



New tools to better model behavior under risk and uncertainty: an overview

Alain Chateauneuf, Michèle Cohen, Isaac Meilijson

► To cite this version:

Alain Chateauneuf, Michèle Cohen, Isaac Meilijson. New tools to better model behavior under risk and uncertainty: an overview. 1997. halshs-04406025

HAL Id: halshs-04406025

<https://shs.hal.science/halshs-04406025>

Submitted on 30 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

CAHIERS ECO & MATHS

97.55

NEW TOOLS TO BETTER MODEL BEHAVIOR UNDER RISK
AND UNCERTAINTY : AN OVERVIEW

Alain CHATEAUNEUF

Michèle COHEN

Isaac MEILIJSON

CERMSEM & CEME

New tools to better model behavior under risk and uncertainty : An overview

Alain Chateauneuf¹

Michèle Cohen²

Isaac Meilijson³

October, 8, 1997

Abstract

Normative models of behavior under risk in the framework of expected utility (EU) or under uncertainty in the framework of subjective expected utility (SEU) are very limited. In this survey paper, it is shown that non-expected utility (non-EU) models based on the Choquet integral allow for much more diversified behavior, both under risk and under uncertainty. Some illustrations by economic applications to insurance and finance are provided.

Keywords : Non-additive expected utility, risk aversion, uncertainty aversion, insurance, finance.

Résumé

Les modèles normatifs de décision dans le risque ou dans l'incertain basés sur l'espérance d'utilité (EU) sont assez rigides. Dans cet article de synthèse, nous montrons que les modèles d'utilité non espérée (non-EU) basés sur l'intégrale de Choquet permettent des comportements beaucoup plus diversifiés, à la fois dans le risque et dans l'incertain. Nous illustrons ensuite cette diversité de comportements à l'aide de quelques applications en assurance et en finance..

Mots clés : Espérance d'utilité non-additive, aversion pour le risque, aversion pour l'incertain, assurance, finance.

¹CERMSEM, Université de Paris I, Maison des Sciences Economiques, 106 - 112 Boulevard de l'hôpital, 75647 Paris Cedex 13, France. E-mail : chateaun@univ-paris1.fr

²CEME, Université de Paris I, Maison des Sciences Economiques, 106 - 112 Boulevard de l'hôpital, 75647 Paris Cedex 13, France. E-mail : cohenmd@univ-paris1.fr

³School of Mathematical Sciences, Raymond and Beverly Sackler, Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

New tools to better model behavior under risk and
uncertainty: An overview
Nouveaux outils pour une meilleure modélisation des
comportements dans le risque et dans l'incertain :
une synthèse

*Alain Chateauneuf, CERMSEM, Université de Paris I,
Michèle Cohen, C.E.M.E., Université de Paris I,
Isaac Meilijson, School of Mathematical Sciences, Tel Aviv University*

October 8, 1997

Abstract

Normative models of behavior under risk in the framework of expected utility (EU) or under uncertainty in the framework of subjective expected utility (SEU) are very limited. In this survey paper, it is shown that non-expected utility (non-EU) models based on the Choquet integral allow for much more diversified behavior, both under risk and under uncertainty. Some illustrations by economic applications to insurance and finance are provided.

Résumé

Les modèles normatifs de décision dans le risque ou dans l'incertain basés sur l'espérance d'utilité (EU), sont assez rigides.

Dans cet article de synthèse, nous montrons que les modèles d'utilité non espérée (non-EU) basés sur l'intégrale de Choquet permettent des comportements beaucoup plus diversifiés, à la fois dans le risque et dans l'incertain. Nous illustrons ensuite cette diversité de comportements à l'aide de quelques applications en assurance et en finance.

1 Introduction

Normative behavior under risk is structurally very limited in the framework of the Expected Utility (EU) model. For example, for an EU decision maker, weak risk aversion in the sense of preferring to any random variable its expected value, implies automatically aversion to any mean preserving increase in risk (MPIR).

In the same spirit, behavior under uncertainty is very rigid in the framework of Subjective Expected Utility (Savage, 1954), since this model differs from EU theory under risk only by its interpretation of the underlying probability distribution.

Our main purpose in this survey paper is to show that the stream of non-EU models based on Choquet integration allows for much more diversified modes of behavior, both under risk and under uncertainty.

The paper is organized as follows. After introducing the framework and the notations common both to risk and to uncertainty in section 2, we study in section 3 behavior under

and detailed. In EU, the notion of increase in risk defined by Rothschild and Stiglitz (1970) is dominant even if it leads to some paradoxes or counter-intuitive examples in portfolio choice, partial insurance, measurement of inequality and other economic issues. We will first define the different notions of increase in risk to be used in this survey, describe their main properties and characterizations and then define the associated notions of risk aversion, postulated independently of any model. In the EU model all of these notions of increase in risk lead to a sole characterization of risk aversion, concavity of u . In contrast, in the Rank Dependent Expected Utility (RDEU) model they lead to different characterizations of risk aversion, even in the particular case of Yaari's model, in which the utility on outcomes is linear.

The impact of these new tools and of the plurality of types of risk aversion will be illustrated with some recent economic applications to insurance linked with Jewitt's Location-independent Risk order (Jewitt (1989) and Landsberger and Meilijson (1994)), a refinement of Rothschild and Stiglitz (1970)'s criterion.

Section 4 focusses less formally on behavior under uncertainty, with a description of some economic applications, far from forming an exhaustive list. The Choquet Expected Utility (CEU) models, the dominant non-EU models under uncertainty, will be introduced stressing features of their axiomatization related to hedging effects. The greater flexibility of CEU models when compared to EU or SEU, will be seen to lead to a more flexible model of actual behaviors, through economic applications to portfolio choice, optimal risk sharing rules and asset pricing. More explicitly, CEU models offer a simple theoretical foundation for the existence of a range of prices at which an investor has no position in a given asset; full insurance at equilibrium may occur in a re-insurance market in case of non-aggregate risk, as long as the agents conceive as possible at least one but possibly several common probability distributions; finally, CEU models offer price functional consistent with the observed violation of the put-call parity.

2 Notations

Let S be the set of states of the world, \mathcal{A} a σ -algebra of subsets of S and \mathbb{V} the set of all bounded \mathcal{A} -measurable real functions on S . An element X of \mathbb{V} , or *act*, can be interpreted as an asset yielding payoff $X(s) \in \mathcal{R}$ if $s \in S$ occurs. Let \succeq be the preference relation of a decision maker on \mathbb{V} . As usual, $X \succeq Y$ reads X is preferred to Y , $X \succ Y$ means X is strictly preferred to Y and $X \sim Y$ means that the decision maker is indifferent between X and Y . The relation \succeq is assumed to be a total preorder, i.e. reflexive, transitive and complete.

Let us first recall the definition of comonotone functions (Schmeidler (1986 and 1989), Yaari (1987)). It will be used extensively both under risk and under uncertainty.

Definition 1 *Two real-valued functions X and Y in \mathbb{V} are comonotone if for any s and s' in S , $[X(s) - X(s')][Y(s) - Y(s')] \geq 0$.*

For example, for any random variable $U \in \mathbb{V}$ and any two non-decreasing real functions g and h on \mathcal{R} , the \mathcal{A} -measurable functions $g(U)$ and $h(U)$ are comonotone.

3 Behavior under risk and economic applications

We assume that risk prevails and describe it through a set of states of nature $S = [0, 1]$ endowed with the uniform probability measure P on its Borel σ -field. Any X of \mathbb{V} is a random variable and has then a probability distribution: Let F_X denote the cumulative distribution function of X ($F_X(x) = P\{X \leq x\}$) and $E(X)$ its expected value. Let \mathcal{D} be the set of cumulative probability distribution functions of random variables in \mathbb{V} . Moreover, define F^{-1} from $(0, 1]$ into \mathcal{R} by $F^{-1}(p) = \inf\{z \in \mathcal{R} | F(z) \geq p\}$, the highest gain among the least favourable $p\%$ of the outcomes.

