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On the Probability of the Ostrogorski Paradox

William V. Gehrlein and Vincent Merlin

Abstract The notion of the existence of a Strict Ostrogorski Paradox presents an interesting phenomenon that could lead to some very unsettling outcomes in group decision-making situations. This phenomenon cannot be observed in two-issue situations, and when three-issue situations are considered, the probability that such an outcome will be observed never reaches a likelihood of as much as two percent for large electorates, regardless of the propensity of voters to align their preferences on issues with the standards of political parties. The probability of observing a Strict Ostrogorski Paradox in four-issue situations is nearly zero when using an assumption that exaggerates the likelihood that such a paradoxical outcome will be observed. We conclude that it is very unlikely that a Strict Ostrogorski Paradox would ever be observed in any real voting situation with a large electorate

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1 Introduction

Suppose that there are two parties {R,L} that have opposing positions on each of m different issues that are being considered. Each of n voters has preferences on the individual issues that are in agreement with the position of either Party R or of Party L, but a given voter does not necessarily agree with the position of the same party on every issue. However, a given voter will be considered to have an overall alignment with a particular party whenever that voter is in agreement with that party's positions over a majority of the issues that are being considered. If a voter has preferences that agree with both Party R and Party L on $m/2$ issues each when m is even, then there is a

tie and that voter is not considered to be aligned with either party. The majority party is the one with which the greater number of voters have an overall alignment, excluding voters that are not aligned with either party. Each issue is voted on individually, and the outcome of the vote will be in agreement with the position of either Party R or Party L, based on the outcome of majority rule voting on the issue. Voters are assumed to vote sincerely on each of the individual issues.

We are interested in developing representations for the probability, PA_m^k , that the voting results on the m different issues produce exactly k outcomes on issues that are in agreement with the positions of the majority party. PA_m^m is the probability that there is complete agreement between voting outcomes on issues and the majority party positions. PA_m^0 is the probability that there is complete disagreement between voting outcomes on issues and the positions of the majority party. The situation of complete disagreement is referred to as an occurrence of a *Strict Ostrogorski Paradox*, loosely following some terminology that is introduced in Deb and Kelsey (1987). The general problem of such disagreement was originally presented in Ostrogorski (1902) and it was discussed in Daudt and Rae (1976). This phenomenon is often referred to in research that is related to paradoxical outcomes that can produce problems in group decision making. For example, see Brams *et al.* (1998), Nurmi (1999), Saari and Sieberg (2001), List (2005) and Gehrlein (2006). Laffond and Laine (2006) consider restrictions on voters' preferences that preclude the existence of an occurrence of Ostrogorski's Paradox. Mbih and Valeu (2016) derive the likelihood of related paradoxes, Ostrogorski's Paradox and Anscombe's Paradox, under the Impartial Anonymous Culture assumption for $m = 3$ issues, as a function of n . They also provide estimations for these paradoxes under the Impartial Culture assumption via Monte-Carlo simulations for small values of n .

Values of PA_m^k for $0 < k < m$ represent probabilities of increasingly less paradoxical outcomes as k increases, with partial agreement between voting outcomes and the positions of the majority party. Attention is focused on developing these representations for the limiting case in voters, with $n \rightarrow \infty$, when voters form their preferences on each of the issues independently of the preferences of other voters. The study begins by considering specific cases with a small number of issues, and then considers general results.

2 The Case of Three Issues

There are $2^3 = 8$ different possible combinations of voters' preference agreement with party positions on issues with $m = 3$. Let q_i denote the probability that a randomly selected voter will have the associated i^{th} such combination for $1 \leq i \leq 8$, as shown in Table 1.

Table 1 Feasible voter preferences for sequential elections on three issues.

