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## FRAGMENTS OF A HISTORY OF THE CONCEPT OF IDEAL.

### Poncelet's and Chasles's reflections on generality in geometry and their impact on Kummer's work with ideal divisors

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The English of this article will be polished after the referee process

For Jean-Pierre Bourguignon, as a token of friendship\*

#### 1. Introduction

The first piece of evidence we have about Ernst Eduard Kummer's (1810- 1893) introduction of "ideal prime factors of a complex number" (idealer Primfactor einer complexen Zahl) in number theory is a letter he wrote to Leopold Kronecker (1823-1891) on October 18, 1845.<sup>1</sup> The letter clearly shows a Kummer who just made a breakthrough and writes to his correspondent in order to put his ideas in order (Weil, ed., vol. 1, 1975, 94). Four months later, in March 1846, the theory, in the context of which ideal factors were introduced, had taken shape and was made public for the first time, through an "excerpt from his latest researches in

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\* Acknowledgements: It is a pleasure to dedicate this article to Jean-Pierre Bourguignon, without whom I might never have known that the mathematics section of CNRS intended to open hirings to history of mathematics. Without this piece of information, I might never have continued research in the history of mathematics. In this context, I find it appropriate to dedicate to him an article showing how the historical and philosophical reflections developed by mathematicians led to significant advancements in mathematics. I have been interested in the history of geometry in France in the first half of the 19<sup>th</sup> century since the 1980s thanks to Serge Pahaut's (1945-2023) invitation to work with him on duality. His passing away just as I was completing this article infuses great sadness with some of its pages. My interest in the notions of ideality was spurred by the ANR project "Ideals of proof" that Mic Detlefsen (1948-2019) developed at the University Paris Diderot between 2007 and 2011. In this context, I gave a joint talk with Bruno Belhoste on Poncelet's ideal elements (titled: "Poncelet's ideal elements in geometry: between Carnot and Chasles", March 5, 2009), which we intend to publish. The inspiration for this article derives from my first encounter with Ernst Kummer's writings on number theory, in the context of a talk given at our research group (REHSEIS at the time, SPHERE, now) by Jacqueline Boniface on the topic. As we incited speakers to do at the time, she handed out pages of Kummer's original text. I immediately recognized that these pages had been inspired by Michel Chasles's "principle on contingent relationships". This formed part of the results that we published in (Chemla, Chorlay, Rabouin eds. 2016), which concluded the research project carried out between 2004 and 2016, about the epistemic and epistemological value of generality. In the book, (Chemla 2016) analyzed Chasles's understanding of generality in geometry, and mentioned the impact of his reflections, notably those relating to the aforementioned principle, on Kummer's work in number theory. (Boniface 2016) described Kummer's creation of ideal numbers, mentioning the analogy with geometry. The prologue to the book (Chemla, Chorlay, Rabouin 2016) took this case as an example of a circulation of scientists' reflections about an epistemological value from one context to another context. This article builds on all these previous researches, and I am in particular grateful to Jacqueline Boniface for having introduced me to Kummer's work in number theory. I also owe a huge gratitude to Harold Edwards, since without his book (Edwards 1977), I might not have been able to enter into the made of Kummer's mathematical world. Last, but not least, since 2022, Bruno Belhoste and I have begun a joint project on Poncelet's works. Our common work has helped me refine my understanding on Poncelet's philosophical and geometrical reflections.

<sup>1</sup> Kurt Hensel (1861-1941) selected and published part of the letters that Kummer sent to Kronecker between 1842 and 1865 in (Vorstand der Berliner mathematischen Gesellschaft (Direction of the Berlin mathematical society) [i.e.: E. Jahnke, R. Güntzsche, and C. Färber]. ed. 1910, 46-102). By "part", I mean in particular that Hensel deleted whole passages of the letters, replacing them by the sign "...". The letter from October 18, 1845, is on pp. 64-68. Unfortunately, Hensel did not reproduce it entirely. The original document seems to be lost (Edwards 1978). The whole volume in which these letters were published was reprinted in (Weil 1975, 30-133). The letter in question is reprinted in (Weil 1975, 94-97). See also (Weil 1975, 5). In what follows, I will quote the reprints of Kummer's publications from this edition.

number theory” that, upon Kummer’s instruction, Johann Peter Gustav Lejeune-Dirichlet (1805-1859) communicated to the Berlin Academy (Kummer 1846 (March), 87). The following year, this publication was reprinted, in Crelle’s journal, under the title “Zur Theorie der complexen Zahlen,” and with some changes.<sup>2</sup> Two features of this first publication (Kummer 1846, reprinted as Kummer 1847b) are essential for the purpose of this article.

First, in March 1846, Kummer published only an overview of this theory, without any proof. This might be due to the fact that this article reflected an academic presentation, in which Kummer simply gave a non-technical presentation of his main ideas. Except for issues of chronology, the historiography has paid less attention to this text than to later, more complete, publications that Kummer made on the same topic. However, I will argue that this overview sheds light on the process through which Kummer achieved his breakthrough. Several kinds of clues support this view—notably the terminology Kummer uses and the structure of the article. In contrast with the March 1846 presentation, some months later, in (Kummer 1847a)—dated from September 1846—and, some years later, in (Kummer 1851)—in which Kummer published his theory with the intention of giving “an entire and continuous treatise” (“*Traité entier et continu*”)—,<sup>3</sup> he gave a uniform presentation, which did not reflect as clearly as in the 1846 presentation elements from which he drew inspiration in his introduction of ideal divisors.

Secondly, to situate what he meant by “ideal complex numbers” in the context of the mathematics of his time, in the 1846 publication, Kummer sketched several comparisons. He presented the uniform decomposition into prime factors that his ideal factors allowed him to achieve as underlain by the same idea as the uniform decomposition of polynomials of the type  $P(X)$  into linear factors, made possible by the introduction of complex numbers (in the ordinary sense of the term today, which, as we shall shortly see, differed from Kummer’s use of the expression). Kummer also compared the need for his new type of factors with the need that led Gauss to introduce complex numbers of the type  $a + bi$  in his research about biquadratic residues. These comparisons are not surprising, and the historiography has mainly dwelled on them. The historiography has also given pride of place to the comparison between the factorization of a “complex number” into prime (sometimes ideal) cyclotomic factors and the decomposition of chemical substances into elements. In fact, Kummer introduced the latter comparison only later (our earliest evidence for this last comparison dates from June 1846), to illustrate the procedure with which he made ideal factors visible and how they combined with each other in actual complex numbers.<sup>4</sup> However, the first publication of March 1846 made a third comparison, which, to my knowledge, has remained unnoticed until our collective work on generality (Boniface 2016, Chemla 2016).<sup>5</sup> Indeed, shifting the point of the comparison to the issue of the definition of his new entities, Kummer drew a

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<sup>2</sup> (Kummer 1847b), reprinted in (Weil, ed., vol. 1, 1975: 203-210). The meaning of “complex numbers” in this title will be clarified in the next section. In this latter publication, Kummer’s first communication to the Berlin Academy was erroneously asserted to date from 1845. The mistake was pointed out in (Edwards 1975, 224) and (Weil, ed., vol. 1, 1975, 4). The changes that occurred between the two versions were not yet described systematically. In this article, I will point out some of them when they occur in the passages of the article that I quote. An English translation of the version in the Crelle journal was published decades ago (Cope 1929). Edwards (1975, 224, fn 22) indicates that it is flawed. I provide here a new English translation of substantial parts of Kummer (1846). I plan to publish a complete English translation of this text, which will indicate systematically the differences between the 1846 and the 1847 versions.

<sup>3</sup> (Kummer 1851, 378), reprinted in (Weil, ed., vol. 1, 1975, 363-484). The quotation comes from p. 364. Note that (Kummer 1847a), reprinted in (Weil, ed., vol. 1, 1975, 211-251), was published in Crelle’s journal immediately after (Kummer 1847b).

<sup>4</sup> See the subsequent extant letter that Kummer wrote to Kronecker on June 14, 1846 (Weil, ed., vol. 1, 1975, 98). We return to this letter below. The analogy with chemistry recurs in (Kummer 1847a, 359-361), and in (Kummer 1851, 447-448), see also, respectively, (Weil, ed., vol. 1, 1975, 243-245, 433-434).

<sup>5</sup> I should mention a passing remark in (Avigad 2006: 208). However, Avigad does not attempt to understand what lies behind this remark. Moreover, he only associates the reference to geometry to Poncelet, whereas we will argue that the reference to Chasles’s understanding of ideal elements in geometry is also important.

comparison between his “true (usually ideal) prime factors” (Kummer 1846, 88) and the “chords common to two circles,” which are “ideal” when the circles do not intersect.

The expression “ideal common chord” unmistakably evokes Jean-Victor Poncelet (1788-1867), who introduced it, notably in his *Traité des propriétés projectives des figures* (1822)—a book that played a major role in the emergence of projective geometry. Given that in his first publication on the topic, Kummer made this comparison explicit, it is thus striking to note that, to refer to his new type of “prime factors of a complex number,” he chose precisely the same term of “ideal” that Poncelet had used about two decades earlier. This terminological choice, and Kummer’s explicit parallel in 1846—in relation to the issue of shaping the correct definition—between his approach and the new geometry that French mathematicians like Gaspard Monge (1746-1818), Jean-Victor Poncelet and Michel Chasles (1793-1880) had developed, both point towards the same conclusion. They both seem to suggest that this new geometry played a part in the reflections that led Kummer to introduce the “ideal complex numbers.”

My claim is that this role is most visible in the 1846 publication, since its structure meshes with the way in which the text places emphasis on the issue of definition, and is correlated with Kummer’s reference to the “ideal” elements introduced by the new geometry. To my knowledge, the reference to this geometry disappears from Kummer’s later publications on the topic, except for the 1851 publication in French (Kummer 1851, 430, see also Weil, ed., vol. 1, 416), in which it recurs. We will see that this latter text yields evidence that converges towards the same conclusion about the impact that the new geometry had on Kummer’s conception of ideal divisors, even though this 1851 article no longer mentions the use of the term “ideal” in geometry.

In brief, in this essay, I argue for the following theses. First, Kummer’s concept of “ideal prime factors of a complex number” drew inspiration precisely from both Poncelet’s introduction of ideal elements in geometry and the reconceptualization for them that Michel Chasles put forward in 1837. That is to say, the idea of ideality in number theory derives from the introduction of ideal elements in the new geometry. This is where the term “ideal” comes from. Secondly, the introduction of ideal elements in geometry as well as the subsequent reconceptualization of what was in play with these elements were linked to philosophical reflections on generality that practitioners of geometry in France developed to devise a new approach to geometry, which would eventually become projective geometry. These philosophical reflections circulated as such and played a key part in the advancement of other domains, including in Kummer’s major innovation in the context of number theory. I claim that Kummer was aware of Poncelet’s work on ideal elements—I return to this issue below—and that he also read Chasles’s writings on the topic, which were instrumental for the development of his new theory. My argumentation will rely in an essential way on the following facts: The structure of the 1846 publication introducing these ideal factors, as well as the terminology about definitions Kummer uses in it, in relation to ideal elements, yield striking evidence about the process through which Kummer progressively shaped his definition of “ideal prime factors.” What is more, the elements of this process that we can restore highlights that Kummer followed Chasles’s recommendations about how to elaborate general and uniformly valid definitions in the new geometry. In other words, the 1846 publication shows that, to yield the uniform generality in the decomposition of complex numbers into prime factors that Kummer sought to achieve, he followed a strategy inspired by actors involved in the development of the new geometry. Establishing these facts will enable us to see that the comparison that Kummer drew between ideal elements in both contexts is an essential one, which explains how and why the origin of the term “ideal” lies in the new geometry. This case study will thus illustrate how mathematicians’ historical and philosophical reflections can be crucial to their mathematical work.

To establish these theses, I will mainly rely on the four pieces of Kummer’s writings that I mentioned above: his two letters to Kronecker from October 18, 1845 and from June 14, 1846; the March 1846 text communicated to the Berlin Academy; and the 1851 publication in French.

## 2. The first public introduction of “ideal complex numbers”

Let us begin by examining in greater detail how, in 1846, Kummer (1846: 87) introduced his results as well as his motivations, in the first public and formal presentation that he gave of his “theory of complex numbers”— an expression that also gave its title to the publication of this text in 1847 (Kummer 1847b).

Kummer defines the notion of “complex number” that is central to this work as early as in the second paragraph of his presentation (Kummer 1846: 87-88). For him,  $\alpha$  denoting an imaginary root of the equation  $\alpha^\lambda = 1$ , in which  $\lambda$  is a prime integer, a “complex number”  $f(\alpha)$  —or alternatively a “complex integral number (complexe ganze Zahl)” — is an expression of the following kind:

$$f(\alpha) = a + a_1\alpha + a_2\alpha^2 + \dots + a_{\lambda-1}\alpha^{\lambda-1},$$

in which the  $a_i$  are integers.

Kummer’s project, whose 1846 presentation announces the completion, consisted in decomposing any complex number of this type (and notably prime integers) into prime factors of the same type. In other words, Kummer aimed at developing a theory of divisibility for these numbers identical to that holding for integers. However, to achieve this goal Kummer identifies a key difficulty, which he depicts as follows:<sup>6</sup>

Such a complex number can either be decomposed into factors of the same kind, or also cannot. In the first case, this number is composite, whereas in the other, the number was *until now called a complex prime number*. However, I have now noticed that, even if  $f(\alpha)$  *can in no way be decomposed into complex factors*, it yet does *not have the true nature of a complex prime number*, because it usually *lacks the first and most important property of prime numbers*: that is, that the product of two prime numbers is not divisible by any prime number different from them. (Eine solche complexe Zahl kann entweder in Factoren derselben Art zerlegt werden oder auch nicht, im ersten Falle ist sie eine zusammengesetzte,<sup>7</sup> im anderen Falle ist sie *bisher eine complexe Primzahl genannt* worden. Ich habe nun aber bemerkt, dass wenn auch  $f(\alpha)$  *auf keine Weise in complexe Factoren zerlegt werden kann*, sie darum<sup>8</sup> *noch nicht die wahre Natur einer complexen Primzahl hat*, weil sie schon *der ersten und wichtigsten Eigenschaft der Primzahlen gewöhnlich*<sup>9</sup> *ermangelt*, nämlich dass das Product zweier Primzahlen durch keine von ihnen verschiedene Primzahl theilbar ist)

<sup>6</sup> The quotation follows the version in (Kummer 1846: 88). My emphasis

<sup>7</sup> Here, (Kummer 1847b: 319) has an additional “Zahl.” I do not indicate the differences in punctuation between the two publications.

<sup>8</sup> Here, instead of “darum” (Kummer 1847b: 319) has “deshalb” (“by way of consequence”).

<sup>9</sup> In (Kummer 1847b: 319), “gewöhnlich” features after “schon.”