If X and Y are comonotone, then $cov(X, Y) \geq 0$ for any probability distribution on (S, \mathcal{A}) (at least if all singletons belong to \mathcal{A}), (see Chateauneuf, Kast and Lapied(1996)).

The distribution (or *law*) of a *discrete* random variable Z will be denoted by

$$\mathcal{L}(Z) = (x_1, p_1; \dots; x_k, p_k; \dots; x_n, p_n) \quad (1)$$

with $x_1 < x_2 < \dots < x_n$, $p_i \geq 0$ and $\sum p_i = 1$. For convenience of exposition, the distribution $\mathcal{L}(Z)$ will also be expressed in the equivalent form :

$$\mathcal{L}(Z) = (x_1, 1 - v_1; x_2, v_1 - v_2; \dots; x_{n-1}, v_{n-2} - v_{n-1}; x_n, v_{n-1}) \quad (2)$$

where $v_i = \sum_{j=i+1}^n p_j$.

3.1 Definition and properties of different notions of increase in risk

In what follows, notions of increase in risk will only compare random variables with the *same mean*.

3.1.1 Justification of different notions of increase in risk

There exist many economic situations where the notion of increase in risk in the sense of Rothschild and Stiglitz (1970) does not seem to be well fitted to the problem; some of these gave rise to other notions of increase in risk. Let us consider the following example of Landsberger and Meilijson: The random variables X and Y take the values indicated respectively in the second and the third rows with probabilities indicated in the first row:

	1/1000	1/1000	498/1000	498/1000	1/1000	1/1000
X	-2.10^6	0	-10^3	10^3	0	2.10^6
Y	-2.10^6	-2.10^6	0	0	2.10^6	2.10^6

It is easy to check that Y is a mean preserving increase in risk with respect to X , in the sense of Rothschild and Stiglitz¹ (1970). However, exchanging X for Y makes more than 99% of the distribution move towards the mean. Such a denomination of increase in risk for this case would seem very inappropriate for some decision makers. Because of this kind of problem, and also because of counter-intuitive economic examples with the definition of increase in risk given by Rothschild and Stiglitz (1971), some authors built other possible definitions of increase in risk. This survey will focus on four of them. To illustrate their differences, a simple canonical example of each is presented before the formal definition, that may not be intuitive at first sight.

¹A formal definition of MPIR is given in the next subsection.

3.1.2 Different notions of canonical spread (Generalization of transfers à la Pigou Dalton)

Let Z be a discrete random variable with

$$\mathcal{L}(Z) = (x_1, p_1; \dots; x_k, p_k; \dots; x_n, p_n), \quad (3)$$

where $x_1 < \dots < x_k < x_{k+1} < \dots < x_n$.

We want to analyze the effect of perturbing the distribution of Z by a transfer à la Pigou-Dalton, that preserves the mean of the distribution. Let $\mathcal{L}(Z^*)$ take the following increasing values with the same probabilities p_1, p_2, \dots, p_n

$$\mathcal{L}(Z^*) = (x_1, p_1; \dots; x_{i-1}, p_{i-1}; x_i - \alpha, p_i; \dots; x_{j-1} - \alpha, p_{j-1}; x_j, p_j; \dots, \quad (4)$$

$$x_{k-1}, p_{k-1}; x_k + \beta, p_k; \dots; x_{l-1} + \beta, p_{l-1}; x_l, p_l; \dots, x_n, p_n) \quad (5)$$

Under this type of spread, all the probabilities and the order of the outcomes remain unchanged, α and β are such that the expected value remain the same :

$$\alpha \sum_{r=i}^{r=j-1} p_r = \beta \sum_{r=k}^{r=l-1} p_r.$$

Such a spread can be conveniently parametrized by specifying i, j, k, l :

General spread: $1 \leq i \leq j \leq k \leq l \leq n$; This general spread allows the extreme outcomes not to be spread out. A *canonical* example can be :

$$\mathcal{L}(Z^*) = (x_1, p_1; x_2 - \epsilon p_4, p_2; x_3, p_3; x_4 + \epsilon p_2, p_4; x_5, p_5)$$

Extreme spread: $1 = i < j \leq k < l = n$; In this extreme spread, the extreme outcomes are spread out. A *canonical* example can be :

$$\mathcal{L}(Z^*) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3)$$

Left spread : $1 = i < j \leq k < l \leq n$; In this "spread of losses", the minimal outcome is spread out, but the maximal outcome is not. A *canonical* example can be :

$$\mathcal{L}(Z^*) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)$$

Right spread : $1 \leq i < j \leq k < l = n$; In this "spread of gains", the maximum outcome is spread out, but the minimal outcome is not. A *canonical* example can be :

$$\mathcal{L}(Z^*) = (x_1, p_1; x_2 - \epsilon p_4, p_2; x_3, p_3; x_4 + \epsilon p_2, p_4)$$

3.1.3 Definition of the different notions of increase in risk

Each of the above canonical spreads is a particular case of a specific notion of increase in risk. These notions will be defined first by integral conditions, expressed in terms of their integrated distribution functions. Other characterizations which may prove to be more intuitive and meaningful will be presented later.

For X and Y with the same mean,

Definition 2 Y is a general mean preserving increase in risk (MPIR) of X if $\int_{-\infty}^t F_Y(x) dx \geq \int_{-\infty}^t F_X(x) dx$ for all $t \in \mathcal{R}$.

This notion was introduced to economics by Rothschild and Stiglitz (1970), (see also Schmeidler for a historical prospective). X is said to dominate Y in second order (X DS2 Y), although second order dominance is a more general notion, allowing unequal means. It is easy to check that each of the spreads above is a particular case of the notion of mean preserving increase in risk.

Definition 3 Y is a monotone mean preserving increase in risk with respect to X if $E(Y) = E(X)$ and $F_Y^{-1}(q) - F_Y^{-1}(p) \geq F_X^{-1}(q) - F_X^{-1}(p)$ for all $0 < p < q < 1$.

The interquantile intervals are shorter for X than for Y .

This concept, that Quiggin (1992) defined as the statement of Proposition 4, is already known in Statistics since the 70's under the denomination of Y more dispersed than X in the sense of Bickel and Lehmann (1976, 1979), although the latter don't require equality of means. The connection with Quiggin's notion is detailed in (3.1.6). It is easy to check that canonical extreme spread is a particular case of monotone mean preserving increase in risk.

Remark 1 *Eeckhoudt and Hansen (1980) and Meyer and Ormiston (1985) have an economic application of the corresponding reduction of risk for a producer, that they call "price band stabilization".*

Definition 4 Y is a left monotone mean preserving increase in risk of X if $\int_{-\infty}^{F_Y^{-1}(p)} F_Y(x) dx \geq \int_{-\infty}^{F_X^{-1}(p)} F_X(x) dx$ for all $p \in (0, 1)$.

This definition is due to Jewitt² (1989) under the name of *Location-independent Risk*. The motivation of Jewitt was to find a notion of increase in risk that models coherent behavior in a context of partial insurance (details are given in section 3.3.3.)

It is instructive to notice that in the integral definition above, the upper limits of integration are arbitrary quantiles corresponding to equal cumulative probabilities p . If the upper limits of integration are taken to be arbitrary but *equal* to each other, the corresponding integral condition becomes (the usual) Definition 2 of MPIR or Second Degree dominance.

Definition 5 Y is a right monotone mean preserving increase in risk of X if $E(Y) = E(X)$ and $\int_{F_Y^{-1}(p)}^{+\infty} F_Y(x) dx \geq \int_{F_X^{-1}(p)}^{+\infty} F_X(x) dx$ for all $p \in [0, 1]$.

It is easy to check that canonical left (resp. right) spread is a particular case of mean preserving left (resp. right) monotone increase in risk.