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
Issue 1	R	R	R	R	L	L	L	L
Issue 2	R	R	L	L	R	R	L	L
Issue 3	R	L	R	L	R	L	R	L
Overall	R	R	R	L	R	L	L	L

The probability of complete voter agreement with the positions of Party R (L) for a randomly selected voter is q_1 (q_8) in Table 1. Let $a = q_1 + q_8$ measure the probability that a randomly selected voter shows complete agreement with the issue positions of one of the parties. Such voters can be viewed as staunch supporters of one party's positions. There is greater degree of complete voter agreement with party positions as a increases, but the preferences of the voters will also reflect an increasing degree of polarization as a increases, since we further assume that $q_1 = q_8 = a/2$ so that neither party has any advantage over the other party. The motivation behind this party parity assumption will be explained in detail later. The remaining six rankings represent more moderate voters who show less than complete agreement with any party, with position agreement with some party on two issues and disagreement on one issue. The parity concept is also used here, since it is assumed that $q_i = (1-a)/6$ for $2 \leq i \leq 7$.

We start the development of a representation for the limiting probability of complete agreement, $PA_3^3(a)$, as $n \rightarrow \infty$ for a specified value of a by defining a binary variable Y_i^j for the j^{th} voter, to denote the party association of that voter's preference on the i^{th} issue. For Issue 1:

$$Y_1^j = \begin{aligned} &+1: q_1 + q_2 + q_3 + q_4 \\ &-1: q_5 + q_6 + q_7 + q_8. \end{aligned} \quad (1)$$

This definition of Y_1^j requires that Issue 1 must have an outcome that is in agreement with Party R by majority rule for n voters whenever $\sum_{j=1}^n Y_1^j > 0$, or when the

average value $\overline{Y_1^j} > 0$. The definitions of the q_i probabilities also lead to the expected values $E(Y_1^j) = 0$ and $E(Y_1^{j^2}) = 1$ with party parity.

Issues 2 and 3 have corresponding binary variables Y_2^j and Y_3^j with

$$\begin{aligned} Y_2^j &= +1: q_1 + q_2 + q_5 + q_6 \\ &\quad -1: q_3 + q_4 + q_7 + q_8 \\ Y_3^j &= +1: q_1 + q_3 + q_5 + q_7 \\ &\quad -1: q_2 + q_4 + q_6 + q_8. \end{aligned} \tag{2}$$

As above, $E(Y_2^j) = E(Y_3^j) = 0$ and $E(Y_2^{j^2}) = E(Y_3^{j^2}) = 1$.

Variable Y_4^j is then defined to account for the party alignment of voter j , with

$$\begin{aligned} Y_4^j &= +1: q_1 + q_2 + q_3 + q_5 \\ &\quad -1: q_4 + q_6 + q_7 + q_8. \end{aligned} \tag{3}$$

Party R will be the majority party if $\sum_{j=1}^n Y_4^j > 0$, or $\overline{Y_4^j} > 0$. We also note that

$E(Y_4^j) = 0$ and that $E(Y_4^{j^2}) = 1$ with the adopted definition of the q_i probabilities.

Issues 1, 2 and 3 will then each have outcomes that are in majority rule agreement with Party R, while Party R is the majority party with the joint probability that $\overline{Y_i^j} > 0$ for each $1 \leq i \leq 4$. The q_i definitions make this equivalent to the joint probability that $\overline{Y_i^j} \sqrt{n} > E(\overline{Y_i^j} \sqrt{n})$ for $1 \leq i \leq 4$ since $E(Y_i^j) = E(\overline{Y_i^j}) = 0$ for $1 \leq i \leq 4$. The Central Limit Theorem requires that the distribution of the $\overline{Y_i^j} \sqrt{n}$ variables is multivariate normal as $n \rightarrow \infty$, and the corresponding correlation terms

for these variables are obtained directly from the $E(Y_i^j Y_k^j)$ of the original variables, since $E(Y_i^j) = 0$ and $E(Y_i^{j^2}) = 1$ for all $1 \leq i \leq 4$, with:

$$\begin{aligned}
E(Y_1^j Y_2^j) &= q_1 + q_2 - q_3 - q_4 - q_5 - q_6 + q_7 + q_8 = (4a - 1)/3. \\
E(Y_1^j Y_3^j) &= q_1 - q_2 + q_3 - q_4 - q_5 + q_6 - q_7 + q_8 = (4a - 1)/3 \\
E(Y_2^j Y_3^j) &= q_1 - q_2 - q_3 + q_4 + q_5 - q_6 - q_7 + q_8 = (4a - 1)/3 \quad (4) \\
E(Y_1^j Y_4^j) &= q_1 + q_2 + q_3 - q_4 - q_5 + q_6 + q_7 + q_8 = (2a + 1)/3 \\
E(Y_2^j Y_4^j) &= q_1 + q_2 - q_3 + q_4 + q_5 - q_6 + q_7 + q_8 = (2a + 1)/3 \\
E(Y_3^j Y_4^j) &= q_1 - q_2 + q_3 + q_4 + q_5 + q_6 - q_7 + q_8 = (2a + 1)/3.
\end{aligned}$$

Since the probability of observing any specific value, including zero, in a continuous distribution is equal to zero, the limiting probability as $n \rightarrow \infty$ that Issues 1, 2 and 3 have outcomes that are in majority rule agreement with Party R, while Party R is the majority party is equivalent to the four-variate normal positive orthant probability, $\Phi_4(\mathbf{R}_1)$, that $\overline{Y_i^j} \sqrt{n} \geq E(\overline{Y_i^j} \sqrt{n})$ for $1 \leq i \leq 4$ with correlation

matrix \mathbf{R}_1 :

$$\mathbf{R}_1 = \begin{bmatrix} 1 & \frac{4a-1}{3} & \frac{4a-1}{3} & \frac{2a+1}{3} \\ - & 1 & \frac{4a-1}{3} & \frac{2a+1}{3} \\ - & - & 1 & \frac{2a+1}{3} \\ - & - & - & 1 \end{bmatrix}. \quad (5)$$

The probability of having complete agreement for the case of three issues with two parties and a given a is therefore obtained from $PA_3^3(a) = 2\Phi_4(\mathbf{R}_1)$, and some results follow immediately. Since all correlation terms in \mathbf{R}_1 increase as a increases, a result from Slepian (1962) can be used to show that $PA_3^3(a)$ does not decrease as a increases. The correlation terms in \mathbf{R}_1 fit the special case for four-variate normal

positive orthant probabilities in Gehrlein (1979) and it leads to a representation for $PA_3^3(a)$, with

$$PA_3^3(a) = \frac{1}{8} + \frac{3}{4\pi} \left\{ \sin^{-1} \left(\frac{2a+1}{3} \right) + \sin^{-1} \left(\frac{4a-1}{3} \right) \right\} + \frac{3}{2\pi^2} \int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}} \sin^{-1} \left(\frac{4a-1-3z^2}{4a+2-6z^2} \right) dz. \quad (6)$$

For the specific case with $a = 1$, an exact result is obtained from direct integration of (6), with $PA_3^3(1) = 1$. Table 2 lists values of $PA_3^3(a)$ for each value of $a = .00(.10)1.00$ that were obtained by numerical integration from (6). The computed value for $a = .25$ is also included since this corresponds to the situation in which all voter preference combinations are equally likely to be observed, which is referred to as the *Impartial Culture Condition (IC)* in the general literature.

Table 2 Computed values of $PA_3^3(a)$, $PA_3^2(a)$, $PA_3^1(a)$ and $PA_3^0(a)$.