In other words, unlike prime numbers in ordinary arithmetic, which serves here as a model for the development of the theory, for complex numbers, Kummer identifies the need to distinguish between indecomposability and primality—a distinction that, he asserts, had been so far overlooked. It is precisely to bridge the gap between these two notions in the case of complex numbers, and thereby achieve uniform decomposition, that Kummer suggests introducing “ideal complex numbers.” Indeed, with respect to these indecomposable complex prime numbers that nevertheless do not have the property of prime numbers pointed out, Kummer goes on:

Numbers  $f(\alpha)$  of this type *still have much more the nature of composite numbers even though they are not decomposable* into complex factors; but the *factors* are consequently *not actual*, but *ideal complex numbers*. (Es haben vielmehr solche Zahlen  $f(\alpha)$ , *wenngleich sie nicht in complexe Factoren zerlegbar sind, dennoch die Natur der zusammengesetzten Zahlen, die Factoren aber sind alsdann nicht wirkliche, sonder<sup>10</sup> ideale complexe Zahlen.*) (Kummer 1846: 88, with my emphasis)

I translate “wirklich” here by “actual”, by which Kummer means complex numbers that can be expressed in the form introduced above ( $f(\alpha) = a + a_1\alpha + a_2\alpha^2 + \dots + a_{\lambda-1}\alpha^{\lambda-1}$ ). On this basis, one thus understands why Kummer opens his communication to the Berlin academy with the following words:

I have succeeded to essentially *complete* and *simplify* the *theory of the complex numbers* that are made up of higher roots of the unity (...); and I did this by introducing a very specific type of *imaginary divisors*, which I call *ideal complex numbers* (...)(Es ist mir gelungen die Theorie derjenigen complexen Zahlen, welche aus höheren Wurzeln der Einheit gebildet sind, (...), wesentlich<sup>11</sup> zu *vervollständigen* und zu *vereinfachen*, und<sup>12</sup> zwar durch Einführung einer ganz<sup>13</sup> eigenthümlichen Art imaginärer Divisoren, welche ich ideale complexe Zahlen nenne (...).)<sup>14</sup>

It is noteworthy that for Kummer, the double effect of the introduction of “ideal complex numbers” is to “complete”, and to “simplify” the theory—we will see the embodiment of these epistemological values below. More importantly for us, note that these “ideal complex numbers” are introduced as “divisors.” In other words, it is not directly, but indirectly—through a specific relation that they have with other *actual* numbers—that they are brought to light. This detail will prove meaningful. In addition, it is interesting that Kummer first refers to these “divisors” as “imaginary,” but then discards this term to introduce the expression of “ideal complex numbers” instead. In fact, as we will see, these “ideal complex numbers” are *not* imaginary, if by this term, we mean “complex numbers” in the sense in which the expression is used today. What this tells us about Kummer’s notion of imaginarity would be interesting to pursue, but we will not follow this line of inquiry. We will rather focus on what the choice of the term “ideal” indicates.

<sup>10</sup> Sic. This typo is corrected into “sondern” in (Kummer 1847b: 319). In this quotation, Kummer only emphasizes “*ideale complexe Zahlen*”. All other italicized formulations are my emphasis.

<sup>11</sup> The adverb *wesentlich* (significantly, essentially) is deleted in (Kummer 1847b).

<sup>12</sup> Sic, and corrected into “und” in (Kummer 1847b)

<sup>13</sup> The adverb *ganz* (very, wholly) is deleted in (Kummer 1847b).

<sup>14</sup> (Kummer 1846: 87). Again, my emphasis, except for the expression “*ideal complex numbers*.”

Indeed, Kummer first justifies the introduction of such “complex numbers” through two comparisons. He writes:

Underlying the introduction of such ideal complex numbers is the same *simple* idea as in the introduction of imaginary formulas in algebra and analysis, that is to say, in the decomposition of integral rational functions into their *simplest* factors—the linear ones. Moreover, it is also the same need through which, forced, Gauss first introduced the complex numbers of the form  $a + b\sqrt{-1}$  in the investigations on the biquadratic remainders, because in this case, all prime factors of the form  $4m + 1$  showed the nature of composite numbers. (Der Einführung solcher idealen complexen Zahlen liegt derselbe einfache Gedanke zu Grunde als<sup>15</sup> der Einführung der imaginären Formeln in die Algebra und Analysis, namentlich bei der Zerfällung der ganzen rationalen Functionen in die<sup>16</sup> einfachsten Factoren, welche die lineären sind.<sup>17</sup> Ferner ist es auch dasselbe Bedürfniss, durch welches genöthigt<sup>18</sup> Gauss bei den Untersuchungen über die biquadratischen Reste, weil hier alle Primfactoren von der Form  $4m + 1$  die Natur zusammengesetzter Zahlen zeigten,<sup>19</sup> die complexen Zahlen von der<sup>20</sup>  $a + b\sqrt{-1}$  zuerst einführte.) (Kummer 1846: 88, my emphasis).

The two examples of introduction of new entities that Kummer puts forward as comparable to his innovation have in common with his “ideal complex numbers” the facts that, in the former case, “the *simplest* factors” were identified, and, in the latter, numbers that might have been conceived as prime were in fact “composite.” These two comparisons are followed immediately by a third one, in which precisely the term “ideal” is introduced. Let us first read it, before we point out the key clues that the text contains:

To achieve now a *lasting definition* of the *true (usually ideal)* prime factors of the complex numbers, it was *necessary to find out the properties* of the prime factors of complex numbers *that would persist/remain in all circumstances*, which would be absolutely *independent* of the *contingency/accidental circumstances* of whether the *actual* decomposition takes place or not, *more or less precisely like, when in geometry*, one speaks of the *chord common to two circles*, also when the *circles do not intersect* each other, *one looks for an actual definition of this ideal common chord that fits for all situations of the circles.*

(Um nun zu einer *festen Definition* der *wahren (gewöhnlich idealen)* Primfactoren der complexen Zahlen zu gelangen, war es nöthig, *die unter*<sup>21</sup> *allen Umständen bleibenden Eigenschaften* der Primfactoren complexer Zahlen *aufzusuchen*,<sup>22</sup> welche von der *Zufälligkeit*, ob die *wirkliche* Zerlegung Statt habe, oder nicht, ganz *unabhängig* wären: *ohnegefähr*<sup>23</sup> *ebenso wie wenn in der Geometrie* von der *gemeinschaftlichen Sehne zweier Kreise* gesprochen wird, *auch*<sup>24</sup> *wenn die Kreise*

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<sup>15</sup> Here, instead of “als,” (Kummer 1847b: 319) has “, wie...”

<sup>16</sup> Here, instead of “die,” (Kummer 1847b) has “, ihre.” I translate according to the more recent version.

<sup>17</sup> Here, instead of “welche die lineären sind” (Kummer 1847b) has “die lineären.”

<sup>18</sup> Here, in 1847b, Kummer added a comma.

<sup>19</sup> What is here between commas was put between brackets in (Kummer 1847b: 319)

<sup>20</sup> Here, (Kummer 1847b: 319) added the noun “Form”. I translate according to the more recent version.

<sup>21</sup> Here, Kummer 1846 has “nnter,” which is corrected into “unter” in (Kummer 1847b: 320).

<sup>22</sup> In (Kummer 1847b: 320), this verb “aufzusuchen” (to seek) is rewritten into “zu ermitteln” (to identify, to determine).

<sup>23</sup> In (Kummer 1847b: 320), this adverb is rewritten into “ungefähr.”

<sup>24</sup> In (Kummer 1847b), “dann” is added here, which adds emphasis “also then...”.

*sich nicht schneiden, eine wirkliche Definition dieser idealen gemeinschaftlichen Sehne gesucht wird,<sup>25</sup> welche für alle Lagen der Kreise passt.*) (Kummer 1846: 88, my emphasis)

As I explained in the introduction, this third comparison that Kummer draws has mainly been overlooked in the secondary literature. However, it contains key clues about the source on which Kummer has drawn to introduce the “ideal complex elements” and also about the reflections that have inspired the procedure he followed to this effect.

Firstly, Kummer here establishes a parallel between his “ideal complex numbers” and the “ideal chord common to two circles”—a concept introduced by Jean-Victor Poncelet in relation to a reflection on how to increase generality in geometry. The use of the same term “ideal” in geometry, which is put forward precisely in the first public presentation of the theory and which occurs in none of the other contexts mentioned, is certainly striking. Kummer further makes explicit the comparison. The situations in which the decomposition of complex numbers into prime factors “actually” “takes place”—namely, one can exhibit “actual” complex numbers that are the prime factors—correspond, in geometry, to the situations in which the two circles intersect”—namely, one can exhibit “actual” points of intersection, which define the common chord. By contrast, the situations in which the decomposition is not “actual”—that is, for which ideal factors need to be introduced—correspond, in geometry, to the situations in which two circles do not (actually) intersect—that is, cases in which, despite the absence of real intersection points, Poncelet still considers the common chord, but as an “ideal” one. To understand this parallel better, in the next section (Section 3), we will outline the main characteristics of Poncelet’s treatment of ideal elements in his *Traité des propriétés projectives des figures* (1822).

Secondly, it is no less striking that in this third comparison, the emphasis is placed on an issue different from what was pointed out in the first two comparisons. Indeed, now, Kummer lays stress on the issue of how to define “the true (usually ideal) prime factors.” For this, he insists that the definition he looks for should be one that would be uniformly valid for all possible prime factors. Indeed, to achieve this definition, he makes explicit that he looks for “*the properties of the prime factors of complex numbers that would persist/remain in all circumstances, which would be absolutely independent of the contingency/accidental circumstances of whether the actual decomposition takes place or not.*” Note again that, the introduction of “ideal complex numbers” is wholly dependent on the part they play, as *factors*, in the theory of divisibility. In Section 4, I will show that this way of shaping definitions for situations such as the one Kummer is facing, as well as the related terminology Kummer uses here, both derive from specific pages of Michel Chasles’s 1837 *Aperçu Historique*. In these pages, Chasles gets rid of Poncelet’s notion of ideality and, as an alternative, offers a way of dealing with “ideal common chords” that is different from Poncelet’s.

As a result, we will see that Kummer’s “theory of complex numbers” has roots in both Poncelet’s and Chasles’s reflections on generality in geometry, which he synthesizes.

### 3. Poncelet’s introduction of “ideal” elements in geometry

Let us focus now on the meaning that Poncelet gave to ideal elements, to begin with, in his *Traité des propriétés projectives des figures*, published in 1822 (see Section 3.2). To do

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<sup>25</sup> In (Kummer 1847b: 320), the structure of the sentence “*ebenso wie wenn in der Geometrie von der gemeinschaftlichen Sehne zweier Kreise gesprochen wird, (...) eine wirkliche Definition dieser idealen gemeinschaftlichen Sehne gesucht wird...*” is rewritten into “*eben so, wie man, wenn in der Geometrie von der gemeinschaftlichen Sehne zweier Kreise gesprochen wird, (...) eine wirkliche Definition dieser idealen gemeinschaftlichen Sehne sucht.*”



this, and also to examine how Kummer had access to Poncelet's ideas on this issue, we will consider the treatise in the broader context of Poncelet's geometrical writings.

### 3.1 Poncelet's geometrical manuscripts and publications

At the end of his life, between 1862 and 1866, Poncelet published or republished most of his geometrical works in four volumes: two of these volumes were titled *Applications d'analyse et de géométrie qui ont servi de principal fondement au Traité des propriétés projectives des figures* (Poncelet 1862 (volume 1) and 1864 (volume 2)), and the other two, *Traité des propriétés projectives des figures; ouvrage utile à ceux qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain* (Poncelet 1865 (volume 1) and 1866 (volume 2)). I will provide some information on these publications here, insofar as this will be useful in determining which of Poncelet's writings were available to Kummer and his contemporaries before 1846. These pieces of information will also make clear which documents are available to us to understand the formation of Poncelet's conception of ideal elements that was instrumental for Kummer.<sup>26</sup>

The first volume of works Poncelet published in 1862 (Poncelet 1862) mainly contains seven notebooks that he had never published before and that he had written in Saratoff, during the year he spent there as war prisoner. Indeed, on November 18, 1812, as Poncelet participated in the Grande Armée for the Russia campaign, he was taken prisoner, being subsequently held in Saratoff from March 1813 till June 1814. To these manuscripts, which in 1862 Poncelet was making public, he added documents and notes, some of which had been written while he was a student at the Ecole Polytechnique between 1807 and 1810.<sup>27</sup>

The war ended in June 1814. Poncelet could thus leave Saratoff, and he reached Metz in September of the same year. In the second volume of the same publication (Poncelet 1864), Poncelet further makes available unpublished drafts in which, between 1814 and 1820, we see the mathematician maturing his approach to geometry, in what spare time was left to him by his life as an officer in Metz. To these manuscripts, Poncelet adds articles that he had published prior to the publication of his 1822 treatise, as well as documents attesting notably to discussions with fellow mathematicians.

The third volume (Poncelet 1865) offered a new edition of the 1822 *Traité des propriétés projectives des figures*, “revised, corrected and with new annotations” (“revue, corrigée et augmentée d’annotations nouvelles”, Poncelet 1865), whereas in the fourth volume, Poncelet (1866) included annotated versions of articles and memoirs that had been published after the treatise—notably in the *Journal für die reine und angewandte Mathematik*, which August Crelle edited in Berlin at the time—as well as other documents and reflections.

When does the expression “ideal chord,” mentioned by Kummer first occur in Poncelet's writings? Despite scattered occurrences of this term, as well of the expression of “ideal secant” in (Poncelet 1862),<sup>28</sup> as far as I can tell, the first work in which Poncelet foregrounds these

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<sup>26</sup> For a more detailed biographical sketch (with all due references) as well as a first approach to the development of Poncelet's ideas about ideal elements, see (Belhoste and Chemla 2023 Forthcoming). In this other publication, we also give a larger bibliography and an outline of the historiography of ideal elements. I will focus here on what is useful in relation to the topic of this article.

<sup>27</sup> In (Belhoste and Chemla 2023 Forthcoming), the approach to Poncelet's introduction of ideal elements draws mainly on a comparison between the Saratoff notebooks and the 1822 treatise. Our goal is to shed light on the mathematical problems to which Poncelet constantly returned, notably between 1813 and 1822, and on how his reflections on these problems and his changing solutions are correlated with his introduction of, in particular, “ideal secants” and “ideal chords”. In this other publication, we discuss the distinction between “secants” and “chords,” and its mathematical importance for Poncelet. As we show below, both Chasles and Kummer speak of “ideal chords,” but they refer to “ideal secants.” I will thus not dwell on the opposition here.