Remark 2 *When X and Y are discrete, with the same mean, it has been proved that each different increase in risk can be obtained by a finite sequence of corresponding Pigou-Dalton transfers. For more details on left and right monotone increase in risk, see Jewitt (1989), Landsberger and Meilijson (1994), Chateauneuf, Cohen and Meilijson (1997a).*

3.1.4 Comparison of these different notions of increase in risk

For X and Y in \mathbb{V} with $E(X) = E(Y)$, the following implications are obvious:

$$\begin{array}{ccccc}
 & & Y \text{ Left monotone MPIR of } X & & \\
 Y \text{ monotone MPIR of } X & \Rightarrow & & \Rightarrow & Y \text{ MPIR of } X \\
 & & Y \text{ right monotone MPIR of } X & &
 \end{array}$$

²In Jewitt, the notion is given for X and Y with possibly unequal means.

3.1.5 Characterizations in terms of quantiles

Re-arranging the preceding formula, the different reductions of risk can be interpreted in terms of quantiles :

Proposition 1 (*Lorentz characterization of Second Degree dominance.*) X is less risky than Y in the sense of Rothschild and Stiglitz (or X DS2 Y) if and only if $\frac{1}{p} \int_0^p F_X^{-1}(t) dt \geq \frac{1}{p} \int_0^p F_Y^{-1}(t) dt$ for every $p \in [0, 1]$.

The expected position for the $p\%$ smallest values are greater for X than for Y .

Proposition 2 X is left monotone less risky than Y (or less risky in the sense of Jewitt) if and only if $\frac{1}{p} \int_0^p [F_X^{-1}(t) - F_Y^{-1}(t)] dt$ is a non-increasing function of $p \in [0, 1]$.

The additional expected gain of X upon Y for the $p\%$ smallest gains is non-increasing in p .

Proposition 3 X is right monotone less risky than Y (or less risky in the sense of Dual-Jewitt) if and only if $\frac{1}{1-p} \int_p^1 [F_X^{-1}(t) - F_Y^{-1}(t)] dt$ is a non-increasing function of $p \in [0, 1]$

The additional expected gain of X upon Y for the $(1-p)\%$ highest positions is non-decreasing in $(1-p)$ (or non-increasing in p).

Proposition 4 X is monotone less risky than Y (or less dispersed in the sense of Bickel and Lehmann) if and only if $F_X^{-1}(p) - F_Y^{-1}(p)$ is non-increasing in p .

3.1.6 Characterization of the different notions of increase in risk in terms of the addition of noise

With the usual notation $Y_1 \stackrel{d}{=} Y_2 \Leftrightarrow \mathcal{L}(Y_1) = \mathcal{L}(Y_2)$, noise characterizations are as follows.

Proposition 5 (i) Y is a general MPIR of X if and only if $Y \stackrel{d}{=} X + Z$, where Z is such that $E(Z|X) = 0$ a.s. (or equivalently $E(Y|X) = X$);

(ii) Y is a monotone increase in risk of X if and only if $Y \stackrel{d}{=} X + Z$, where Z is such that $E(Z) = 0$ and X and Z are comonotone;

(iii) Y is a left monotone increase in risk of X if and only if $Y \stackrel{d}{=} X + Z$, where Z is such that $E(Z) = 0$ and $E(Z|X \leq F_X^{-1}(p))$ is non-decreasing in $p \in (0, 1)$;

(iv) Y is a right monotone increase in risk of X if and only if $Y \stackrel{d}{=} X + Z$, where Z is such that $E(Z) = 0$ and $E(Z|X \geq F_X^{-1}(p))$ is non-decreasing in $p \in (0, 1)$.

Part (ii) of this proposition is the definition given by Quiggin (1992). The equivalence between this property stated as a definition by Quiggin and the definition of monotone increase in risk given here, due to Bickel and Lehmann (1976, 1979), has been found in Landsberger and Meilijson (1994). (see also Cohen (1995), Chateauneuf, Cohen and Kast (1997)).

3.1.7 Examples of the different notions of mean preserving increases in risk

To better understand all of these notions, we will compare, in the following table, some examples of the different ways of reducing risk by replacing the random variable Y by random variables X_i , where each X_i is less risky than $Y = X_i + Z_i$.

Probability	1/5	1/5	1/5	1/5	1/5	
Y	-2000	-1000	0	+1000	+2000	
$X1$	0	0	0	0	0	Complete reduction of risk
$Z1 = Y - X1$	-2000	-1000	0	+1000	+2000	
Y	-2000	-1000	0	+1000	+2000	
$X2$	-1250	-500	0	500	+1250	Monotone reduction of risk
$Z2 = Y - X2$	-750	-500	0	+500	+750	
Y	-2000	-1000	0	+1000	+2000	
$X3$	-1250	-750	0	0	+2000	Left monotone reduction of risk
$Z3 = Y - X3$	-750	-250	0	+1000	0	
Y	-2000	-1000	0	+1000	+2000	
$X4$	-2000	0	0	+750	+1250	Right monotone reduction of risk
$Z4 = Y - X4$	0	-1000	0	+250	+750	
Y	-2000	-1000	0	+1000	+2000	
$X5$	-2000	0	0	0	+2000	Rothschild-Stiglitz reduction of risk
$Z5 = Y - X5$	0	-1000	0	+1000	0	

3.1.8 Characterization in terms of comparing functionals

Let \mathcal{F} denote the set of non-decreasing functions f from $[0, 1]$ to $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. The notion of a function f *star-shaped* at a point m is :

Definition 6 A function $f \in \mathcal{F}$ is *star-shaped* at m , if $\frac{f(m)-f(p)}{m-p}$ is a non-decreasing function of p on $[0, m) \cup (m, 1]$.

Proposition 6 For X and Y in \mathbb{V} with respective cumulative distribution functions F and G with the same finite mean,

- (i) Y is a general increase in risk of X if and only if for every convex $f \in \mathcal{F}$, $\int f(P\{X > x\})dx \geq \int f(P\{Y > x\})dx$;
- (ii) Y is a monotone increase in risk of X if and only if for every $f \in \mathcal{F}$ with $f(p) \leq p$ for all $p \in [0, 1]$, $\int f(P\{X > x\})dx \geq \int f(P\{Y > x\})dx$;
- (iii) Y is a left monotone increase in risk of X if and only if for every $f \in \mathcal{F}$ that is star-shaped at 1, $\int f(P\{X > x\})dx \geq \int f(P\{Y > x\})dx$;
- (iv) Y is a right monotone increase in risk of X if and only if for every $f \in \mathcal{F}$ that is star-shaped at 0, $\int f(P\{X > x\})dx \geq \int f(P\{Y > x\})dx$.

Remark 3 The properties stated in this theorem are independent of any model of choice. It will be seen in the next section that the functional $V(X) = \int f(P\{X > x\})dx$ characterizes

a Yaari DM with perception function f , in the sense that this DM prefers X to Y if and only if $V(X) \geq V(Y)$.

3.2 Definition of the corresponding notions of risk aversion and their characterization in the various models

3.2.1 Definition of different notions of Risk Aversion (RA)

Each concept of risk aversion will now be defined as aversion to the corresponding concept of increase in risk introduced in 3.1.3.

Definition 7 (i) A DM is strongly risk averse (respectively, strongly risk seeking) if for any X and Y in \mathbb{V} such that Y is a MPIR of X , the DM prefers X to Y (respectively, Y to X).

(ii) A DM is monotone risk averse (respectively, monotone risk seeking) if, for any X and Y in \mathbb{V} such that Y is a monotone MPIR of X , the DM prefers X to Y (respectively, Y to X).

(iii) A DM is left monotone risk averse (respectively, left monotone risk seeking) if for any X and Y in \mathbb{V} such that Y is a left monotone MPIR of X , the DM prefers X to Y (respectively, Y to X).

(iv) A DM is right monotone risk averse (respectively, monotone risk seeking) if for any X and Y in \mathbb{V} such that Y is a right monotone MPIR of X , the DM prefers X to Y (respectively, Y to X).

