one can isolate a way in which things cannot be discerned one from the other, one has by the same token a relation under which they are identical (under a certain point of view) and can be substituted one by the other (by preserving the chosen point of view). I will come back to this general strategy which was the core of the new *ars inveniendi* Leibniz devised in those years for mathematics, but before that, let me first briefly mention two related aspects.

First, it is easy to see that this indiscernibility relation also applies in infinitesimal techniques whenever one can render the difference between two entities as small as one wishes (i.e. one renders them "indiscernible" under the given point of view, which corresponds to a certain order of magnitude). Realizing this fact was a crucial issue for Leibniz, as he explains in his *Quadratura*, because it allowed for a systematic translation between traditional *reductio ad absurdum* and direct demonstrations in which one shows that "the difference between two quantities is null, and hence they are equal" (*inter duas quantitates nullam esse differentiam, adeoque eas esse aequales*, prop. VII, schol.)<sup>27</sup>. After abstract algebra and *analysis situs*, this is yet another domain in which Leibniz reduced

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<sup>23</sup> Congruence being defined in this manner: "I say that those things are 'congruent' which are within the same boundaries". (A VI 3, 482; transl. from Parkinson: *De Summa Rerum: Metaphysical Papers*, Yale 1992, p. 37)

<sup>24</sup> Later Leibniz would refine this definition by adding the possibility of transforming one thing into another by decomposing them into congruent parts (or even "fictional" parts).

<sup>25</sup> A VII, 6, 486.

<sup>26</sup> *Elementa nova matheseos universalis* (A VI, 4, 513-524).

<sup>27</sup> A VII, 6, 200.

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the truths to a basic equivalence relation defined by a form of indiscernibility and rules for substitution (giving birth each time to a calculus). Not only do we see here how the “reduction to identicals” was put into practice by Leibniz, but we also see its central role in his mathematical invention since the three above mentioned calculi are amongst his most celebrated achievements in this field<sup>28</sup>.

The other point I would like to emphasize in the letter to Gallois is the reference to metaphysics.<sup>29</sup> This indicates another route to the central role of identicals and indiscernibility which I have not mentioned yet. It shows, in particular, that the reduction to identicals should not be considered merely as a default method (in the absence of simple elements of thought), but in fact as the only way one can deal with “incomplete” entities. Let me quickly explain how Leibniz came to realize this.

In the same folios in which he wrote the *De Elementis cogitandi*<sup>30</sup>, there is a very interesting piece in which Leibniz returns to a problem he was interested in when he was young, that of the principle of individuation. This interest was strongly related to his work in physics, in which he isolated the importance of the equivalence (or “equipollence”) between the cause and the full effect. Indeed, what this conservation principle implies is that it is always possible to come back from a complete knowledge of the effect to the cause (since they are equivalent!). But this principle seems to be violated in some situations. As an example, Leibniz takes a square, which can be obtained by the composition of different basic figures, such as two triangles or two rectangles. The first consequence of this situation seems to be that, in reality, the two results should not be said identical because they involve different material processes and this means that matter is not reduced to the pure geometrical extension, but has to include the “memory” of these different processes. This does not apply, however, to mathematical objects in and of themselves since they are considered in abstraction from matter. The fragment is not conclusive and ends with a dilemma: either we accept that there are perfectly similar entities and the way to discern them would have to be outside of them; or one refuses the perfect similarity of things and this means that matter has to be accompanied by something related to the mind.

Another text from the period shows that Leibniz realized that this dilemma was in fact related to two different kinds of notions. In December of the same year, he wrote the following note, which constitutes one of the first occurrences of the distinction between complete and incomplete entities:

“A Substance, or a complete Being, is for my that which involves everything, or that whose perfect understanding is in need of nothing else. This is not the case of figures, since in order to understand from what a figure of a certain sort, one needs to refer to motion.

Any complete Being cannot be produced except by one way: it is a sufficient sign to see that Figures, which can be produced in several different ways, are not Complete Being.” (A VI, 3, 400)

One immediate consequence of this is that incomplete entities are defined by some kind of indiscernibility, expressed by a form of “perfect similarity”, as expressed later in the *Principia logico-metaphysica*.