<sup>28</sup> See (Belhoste and Chemla 2023 Forthcoming). We do not know how exactly Poncelet published his notebooks, and in particular how much change he brought to them. Accordingly, we need to be cautious when expressions such as these occur in these notebooks without being used in a systematic way.

two concepts in its structure is the memoir that he presented to the Academy of science, on May 1, 1820 (Poncelet 1865: 365). Its text was published as the fifth notebook in (Poncelet 1865: 365-454), under the title “Essai sur les propriétés projectives des sections coniques (Essay on the projective properties of conic sections),” and the first section of this memoir was entitled precisely “Notions préliminaires sur les cordes ou sécantes idéales des sections coniques (Preliminary notions on ideal chords and secants of conic sections).” However, when Kummer presented his own theory in 1846, none of these texts were published yet. Ironically, it seems that Augustin-Louis Cauchy’s report on this 1820 memoir, which, to Poncelet’s great despair, criticized a key part of his memoir to which we return, was the first publication discussing the notion of “ideal chord” (Cauchy 1820).

Poncelet’s 1820 memoir was part of a first draft of the 1822 *Traité des propriétés projectives des figures* (Poncelet 1865: 365). The treatment of “ideal chords” and “ideal secants” that the memoir put forward testifies to ideas about ideality that are quite similar to those presented in the treatise. Let us thus first examine Poncelet’s project in this treatise and which part it devoted to ideal elements.

### 3.2 *Ideal elements in the 1822 Traité des propriétés projectives des figures*

As is well known, and as Poncelet explains in the “Introduction” to his major work in geometry, his main project grew out of key questions: Why, he asked, is analysis able of proceeding uniformly and of achieving general results, when it is applied to geometrical issues, and why, by contrast, is “the ordinary or ancient geometry” deprived of the same properties? How could one change geometry to make it enjoy these properties?<sup>29</sup> The treatise thus aims at offering general means to “improve rational geometry” and notably to increase significantly the generality that can be achieved by proceeding geometrically.<sup>30</sup>

A first general method of this kind, to which Poncelet (1822) gives pride of place in the very title of the treatise, consists in concentrating on the properties of a figure that are invariant by central projection—these are the properties he calls “projective.” Indeed, properties of this kind can be proved by reference to a particular figure, and they can subsequently be stated with full generality for any figure that derives from this particular one by central projection. Poncelet specifies that these properties can be “descriptive” (that is, depending only on the situations of lines with respect to each other) as well as “metrical.”<sup>31</sup>

In contrast with the latter kind of reasoning, which proceeds from the particular to the general, Poncelet introduces a second principle, which will enable to conclude that a property

<sup>29</sup> (Poncelet 1822: xix-xx). The third manuscript that was published in (Poncelet 1864: 167-295) under the title “Sur la loi des signes de position en géométrie, la loi et le principe de continuité (On the law of signs of position in geometry and on the law and principle of continuity)” was written during the winter 1815-1816. This notebook (cahier) nevertheless has marks of changes that date to 1817. In it, Poncelet’s reflection seems to bear rather on how to use “algebraic analysis” and “rational geometry” conjointly (see notably (Poncelet 1894: 294)). The article (Poncelet 1817-1818), published on November 1, 1817, appears to me to follow this idea. To be sure, this was what Monge encouraged students in the Ecole Polytechnique to do (Belhoste and Taton 1992). It seems that, by contrast, we can note a shift in Poncelet’s reflections with the fourth manuscript (“cahier”) entitled “Considérations philosophiques et techniques sur le principe de continuité dans les lois géométriques (Philosophical and technical considerations on the principle of continuity in geometrical laws)” (Poncelet 1864: 296-362). Indeed, in contrast with the previous manuscript, this one, which Poncelet dates from the winter 1818-1819, starts with the statement of a project that is now quite similar to the one that motivates the 1822 treatise (Poncelet 1864: 296-298). As we will see, this shift is correlated with a new and massive use of the adjective “ideal,” with, however, a meaning different from the meaning that appears in 1820 and will be the one used in the 1822 treatise.

<sup>30</sup> (Poncelet 1822: xxxiii). In a text written in 1823, presented at the Académie des sciences in March 1824, and published in *Crelle’s Journal* (Poncelet 1828), the geometer returns to these principles, puts forward additional principles, and adds further explanation.

<sup>31</sup> (Poncelet 1822: xxxiii). In (Poncelet 1828), this principle is promoted as the first one, while in (Poncelet 1822), it features as the second one. (Poncelet 1828) discusses further both the descriptive and metrical properties of this kind, devoting a “second note” to the latter. Poncelet’s systematic consideration of metrical projective properties has been largely ignored in the secondary literature (Belhoste and Chemla 2023 Forthcoming).

established on a general state of a figure holds true for any other general state of the same figure.<sup>32</sup> When introducing it, Poncelet refers to it as “the *principle* or the *law of continuity* of mathematical relations of the abstract and figurate magnitude (*le principe ou la loi de continuité* des relations mathématiques de la grandeur abstraite et figure).” (Poncelet 1822: xxiii. Emphasis is his). As we will see, the full expression is meaningful, but, to refer to the principle in a more convenient way, we will shorten it below into “the principle of continuity.” The introduction of ideal elements will be intimately related to the reflections Poncelet devotes to this principle between 1815 and 1822. In the third and fourth notebooks, composed between 1815 and 1818 (see footnote 29), Poncelet discusses, under the same name, principles of this kind. This is also precisely the principle criticized by Cauchy’s report of the 1820 memoir—which forms the fifth manuscript in (Poncelet 1862). We leave for a future publication a complete examination of the evolution of the principle, with its transformations that can be identified through Poncelet’s manuscripts, focusing here simply on what sheds light on Poncelet’s notion of “ideal elements” that will be taken up by Kummer.

In the introduction to his 1822 treatise, Poncelet devotes many pages to a discussion of the “*principle or the law of continuity* of mathematical relations of the abstract and figurate magnitude.” The principle draws in an essential manner on a way of conceiving a figure, with which Poncelet begins as follows:

Let us consider an *arbitrary figure*, in a *position* that is *general* and in a certain sense *indeterminate*, among all those that the figure could take without *violating the laws*, the *conditions* and the *link* that *persist* between the various *parts* of the *system*; (Considérons une *figure quelconque*, dans une *position générale* et en quelque sorte *indéterminée*, parmi toutes celles qu'elle peut prendre *sans violer les lois*, les *conditions*, la *liaison* qui *subsistent* entre les diverses *parties* du *système*;) (Poncelet 1822: xxii, my emphasis)

Here, Poncelet introduces two terms to refer to the object that is taken as his subject of inquiry. At the beginning of the quotation, he refers to this object as a “figure,” which can have several “positions.” The second half of the quotation makes clear what a “position” is, by reference to the object now taken as a “system,” which is composed of parts—for instance, a line and a curve in a plane. Indeed, these two parts of the system can intersect each other or not: these are two “general” positions of the figure, in contrast to a position in which the line would be tangent to the curve. In other words, the figure for Poncelet is not the figure of the ancient geometry, but it is composed of parts and encompasses any configuration that the different parts can take.<sup>33</sup> However, Poncelet adds a key restriction, when he specifies that, in any position considered, “the laws, the conditions and the link that persist between the various parts of the system” should not be “violat[ed].”<sup>34</sup> What does he mean by these terms?

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<sup>32</sup> On this opposition, and the relationship between Lazare Carnot’s principle of correlation between figures and Poncelet’s principle of continuity, see (Chemla 1998 and 2016).

<sup>33</sup> In the 1820 memoir (Poncelet 1864: 387), he refers to the “intention that one has to extend the conception of a figure, presently geometric and possible, to all the states that this figure can take, even to those in which certain objects lose their absolute existence (la volonté qu’on a d’étendre la conception d’une figure, actuellement géométrique et possible, à tous les états que peut prendre cette figure, même à ceux où certains objets perdent leur existence absolue.)” Poncelet (1822: 50) repeats this sentence almost *verbatim*, with two exceptions: instead of “all the states that a figure can take”, he writes “the various states through which a figure can go.” In addition, instead of “certain objects lose their absolute existence,” he specifies “their absolute and real existence”.

<sup>34</sup> Earlier, for instance in the fourth manuscript of winter 1818-1819, Poncelet (1864: 300) would speak of “comparing a figure with all those that may be supposed to result from it through the progressive and continuous

For the example of the curve and the line in a plane precisely, in a letter to Terquem from October 14, 1819, in which he replies to objections made to his principle of continuity, Poncelet clarifies what he means by “conditions.” The fact that the line and the curve be in the same plane implies that an equation between the “constants” defining these parts holds for them to intersect, and this “condition” is violated when the line does not belong to the plane of the curve (Poncelet 1864: 549). Note that this example highlights the fact that, for Poncelet, the condition “persists” whether the intersections are real or not.

Poncelet (1822: 95) further gives the example of a “link,” when he mentions the relation of involution that exists between the six points that a transversal the system of a conic section and a quadrilateral inscribed in it. Again, this link is conceived as persisting, even when the intersections are no longer real.<sup>35</sup>

It is precisely for figures conceived in this way that the principle of continuity is meaningful. It even, in fact, underlies this very conception of the figure. Poncelet formulates the principle immediately after, as follows:

“suppose that, on the basis of these data, one *found one or more relations or properties*, either *metrical*, or *descriptive*, that belong to the figure, using the ordinary explicit reasoning, that is to say, using this way of proceeding that, in certain cases is the only one considered as rigorous. Is it not obvious that if, keeping the same data, one makes the *figure vary by imperceptible degrees*, or one applies to some parts of this figure a *continuous motion* (in fact arbitrary), is it not obvious that the properties and relations that were *found for the first system* will *continue to apply* for its *successive states*, provided, however, that one pays attention to the particular changes that might have occurred, as when some magnitudes have vanished, or have changed direction or sign, etc.—changes that will always be easy to recognize *a priori* and using safe rules?

(supposons que, d'après ces données, on ait *trouvé une ou plusieurs relations ou propriétés*, soit *métriques*, soit *descriptives*, appartenantes (sic) à la figure, en s'appuyant sur le raisonnement explicite ordinaire, c'est-à-dire par cette marche que, dans certains cas, on regarde comme seule rigoureuse. N'est-il pas évident que si, en conservant ces mêmes données, on vient à faire varier la figure primitive par degrés insensibles, ou qu'on imprime à certaines parties de cette figure un mouvement continu d'ailleurs quelconque, n'est-il pas évident que les propriétés et les relations, trouvées pour le premier système, demeureront applicables aux états successifs de ce système, pourvu toutefois qu'on ait égard aux modifications particulières qui auront pu y survenir, comme lorsque certaines grandeurs se seront évanouies, auront changé de sens ou de signe, etc., modifications qu'il sera toujours aisé de reconnaître *a priori* et par des règles sûres?)” (Poncelet 1822: xxii, my emphasis, except for “metrical”, “descriptive” and “a priori”, which are emphasized by Poncelet)

The principle thus asserts that, having established properties or relations through reliance on a general position of a figure, conceived along the lines discussed above, the same

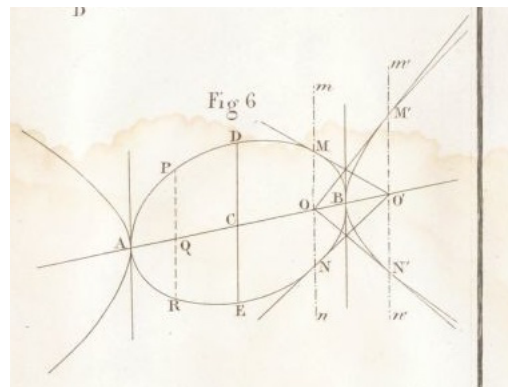
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motion of some of the parts that enter in the figure, without violating the link and the dependence that were primitively established between these parts (de comparer une même figure avec toutes celles qui peuvent être censées en résulter par le mouvement progressif et continu de certaines des parties qui y entrent, sans violer la liaison et la dépendance primitivement établies entre elles.)” The same expression recurs in the 1820 memoir (Poncelet 1820: 451).

<sup>35</sup> This idea is expressed quite clearly, e.g., in the fourth manuscript of the winter 1818-1819 (Poncelet 1864: 341-342).

properties or relations “continue to apply,” even when the figure is in another general position. This use of the verb “continue” here reflects another expression used by Poncelet elsewhere, when, as we have seen, to refer to the principle, he speaks of the “*principle of the continuity or permanence of mathematical relations of the figurate magnitude* (principe de la continuité ou permanence des relations mathématiques de la grandeur figurée)” (Poncelet 1864, fourth manuscript: 319). In other words, the idea of “continuity” here connotes the motion applied to the system that ensures that the validity of the properties and relations established persists—this is, the meaning that Cauchy attributes to the term, when he criticizes the principle in his report (Cauchy 1820: 72-73). However, for Poncelet, “continuity” simultaneously connotes the persistence of the properties and of the relations between its parts that enter into the definition of the figure. Their “permanence” does not mean that they stay identical, since their formulation changes in relation to transformations in the figure.

These assertions can be illustrated using the secants and the chords of a conic section—the very context in which precisely Poncelet introduces precisely the notion of “ideal chord.”<sup>36</sup>



**Figure 1: Poncelet’s Figure 6, *Traité des propriétés projectives des Figures*, 1822, plate 1**

The line  $mn$  being drawn in the plane of a conic section, Figure 1 shows that it intersects the conic section in  $M$  and  $N$ . The middle point  $O$  of the chord  $MN$  thereby defined belongs to the diameter  $AB$  that is conjugate to the direction of  $mn$ , as do the centers of all the chords that lines parallel to  $mn$  introduce in the conic section (e.g.,  $DE$ , or  $QR$ ). Moreover, the tangents to the conic section at  $M$  and  $N$  meet in  $O'$ , which is likewise on the conjugate diameter  $AB$ . What is more,  $A$ ,  $B$ ,  $O$  and  $O'$  satisfy the following relation:

$$\frac{O'A}{O'B} = \frac{OA}{OB}$$

Even though  $O$  and  $O'$  were constructed by reference to the points  $M$  and  $N$ , these points “persist” even if the line  $mn$  is continuously moved parallel to itself till it becomes  $m'n'$ . From the point  $O'$ , in which the line in its “new position” meets with the diameter  $AB$ , drawing the two tangents to the conic section,  $O'M$  and  $O'N$ , defines the chord  $MN$ . Its intersection with  $AB$ ,  $O$ , satisfy the same relation as above with the points  $A$ ,  $B$  and  $O'$ . Poncelet thereby points out the “relation” between the elements  $mn$ ,  $m'n'$ ,  $O$  and  $O'$  that holds for the figure: Given any of them, the other three are determined. Poncelet continues:

<sup>36</sup> (Poncelet 1822: 26ff). See (Belhoste and Chemla 2023 Forthcoming) for greater detail.