3.2.2 Comparison of the different notions of risk aversion

Let us first recall the Arrow-Pratt notion of risk aversion, to be called *weak* risk aversion and often referred to as aversion to global risk. This notion is expressed independently of any model of choice between lotteries.

Definition 8 A DM is a weakly risk averse (respectively, a weakly risk seeking) if for any $X \in \mathbb{V}$, the DM prefers $E(X)$ to X (respectively, X to $E(X)$).

It is easy to check the following implications:

$$\begin{array}{ccccc} & & \text{Left monotone RA} & & \\ \text{Strong RA} \Rightarrow & & & \Rightarrow \text{Monotone RA} & \Rightarrow \text{Weak RA} \\ & & \text{Right monotone RA} & & \end{array}$$

3.2.3 Characterization of risk aversion in the EU model

Let us recall the essentials of the Expected Utility model:

Definition 9 A DM behaves in accordance with the Expected Utility (EU) model, if the DM's preferences are represented by the functional³

$E[u(Z)] = \int u(x)dF(x) = - \int u(x)d(1-F(x))$, characterized by a continuous, increasing utility function $u: \mathcal{R} \rightarrow \mathcal{R}$. This utility function is uniquely determined up to a positive affine transformation.

³These formulas are helpful to compare the EU functional with the RDEU functional.

When Z is a discrete random variable (see formula (2)), $E[u(Z)]$ can be written as:

$$\begin{aligned} E[u(Z)] &= u(x_1) + v_1[u(x_2) - u(x_1)] \\ &\quad + v_2[u(x_3) - u(x_2)] + \cdots + v_{n-1}[u(x_n) - u(x_{n-1})] \end{aligned} \quad (6)$$

The EU model yields the following very rigid result.

Theorem 7 *For an EU DM, each of the previously defined notions of risk aversion is characterized by the same condition: concavity of the utility function u .*

This result can be interpreted in the following way: A DM who behaves in accordance with EU theory cannot be weakly risk averse without being risk averse for all the other types of risk aversion. As will be seen, this is not the case in the Rank-Dependent Expected Utility model. Because of this result, in EU theory, it is common to speak of a *risk averse EU DM*, without specifying the type of risk aversion.

3.2.4 Characterization of the various notions of risk aversion in the RDEU model

Definition of the RDEU model and Yaari's Dual Theory The RDEU (Rank Dependent Expected Utility) model is due to Quiggin (1982) under the denomination of Anticipated Utility. Variants of this model are due to Yaari (1987), Segal (1987) and Allais (1988). More general axiomatizations can be found in Wakker (1994), Chateauneuf (1997).

Definition 10 *A DM behaves in accordance with the rank-dependent expected utility (RDEU) model if the DM's choices between lotteries are characterized by two functions u and f : a continuous, non-decreasing utility function $u : C \rightarrow \mathcal{R}$ (that plays the role of utility on certainty) and an increasing probability-perception function $f : [0, 1] \rightarrow [0, 1]$ that satisfies $f(0) = 0, f(1) = 1$. Such a DM prefers the random variable X to the random variable Y if and only if $V(X) \geq V(Y)$, where the functional V is given by*

$$V(Z) = V_{u,f}(Z) = - \int_{-\infty}^{\infty} u(x) df(P(Z > x)) = - \int_{-\infty}^{\infty} u(x) df(1 - F(x)) \quad (7)$$

$$= \int_{-\infty}^0 [f(P(u(Z) > t)) - 1] dt + \int_0^{\infty} f(P(u(Z) > t)) dt. \quad (8)$$

When Z is discrete, (see (2)), $V(Z)$ can be written as

$$\begin{aligned} V(Z) &= - \int u(x) df(1 - F(x)) = u(x_1) + f(v_1)[u(x_2) - u(x_1)] \\ &\quad + f(v_2)[u(x_3) - u(x_2)] + \cdots + f(v_{n-1})[u(x_n) - u(x_{n-1})] \end{aligned} \quad (9)$$

The DM takes for sure the utility of the worst outcome $u(x_1)$ and weights the additional possible increases of utility $u(x_i) - u(x_{i-1})$ by his personal transformation $f(v_i)$ of the probability v_i of having at least x_i . According to this interpretation of the formula, if the decision maker is such that $f(p) \leq p$, comparing (3,8) with (3,5), it means that he underestimates all the additional utilities of gains. In this sense, we will call him *pessimist under risk*.

*If the perception function f is the identity function $f(v) \equiv v$, then $V(Z) = V_{u,I}(Z)$ is the expected utility $E[u(Z)]$ of the random variable.

*The *Yaari* functional (see Yaari (1987)) is the special case $V(Z) = V_{I,f}(Z)$, in which the utility u is the identity function $u(x) \equiv x$.

*If both perception and utility are identity functions, then $V(Z) = V_{I,I}(Z)$ is simply the expected value $E[Z]$ of the random variable.

Characterization of different notions of risk aversion in Yaari's model We will focus first on Yaari's model because this model, although based as EU theory, on only one function f , is flexible enough to discriminate between the different kinds of risk aversion.

Theorem 8 (i) A Yaari DM is strong risk averse if and only if his perception function f is convex (see Yaari, 1987).

(ii) A Yaari DM is monotone risk averse if and only if his perception function f is majorized by the identity function. (see Yaari 1987, Quiggin 1992).

(iii) A Yaari DM is left monotone (respectively right monotone) risk averse if and only if his perception function f is star-shaped at 1 (respectively star-shaped at 0) (see Chateauneuf, Cohen and Meilijson 1997a).

As a conclusion to this theorem, the denomination of Yaari model as Dual Theory under risk, is not so appropriate since this model is more flexible than EU model : it allows more diversified behaviors.

Characterization of different notions of increase in risk using Yaari model According to the remark in 3.1.8., the theorem 6 can now be restated in terms of Yaari's decision makers:

Theorem 9 For two assets X and Y with respective distributions functions F and G , and with the same finite mean :

(i): Y is a MPIR of X if and only if all the Yaari's Decision Makers with convex probability perception function prefer X to Y .

(ii): Y is a monotone MPIR of X if and only if all the Yaari's Decision Makers with probability perception function majorized by the identity function prefer X to Y .

(iii) : Y is a left monotone MPIR of X if and only if all the Yaari's Decision Makers with probability perception function star shaped at 1 prefer X to Y

(iv) Y is a right monotone MPIR of X if and only if all the Yaari's Decision Makers with probability perception function star shaped at 0 prefer X to Y .

Part (i) is proved in Yaari (1987), (see also Chateauneuf, 1994). Part (ii) can be found in Quiggin, 1992. Parts (iii) and (iv) are proved in Chateauneuf, Cohen and Meilijson, 1997a.

Characterization of different notions of risk aversion in the RDEU model Each of the four different notions of aversion to risk considered in this survey has a specific characterization⁴. Gathering several results in different papers, we get the following results:

⁴Machina (1982a, 1982b) was the first to notice that the equivalence between different notions of risk aversion in the EU model does not carry over to generalized models.

Theorem 10 Let the two functions u and f characterizing a RDEU decision maker be differentiable. Then

- (i) The DM is strongly risk averse if and only if the perception function f is convex and the utility function u is concave (see Chew, Karni and Safra, 1987).
- (ii) The DM is left monotone risk averse if and only if the perception function f is star-shaped at 1 and the utility function u is concave (see Chateauneuf, Cohen and Meilijson, 1997a).
- (iii) The DM is right monotone risk averse if and only if the perception function f is star-shaped at 0 and the utility function u is concave (see Chateauneuf, Cohen and Meilijson, 1997a).

The case of monotone risk aversion is worth treating separately because monotone risk aversion does not require concavity of u .

This treatment will be based on two indices $P_f = \inf_{0 < v < 1} [\frac{1-f(v)}{f(v)} / \frac{1-v}{v}]$, called index of pessimism, which is ≥ 1 as soon as $f(p) \leq p$, and $G_u = \sup_{y \leq x} u'(x)/u'(y)$, called index of greediness, which always satisfies $G_u \geq 1$, and where the value 1 corresponds exclusively to concavity.