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<sup>28</sup> For more examples, see D. Rabouin: “*Analytica Generalissima Humanorum Cognitionum*. Some reflections on the relationship between logical and mathematical analysis in Leibniz”, in: *Studia Leibnitiana* 45/1 (2013), pp. 109-130.

<sup>29</sup> The passage is introduced by the sentence: “Il n’est pas si aisé qu’on pense, de donner des véritables démonstrations en métaphysique. Cependant il y en a et de très belles. On n’en sauroit avoir avant que d’avoir établi de bonnes définitions qui sont rares. Par exemple il n’y a personne qui ait bien défini ce que c’est que semblables.”

<sup>30</sup> LH IV 1,8 fol. 1-2.

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#### IV. A new way of doing mathematics

We now have all the ingredients to support the view I would like to advocate. Far from being synonymous to the sterile reduction of mathematical truths to logical tautologies, the “reduction to identicals” should be seen as a new way of doing mathematics based on the centrality of equivalence relations in the definition of mathematical objects. Moreover, this mathematical practice was developed by Leibniz *before* the creation of the great logical calculi from the middle of the 1680s. He devised it when he realized that any abstract object is grounded on a form of indiscernibility, hence on a mechanism of substitution allowing for successful computations, even in the absence of a complete analysis into simple elements of thought.

In order to substantiate this claim, let me make a series of important remarks. First, it is very crucial to note that Leibniz refers to “identicals” *in the plural form*. In some texts, he gives lists of them, which amply support the claim that they correspond to various equivalence relations. For example, in a letter to Foucher from 1675 he gives as examples “that two contradictories cannot exist and that at any given time a thing is as it is; that it is, for example, equal to itself, as great as itself, similar to itself, etc.” (A II, 1, 387)<sup>31</sup>.

How is it, however, that the second letter to Clarke only mentions the logical principle of identity? It is because this principle does not have the meaning we tend to give to it nowadays, that is to say a property of logical identity. In a way, we already know that, since the definition of logical coincidence was a *consequence* of a mathematical practice in which the reduction to identicals was *already* a central issue. But how is it that  $A \text{ is } A$  remains for Leibniz the paradigm of “identicals”? It is because it formulates a property of all equivalence relations, the one nowadays we call “reflexivity”. This is particularly clear in the opening of the *Principia logico-metaphysica* from 1689:

“*First truths* are those which predicate something of itself or deny the opposite of its opposite. For example,  $A \text{ is } A$ , or  $A \text{ is not non-}A$ ; if it is true that  $A \text{ is } B$ , it is false that  $A \text{ is not } B$  or that  $A \text{ is non-}B$ . Likewise, *everything is what it is; everything is similar or equal to itself; nothing is greater or less than itself*. These and other truths of this kind, though they may have various degrees of priority, can nevertheless all be grouped under the one name of identicals” (A VI, 4, 1644; transl. Loemker modified 268).

As I have already mentioned, another feature of the “identicals” which Leibniz identified very early is their association with a regime of indiscernibility or of substitution (a certain point of view being preserved in the substitution). This is particularly clear in the following list of definitions Leibniz formulated at the end of 1680:

*Identical* is that which can be substituted anywhere for another. If one can be substituted for the other, then the other for the one in return.

*Equal* are those whose quantity is same in quantity, i.e. which can be substituted for each other while preserving quantity.

*Similar* are those which can be substituted by preserving quality, i.e. in such a way that they cannot be discerned except by observing them simultaneously.

*Congruent* are those which can be contained within the same limits; these, if they are of the same matter, that is, if their parts behave with each other as similar, are similar. Congruents are always equal.

*Coincident* are, in fact, identical things that nevertheless appear to be diverse. (A VI, 4, 406).

In a later text entitled *Elementa nova matheseos universalis*, Leibniz explains that this approach the basis of a new “universal mathematics”, much more powerful than the one devised by Vieta and Descartes, capable of accommodating all kinds of relations, each defined by a kind of indiscernibility.<sup>32</sup>

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<sup>31</sup> Transl. Loemker, p. 151. Similar lists in A II, 1, 420-421; A II, 1, 599 or GM VII, 20 (where Leibniz states that the list of such truths is potentially infinite).