“since the points  $O$  and  $O'$ , considered as belonging to the line  $mn$ , *subsist independently of the reality or the non-reality* of the intersections  $M$  and  $N$  of this line with the curve, there is no reason to neglect them in one case and not in the other; nor is there reason to neglect the line itself, since, when it becomes exterior to the curve, it does not cease to *maintain certain dependences* with it. (...les points  $O$  et  $O'$ , considérés comme appartenant à la droite  $mn$ , *subsistent indépendamment de la réalité ou de la non-réalité* des intersections  $M$ ,  $N$  de cette droite et de la courbe, il n'y a aucune raison de les négliger plutôt dans un cas que dans l'autre, non plus que cette droite elle-même, puisqu'en devenant extérieure à la courbe elle ne cesse pas de *conserver certaines dépendances* avec elle.)” (Poncelet 1822: 27, my emphasis)

To put it differently, the same geometrical entities can be seen from different viewpoints. Viewing  $O$  as the center of the chord  $MN$  might lead us to consider it vanishes when  $M$  and  $N$  vanish. However, points such as  $O'$  are tied to the figure by other *relations* that highlight that these points  $O'$  *subsist*. What is more, as a result, the part of the system that the line  $m'n'$  constitutes has *enduring relationships* with the other part of the system constituted by the curve, even when they do not meet. Since the relationships persist, Poncelet chooses to use terms that reflect this “continuity” to refer to the geometrical objects. This leads him to discard the usual name of  $m'n'$  as the “polar line of  $O$ ,” and to refer to it rather as an “ideal secant” of the conic section (Poncelet 1822: 27). Similarly, he refers to  $M'N'$  as an “ideal chord,” whereas, for him,  $M'$  and  $N'$  are imaginary points.<sup>37</sup>

Why, one might ask, does Poncelet here distinguish between “ideal” and “imaginary”? We have reached the crux of our argument. Let us read the clearcut explanation that Poncelet develops right away:

“In general, one could designate, with the adjective *imaginary*, any object that, from absolute and real that it was in a given figure, would have become entirely impossible or inconstructible in the *correlative* figure: ‘the one that is supposed to derive from the first one<sup>38</sup> by the progressive and continuous motion of some of its parts, without violating the primitive laws of the system;’ the epithet *ideal* would serve to designate the specific mode of existence of an object which, while remaining in contrast (KC: with the previous case) real in the transformation of the primitive figure, would nevertheless cease to depend in an absolute and real manner on other objects which define it graphically, because these objects would have become imaginary. Since, in the same way as one already has in geometry names to express various modes of existence that one wants to compare, like *infinitely small* and *infinitely large*, one must also have names to express those of non-existence, to give accuracy and precision to the language of geometric reasoning.

By comparison with all the definitions that one might want to substitute for them, these definitions have the advantage that they can be extended directly to any point, line and surface (...); they are useful to shorten the discourse and broaden the object of geometric conceptions; lastly, they allow us to establish a contact, if not always real, at least fictive, between figures that at first sight seem to have no

<sup>37</sup> For a more detailed analysis of differences between these two types of ideal elements, see (Belhoste and Chemla 2023 Forthcoming).

<sup>38</sup> Note of the translator: that is, the primitive figure. This terminology comes from Lazare Carnot’s geometry of position (see Chemla 1998).

relationship with each other, and to discover easily the relations and properties that are common to them.

(En général on pourrait désigner par l'adjectif *imaginaire* tout objet qui, d'absolu et réel qu'il était dans une certaine figure, serait devenu entièrement impossible ou inconstructible dans la figure *corrélative*: 'Celle qui est censée provenir de la première par le mouvement progressif et *continu* de quelques parties, sans violer les lois primitives du système;' l'épithète *idéal* servirait à désigner le mode particulier d'existence d'un objet qui, en demeurant au contraire réel dans la transformation de la figure primitive, cesserait cependant de dépendre d'une manière absolue et réelle d'autres objets qui le définissent graphiquement, parce que ces objets seraient devenus imaginaires. Car, de même qu'on a déjà en Géométrie des noms pour exprimer les divers modes d'existence qu'on veut comparer, tels que *infinitement petits*, *infinitement grands*, il faut aussi en avoir pour exprimer ceux de la non-existence, afin de donner de la justesse et de la précision à la langue du raisonnement géométrique.

Ces définitions ont, sur toutes celles qu'on pourrait leur substituer, l'avantage de pouvoir s'étendre directement à des points, des lignes et des surfaces quelconques; (...) elles servent à abrégier le discours et à étendre l'objet des conceptions géométriques; enfin elles permettent d'établir un point de contact, sinon toujours réel, au moins fictif, entre des figures qui paraissent, au premier aspect, n'avoir aucun rapport entre elles, et de découvrir sans peine les relations et les propriétés qui leur sont communes.) ” (Poncelet 1822: 28. Poncelet's emphasis is in italics; I underline what I emphasize).

In other words, Poncelet distinguishes between “imaginary” and “ideal” as two modes of non-existence. The former can be illustrated by the points of intersection between the secant  $m'n'$  and the conic section, while the latter is illustrated precisely by the “ideal secant”  $m'n'$  and the “ideal chord”  $M'N'$ : they can still be constructed, although their dependence on the curve has become imaginary. As the final part of the quotation makes clear, the promotion of these terms derives from Poncelet's reflections on how to use language in science as well as on how to instill generality in geometry. Here, the point is to use general enough concepts and general practices with figures and properties.

Moreover, clearly, Poncelet defines here ideal and imaginary elements with full generality, and he insists that the definitions apply broadly. In the treatise, he will qualify as “ideal” various types of geometrical elements. However, “ideal secants” and “ideal chords” remain the key ideal elements that he uses. Moreover, in the 1820 memoir, even though the notion of ideality is clearly the same, “ideal secants” and “ideal chords” are the only “ideal elements” put into play.<sup>39</sup>

### ***3.3 Ideal elements as an outcome of Poncelet's philosophical work***

The previous quotation is interesting for an additional reason. Indeed, it highlights that Poncelet conceives of the distinction between ideal and imaginary as that between different modes of non-existence, and that he opposes them in this respect to “infinitely small” and “infinitely large”, understood as “names” expressing “various modes of existence.” This way of formulating the opposition is striking because the various manuscripts published by Poncelet show that Poncelet varied significantly in the appreciation of the understanding of and the relationship between these notions. I will limit myself to some remarks on this point,

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<sup>39</sup> See the discussion in (Poncelet 1864: 367-370), which is quite similar to what I have just quoted.

to the effect of showing that Poncelet's 1822 terminology was the outcome of a philosophical work, which seems to have reached a stable state in 1822.

The fourth manuscript published in (Poncelet 1864) and completed during the winter 1818-1819, which was devoted to the principle of continuity, contains very important pieces of evidence in this respect. To begin with, clearly for Poncelet, the idea of making use of “beings of non-existence (*êtres de non-existence*)” in mathematics is due to the practice of analysis. Indeed, he writes:

“the use and routine of algebraic Analysis over the last two centuries (...) [have] *introduce*[ed] and *enshrine*[ed], in the ordinary language of Geometry, certain *expressions* which recall and characterize these *beings of non-existence*, and [they] even make us consider these *fictitious* beings *as real objects* in reasoning and conception. (L'usage et l'habitude de l'Analyse algébrique pendant deux siècles (...) [ont] *introdui*[t] et consacré], dans le langage ordinaire de la Géométrie, certaines *expressions* qui rappellent et caractérisent les *êtres de non-existence*, et font même considérer ces êtres *fictifs* comme des *objets réels* dans le raisonnement et la conception.)” (Poncelet 1864: 344, my emphasis).

The example that Poncelet gives to illustrate this statement is highly meaningful for us. It reads as follows:

This is how the *infinitely large* and *infinitely small* [entities], whose existence is purely hypothetical, have already been introduced into Geometry; For, when, e.g., we admit that two parallel straight lines meet at infinity, we *ideally* give an *indefinite existence* to their common point of intersection, and thereby also establish the idea of *continuity* in the possible *motion* of this point. On the other hand, the word *infinite* recalls the *geometrical non-existence* of the object designated, and it indicates that the *assumed existence* is purely *ideal*. (C'est ainsi que l'on a déjà reçu en Géométrie les *infiniment grands* et les *infiniment petits*, dont l'existence est purement hypothétique; car, lorsqu'on admet, par exemple, que deux droites parallèles se rencontrent à l'infini, on donne *idéalement* une existence indéfinie à leur point commun d'intersection, et par là aussi on établit l'idée de la *continuité* dans le mouvement possible de ce point. D'un autre côté, le mot *infini* rappelle la *non-existence géométrique* de l'objet désigné, et indique que l'existence supposée est purement *idéale*.)” (Poncelet 1864: 344-345, Poncelet emphasizes infinitely large, infinitely small, and infinite. The other italicized words are my emphasis)

From this quotation, we will retain only three points.

First, the way in which Poncelet uses the term “continuity” in relation to the point at infinity does seem to combine the idea of the continuity of the “motion” as well as that of the continuity of the relationship between lines (namely, that they intersect in one and a single point), thereby confirming what we have seen above.

Secondly, in contrast to Poncelet's 1822 quotation with which we have concluded the previous section, here, in Poncelet's view, the point at infinity, which is the point of intersection of parallel lines, has no “geometric existence.” In other words, between 1818-1819 and 1822, Poncelet has recast his idea about existence and non-existence in geometry.<sup>40</sup>

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<sup>40</sup> The issue of “existence” becomes very prominent in Poncelet's reflections starting from the fourth notebook, composed during the winter 1818-1819.



Thirdly, in the same quotation, Poncelet uses the term “ideal” in a way completely different from the use that is evidenced from his 1820 memoir onwards. Indeed, in 1818-1819, the adjective “ideal” is applied to the type of existence that is given to a point at infinity. A few pages later, Poncelet has exactly the same use of the adjective “ideal,” when analyzing the sentence “these two points have become imaginary”: in his view, the expression is introduced to “maintain for these points an existence of sign, at least *ideal*, in discourse and conception” (Poncelet 1864: 350, my emphasis).<sup>41</sup> Still in the same work, Poncelet employs the same term “ideal” to refer to the nature of the relations and properties that are claimed by continuity, when they bring into play elements of the two latter types.

Exactly the same use of “ideal” as well as the connection between this use and the principle of continuity are also manifest in a letter that he writes on November 11, 1818—that is, more or less at the same time as he is working on this fourth notebook. Indeed, he writes the following to Olry Terquem (1782-1862), one of his friends with whom he communicates about his mathematical ideas: “The axiom examined so far, when considered from a certain perspective, is in the end nothing more than the *principle of permanence or indefinite continuity of the mathematical laws of magnitudes that are variable by imperceptible succession*, a continuity that, for certain states of the same system, often subsists only in a purely abstract and *ideal* fashion (L'axiome jusqu'ici examiné n'est au fond, quand on le considère sous un certain point de vue, que le *principe de permanence ou continuité indéfinie des lois mathématiques des grandeurs variables par succession insensible*, continuité qui pour certains états d'un même système ne subsiste souvent que d'une manière purement abstraite et *idéale*.)” (Poncelet 1864: 533, except for “ideal”, Poncelet’s emphasis)

To conclude, the term “ideal,” which is consistently used in this way in the fourth notebook, changes meaning in Poncelet’s writings between 1818-1819 and 1820. In 1818-1819, “ideal” qualifies the way in which non-existent elements are nevertheless assumed to exist, and it also qualifies the way in which relations and properties extend to them. In contrast, starting from 1820, “ideal” refers to “elements” that are real, but whose connection to the other parts of the figure has become imaginary. Moreover, this change in the meaning of the term “ideal” is, as we have seen, correlated with the change in the understanding of the nature of the infinite magnitudes or elements at infinity

What is striking is that, still in the same notebook from the winter of 1818-1819, Poncelet formulates a research program for metaphysics, as follows:

“(…) just as we have *names* to express the *various modes of existence* that we want to *compare*, we must also have *names to express the various modes of non-existence*, in order to give both accuracy and precision to the language of geometrical reasoning. In addition, the *metaphysical notions* themselves have their true *source* in the *perseverance* to apply the idea of *continuity* to a purely *ideal* state of a system, and to *extend to it conceptions and laws that belonged only to the primitive state of this system, which was real*.

From these ideas, we can see how we should go about tracing a faithful and complete *table of all the metaphysical notions* that belong to *figurate magnitude*. My intention is not to undertake this [table]. It would require more boldness and talent than I possess to dare deliver such a work to the judgment of the severe

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<sup>41</sup> In fact, in the previous manuscript (the third notebook), written between 1815 and 1817, Poncelet used “ideal” as interchangeable with “imaginary” and “inconstructible” and in parallel with infinitely small and large entities. From a given perspective that we need not make explicit, he asserts: “The same thing will still hold true for the whole interval in which these distances remain inconstructible, ideal, imaginary, and that in which they vanish or become infinite momentarily, transiently (La même chose aura lieu encore pour tout l'intervalle où ces distances resteront inconstructibles, idéales, imaginaires, et celui où elles s'évanouissent ou deviennent infinies momentanément, transitoirement.)” (Poncelet 1864: 198).

criticism of geometers: it will suffice for me to have indicated to those more skillful than I am the road that must be followed, and to have taken the first steps along it.

((...) de même qu'on a des *noms* pour exprimer les *divers modes d'existence* qu'on veut *comparer*, il faut aussi en avoir pour *exprimer ceux de la non-existence*, afin de donner à la fois de la justesse et de la précision à la langue du raisonnement géométrique. Enfin, les *notions métaphysiques* elles-mêmes ont leur *source véritable* dans la *persévérance* qu'on a d'appliquer l'idée de la *continuité* à un état purement *idéal* d'un système, et d'y *étendre les conceptions et les lois qui n'appartenaient qu'à l'état primitif et réel de ce système*.

D'après ces idées, on voit comment il faudrait s'y prendre pour parvenir à tracer le tableau fidèle et complet de *toutes les notions métaphysiques* qui appartiennent à la *grandeur figurée*; mon intention n'est pas de l'entreprendre, il faudrait plus de hardiesse et de talent que je n'en possède pour oser livrer un tel travail au jugement de la critique sévère des géomètres: il me suffira d'avoir indiqué à de plus habiles que moi la route certaine à suivre, et d'y avoir fait les premiers pas. (Poncelet 1864: 345-346, emphasis is mine)

The way in which this research program in metaphysics is formulated during the winter 1818-1819 has striking echoes with the long quotation from the 1822 treatise with which we have concluded the previous section. Both texts even share the same sentence on the principles and the epistemological values that should be respected to shape the language to be used in geometry. In the text just above, the term “ideal” still has its meaning from 1818-1819, whereas in the 1822 text, the term takes the new meaning that would remain the one Poncelet would systematically use after 1820. We thus see that the shift of meaning of the term “ideal” and the new understanding of the modes of non-existence as well as of those of existence derive from Poncelet’s pursuit of the inquiry into metaphysics, for which he was formulating the main questions in 1818-1819.

To conclude, when Kummer discarded the term “imaginary” and opted instead for the term “ideal,” he was making a distinction that Poncelet had made before him, as the outcome of the pursuit of metaphysical inquiry. This naturally leads us to the question of the pieces of evidence we have for the reception of Poncelet’s geometrical theories in German-speaking countries.