Theorem 11 A RDEU DM with probability perception function f and differentiable utility u is monotone risk averse if and only if $P_f \geq G_u$ (see Chateauneuf, Cohen and Meilijson, 1997b).

Remark 4 The most significant feature of this result is that a DM does not need to have a concave utility function u to be monotone risk averse.

Synthesis of the results The following table summarizes the main results.

Risk aversion Characterization	Expected Utility u	Dual Theory (Yaari) f	Rank-Dependent EU (u, f)
Weak RA	u concave	$f(p) \leq p$	(1)
Monotone RA	u concave	$f(p) \leq p$	$P_f \geq G_u$ (2)
Left monotone RA	u concave	$\frac{1-f(p)}{1-p}$ increasing	$\frac{1-f(p)}{1-p}$ increasing and u concave (3)
Right monotone RA	u concave	$\frac{f(p)}{p}$ increasing	$\frac{f(p)}{p}$ increasing and u concave (4)
Strong RA	u concave	f convex	f convex and u concave

(1) : In this case, there is no known characterization but sufficient conditions, not implying concavity of u (see Chateauneuf and Cohen, 1994).

(2) holds even if u is not differentiable (see Chateauneuf, Cohen and Meilijson, 1997b), in which case G_u becomes more complex:

$$G_u = \sup_{x_1 < x_2 \leq x_3 < x_4} \left[\frac{u(x_4) - u(x_3)}{x_4 - x_3} / \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right] \quad (10)$$

(3) and (4) : In these cases, if f is not differentiable at 1 (a very strong *certainty effect*), u need not to be concave (see Chateauneuf, Cohen and Meilijson (1997a) for details and proofs).

3.3 Economic applications

Obviously, a more flexible model such as the RDEU model can be expected to explain more diversified economic behaviors. We present a short list of economic applications of RDEU behaviors, without any claim of exhaustivity.

3.3.1 Optimality of insurance with deductible for all left monotone risk averse insurees

In the framework of EU model, Arrow (1965) proved that for a given premium, the optimal insurance contract for a EU risk averse decision maker is a contract with deductible. Gollier and Schlesinger (1996) obtained a nice generalization of this result by proving that this result holds also under strong aversion, whatever be the DM's decision model under risk.

Vergnaud, 1997, in this volume, refined this result by proving that for any left monotone risk averse DM (not necessarily strongly risk averse), whatever be the decision model under risk, the optimal contract for a given premium is a deductible policy. This last result is important since, as shown above, strong risk aversion is disputable in some situations (see Example 1), while Jewitt's refinement appears to be better adapted to insurance. This adds further justification to RDEU models that allow the DM to be left monotone risk averse without being strongly risk averse, which is impossible in the EU model.

As an illustration, the example given by Vergnaud (see *infra* in this volume), an example of a preference pattern between lotteries not explainable by a EU model, can be explained within the Yaari model with a (non-convex) probability perception function f star-shaped at 1. In fact, it can be readily seen that the continuous, piece-wise linear perception function f with break-points $f(0) = 0; f(0.1) = 0.1; f(0.2) = 0.15; f(0.8) = 0.75; f(0.9) = 0.8; f(1) = 1$ is such a solution. f is star-shaped at 1, non-convex and the associated Yaari functional $V()$ is such that $V(L) - V(L') = 150$, $V(L'') - V(L) = 50$ and $V(M) - V(M'') = 150$.

3.3.2 Computation of deductibles with the Yaari model

Once the optimality of a deductible policy for the insuree has been established, the question of computing his optimal level of deductible has to be settled. In a very stimulating paper, Doherty and Eeckhoudt (1995) have shown that this question is very tractable in Yaari's model, at least for strongly risk averse insurees (see their formula (4)).

The theorem below (see Chateauneuf, Cohen and Meilijson, 1997a) proves that this remains true for insurees merely assumed to be left monotone risk averse.

Consider a DM endowed with an initial wealth $w \in \mathcal{R}^+$, facing an insurable risky loss L with distribution function $F : F(l) = P(L \leq l), \forall l \in \mathcal{R}$, supported by the interval $[0, w]$.

Theorem 12 *A left monotone risk averse insured Yaari DM will purchase full insurance if $(1+m)(1-F(0)) - [1-f(F(0))] < 0$.*

Otherwise, \bar{d} is an optimal level of deductible if and only if it satisfies :

$$(1+m)(1-F(\bar{d}_-)) - [1-f(F(\bar{d}_-))] \geq 0 \geq (1+m)(1-F(\bar{d})) - [1-f(F(\bar{d}))] \quad (11)$$

Remark 5 *If F is continuous, (11) reduces to the simple equation :*

$$(1+m)(1-F(\bar{d})) - [1-f(F(\bar{d}))] = 0.$$

Remark 6 *An important feature of this result is that in such a model, left monotone risk averse decision makers (and naturally also strongly risk averse DM's, see Doherty and Eeckhoudt, 1995), can buy full insurance even if the premium is not fair, as long as they are pessimistic enough and the loading rate is not too high: $f(F(0)) < F(0) - m(1-F(0))$.*

3.3.3 Partial insurance

The following proposition is, in fact, the definition by Jewitt (1989) of his notion of *Location-independent Risk* (i.e., the notion of left monotone increase in risk).

Proposition 13 *Y is a left monotone increase in risk of (or Location-independent riskier than) X if and only if for any two utility functions u and v with v more concave than u (i.e., Arrow-Pratt EU more risk averse), $Eu(Y) \leq Eu(X - \pi)$ implies $Ev(Y) \leq Ev(X - \pi)$.*

Remark 7 *We have chosen not to give this result as a definition (as Jewitt did), because this notion of increase in risk is model-free, that is, can be defined independently of the EU model or of any other model.*

This result can be translated to the language of partial insurance. Consider the following counter-intuitive example, in the context of partial insurance: If an EU risk averse insuree is ready to exchange an asset Y for a less risky one X (in the sense of Rothschild and Stiglitz)) at a price π , a EU more risk averse DM should be ready to pay *at least* π to exchange Y for X . However, this is not always true (see Rothschild and Stiglitz, 1971). Jewitt justifies his definition of left monotone increase in risk as the kind of increase in risk making this intuitive result always true.

3.3.4 Portfolio choice

Due to the new notions of increase in risk and the RDEU model, numerous comparative statics results concerning portfolio choice become much more intuitive (see Quiggin, 1991, Ormiston and Quiggin, 1994, Ormiston and Schlee, 1994, Cohen, 1995).

4 Behavior under uncertainty and economic applications

The preceeding section dealt with risk, i.e., with probabilized uncertainty. We hope to convince readers that non-additive models may also show promise in situations of uncertainty, i.e., in situations when there does not exist a given objective probability distribution on the set S of states of the world, available to the decision maker.

The first sub-section contains a review of some non-additive models, more precisely those models directly connected with the Choquet integral (Choquet, 1953), which have been

proposed (see e.g. Schmeidler, 1986 and 1989, Gilboa, 1987, Wakker, 1990) as pertinent substitutes to Subjective Expected Utility theory. The reader will find in the paper of Jeleva (1997, in this volume) a presentation of the related model of Jaffray (1989a and 1989b).

Sub-section 4.2 reviews some applications to portfolio choice, optimal risk-sharing rules and asset pricing.

4.1 Choquet Expected Utility (CEU) models

These are models that quantify preference relations \succeq on \mathbb{V} . A proper definition of this models requires the definition and some properties of a *capacity* (see Choquet, 1953):

Definition 11 A capacity v is a monotone set function ($\forall A, B \in \mathcal{A}, A \subset B \Rightarrow v(A) \leq v(B)$) from \mathcal{A} to $[0, 1]$ satisfying $v(\Phi) = 0, v(S) = 1$.

A capacity v is convex if $\forall A, B \in \mathcal{A}, v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$, and is concave if $\forall A, B \in \mathcal{A}, v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$.