<sup>32</sup> A VI, 4, 513-524. See D. Rabouin (ed.): *Leibniz. Ecrits sur la mathématique universelle*, Paris 2018, for a French translation and a commentary of this text.

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Very early, Leibniz also realized that the two properties he isolated for “identicals” (reflexivity and substitution) were sufficient to demonstrate what we now call the “transitivity” of these relations and which was at the time formulated in the form of the Euclidean axiom “if things are equal to a same thing, they are equal between themselves”. As we saw, he provided a purely formal proof of this fact in the case of equality as early as 1672. Later, he identified this property as characteristic to all identicals: “these definitions of coincidence, congruence, equality and similarity lead to some inferences: things which are equal, similar, congruent, coincident, with the same thing are so between themselves too”<sup>33</sup>. In fact, he reached such a level of abstraction that he was able to devise a purely formal derivation of properties attached to any identity (*identitas*)<sup>34</sup>.

Not only have we seen that this approach corresponds to the way in which Leibniz explored new mathematical territories such as differential calculus and analysis situs, but also that he wrote a number of texts dedicated to this general architecture of mathematics in which he saw a way to develop a new *mathesis universalis*.<sup>35</sup> The formal axiomatic approach of basic mathematical relations he developed was explicitly articulated: “Beyond the axioms and theorems of Euclid about magnitude and proportion, I have discovered things much more important and of broader use about *Coincidence, Congruence, Similitude, Determination, Cause and Effect* or *power, relations in general, the container and the contained*” (*De arte characteristica ad perficiendas scientias ratione nitentes* (1688?; A VI 4 909-915; GP VII 198-203)<sup>36</sup>.

For lack of space I will stop here the description of this strategy and the study of its development in later periods. The point I would like to emphasize is that, at any rate, the practice I have described not only occurs before the creation of the great logical calculi from the middle of the 1680s, but clearly served as the model for it. This is the main thing I wanted to show in this paper: the reduction to identicals cannot be interpreted as a way to make mathematics *depend* on logical calculi<sup>37</sup>, as the logicist wanted. But neither is it true that it was just a dream which was never put into practice, as claimed by the “conceptual realists” (following Mahnke’s terminology).<sup>38</sup> The truth lies here in the middle: Leibniz certainly devised a certain way of doing mathematics, which was completely new for the time and certainly rests on something he saw as “logical analysis” (in the sense of the “analysis of notions and truths”). By doing so, however, he was more in line with the modern axiomatic approach than with the logicist program.

A last comment, to finish with: one should not conclude from what precedes that Leibniz was not aware of the existence of other types of relations beyond “identicals”. He also was very interested in order relations and in functional correspondences (which he calls *determinatio*). On this path, he realized that there was also relations between relations, for example similitude between similitudes. What I described in the preceding study is just the beginning of a story that needs to be continued.

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<sup>33</sup> *La Caractéristique géométrique*, texte établi, introduit et annoté par Javier Echeverría, traduit, annoté et postfacé par M. Parmentier, Paris 1995, p. 187.

<sup>34</sup> A VI, 4, 815. For a similar proof of the property which we call nowadays « symmetry », see A VI, 4, 831.

<sup>35</sup> See Martin Schneider: “Funktion und Grundlegung der Mathesis Universalis im Leibnizschen Wissenschaftssystem”, in: A. Heinekamp (ed.), *Questions de logique, Studia Leibniana Sonderheft 15*, Steiner 1988.

<sup>36</sup> English translation from *G. W. Leibniz: The Art of Controversies*, transl. and ed., with an introd. essay and notes by Marcelo Dascal, Berlin 2008, p. 265.

<sup>37</sup> In fact, there is also a stronger argument I could not develop here: in edifying his logical calculi, Leibniz realized that they depart from the usual calculus on magnitude because of the axiom of idempotence (which he was the first to identify as such). So in this sense, although Leibniz acknowledges a higher level, that of *ars combinatoria*, in which these different calculi could enter, it is not possible to make mathematical calculi depends on logical calculi in the strict sense.

<sup>38</sup> Dietrich Mahnke: *Leibnizens Synthese von Universalmathematik und Individualmetaphysik*, in: *Jahrbuch für Philosophie und Phänomenologische Forschungen*, Halle 1925, p. 305-612.