### **3.4 The reception of Poncelet’s ideal elements in Germany**

As early as in the first issue of the *Journal für die reine und angewandte Mathematik*, in 1826, the editor August Crelle published a short review of Poncelet’s *Traité des Propriétés Projectives des Figures*.<sup>42</sup> The review insists on the fruitfulness of the use of projections in geometry, and on the interest of the theorems that Poncelet obtained. However, it mentions neither the principle of continuity nor ideal elements.

The situation would change radically with the third volume of the journal, published in 1828. First, Poncelet could publish his first geometrical contribution to the journal and present

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<sup>42</sup> (Crelle 1826: 96). The review is anonymous. However, Poncelet (1866: 407) attributes it to Crelle. As early as 1808, Poncelet, who was a first year student at the Ecole Polytechnique, had the opportunity of making Alexandre von Humboldt’s (1769-1859) acquaintance (Poncelet 1866: 406). He attributes to the latter the fact that Crelle’s *Journal* opened its pages to the publications of his articles (Poncelet 1866: 407), and, to begin with, the publication of an article of “applied mathematics” in 1827, and then that of (Poncelet 1828). Crelle was a personal friend of Poncelet, their acquaintance dating to before 1830 (Poncelet 1866: 407).

an overview of his theory (Poncelet 1828). There, he outlined his ideas about the principle of continuity and his motivations to introduce it (Poncelet 1828: 215). He further added:

“The consequence of admitting continuity was the theory of *ideal secants* and *ideal chords*, and of any object which, without ceasing to exist effectively in the transformations of a single figure, has nevertheless ceased to depend in a purely geometrical manner on other objects to which it related, and which defined or constructed it in the primitive figure. (La conséquence de l'admission de la continuité a été la théorie des *sécantes* et des *cordes idéales* et, de tout objet qui, sans cesser d'exister effectivement dans les transformations d'une même figure, a pourtant cessé de dépendre d'une manière purement géométrique, d'autres objets auxquels il se rapportoit, et qui le définissoient ou le construisoient dans la figure primitive.) (Poncelet 1828: 215-216, Poncelet's emphasis is italicized, mine is expresses through underlining terms)

The conception of ideal elements presented here is exactly the same as that to which Poncelet adhered since 1820. Moreover, the conception is presented in a fully general manner, and not only for ideal secants and chords. One might nevertheless notice that Poncelet did not introduce it in contrast with imaginary elements, as he had done in his treatise.

The outline of Poncelet's theory was formulated in French. However, in the same issue, Carl Gustav Jacobi (1804-1851) published an article in German (Jacobi 1828), in which he referred to Poncelet's book as “the famous work *Traité des propriétés projectives des figures*” (Jacobi 1828: 377), quoting Poncelet's treatise abundantly. Indeed, Jacobi (1828) made use of several theorems that Poncelet has established in his book. What is more, in a footnote, Jacobi develops still in German a full explanation for the introduction of “ideal secants”—not chords. Poncelet in fact had made Jacobi's acquaintance in Paris no later than 1829, and writes some recollections of their mathematical conversation on the topic (Poncelet 1864: 485-488).

In the years following the publication of Jacobi's article, several German works about geometry mention “ideal secants” and “ideal chords.” For instance, in 1830, Christoph Gudermann (1798-1852) publishes his *Grundriss der analytischen Sphaerik*. In it, he mentions an opposition between real and ideal chords (for him, *Durchschnittsehne*) as if it were common knowledge, without any specific explanation (Gudermann 1830: 138). Another example is a book related to the lectures in descriptive geometry that Guido Schreiber (1799-1871) gave in the Polytechnic school of Karlsruhe. In it, the author introduces ideal chords (for him, *Durchschnittslinie*), with an explanation and a reference to Poncelet (Schreiber 1839: 200, §570). Again, later, in 1844, Franz Seydewitz (1807-1852), an *Oberlehrer* in the *Gymnasium* of Heiligenstadt, publishes an article in which he makes an extensive use of ideal elements—also in the context of double-column presentation of dual propositions—, making in conclusion an indirect reference to Poncelet (Seydewitz 1844: 258).

It is thus not surprising that Kummer knew about “ideal chords”—maybe through reading German works making use of them, or maybe through discussions with Jacobi. Why, however, did he choose to illustrate the notion of ideality in geometry using “ideal chords”? We return to this point in the next section.

However, more importantly for us, to understand which meaning Kummer attached to “ideal chords,” and which inspiration he drew from them, we now need to turn to Michel Chasles's reconceptualization of Poncelet's ideas about continuity and ideality.

#### 4. Michel Chasles's on contingent and permanent properties

Indeed, in 1837, Michel Chasles published a historical overview of the development of methods in geometry. In this essay, he reviewed the history of geometry from Euclid's *Elements* to the new geometry of his time, his main goal being to highlight how the methods newly acquired by geometry could now compete with the powerful methods that had hitherto been the preserve of analysis applied to geometry, especially from the viewpoint of generality (Chasles 1837, Chemla 2016).

In this context, Chasles addresses precisely the type of proof that made use of what Poncelet has introduced under the name of “principle of continuity,” referring to it as “a new mode of demonstration” (Chasles 1837: 197). Like Poncelet (1864: 309-311), and by reference to exactly the same example, Chasles attributes to Monge the use of such proofs in mathematical practice (Chasles 1837: 197-198). Chasles further makes clear that in contrast to other disciples of Monge's, who followed his practice without justifying it, Poncelet was the first one who made the underlying principle explicit, even though he failed to prove it rigorously (Chasles 1837: 199-200). Chasles evokes and appears to embrace Cauchy's criticism thereof.

Accordingly, Chasles will offer an alternative analysis of the principle underlying Monge's practice. The formulation that he offers right away for this principle does not make any reference to “continuity” (Chasles 1837: 198). A few pages later, he clarifies that he prefers not to make use of the notion of “continuity,” arguing that this would bring in the “idea of the infinite.”<sup>43</sup> Nor does Chasles make use of any “ideal” elements. Indeed, he formulates his principle, making use of new concepts, some of which feature in the name he chooses for this principle: “the principle of contingent relationships” (Chasles 1837). Since this reformulation and Chasles's related reflections will play a key role in Kummer's introduction of ideal elements, I will now examine them in detail.

Like Poncelet, Chasles considers figures as made of parts that could be, e.g., “points, planes, lines or surfaces” (Chasles 1837: 198). On this basis, he begins by introducing the key notion of “circumstances of construction.” (Chasles 1837: 199-200). This notion allows Chasles to distinguish, for the same figure, between various dispositions that these elements can have with respect to one another. Some of these dispositions derive from “particular circumstances of construction” (Chasles 1837: 199). If we consider the example of a figure consisting of two circles in a plane, this is the case when these circles are tangent to each other. However, other dispositions derive from “general circumstances of construction” (Chasles 1837: 200). The latter can be illustrated by the case when the two circles cut each other, or by that in which they do not cut each other. In other words, a figure can present different cases, which, Chasles emphasizes, have the “same generality” (Chasles 1837: 199). To analyze the relationship between the same figure drawn in different general circumstances of construction, Chasles introduces the following fundamental opposition:

“(…) a figure can present *in its most general construction* two cases; in the first one, some parts (points, planes, lines, or surfaces), on which the general construction of the figure does not necessarily depend, but that are contingent or accidental consequences of it, are real and palpable; in the second case, these same parts no longer appear; they have become imaginary; and yet the general conditions of construction of the figure remained the same. ((…) une figure peut présenter dans sa construction la plus générale deux cas différents (sic); dans le premier, certaines parties (points, plans, lignes ou surfaces) d'où ne dépend pas nécessairement la construction générale de la figure, mais qui en sont des

<sup>43</sup> On how Chasles deals with elements at infinity, see (Chasles 1837: 746).

conséquences *contingentes* ou accidentelles, sont réelles et palpables; dans le second cas, ces mêmes parties n'apparaissent plus; elles sont devenues imaginaires; et cependant les conditions générales de construction de la figure, sont restées les mêmes.” (Chasles 1837: 198, italicized terms are Chasles’s emphasis, whereas I emphasized through underlining)

Some parts in the figure are “real” in one case, whereas they “become imaginary” in the other. These are, for instance, the two points of intersection in the plane figure consisting of two circles. Chasles refers to such parts as “contingent or accidental,” to oppose them to the parts of the figure to which he refers as the “*integrant and permanent parts* of the figure” (Chasles 1837: 200). The latter parts “belong to its general construction,” independently of the case considered, and “they always real” (Chasles 1837: 200).

Even though the distinction between the various “general circumstances of construction” is important, Chasles insists, it vanishes in the context of an application of the analysis to geometry, which does not discriminate between these cases when establishing “theorems concerning the *integrant and permanent parts* of the figure.” Such results consequently apply to all cases, independently of the reality or imaginary character of the “contingent parts” (Chasles 1837: 200, his emphasis). For Chasles, reasonings of this kind, which “rely on the general procedures of analysis” justify—with some caveats—*a posteriori* the “principle of contingent relationships,” which underlies Monge’s method and which he formulates as follows:

“To define this method, we will say that it consists in considering the figure on which we have to prove some general property, in general circumstances of construction, where the presence of certain points, planes or lines, which in other circumstances would be imaginary, makes the demonstration easier. We then apply the theorem that we have proved in this way to the cases of the figure where these points, planes and lines would be imaginary; that is, we consider the theorem as true in all the general circumstances of construction that the figure to which it refers may present. (Pour définir cette méthode, nous dirons qu'elle consiste à considérer la figure, sur laquelle on a à démontrer quelque propriété générale, dans des circonstances de construction générale [NoT : elsewhere, Chasles speaks of “circonstances générales de construction”], où la présence de certains points, de certains plans ou de certaines lignes, qui dans d'autres circonstances seraient imaginaires, facilite la démonstration. Ensuite, on applique le théorème qu'on a ainsi démontré aux cas de la figure où ces points, ces plans et ces droites seraient imaginaires; c'est-à-dire, qu'on le regarde comme vrai dans toutes les circonstances de constructions générales que peut présenter la figure à laquelle il se rapporte).” (Chasles 1837: 198).

In other words, with Chasles’s terminology, Monge’s method consisted in proving a theorem about a figure by reliance on one of its states drawn in general circumstances of construction. The theorems considered bore only on permanent parts of the figure. However, for such theorems, the practitioner would choose, between these different cases, one that would make the proof easier, owing to the fact that some contingent parts would be real. The truth of the theorem would actually be established for this case. However, once this proof is carried out, the theorem could be claimed to be valid indiscriminately for the figure drawn in any of its general circumstances of construction.<sup>44</sup>

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<sup>44</sup> Chasles (1837: 203 and 205) gives alternative formulations of the principle.

However, Chasles considers this practice as unsatisfactory insofar as its validity rests on using analytical reasoning *a posteriori*. In order to remain closer to the “rigorous principles of the Ancients,” Chasles (1837: 205) invites the practitioner to go further and, he says, deeper. Indeed, once a result established in this way, he suggests one might look for another proof as follows:

“Indeed, after having discovered a truth using Monge’s somewhat superficial method, which *grasps and makes use of some external and palpable circumstance*, which, however, is *fortuitous and ephemeral*, one will have (in order to *establish this truth on reasons that are permanent and independent of variable circumstances of construction of the figure*) to get to the bottom of things and make use not any longer, like Monge, of *secondary and contingent properties*, which, *in some cases, suffice to define* various parts of the figure, but *rather of properties of these same parts* of the figure that are *intrinsic and permanent*. (Il faudra en effet, après avoir découvert quelque vérité par la méthode, en quelque sorte superficielle, de Monge, qui *s’empare et tire parti de quelque circonstance externe et palpable, mais accidentelle et fugitive*, il faudra, dis-je, pour *établir cette vérité sur des raisons permanentes et indépendantes des circonstances variables de construction* de la figure, aller au fond des choses et faire usage *non plus* comme Monge, des *propriétés secondaires et contingentes* qui suffisent, dans certains cas, pour définir diverses parties de la figure, mais bien des *propriétés intrinsèques et permanentes de ces mêmes parties de la figure*) (Chasles 1837: 205, emphasis is mine)

The description of an alternative proof that would rely only on “permanent reasons” requires a reflection on what counts as “permanent elements.” In this context, Chasles introduces a new element that will be crucial for my argument. Chasles points out the fact that we might define the *same parts* of a figure in two ways, essentially distinct from the perspective of the proof sought for. The same parts might be “defined” using “contingent properties.” However, they might also enjoy “intrinsic and permanent properties.” Chasles adds specification on these properties:

“By intrinsic and permanent properties, we mean those that in all cases could be used to define and construct parts of the figure that we have called *intégrant* or *principale*, whereas the *secondary* and *contingent* properties are those that can disappear and become imaginary in some circumstances of construction of the figure.” (Nous entendons par propriétés intrinsèques et permanentes celles qui serviraient, dans tous les cas, à la définition et à la construction des parties de la figure que nous avons appelées *intégrantes* ou *principales* ; tandis que les propriétés *secondaires* et *contingentes* sont celles qui peuvent disparaître et devenir imaginaires dans certaines circonstances de construction de la figure.) (Chasles 1837: 205. Chasles’s emphasis is in italics, mine is in the underlined passages)

In other words, some parts of the figure might appear to be contingent, because we have chosen a contingent property to “define and construct them.” However, one might as well identify a permanent property that they fulfil, and that might highlight the fact that these parts are actually permanent. This remark implies that viewing an object as permanent is the outcome of mathematical work, and is not immediately given. For the second type of proof that Chasles has in mind, a task of this kind is indeed essential.

It is in the context of this discussion, about the change of definition of an object from a contingent property to a permanent one, that Chasles chooses to illustrate his point using the figure of two circles in a plane—that is, precisely the example that Kummer mentions in his 1846 article. Chasles’s formulation of this example reads as follows:

“The theory of circles drawn on a plane gives us an example of this distinction that we make between *accidental* properties and *permanent* properties of a figure. The system of two circles always includes the existence of a certain line, whose consideration is very useful for the whole theory. When the two circles intersect, this line is their *common chord*, and this *very circumstance suffices to define and construct* it; here is what we call *one of its contingent or accidental properties*. But when the two circles do not intersect, this property disappears, even though, nevertheless, the line still exists and that its consideration is still most useful in the theory of circles. One must define and construct this line using another of its properties, which would hold in all the cases of general construction of the figure, which is the system of two circles. This will be one of its permanent properties (...here Chasles begins a discussion of the various permanent properties that can be used, starting from that of considering the line as the radical axis of the two circles). (La théorie des cercles tracés sur un plan nous offre un exemple de cette distinction que nous faisons entre les propriétés *accidentelles*, et les propriétés *permanentes* d'une figure. Le système de deux cercles comporte toujours l'existence d'une certaine droite, dont la considération est fort utile dans toute cette théorie. Quand les deux cercles se coupent, cette droite est leur *corde commune*, et cette seule circonstance suffit pour la définir et la construire ; voilà ce que nous appelons une de ses propriétés contingentes ou accidentelles. Mais quand les deux cercles ne se coupent pas, cette propriété disparaît quoique la droite pourtant existe toujours, et que sa considération soit encore extrêmement utile dans la théorie des cercles. Il faut donc définir cette droite et la construire par quelqu'une de ses autres propriétés, qui ait lieu dans tous les cas de construction générale de la figure, qui est le système des deux cercles. Ce sera une de ses propriétés permanentes (...).” (Chasles 1837: 205-206. Chasles’s emphasis is in italics, mine is in the underlined passages)

Interestingly, in his *Aperçu Historique*, Chasles refers to this line as a “common chord,” although for Poncelet, it would be a “secant” (see also Chasles 1837: 207). Yet, the very use of the term “common chord” indicates that he has Poncelet’s terminology in mind. Setting aside the fact that Chasles does not refer to either “continuity” or “ideal”, with the example of the “common secant,” we can establish a dictionary between Poncelet’s terms and Chasles’s. This will be useful when we turn to Kummer’s “ideal divisors.”