Definition 12 The Choquet Expected Utility (C.E.U.) model stipulates that the DM ranks assets X with the help of a utility-on-wealth function $u: \mathcal{R} \rightarrow \mathcal{R}$, that is continuous, strictly increasing and cardinal (i.e., defined up to a strictly positive affine transformation) and the help of a personal evaluation set function v that is a capacity. The ranking of assets X is performed through $I(u(X)) = \int u(X)dv$, the Choquet integral of $u(X)$ with respect to the capacity v , defined for $X \in \mathbb{V}$ by

$$\int u(X)dv = \int_{-\infty}^0 [v(u(X) > t) - 1] dt + \int_0^{\infty} [v(u(X) > t)] dt \quad (12)$$

The similarity with the RDEU model under risk (see formula 3.7) is clear, once it is noticed that if f is a probability perception function, $f(P)$ is a capacity.

Remark 8 Notice that the axiomatic of RDEU can be deduced from the corresponding axiomatic under uncertainty when merely assuming a first stochastic dominance axiom⁵ i.e. $P(X > t) \geq P(Y > t), \forall t \in R$, entails $X \succeq Y$.

It should be noted that if v proves to be an additive probability P , $\int u(X)dv$ reduces to the mathematical expectation of $u(X)$ with respect to P . In this sense, Choquet Expected Utility is a true generalization of Expected Utility theory.

As for RDEU in section 3, the Choquet formula can be expressed in a simple and interpretable manner for step functions, i.e., for assets with a finite number of payoffs.

To this effect, let $X = \sum_1^n x_i A_i^*$ with $x_1 < \dots < x_i < \dots < x_n$, where $(A_i)_{1 \leq i \leq n}$ is a partition of S and A_i^* is the characteristic function of A_i , i.e., the asset offering a payoff 1 if $s \in A_i$ occurs, and 0 otherwise. In this case,

$$\begin{aligned} \int u(X)dv &= u(x_1) + v(X \geq x_2)(u(x_2) - u(x_1)) + \dots \\ &+ v(X \geq x_i)(u(x_i) - u(x_{i-1})) + \dots + v(X \geq x_n)(u(x_n) - u(x_{n-1})) \end{aligned}$$

The DM takes for sure the utility of the worst outcome $u(x_1)$ and weighs the additional possible increases of utility $u(x_i) - u(x_{i-1})$ by the DM's personal estimation $v(X \geq x_i)$ of the occurrence of the event $X \geq x_i$.

⁵See Wakker, 1990, for a justification of this result.

Another way to write this formula is

$$\begin{aligned} \int u(X)dv &= (1 - v(X \geq x_2))u(x_1) + \dots + (v(X \geq x_i) - v(X \geq x_{i-1}))u(x_i) + \dots \\ &\dots + (v(X \geq x_n) - v(X \geq x_{n-1}))u(x_n) . \end{aligned} \quad (13)$$

Thus, the DM values the “utility of X ” as a weighted average of the utility of the payoffs $u(x_i)$, with weights $v(X \geq x_i) - v(X \geq x_{i-1})$ that are positive and sum up to 1; if v is a probability P , these weights are nothing else than $P(X = x_i)$.

A central feature of the CEU model is that it aims to take into account possible hedging effects. A crucial axiom for axiomatizations of CEU (see e.g. Gilboa, 1987, Chew and Wakker, 1996) is the comonotone sure thing principle, a weakening of the sure thing principle, which can be stated as:

Axiom 1 Let $X = \sum_{i=1}^n x_i A_i^*$ and $Y = \sum_{i=1}^n y_i A_i^*$, where $\{A_i\}$ is a partition of S and $x_1 \leq \dots \leq x_i \leq \dots \leq x_n$; $y_1 \leq \dots \leq y_i \leq \dots \leq y_n$ are such that $x_{i_0} = y_{i_0}$ for some $1 \leq i_0 \leq n$. Then $X \succeq Y$ implies $X' \succeq Y'$ for acts X' and Y' obtained from the acts X and Y by merely replacing the i_0 'th common outcome by any other common outcome that preserves the rank i_0 in each of the two sequences x and y .

This axiom expresses that as long as assets remain *comonotone* (i.e., no hedging effect happens), there is no reason to change the direction of preferences when a common outcome is modified.

Note, however, that even jointly with standard axioms, the comonotone sure-thing principle fails to fully characterize CEU.

Things are much more clear-cut if marginal utility of wealth is assumed to be constant ($u(x) = x, \forall x \in R$). By the way, this entails that $\forall X \in \mathbb{V}, I(X) = \int X dv$ can be interpreted as the certainty equivalent of X . In such a case (see e.g. Chateauneuf, 1994), CEU is mainly characterized through the following *comonotone Independence Axiom*:

Axiom 2

For $X, Y, Z \in \mathbb{V}$, X and Y comonotone, Y and Z comonotone, then,

$$X \sim Y \Rightarrow X + Z \sim Y + Z$$

This axiom can be easily interpreted as : the direction of preferences has to be maintained under adding payments, provided hedging is not involved.

Such an axiom allows for taking into account hedging effects, as the following *uncertainty aversion axiom* :

Axiom 3 [For $X, Y, Z \in \mathbb{V}, X \sim Y, Z$ comonotone with Y] $\Rightarrow X + Z \succeq Y + Z$

Since Z is not a hedge against Y but may be a hedge against X , then $X + Z$ displays a reduction of uncertainty with respect to $Y + Z$.

Note that Axiom 3 entails Axiom 2, hence in case of uncertainty aversion, Axiom 3 can be substituted to Axiom 2.

Under Axiom 3, v proves to be convex and this is a nice property, since as shown by Schmeidler (1986), if v is convex, then, $I(X) = \int X dv$ is equal to $\min \{ \int X dP, P \in \text{core}(v) \}$, where

$core(v) = \{\text{additive probabilities measures } P \text{ on } \mathcal{A} \text{ s.t. } P(A) \geq v(A), \forall A \in \mathcal{A}\}$

This offers an attractive interpretation of uncertainty aversion in terms of pessimism: the DM conceives as possible any probability distribution in the core of v , and by pessimism, feels that the worst of these distributions (in terms of mathematical expectation) will happen.

4.2 Some applications

4.2.1 Optimal portfolio choice

A major economic application of the Choquet Expected Utility model was obtained by Dow and Werlang (1992), when revisiting the basic portfolio problem. Let $w > 0$ be the initial certain wealth of a DM facing the question of buying or selling short an uncertain asset $X \in \mathbb{V}$, whose price is $p > 0$. Under EU theory, the agent is assumed to assign probabilities $P(A)$ to events A and to rank assets $X \in \mathbb{V}$ by comparing expected utilities of X , with a concave utility function u - concavity of u conveying, in the framework of EU model, uncertainty aversion. According to a famous result of Arrow (1965), an EU agent will invest in an asset X if and only if the expected value $E_P(X)$ of this asset exceeds the price, and will wish to sell the asset short if and only if the expected value is lower than the price of the asset. Consequently, the investor will have no position in the asset if and only if the price is exactly $E_P(X)$.

Dow and Werlang (1992) used Choquet Expected Utility theory to show a result intuitively plausible and compatible with observed investment behavior: there is a range of prices at which the investor has no position in the asset. At prices below these, the investor holds a positive amount of the asset and at higher prices he holds a short position.

More precisely, the agent is assumed to rank assets X by performing the Choquet Expected Utility $I(u(X))$ of X , with concave u and convex v . This expresses uncertainty aversion, since $I(u(X))$ can then be expressed by the following formula (see Schmeidler, 1986, Shapley, 1971):

$$I(u(X)) = \min_{P \in core(v)} (E_P(uX))$$

In fact (see Chateauneuf, Dana and Tallon, 1997), u concave and v convex is equivalent to the well-known axiom of convexity of preferences: $\forall X, Y \in V, \forall \alpha \in [0, 1], X \sim Y \Rightarrow \alpha X + (1 - \alpha)Y \succeq X$.