Indeed, the parts of a system that for Poncelet were imaginary elements were, for Chasles, contingent parts of the figure.<sup>45</sup> Moreover, Poncelet’s real parts would be either permanent or contingent. However, Chasles had a different approach to the geometrical objects to which Poncelet would refer as “ideal elements.” For him, they were permanent elements for which only a contingent definition had been put forward. Accordingly, a permanent property should be found for them. This is precisely what we will see Kummer doing to define the prime divisors of complex numbers in a fully general and uniform way.

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<sup>45</sup> For Chasles’s interpretation of imaginaries, see (Chasles 1837: 207).

## 5. How Kummer's ideal divisors resembled ideal chords

### 5.1 Kummer reader of Chasles' *Aperçu historique*

How, one may ask, did Kummer have access to Chasles's approach to these issues? In fact, as early as 1839—that is, only two years after the publication of the book—, Chasles's *Aperçu historique* was translated into German by Ludwig Adolph Sohncke ((1807-1853), one of Jacobi's students.<sup>46</sup> The German translation of the page about circles just quoted reads as follows

“Die Theorie der Kreise in einer Ebene bietet uns ein Beispiel von diesem Unterschied dar, welchen wir zwischen den *zufälligen* und den *bleibenden Eigenschaften* einer Figur gemacht haben. Das System zweier Kreise gestattet immer das Vorhandensein einer gewissen geraden Linie, deren Betrachtung in dieser ganzen Theorie von Nutzen ist. Wenn beide Kreise sich schneiden, so ist diese Gerade ihre *gemeinschaftliche Sehne*, und dieser einzige Umstand reicht hin, um sie zu erklären (KC: “explain,” instead of “define”) und zu construiren; dieses ist eine von den Eigenschaften, welche wir *zufällige* (KC: a single adjective translating *contingent* or *accidental*) genannt haben. Wenn aber die beiden Kreise sich nicht schneiden, so verschwindet diese Eigenschaft, obgleich die Gerade dennoch immer besteht und ihre Betrachtung von ausserordentlichem Nutzen in der Theorie des Kreises ist. Man muss daher diese Gerade definiren und construiren durch irgend eine andre ihrer Eigenschaften, welche stattfindet in allen Fällen der allgemeinen Construction der Figur, welche hier das System der beiden Kreise ist. Diese wird eine ihrer *permanenten* Eigenschaften sein.”  
(Chasles as translated by Sohncke 1839: 201)

Sohncke's translation is excellent, as can be assessed from the small divergences between the German text and the original one—quoted above—that I have indicated between brackets. If, now, we compare the German translation of Chasles's discussion of the figure of the two circles with what Kummer says when he makes his analogy between his introduction of ideal divisors and what happens in geometry, a striking phenomenon appears. To highlight the phenomenon, let us repeat the quotation of Kummer's comparison that I have translated above:

To achieve now a lasting definition of the true (usually ideal) prime factors of the complex numbers, it was necessary to find out the properties of the prime factors of complex numbers that would persist/remain in all circumstances, which would be absolutely independent of the contingency of whether the actual decomposition takes place or not, more or less precisely like, when in geometry, one speaks of the chord common to two circles, also when the circles do not intersect each other, one looks for an actual definition of this ideal common chord that fits for all situations of the circles.

(Um nun zu einer festen Definition der wahren (gewöhnlich idealen) Primfactoren der complexen Zahlen zu gelangen, war es nöthig, die unter allen Umständen bleibenden Eigenschaften der Primfactoren complexer Zahlen aufzusuchen, welche von der Zufälligkeit, ob die wirkliche Zerlegung Statt habe, oder nicht, ganz unabhängig wären: ohngefähr ebenso wie wenn in der Geometrie von der gemeinschaftlichen Sehne zweier Kreise gesprochen wird, auch wenn die Kreise

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<sup>46</sup> (Lorey 1916: 62).



sich nicht schneiden, eine wirkliche Definition dieser idealen gemeinschaftlichen Sehne gesucht wird, welche für alle Lagen der Kreise passt.) (Kummer 1846: 88, underlining is mine)

I have underlined all the terms that appear in both contexts. The comparison shows clearly how many terms Kummer has taken from this page that Chasles inserted in his *Aperçu historique*. Before I explain the importance of this remark, let me begin by establishing why we can be sure that Kummer did read Chasles's *Aperçu historique*.

Two hints clearly support this conclusion. The first one lies in the fact that, to refer to the secant line, Kummer, like Chasles, uses the term “common chord”, translated by Sohncke as “*gemeinschaftlichen Sehne*.” We have seen that other German authors used different terms for the “common chord:” for this term too, Kummer uses Sohncke's terminology. However, more importantly, as I have pointed out above, here Chasles uses the term “chord,” although he actually speaks of a common “secant”—the whole line that the radical axis constitutes. We note exactly the same phenomenon in Kummer's comparison.

The second hint derives from a letter that Kummer wrote to Kronecker on 14 June, 1846 (Weil, ed., vol. 1, 1975, 98-99). This is the second extant letter in which Kummer comments upon the ideal factors and in which, as I have mentioned in the introduction, he puts forward the comparison between the composition of numbers using ideal factors and the composition of chemical substances. Incidentally, he mentions having discussed his ideal divisors with Jacobi. At the end of this letter, however, he turns to a completely different topic, as follows:

From my own work, I further mention something really pretty. Indeed, I have indeed solved in a fully general way the problem of finding quadrilaterals, whose sides and diagonals are rational. This problem becomes interesting, insofar as CHASLES (wrongly) thinks that the Indians had set themselves this problem and had solved it in their way and further insofar as the punch line of the whole thing comes down to the fact of making a root  $\sqrt{a + bx + cx^2 + dx^3 + ex^4}$  rational, which Euler had already treated in the Algebra and about which JACOBI has later shown that the theory of elliptic function solves this problem (Von meinen eigenen Arbeiten erwähne ich noch eine recht nette Sache. Ich habe nämlich ganz allgemein die Aufgabe gelöst Vierecke zu finden, deren Seiten und Diagonalen rational sind. Dieselbe wird dadurch interessant daß CHASLES (fälschlich) meint die Indier hätten sich diese Aufgabe gestellt und in ihrer Weise gelöst, ferner dadurch daß die Pointe des ganzen darauf hinauskommt eine Wurzel  $\sqrt{a + bx + cx^2 + dx^3 + ex^4}$  rational zu machen, welches EULER schon in der Algebra behandelt und wovon JACOBI später gezeigt hat daß die Theorie der elliptischen Functionen diese Aufgabe löst.) (Weil, ed., vol. 1, 1975: 99).

Kummer's allusion refers to the Note XII of the *Aperçu historique*, titled “Sur la Géométrie des Indiens, des Arabes, des Latins et des Occidentaux au moyen âge” (Chasles 1837: 416-542).<sup>47</sup> The letter, dated from a few months after the presentation at the Berlin academy, thus shows that in the same months as he was working on his theory of complex numbers, Kummer was reading Chasles in great detail.

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<sup>47</sup> Ivahn Smadja has analyzed Kummer's work on these Sanskrit writings in “Part 4: Les quadrilatères de Brahmagupta: Chasles, Kummer, Hankel” (Smadja 2016: 17-31). See his publication in preparation on this topic.

## 5.2 Kummer's search for a permanent definition of prime factors

Let us return to the terms that, as I have underlined above, Kummer (1846) borrows from Chasles's page on the system of two circles. These terms include: "contingency (zufälligkeit)," "permanent property (bleibende Eigenschaft)," "common chord to two circles (gemeinschaftliche Sehne zweier Kreise)," "when the two circles do not intersect each other (Wenn aber die beiden Kreise sich nicht schneiden)," "definition/to define (definiren)," "Umstand (circumstance)."

These terms refer to the issue of "definition" in relation to permanent/contingent properties. We have seen that Kummer put forward his third comparison—the one that pointed out a parallel between "ideal divisors" and "ideal chords"—precisely when he turned to the issue of the definition of divisors. Focusing on these terms highlight a crucial phenomenon: Indeed, these terms are correlated with the very structure of Kummer's first presentation of his ideal divisors in 1846.

On this basis, I will argue that Kummer's 1846 article reflects an important part of the thought-process that led Kummer to introduce his ideal factors. More precisely, Kummer's achievement was inspired by a transposition, to the context of the divisibility of complex numbers by prime factors, of Chasles's prescription about looking for permanent properties to define objects that persist in all circumstances.

Indeed, immediately after the comparison on which we dwelled, Kummer writes:

"Among the *permanent properties* of the complex numbers which are handy to be used as definitions of ideal prime factors, there are several which fundamentally always lead to the same result and among which I have chosen one as the simplest and the most general. (Dergleichen *bleibende Eigenschaften* der complexen Zahlen, welche geschickt sind,<sup>48</sup> als Definitionen der idealen Primfactoren benutzt zu werden, giebt es mehrere, welche im Grunde immer auf dasselbe Resultat führen und von denen ich eine als die *einfachste* und *allgemeinste* gewählt habe.)" (Kummer 1846: 88-89, my emphasis)

Clearly, Kummer has been working on various "permanent properties," with the intention to seek an appropriate "definition" of ideal prime factors.<sup>49</sup> As he makes clear, his choice among those he considered was dictated by two epistemological values that were also crucial for Chasles: "simplicity" and "generality" (Chemla 2016). This work on definitions is reflected in the first presentation of the "theory of complex numbers." Indeed, what is striking against this backdrop is that Kummer's first public presentation (Kummer 1846) has three parts, each devoted to a definition of ideal prime factors, the first definition and then the second one being in turn discarded for their lack of generality and being eventually replaced by the third one. Let us examine them in turn, to make this point clear.

The first definition considered by Kummer is introduced immediately after the last statement quoted, and it focuses on the decomposition of a prime integer  $p$  into complex prime factors, in the case when  $p$  is of the form  $m\lambda + 1$  (we remember that  $\lambda$  is prime and  $\alpha$  is an imaginary root of the equation  $\alpha^\lambda = 1$ ). The introduction of the first case reads as follows:

"If  $p$  is a prime number of the form  $m\lambda + 1$ , in many cases it can be represented as product of the following  $\lambda - 1$  complex factors:  $p = f(\alpha) \cdot f(\alpha^2) \cdot f(\alpha^3) \cdot \dots \cdot f(\alpha^{\lambda-1})$ ;

<sup>48</sup> Here "um" was added in (Kummer 1847b: 320)

<sup>49</sup> Kummer's first letter about ideal divisors from October 18, 1845 contains a similar hint (Weil (ed.) 1975: 96), without, however, speaking of "permanent property."

however, in cases when a decomposition into actual complex prime factors is not possible, then ideal prime factors must be introduced, to get the same decomposition. (Ist  $p$  eine Primzahl von der Form  $m\lambda + 1$ , so lässt es<sup>50</sup> sich in vielen Fällen als Product von folgenden  $\lambda - 1$  complexen Factoren darstellen:  $p = f(\alpha) \cdot f(\alpha^2) \cdot f(\alpha^3) \cdot \dots \cdot f(\alpha^{\lambda-1})$ ,<sup>51</sup> wo aber eine Zerlegung in wirkliche complexe Primfactoren nicht möglich ist, da eben<sup>52</sup> sollen die idealen Primfactoren eintreten, um dieselbe zu leisten.)” (Kummer 1846: 89, my emphasis)

The problem is clearly set: the decomposition into “actual prime factors” occurs in “many cases,” but not in all. The point in introducing “ideal prime factors” is to get “the same decomposition” uniformly for any such  $p$ . We recognize an assumption similar to Poncelet’s credo in the “permanence of laws.” The path followed by Kummer’s presentation for this first set of prime numbers  $p$  that he considers is then to find out an adequate property that “actual prime factors” possess and that can be used as a definition of “prime factors” for all cases of divisors. Kummer thus goes on with the statement of a property of actual factors as follows:

“If  $f(\alpha)$  is an actual complex number and a prime factor of  $p$ , then this complex number has the property that, if, instead of the root of the equation  $\alpha^\lambda = 1$ , one substitutes in  $f(\alpha)$  a certain root of the congruence  $\xi^\lambda \equiv 1, \text{ mod. } p$ , then  $f(\xi) \equiv 0, \text{ mod. } p$ . Therefore also, if the prime factor  $f(\alpha)$  is contained in [KC: contained as factor, that is, divides] a complex number  $\Phi(\alpha)$ , thus  $\Phi(\xi) \equiv 0, \text{ mod. } p$ ; and conversely: if  $\Phi(\xi) \equiv 0, \text{ mod. } p$ , and  $p$  can be decomposed into  $\lambda - 1$  complex prime factors, then  $\Phi(\alpha)$  contains [KC: contains as factor, that is, is divided by] the prime factor  $f(\alpha)$ . (Ist  $f(\alpha)$  eine wirkliche complexe Zahl und<sup>53</sup> Primfactor von  $p$ , so hat sie die Eigenschaft, dass wenn anstatt<sup>54</sup> der Wurzel der Gleichung  $\alpha^\lambda = 1$  eine bestimmte Congruenzwurzel von  $\xi^\lambda \equiv 1, \text{ mod. } p$ , substituirt wird,  $f(\xi) \equiv 0, \text{ mod. } p$ , ist. Darum auch<sup>55</sup> wenn in einer complexen Zahl  $\Phi(\alpha)$  der Primfactor  $f(\alpha)$  enthalten ist, so<sup>56</sup> wird  $\Phi(\xi) \equiv 0, \text{ mod. } p$ ; und umgekehrt: wenn  $\Phi(\xi) \equiv 0, \text{ mod. } p$ , und  $p$  in  $\lambda - 1$  complexe Primfactoren zerlegbar ist, enthält  $\Phi(\alpha)$  den Primfactor  $f(\alpha)$ .)” (Kummer 1846: 89, my emphasis)

In other words, Kummer points out a property of an actual prime factor  $f(\alpha)$  of  $p$ : if  $f(\alpha)$  can be exhibited, then there exists an integer  $\xi$  such that  $\alpha \equiv \xi \pmod{f(\alpha)}$ . Since  $f(\alpha)$  is a prime factor of the prime integer  $p$ , such a  $\xi$  satisfies the equation  $\xi^\lambda \equiv 1, \text{ mod. } p$ . As a result, the test of the divisibility of  $\Phi(\alpha)$  by  $f(\alpha)$  is equivalent to the test of the integer  $\Phi(\xi)$  by  $p$ .<sup>57</sup> Now, Kummer proceeds in exactly the same way as Chasles invited practitioners to proceed, when he suggested defining the ordinary secant no longer as the straight line going through the points of intersection between the circles, but rather using the same property as the “ideal secant”—that is, using the permanent property that they are both the radical axis of the two circles: For Kummer, any non-trivial root of the equation  $\xi^\lambda \equiv 1, \text{ mod. } p$ , gives rise to a test among integers, which enables one to test whether the complex divisor corresponding to

<sup>50</sup> (Kummer 1847b: 320) replaces “es” by “sie”.

<sup>51</sup> (Kummer 1847b: 320) changes the comma into a semi-colon.

<sup>52</sup> (Kummer 1847b: 320) replaces “, so” with “: dann”

<sup>53</sup> (Kummer 1847b: 320) inserts here “ein”.

<sup>54</sup> Instead of “wenn anstatt,” (Kummer 1847b: 320) has here “, wenn statt”.

<sup>55</sup> Instead of “Darum auch” (Kummer 1847b: 320) has here “Also auch,…”.

<sup>56</sup> (Kummer 1847b: 320) deletes “so”.

<sup>57</sup> Precise theorems and proofs for these statements can be found in (Edwards 1977: 89-95).

this root  $\xi$  in the case of numbers  $p$  of the form under consideration divides any given complex number. Accordingly, having observed the aforementioned property of an actual prime factor, Kummer next turns this property into a general definition of any prime factors of  $p$ , including “ideal divisors.” Indeed, Kummer’s subsequent statements read as follows:

“Now, the property  $\Phi(\xi) \equiv 0, \text{ mod. } p$  is one which is such that it in itself *does not depend at all* on the decomposability of the number  $p$  into  $\lambda - 1$  prime factors; *it can therefore be used as a definition*, through which it is decided that the complex number  $\Phi(\alpha)$  contains the ideal prime factor of  $p$  that belongs to  $\alpha = \xi$ , if  $\Phi(\xi) \equiv 0, \text{ mod. } p$ . Each of the  $\lambda - 1$  complex prime factors of  $p$  is in this way replaced by a condition of congruence. This suffices to show that the complex prime factors, whether they be actual or exist only ideally, give to the complex numbers the same defining character. (Die Eigenschaft  $\Phi(\xi) \equiv 0, \text{ mod. } p$ , ist nun eine solche, welche für sich<sup>58</sup> von der Zerlegbarkeit der Zahl  $p$  in  $\lambda - 1$  Primfactoren gar unabhängig ist;<sup>59</sup> sie kann daher<sup>60</sup> als Definition benutzt werden, indem bestimmt wird, dass die complexe Zahl  $\Phi(\alpha)$  den idealen Primfactor von  $p$  enthält, welcher zu  $\alpha = \xi$  gehört, wenn  $\Phi(\xi) \equiv 0, \text{ mod. } p$ , ist. Jeder der  $\lambda - 1$  complexen Primfactoren von  $p$  wird so durch eine Congruenzbedingung ersetzt. Dies reicht hin, um zu zeigen, dass die complexen Primfactoren, sie seien wirklich, oder nur ideal vorhanden, den complexen Zahlen denselben bestimmten Character imprimiren.<sup>61</sup>) (Kummer 1846: 89, my emphasis)

The property identified does not depend on the actual decomposability, in exactly the same way as the property of a line to be the radical axis does not depend on whether the circles intersect or not. Kummer thus replaces the exhibition of actual prime complex divisors of  $p$  by this permanent property, taken as a definition of a prime factor that relies on a congruence test. The congruence  $\xi^\lambda \equiv 1, \text{ mod. } p$ , has  $\lambda - 1$  roots that differ from 1. Each of these roots is associated with a prime divisor of  $p$ .

The definition put forward is that of the divisibility by a factor, and not that of the factor itself. Kummer has identified the property of prime factors of  $p$  that enables him to detect them in all cases: whether they are actual (like a number  $f(\alpha)$ ) or ideal, they satisfy exactly the same property. If we draw on Chasles’s “principle of contingent relationships,” which Kummer is transposing here, this suggests that for Kummer, these ideal factors are “integrant and permanent parts” of the complex number that they divide. Only a belief of this kind can justify looking for permanent properties. However, unlike the case of the radical axis, the ideal divisors cannot be “exhibited.”

One might be tempted to believe that Kummer has now managed to define ideal factors and that his work is thus over. However, he immediately points out that this first “permanent property,” which he has introduced as such, will not solve his problem, since, as he states:

“However, we do *not use* the congruence conditions in the way that is given here as definition of the ideal prime factors, because these conditions would *not be sufficient* to represent several identical/equal ideal prime factors that would

<sup>58</sup> (Kummer 1847b: 320) inserts here “selbst”.

<sup>59</sup> Instead of “gar unabhängig ist,” (Kummer 1847b: 320) has “gar nicht abhängt (sic) (does not depend at all).”

<sup>60</sup> Instead of “daher,” (Kummer 1847b: 320) has “demnach.”

<sup>61</sup> Instead of “imprimiren,” (Kummer 1847b: 320) has “erteilen.” The verb “imprimiren” has the nuance of “imprinting” a similar structure to the numbers, whereas “erteilen/ertheilen” can be translated as “to grant the numbers the same character.”

occur in a complex number, and *because these conditions are too limited* and would give *only* ideal prime factors of the real prime numbers of the form  $m\lambda + 1$ . (In der hier gegebenen Weise aber *gebrauchen wir die Congruenzbedingungen nicht als Definitionen* der idealen Primfactoren, weil dieselbe<sup>62</sup> *nicht hinreichend* sein würden, mehrere gleiche, in einer complexen Zahl vorkommende ideale Primfactoren *vorzustellen*, und *weil sie, zu beschränkt, nur* ideale Primfactoren der realen Primzahlen von der Form  $m\lambda + 1$  geben würden<sup>63</sup>.)” (Kummer 1846: 89, my emphasis)

In effect, in what follows, this first “permanent property” is replaced by a second, and then a third one, which respectively extend it and successively encompass the cases that Kummer notes here as not dealt with if one uses the first definition. The last definition will be fully general. We might thus assume that this structure of the presentation reflects Kummer’s statement quoted above, when he asserts that he has found several permanent properties, which lead to introducing the same entities, and that he has chosen the “most general.” This remark highlights the part played by Chasles’ invitation to concentrate on “permanent properties.” What is more, I argue that structure of the presentation—that is, the introduction of Kummer’s first definition of ideal complex divisors and its subsequent extensions into more general definitions—also appears to reflect part of the process of Kummer’s discoveries with respect to ideal factors.

### 5.3 Kummer’s discovery as a reflection of the impact of geometry

To understand this point, let us consider how Kummer shapes the second definition—which eliminates the previous restriction on  $p$  and deal with all prime integers  $p$  ( $\lambda$  excluded), the third definition then turning to all complex numbers and solving the problem of the multiple factors that can occur among their prime divisors.<sup>64</sup> For this, we will need to introduce a few notations and some mathematical facts.

For any prime integer  $p$  different from  $\lambda$ , we have

$$p^{\lambda-1} \equiv 1 \pmod{\lambda}.$$

Let us call  $f$  the smallest integer for which we have

$$p^f \equiv 1 \pmod{\lambda}$$

and  $e$  the integer such that

$$ef = \lambda - 1.$$

For the second definition to which Kummer turns, he will make use of a tool introduced by Carl Friedrich Gauss (1777-1855) in the *Disquisitiones*: the “periods.” The  $e$  periods of  $f$  terms are defined using a primitive root  $\gamma$  of  $(\mathbb{Z}/\lambda\mathbb{Z})^*$ , as follows:

$$\begin{aligned} \eta_0 &= \alpha^{\gamma^e} + \alpha^{\gamma^{2e}} + \dots + \alpha^{\gamma^{ef}} \\ \eta_1 &= \alpha^{\gamma^{e+1}} + \alpha^{\gamma^{2e+1}} + \dots + \alpha^{\gamma^{ef+1}} \end{aligned}$$

<sup>62</sup> Instead of “dieselbe,” (Kummer 1847b: 320) has “diese.”

<sup>63</sup> Here, (Kummer 1847b: 320) has a typo, since it reads “ $m\lambda - 1$ .” (Kummer 1846: 89) has the correct expression “ $m\lambda + 1$ .”

<sup>64</sup> I rely on (Edwards 1977: 107-133), which I summarize coarsely here. The relevant sections of Edwards (1977) give a detailed and rigorous presentation.

$$\eta_{e-1} = \alpha^{\gamma^{e+(e-1)}} + \alpha^{\gamma^{2e+(e-1)}} + \dots + \alpha^{\gamma^{ef+(e-1)}}$$

The period  $\eta_0$  (Kummer notes it  $\eta$ ) is the one containing  $\alpha$ , and hence it is independent from the primitive root chosen, whereas the other periods are simply permuted with one another if one chooses another primitive period. Several facts will be essential for Kummer's second definition.

First, if  $h(\alpha)$  is a prime complex divisor of a prime integer  $p$ , then one can find  $e$  integers  $u_0, u_1, \dots, u_{e-1}$  such that, pour each integer  $i$  comprised between 0 and  $e-1$ , we have

$$\eta_i \equiv u_i \pmod{h(\alpha)}$$

Second, to test the divisibility of a complex number  $g(\alpha)$  by such a prime complex divisor  $h(\alpha)$ , a key fact consists in noticing that for any  $g(\alpha)$ , one can find  $f$  functions  $\varphi_i$  of the periods (which are noted  $\varphi_i(\eta)$ ) such that

$$g(\alpha) = \varphi_0(\eta) + \alpha\varphi_1(\eta) + \dots + \alpha^{f-1}\varphi_{f-1}(\eta)$$

As a result,

$$\begin{aligned} \varphi_0(\eta) + \alpha\varphi_1(\eta) + \dots + \alpha^{f-1}\varphi_{f-1}(\eta) \\ \equiv \varphi_0(u) + \alpha\varphi_1(u) + \dots + \alpha^{f-1}\varphi_{f-1}(u) \pmod{h(\alpha)} \end{aligned}$$

in which the periods  $\eta_i$  have been replaced by the integers  $u_i$  (using a similar notation for the  $u$ 's as for the  $\eta$ 's).

One can prove that two distinct numbers of the kind  $\varphi_0(u) + \alpha\varphi_1(u) + \dots + \alpha^{f-1}\varphi_{f-1}(u)$  are congruent to each other modulo  $h(\alpha)$  if and only if they are equal, and hence any complex number  $g(\alpha)$ — which, as we have seen, can be written as  $\varphi_0(\eta) + \alpha\varphi_1(\eta) + \dots + \alpha^{f-1}\varphi_{f-1}(\eta)$ —is divisible by the prime complex divisor  $h(\alpha)$  if and only if

$$\begin{aligned} \varphi_0(u) &\equiv 0 \pmod{p} \\ \varphi_1(u) &\equiv 0 \pmod{p} \end{aligned}$$

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$$\varphi_{f-1}(u) \equiv 0 \pmod{p}$$

Consequently, the divisibility by a prime complex factor can be tested using only congruences among integers. The key fact is that the  $e$  periods  $\eta_i$  satisfy an equation of degree  $e$  with integral coefficients.<sup>65</sup> Solving it as a congruence modulo the prime number  $p$  yields  $e$  integers  $u_i$ , which (by a circular permutation among the  $u_i$ ) allow Kummer to define the  $e$  ideal complex divisors of any prime integer  $p$  (except the prime integer  $\lambda$  which is dealt with separately).

On the basis on these notations and facts, let us return to how Kummer introduces the second definition he considers. He begins with an outline of the case in which the decomposition can be carried out, as follows:

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<sup>65</sup> The product of any two periods can be expressed as a linear combination of periods with integral coefficients, and the sum of all periods is -1.

“Every prime factor of a complex number is always at the same time also a prime factor of some real prime number  $q$ , and the constitution of the ideal prime factors depends above all on the exponent to which  $q$  belongs, for the module  $\lambda$ . Let this one be  $f$ , so that  $q^f \equiv 1 \pmod{\lambda}$ , and  $\lambda - 1 = e \cdot f$ . A prime number  $q$  of this kind can never be decomposed into more than  $e$  complex prime factors, which, if the decomposition can actually be carried out, are represented as linear functions of the  $e$  periods with each  $f$  terms. I denote these periods of the roots of the equation  $\alpha^\lambda = 1$  as  $\eta, \eta_1, \eta_2, \dots, \eta_{e-1}$ ; and hence in the order in which each becomes the next one, when  $\alpha$  is turned into  $\alpha^\gamma$ , where  $\gamma$  is a primitive root of  $\lambda$ . (Jeder Primfactor einer complexen Zahl ist immer zugleich auch Primfactor irgend einer realen Primzahl  $q$ , und die Beschaffenheit der idealen Primfactoren ist besonders von dem Exponenten abhängig, zu welchem  $q$  gehört, für den Modul  $\lambda$ , derselbe<sup>66</sup> sei  $f$ , so dass  $q^f \equiv 1 \pmod{\lambda}$ , und  $\lambda - 1 = e \cdot f$ . Eine solche Primzahl  $q$  lässt sich niemals in mehr als  $e$  complexe Primfactoren zerlegen, welche, wenn diese Zerlegung wirklich ausführbar ist, sich als lineäre<sup>67</sup> Functionen der  $e$  Perioden von je  $f$  Gliedern darstellen. Diese Perioden der Wurzeln der Gleichung  $\alpha^\lambda = 1$  bezeichne ich durch  $\eta, \eta_1, \eta_2, \dots, \eta_{e-1}$ ; und zwar in der Ordnung, dass jede in die folgende übergeht, wenn  $\alpha$  in  $\alpha^\gamma$  verwandelt wird, wo  $\gamma$  eine primitive Wurzel von  $\lambda$  ist.)” (Kummer 1846: 89-90, my emphasis)

We can notice that the description of the prime complex divisors starts with the fully general case (prime factors of a complex number) and introduces prime integers as a key case to deal with the prime complex divisors. The discussion of prime integers is now fully general, and does not give any special status to the case dealt with previously. In fact, the first extant letter in which, on October 18, 1845, Kummer announces to Kronecker his breakthrough, enables us to establish that previously, Kummer had treated the cases of distinct types of prime integers separately (again setting the prime integer  $\lambda$  aside). Indeed, about a letter prior to the October 18 one, he writes:

“The notations that I have will already be known to you from the previous letter and otherwise, that is, that  $\lambda - 1 = e \cdot f$  and  $q$  belongs to the exponent  $f$  modulo  $\lambda$  and is as well a prime integer, but no factor of  $N(\eta - \eta_r)$ . Here, too, I no longer distinguish the prime integers  $p$ 's from the  $q$ 's, but take the prime integers that belong to the exponent 1 just as falling under the same law<sup>68</sup> as the others, the periods for these integers being simply with a single term and  $f = 1$ . (Die Bezeichnungen, welche ich habe, werden Ihnen wohl aus dem vorigen Briefe und sonst schon bekannt sein, nämlich  $\lambda - 1 = e \cdot f$  und  $q$  zum Exponenten  $f$  gehörig modulo  $\lambda$  und Primzahl aber kein Factor von  $N(\eta - \eta_r)$ . Ich unterscheide auch hier gar nicht mehr Primzahlen  $p$  von den  $q$ , sondern nehme die zum Exponenten 1 gehörigen Primzahlen als eben in demselben Rechte befindlich als die andern, es sind für diese nur die Perioden eingliedrig und  $f = 1$ .)” (Weil (ed.) 1975: 94, my emphasis)

<sup>66</sup> Here, (Kummer 1847b: 321) starts a new sentence.