Such an axiom does express uncertainty aversion, since as pointed out by Schmeidler (Schmeidler, 1986 and 1989), this axiom means that "smoothing" or averaging potential gains makes the DM better off. Then, Dow and Werlang proved that for any asset X , the reservation prices only depend of the DM's attitudes towards uncertainty. They are respectively

$$I(X) = \min \{E_P(X), P \in core(v)\} \text{ and } -I(-X) = \max \{E_P(X), P \in core(v)\}$$

Hence, the length of the no-trade interval $(I(X), -I(-X))$ is intimately connected with the size of $core(v)$. Obviously, if $core(v)$ reduces to a single probability law, one obtains the result of Arrow (Arrow, 1965).

4.2.2 Optimal risk-sharing rules and equilibria

Assume that the set of states of nature $S = \{1, \dots, j, \dots, n\}$ is finite and consider m agents endowed respectively with initial portfolios $X_i \in \mathbb{V}, X_i \geq 0, i = 1, \dots, m$.

Suppose also that these agents are engaged in exchanges in order to increase their satisfaction. A possible interpretation of such a market is a re-insurance market for m insurance companies, as suggested by Borch (Borch, 1962). Let us confine ourselves here to the case of non-aggregate risk, i.e., $\sum_{i=1}^m X_i(s) = \sum_{i=1}^m X_i(t), \forall s, t \in S^2$.

Under Subjective Expected Utility, where uncertainty aversion is characterized by a concave u , a well-known result is that uncertainty averse decision makers with common belief, i.e., with the same probability distribution P on (S, \mathcal{A}) , are fully insured at equilibrium (i.e., will have ex-post uncertain endowments $Y_i \geq 0$ satisfying $Y_i(s) = Y_i(t), \forall s, t \in S^2$, for any equilibrium price p belonging to R_{++}^n).

In a very stimulating paper, Tallon (Tallon, 1997) proved that this result remains true under CEU theory, as long as the agents conceive as possible at least one common probability distribution.

More precisely, let the agents be uncertainty averse, i.e., let their choices be represented by convex capacities v_i and concave utility functions $u_i, 1 \leq i \leq m$. Assume further that $\bigcap_i \text{core}(v_i) \neq \Phi$. Then there exists an equilibrium price $p \in R_{++}^n$, and at any equilibrium price agents are fully insured.

Notice (see Chateauneuf, Dana and Tallon, 1997) that this result generalizes to the case of non-convex preferences, i.e., to the case when only $\bigcap_i \text{core}(v_i) \neq \Phi$. Note also that in case of a common set of probability beliefs with a convex v (i.e., $\forall i, v_i = v$ where v is convex), equilibrium prices can be identified with probability laws in $\text{core}(v)$.

In the papers of Tallon, 1997a and 1997b, Chateauneuf, Dana and Tallon, 1997 and Dana, 1997, the reader will find extensions of the previous results to the multidimensional case, aggregate risk and risky situations.

4.2.3 Asset pricing

In Chateauneuf, Kast and Lapied, 1996, for any asset X , $I(X)$ (resp. $-I(-X)$) denotes the selling (resp. buying) price of asset X by a broker⁶.

Three axioms on the functional $I : \mathbb{V} \rightarrow \mathcal{R}$ are proposed in order to model the dealer's behavior. The first two ones are standard: requiring on one hand that if an asset Y pays more in all states than an asset X , its price must be higher, and on the other hand assuming no transaction costs on riskless assets, i.e., $I(\alpha S^*) = \alpha, \forall \alpha \in \mathcal{R}$. The third axiom, named "*comonotonicity premium*" stipulates that $\forall X, Y \in \mathbb{V}, I(X + Y) \leq I(X) + I(Y)$, with equality holding if X and Y are comonotone. This axiom aims at taking into account a potential reduction of uncertainty (when the portfolio $X + Y$ is sold instead of X or Y alone), which would induce the dealer to sell $X + Y$ at a discount compared to X and Y . However, there are situations where no hedging effects occur when adding two assets X and Y : this happens when X and Y are comonotone.

These axioms are equivalent to $I(X) = \int X dv$ with v concave.

Such a pricing scheme provides an explanation for possible positive profits for the broker: since $I(X) \geq -I(-X)$, all the more so the dealer behaves as if he conceived as possible all the probabilities P s.t. $P \leq v$ and evaluates the selling price $I(X)$ by $I(X) = \text{Max}\{E_P(X), P \leq v\}$ and the buying price by $-I(-X) = \text{Min}\{E_P(X), P \leq v\}$. Furthermore, as usual in finance, no arbitrage opportunity is left to the consumer (i.e., in-

⁶Notice that this notation is consistent with the fact that selling asset X is the same as buying the short position $-X$.

vestor) in the sense that he cannot form a portfolio yielding positive payments at a strictly negative cost; this can be easily checked from subadditivity, positive homogeneity and positivity of the price function I .

Lastly, such a price functional is consistent with the observed violation of the put-call parity (see e.g. Gould and Galai, 1974, Galai, 1983, and Lefoll, 1994), namely, with $I(P) \leq I(C) + I(-S) + K$, i.e., it is consistent with the observed fact that for a call C and a put P with exercise price K and underlying security S , the price of the put P is smaller than the price of the replicating portfolio $C - S + KS^*$.

4.3 Concluding remarks

This paper has been devoted to present some aspects of research concerning the use of the Choquet integral in decision theory under risk and under uncertainty, and its economic applications. It was our purpose, in particular, to shed some light on a number of economic behaviors not explainable by Expected Utility theory but explainable by Rank Dependent Expected Utility theory or Choquet Expected Utility theory.

We are grateful to an anonymous referee for helpful comments.

References

- [1] Allais, M., "Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine", *Econometrica*, **21**, 1953, p. 503–546.
- [2] Allais, M." The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function", in *Risk, Decision and Rationality*, B. R. Munier (Ed.), Reidel: Dordrech, 1988, p. 233–289.
- [3] Arrow, K.J. "The theory of risk aversion, in Aspects of the theory of risk bearing". Helsinki : Ynjo Jahnsonin Saatio, 1965.
- [4] Bickel, P. J. and E. L. Lehmann. "Descriptive statistics for non-parametric models, III. Dispersion", *Annals of Statistics*, **4**, , 1976, p. 1139–1158.
- [5] Bickel, P. J. and E. L. Lehmann. "Descriptive statistics for non-parametric models, IV. Spread", in *Contributions to Statistics*, Jureckova (Ed.), Reidel, 1979.
- [6] Borch, K. "Equilibrium in a reinsurance market", *Econometrica*, **30**, 1962, p. 424–444.
- [7] Chateauneuf, A. "Modeling attitudes towards uncertainty and risk through the use of Choquet integral", *Annals of Operations Research*, **52**, 1994 ,p. 3–20.
- [8] Chateauneuf, A. "Comonotonicity axioms and RDEU theory for arbitrary consequences", under revision for *Journal of Mathematical Economics*, 1996.
- [9] Chateauneuf, A. and M. D. Cohen. "Risk-seeking with diminishing marginal utility in a non-expected utility model", *Journal of Risk and Uncertainty*, **9**, 1994, p. 77–91.
- [10] Chateauneuf, A., M. D. Cohen and R. Kast. "A review of some results related to comonotonicity", cahiers d'ecomath, n^o.97.32, 1997.
- [11] Chateauneuf, A., M. D. Cohen and I. Meilijson. "Rank-dependent utilities and comonotonicity-based stochastic orders", Working Paper, 1997a.