<sup>67</sup> In (Kummer 1847b: 321), the word is written “lineare.”

<sup>68</sup> The term “Recht (right)” is difficult to translate. To highlight the juridical connotation, I translate as “law.” However, I warn the reader that the translation might be misleading. Kummer simply points out the fact that they are all subject to the same jurisdiction.

Just as in the first public presentation of 1846, Kummer's letter refers to the prime integers that are congruent to 1 modulo  $\lambda$  using the letter  $p$ . Interestingly, in the letter, he refers to the others using the letter  $q$ . Even though  $p$  and  $q$  are still used in the published article,  $p$  keeps designating the first case considered, whereas  $q$  now designates any prime integer (except  $\lambda$ ), following the new insight that motivated the letter. This remark highlights the work that Kummer carried out to increase generality and uniformity in his treatment of divisibility, looking, as he later said, for a permanent property that would hold for all prime complex divisors and would thus allow him to introduce ideal divisors in exactly the same fashion. The remark also sheds light on Kummer's discovery process. On this basis, let us thus return to how the 1846 article subsequently deals with the second definition for all prime complex divisors of prime integers. Kummer first makes a general statement about the periods:

“One knows that the *periods* are the  $e$  roots of an *equation* of degree  $e$ ; and that this equation, interpreted as congruence, for the *module*  $q$ , *always* has  $e$  *real roots of the congruence*, which I denote as  $u, u_1, u_2, \dots, u_{e-1}$ , and which I take in a sequence corresponding to the periods (...). If, now, we denote simply with  $\Phi(\eta)$  the *complex number made up of periods*  $c'\eta + c'_1\eta_1 + c'_2\eta_2 + \dots + c'_{e-1}\eta_{e-1}$ , then there are, among the prime numbers  $q$  that belong to the exponent  $f$ , *always some that can be brought into the form*  $q = \Phi(\eta)\Phi(\eta_1)\Phi(\eta_2)\dots\Phi(\eta_{e-1})$ , in which the  $e$  factors also never allow a further decomposition. (Bekanntlich sind die *Perioden* die  $e$  Wurzeln einer *Gleichung* vom  $e$ ten Grade, und diese als Congruenz aufgefasst,<sup>69</sup> für den Modul  $q$ , hat immer  $e$  reale Congruenzwurzeln, welche ich durch  $u, u_1, u_2, \dots, u_{e-1}$  bezeichne, und welche ich in einer entsprechenden Reihenfolge nehme wie die Perioden,<sup>70</sup> (...). Wird nun die aus Perioden gebildete complexe Zahl  $c'\eta + c'_1\eta_1 + c'_2\eta_2 + \dots + c'_{e-1}\eta_{e-1}$  kurz durch  $\Phi(\eta)$  bezeichnet, so giebt es unter den Primzahlen  $q$ , welche zum Exponenten  $f$  gehören, immer solche, die sich auf die Form  $q = \Phi(\eta)\Phi(\eta_1)\Phi(\eta_2)\dots\Phi(\eta_{e-1})$  setzen lassen,<sup>71</sup> in welcher auch die  $e$  Factoren niemals eine weitere Zerlegung gestatten.)” (Kummer 1846: 90, my emphasis)

In other words, Kummer establishes the link between the  $e$  periods of length  $f$  and  $e$  integers with respect to a given prime integer  $q$  that corresponds to the exponent  $f$ . He then turns to prime integers  $q$  of this kind, whose decomposition into prime factors is actual (I use the term “actual” as a translation of *wirklich*, which Kummer uses above, when he refers to a decomposition into prime factors that can be carried out explicitly). The key point now for him, as it was in the first case, is to find a property of these divisors that does not depend on the “actual” decomposability, which he does in the subsequent sentences as follows:

Let us put, *instead of the periods*, the *roots* of the congruence that correspond to them (...). Hence one of the  $e$  prime factors will always be congruent to 0 for the module  $q$ . If, now, *any complex number*  $f(\alpha)$  contains the prime factor  $\Phi(\eta)$ ,<sup>72</sup>

<sup>69</sup> Here, instead of “, und diese als Congruenz aufgefasst, ...” (Kummer 1847b: 321) has “; und diese, als Congruenz betrachtet, ... (; and this equation, considered as congruence, ...).”

<sup>70</sup> Here, (Kummer 1847b: 321) reads as follows: “und in einer entsprechenden Reihenfolge nehme, wie die Periode (and take in a corresponding sequence, like the periods)”.

<sup>71</sup> Here, (Kummer 1847b: 321) has “bringen lassen”.

<sup>72</sup> In relation to the notation used above by Kummer for the decomposition of  $q$ , we should here rather  $\Phi(\eta_k)$ , to point out the fact that the divisor considered is the one corresponding to a circular permutation of the integers  $u, u_1, u_2, \dots, u_{e-1}$ , which is the order corresponding to  $\eta, \eta_1, \eta_2, \dots, \eta_{e-1}$ .



then it will have the *property* that it becomes congruent to 0 for the module  $q$ , for  $\eta = u_k, \eta_1 = u_{k+1}, \eta_2 = u_{k+2}, \dots$ , etc.. This property, now, (...) is a *permanent property/property that holds also for the prime numbers  $q$  that do not allow a decomposition* into  $e$  actual complex prime factors. This *property could hence* be used as *definition* of the complex prime factors. *However*, it would have the *deficiency that* it would not express the equal ideal prime factors that occur in a complex number. (Setzt man *anstatt*<sup>73</sup> der *Perioden* ihre entsprechenden *Congruenzwurzeln*, (...) so wird immer einer der  $e$  Primfactoren congruent Null, für den Modul  $q$ . Enthält nun *irgend eine complexe Zahl*  $f(\alpha)$  den Primfactor  $\Phi(\eta)$ ,<sup>74</sup> so wird sie die *Eigenschaft* haben, für  $\eta = u_k, \eta_1 = u_{k+1}, \eta_2 = u_{k+2}, \dots$ , etc. congruent Null zu werden, für den Modul  $q$ . Diese *Eigenschaft* nun (...) ist eine *bleibende* auch für *diejenigen Primzahlen  $q$ , welche eine Zerlegung in die  $e$  wirklichen complexen Primfactoren nicht gestatten*, sie *könnte daher als Definition* der complexen Primfactoren benutzt werden<sup>75</sup> würde aber *auch* den *Mangel haben*, dass sie die in einer complexen Zahl vorhandenen gleichen idealen Primfactoren nicht ausdrückte.)” (Kummer 1846: 90, my emphasis).

For any prime integer  $q$  associated to the exponent  $f$ , for which a decomposition holds, Kummer highlights a “property (Eigenschaft)” that is again as “permanent” and as thus “defining” the divisors whether the decomposition be actual or not. We thus see again the same conceptual tools as those introduced by Chasles, which reflects the same assumption: the “ideal divisors” are integrant parts of the numbers they divide. What Kummer explains in the part of his October 18 letter that was quoted above highlights in which respect this way of dealing with any prime integer  $q$  is simply a generalization of the first considerations about the  $p$ 's.

As the end of the quotation makes clear, once again this definition is not completely general, since it cannot encompass the prime divisors that might occur with a multiplicity strictly greater than 1 in the decomposition of a complex number into prime factors. For this, once again, Kummer will look for a still more general property that will hold for all cases. As he writes:

“The definition of ideal complex prime factors that I have chosen, which indeed essentially *agrees* with the one that was indicated here, but is *simpler* and *more general*, rests on the fact, that (...). It would lead me too far here, if I wanted to develop the relationship and the agreement between this definition and those that were indicated above, which were given through conditions of congruences;... (Die von mir gewählte Definition der idealen complexen Primfactoren, welche im wesentlichen<sup>76</sup> zwar mit der hier angedeuteten *übereinstimmt*, aber zugleich<sup>77</sup> *einfacher* und *allgemeiner* ist, beruht darauf, dass (...). Es würde mich<sup>78</sup> hier zu weit führen, wenn ich den Zusammenhang und die Übereinstimmung dieser Definition mit den oben angedeuteten, welche durch Congruenzbedingungen gegeben werden, entwickeln wollte;...)” (Kummer 1846: 90-91, my emphasis)

<sup>73</sup> Here, (Kummer 1847b: 321) has “statt”.

<sup>74</sup> See footnote 71.

<sup>75</sup> In (Kummer 1847b: 321), Kummer modified the punctuation as follows: “Diese *Eigenschaft* nun (...) ist eine *bleibende*: auch für diejenigen Primzahlen  $q$ , welche eine Zerlegung in die  $e$  wirklichen complexen Primfactoren nicht gestatten. Sie könnte daher als Definition”

<sup>76</sup> Kummer (1847b: 321) capitalizes the word.

<sup>77</sup> Kummer (1847b: 321) deletes this adverb.

<sup>78</sup> Kummer (1847b: 322) deletes this pronoun.

We see that, here again, Kummer underlines the values of simplicity and generality on which he had laid stress in his introduction. I will not examine further the third definition that Kummer puts forwards,<sup>79</sup> since I think the general structure of Kummer's procedure is clear: as he made clear at the beginning of his development, he has considered several permanent properties, in his search for a property that would be uniform for all cases. The organization of the article puts forward three such properties in turn, discarding the first two for their lack of generality. However, at each step, the permanent property considered fulfilled the purpose that Chasles had outlined for properties of this kind: the property allowed Kummer to define at the same time actual and ideal divisors—all then considered as integrant components of the numbers they divide. We thus see clearly in which respect Kummer adopted the strategy suggested by Chasles to approach in a uniform way the type of elements that Poncelet had opposed: those, like secants and chords, that had their mathematical identity “really” and those who had it only “ideally”.

In his later (1851) presentation of his theory in French, Kummer would devote a specific section to the “Definition and general properties of ideal factors of a complex number (définition et propriétés générales des facteurs idéaux d'un nombre complexe)” (Kummer 1851: 425). After having defined the “ideal factors” of a prime integer  $q$ , Kummer comments on the value he attached to the introduction of factors of this kind, and he further explains how they compare with previous mathematical achievements. In this context, it is again from the viewpoint of “permanent” and “contingent” properties that he compares the definition of “ideal divisors” that he puts forward with the “permanent” property that defines the radical axis of two circles, in contrast to the “contingent” property that the line might go through the points of intersection of the two circles (Kummer 1851: 430). Despite the fact that the analogy with geometry is only one among three comparisons he makes, only in this case, does Kummer justify his definition on account of the fact that it, too, relies on a permanent property. This remark confirms the part played by the “new geometry” in his own reflections.

## 6. Conclusion

This article has argued that Kummer's introduction of “ideal divisors” owed to both Poncelet's concept of “ideal elements” and to Chasles's reconceptualization of the “principle of continuity” into the “principle of contingent relationships.” Indeed, the use of the term “ideal” as well as the focus on “permanent properties” to “define” “ideal divisors” are central facets of the 1846 public presentation of Kummer's “theory of complex numbers”, and they allow us to interpret the meaning of Kummer's explicit parallel between his work and recent work in the “new geometry.”

In addition to establishing a historical fact, what does this conclusion tell us about Kummer's ideal divisors?

I have insisted that reading Kummer's first public presentation of his theory of complex divisors against the backdrop of Chasles's *Aperçu historique* highlights the part played in Kummer's reflections by Chasles's concepts of “contingent” and “permanent properties,” as well as by the geometer's urge to look for definitions that would rest on permanent properties. However, the very fact that Kummer embraces these ideas sheds light on his conception of these “ideal divisors.” For him, like the ideal secant common to the two

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<sup>79</sup> For the general idea, see (Edwards 1977: 134-137) and the previous proofs in which the key function  $\Psi$  is introduced. However, here, Edwards does not strictly follow Kummer's procedure (see Edwards 1977: 129, footnote).

circles, “ideal divisors” are “integrant” components of the numbers they divide: this is what justifies looking for a permanent property that will define them in all circumstances. This conclusion fits with the claim Kummer makes at the beginning of his 1846 presentation, when he asserts that the “prime factors” of complex numbers are “true (usually ideal).”

The only difference between Kummer and Chasles in this respect is that Kummer keeps referring to the permanent factors that he seeks to identify—as well as to the line that persists when the circles no longer cut each other—using the term “ideal.” This, and the fact that Kummer makes clear that for him these “prime factors” are “true (usually ideal)” suggests a similarity between Poncelet’s and Kummer’s understanding of these permanent elements. Indeed, for both of them these elements are different from “imaginary” elements. We remember that, for Poncelet, in contrast with “ideal” elements, “imaginary” elements are those that cannot be constructed, whereas in Chasles’s reformulation, as we have seen, such elements would be contingent. Moreover, for Poncelet, “ideal” elements were parts of a figure that would subsist in any general position of this figure, but whose relationships with its other parts would have become “imaginary.” It is interesting in this respect that Kummer’s ideal numbers are also only defined not in and of themselves, but only *through their relations* with other complex numbers.

But, here, we may go one step further. Indeed, as early as in the October 18, 1845 letter, Kummer “justifies” the use of the expression “ideal prime factor” by reference to the fact that these factors make “computation” with complex numbers identical to computations with integers. This reference to what we may call “the laws of arithmetic” evoke the full name that Poncelet had given to his key principle: “the *principle of permanence or indefinite continuity of the mathematical laws of magnitudes.*” In both cases, what is at stake is a persistence of the mathematical relationships between the entities that takes the form of a total uniformity. Hence, the values of simplicity and generality, on which Kummer insists, and which were also crucial epistemological values for Chasles (Chemla 2016).

The preceding remarks explain in which respect we might consider that Kummer realized a synthesis between Poncelet’s and Chasles’s approach of “ideal” elements. More precisely, Kummer’s achievement drew on a synthesis of the philosophical reflections underpinning the shaping of the new geometry in the hands of geometers such as Poncelet and Chasles. Indeed, their philosophical reflections were instrumental in fashioning what would later become projective geometry (Chemla 2016). Moreover, it was precisely these reflections that were picked up from geometry into the “theory of complex numbers,” contributing to a major change in this latter context.

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