- [12] Chateauneuf, A., M. D. Cohen and I. Meilijson. "More pessimism than greediness: A characterization of monotone risk aversion in the Rank Dependent Expected Utility model", *cahiers d'ecomath*, n^o.97. 53, 1997b.
- [13] Chateauneuf, A., Dana, R.A. and J.M.Tallon. "Optimal risk-sharing rules with non-additive expected utility", *cahiers d'ecomath*, n^o.97. 54, 1997.
- [14] Chateauneuf, A, Kast, R. and A. Lapied. "Choquet pricing for financial markets. *Mathematical Finance*, **6**, n3, 1996, p. 323-330.
- [15] Chew, S., E. Karni and Z. Safra. "Risk aversion in the theory of expected utility with Rank Dependent preferences", *Journal of Economic Theory*, **42**, 1987, p. 370-381.
- [16] Chew, S. and P.Wakker. "The comonotonic sure-thing principle", *Journal of risk and uncertainty*, **12**, 1996, p. 5-27.
- [17] Choquet, G. "Theorie des capacités", *Ann. Institut Fourier* (Grenoble), **V**, , 1953, p. 131-295.
- [18] Cohen, M. D. "Risk aversion concepts in expected and non-expected utility models", *The Geneva Papers on Risk and Insurance theory*, **20**, 1995, p. 73-91.
- [19] Dana, R.A. "Optimal risk-sharing and pricing rules when agents have non-additive expected utility and homogeneous expectations", Working Paper, 1997.
- [20] Denneberg, D. "*Non-additive measure and integral*", Kluwer Academic Publishers, 1994.
- [21] Diamond, P. and J. Stiglitz. "Increases in risk and risk aversion", *Journal of Economic Theory*, **8**, 1974, p. 337-360.
- [22] Doherty, N, and L. Eeckhoudt. "Optimal insurance without expected utility : the dual theory and the linearity of insurance contracts", *Journal of risk and uncertainty*, **10**, 1995, p. 157-179.
- [23] Dow, J. and S. Werlang. "Uncertainty aversion, risk aversion and the optimal choice of portfolio", *Econometrica*, **60**, 1992, p. 197-204.
- [24] Dupuis, A. and E. Langlais. "The basic analytics of insurance demand and the rank-dependent expected utility model", *Revue Finance*, this issue, 1997.
- [25] Eeckhoudt, L. and P. Hansen. " Minimum and maximum prices, uncertainty and the theory of the competitive firm", *American Economic Review*, **70**, 1980, p. 1064-1068.
- [26] Galai, D. "A survey of empirical tests of option-pricing models", in *Option Pricing Theory and Applications*, ed. M. Brenner, 1983.
- [27] Gayant, J.P. "Constitution d'un portefeuille et espérance non-additive de gains : suggestions prescriptives pour la combinaison optimale d'actifs financiers", *Revue Finance*, this issue, 1997.
- [28] Gould and Galai. "Transaction costs and the relationship between Put and Call prices", *Journal of Financial Economics*, **1**, 1974, p. 105-129.

- [29] Gilboa, I. "Expected utility with purely subjective non-additive probabilities" *Journal of Mathematical Economics*, **16**, 1987, p. 65-88.
- [30] Gollier, C. and H. Schlesinger. "Arrow's theorem on the optimality of deductibles : a stochastic dominance approach", *Economic Theory*, 1996.
- [31] Jaffray, JY. "Généralisation du critère de l'utilité espérée aux choix dans l'incertain régulier", *Recherche Opérationnelle*, **23**, 1989 a, p. 237-267.
- [32] Jaffray, JY. "Linear utility for belief functions", *Operations Research Letters*, **8**, 1989b, p. 107-112.
- [33] Jeleva, M. "Demand for insurance with imprecise probabilities", *Revue Finance*, this issue, 1997.
- [34] Jewitt, I. "Choosing between risky prospects : the characterization of comparative statics results, and location independent risk", *Management Science*, **35**, 1989, p. 60-70.
- [35] Landsberger, M. and I. Meilijson. "Lotteries, insurance and star-shaped utility functions", *Journal of Economic Theory* , **52**, 1990, p. 1-17
- [36] Landsberger, M. and I. Meilijson. "The generating process and an extension of Jewitt's location independent risk concept", *Management Science*, **40**, 1994, p. 662-669.
- [37] Landsberger, M. and I. Meilijson. "Comonotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion", *Annals of Operations Research*, **52**, 1994, p. 97-106.
- [38] Lefoll, J. "Coûts de transaction et parité des options sur actions du SOFFEX", *Journal de la Société de Statistique de Paris*, tome 135, 2, 1994, p. 19-38.
- [39] Machina, M. "Expected utility analysis without the independence axiom", *Econometrica*, **50**, 1982a, p. 277-323.
- [40] Machina, M. "A stronger characterization of declining risk aversion", *Econometrica*, **50**, n°4, 1982b, p., 1069-1079.
- [41] Meyer, J. and M. Ormiston. "Strong increases in risk and their comparative statics", *International Economic Review*, **26**, 1985, p. 425-437.
- [42] Ormiston, M. and E. Schlee. "Comparative statics tests between decision models under risk", . Working Paper.
- [43] Ormiston, M. and J. Quiggin, (1994). Two-parameter decision models and rank-dependent expected utility, *Journal of risk and uncertainty*, **8**, 273-282.
- [44] Quiggin, J. "A theory of anticipated utility", *Journal of Economic Behavior and Organisation*, **3**, 1982, p. 323-343.
- [45] Quiggin, J. "Comparative statics for rank-dependent expected utility", *Journal of risk and uncertainty*, **4**, 1991, p. 339-350.

- [46] Quiggin, J. "Increasing risk: another definition", In *Progress in Decision, Utility and Risk Theory*, A. Chikan (Ed.), Dordrecht: Kluwer, 1992.
- [47] Quiggin, J. and P. P. Wakker. "The axiomatic basis of anticipated utility: a clarification", Working Paper, Australian National University and University of Nijmegen, 1992.
- [48] Ross, S. "Some stronger measures of risk aversion in the small and in the large, with applications", *Econometrica*, **49**, 1981, p. 621-663.
- [49] Rothschild, M. and J. Stiglitz. "Increasing Risk: I. A definition", *Journal of Economic Theory*, **2**, 1970, p. 225-243.
- [50] Rothschild, M. and J. Stiglitz. "Increasing Risk: II. Its economic consequences", *Journal of Economic Theory*, **3**, 1971, p. 66-84.
- [51] Savage, L. "The foundations of statistics. New York. Wiley, 1954, (Second edition 1972, Dover).
- [52] Schmeidler, D. "Integral representation without additivity", *Proceedings of the American Mathematical Society*, **97**, 1986, p. 255-261.
- [53] Schmeidler, D. "Subjective probability and expected utility without additivity" *Econometrica*, **57**, 1989, p. 517-587 (First version: Subjective expected utility without additivity, Forder Institute Working Paper (1982)).
- [54] Segal, U. "Anticipated utility: a measure representation approach", *Annals of Operations Research*, **19**, 1987, p. 359-374.
- [55] Segal, U. "The measure representation : A correction", *Journal of Risk and Uncertainty*, **6**, 1993, p. 99-107.
- [56] Shapley, L.S. "Cores of convex games", *International Journal of Game theory*, **1**, 1971, p. 11-26.
- [57] Tallon, JM. "Risque microeconomique, aversion a l'incertitude et indetermination de l'equilibre", To appear in *Annales d'economie et de statistique*, 1997a.
- [58] Tallon, JM. "Risque microeconomique et prix d'actifs dans un modele d'equilibre general avec esperance d'utilite dependante du rang", *Revue Finance*, this issue, 1997b.
- [59] Vergnaud, J.C. "Analysis of risk in a non-expected utility framework and applications to the optimality of the deductible", *Revue Finance*, this issue, 1997.
- [60] Wakker, P. "Separating marginal utility and risk aversion", *Theory and Decision*, **36**, 1994, p. 1-44.
- [61] Wakker, P. "Under Stochastic Dominance Choquet Expected Utility and Anticipated Utility are identical", *Theory and Decision*, **29**, 1990, 119-132.
- [62] Yaari, M. "The dual theory of choice under risk", *Econometrica*, **55**, 1987, p. 95-115.
- [63] Yaari, M. "Risk aversion without diminishing marginal utility", *International Center for Economics and Related Disciplines*, London, 1984.