A	$PA_3^3(a)$	$PA_3^2(a)$	$PA_3^1(a)$	$PA_3^0(a)$
.00	.0877	.6491	.2632	.0000
.10	.1499	.5971	.2490	.0040
.20	.2097	.5527	.2291	.0085
.25	.2396	.5312	.2187	.0104
.30	.2697	.5098	.2083	.0121
.40	.3314	.4665	.1873	.0148
.50	.3959	.4215	.1662	.0164
.60	.4648	.3735	.1446	.0170
.70	.5407	.3207	.1220	.0166
.80	.6282	.2598	.0971	.0149
.90	.7392	.1823	.0671	.0114
1.00	1.0000	.0000	.0000	.0000

The voting outcomes on all three issues agree with the position of Party R under simple majority rule if $\overline{Y_i^j} \sqrt{n} \geq E \left(\overline{Y_i^j} \sqrt{n} \right)$ for all $i = 1,2,3$, and Party L is the majority party if $\overline{Y_4^j} \sqrt{n} \leq E \left(\overline{Y_4^j} \sqrt{n} \right)$. A representation for this probability of

the Strict Ostrogorski Paradox with three issues, $PA_3^0(a)$, can be obtained quite easily by going through the analysis that was presented above while replacing variable Y_4^j with $-Y_4^j$, which leads to

$$PA_3^0(a) = \frac{1}{8} - \frac{3}{4\pi} \left\{ \sin^{-1} \left(\frac{2a+1}{3} \right) - \sin^{-1} \left(\frac{4a-1}{3} \right) \right\} - \frac{3}{2\pi^2} \int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}} \sin^{-1} \left(\frac{4a-1-3z^2}{4a+2-6z^2} \right) dz \cdot \quad (7)$$

An exact solution of (7) can be obtained for the special case of $a = 1$, with $P_3^0(1) = 0$.

Computed values of $P_3^0(a)$ are listed in Table 2 for each $a = .00(.10)1.00$ that were obtained from numerical integration from (7), along with the value for $a = .25$. These results indicate that the probability of observing a Strict Ostrogorski Paradox is quite small over the range of all possible values of a .

An alternative representation for $P_3^0(a)$ is found by using a different approach to the problem that follows Merlin and Tataru (1997), Saari and Tataru (1999) and Merlin *et al.* (2000, 2002), with

$$PA_3^0(a) = \frac{3}{2\pi^2} \int_0^a \left\{ \frac{2 \cos^{-1} \left(\frac{\sqrt{2t+1}}{\sqrt{8t+1}} \right)}{\sqrt{2+2t-4t^2}} - \frac{\cos^{-1} \left(\frac{t+1}{2t+1} \right)}{\sqrt{2-t-t^2}} \right\} dt \cdot \quad (8)$$

This particular representation for $P_3^0(a)$ is useful, since it can be used to find that the maximum value of $P_3^0(a)$ exists at $P_3^0(.60976) \approx .01702$, which is not equivalent to the IC scenario.

For any a , the sum $PA_3^3(a) + PA_3^0(a)$ is simply obtained as two times the joint probability that $\overline{Y_i^j} \sqrt{n} \geq E \left(\overline{Y_i^j} \sqrt{n} \right)$ for all $i = 1, 2, 3$. This joint probability is the orthant probability of the multivariate normal distribution $\Phi_3(\mathbf{R}_2)$ where all correlation terms in \mathbf{R}_2 are obtainable from \mathbf{R}_1 and are equal to $(4a-1)/3$. Using the three-variate extension of Sheppard's Theorem of Median Dichotomy (see Johnson and Kotz (1972), pg. 92), it is simple to show that

$$PA_3^3(a) + PA_3^0(a) = 2\Phi_3(\mathbf{R}_2) = \frac{1}{4} + \frac{3}{2\pi} \text{Sin}^{-1}\left(\frac{4a-1}{3}\right). \quad (9)$$

3 The Case of Partial Agreement on Three Issues

Party R will be the majority party and Issue 1 will have the only election outcome that is in agreement with the position of Party L if $\overline{Y_1^j} \sqrt{n} \leq E(\overline{Y_1^j} \sqrt{n})$ and if $\overline{Y_i^j} \sqrt{n} \geq E(\overline{Y_i^j} \sqrt{n})$ for $2 \leq i \leq 4$. This is the same as the joint probability that $\overline{Y_i^j} \sqrt{n} \geq E(\overline{Y_i^j} \sqrt{n})$ for $2 \leq i \leq 4$ minus the probability that $\overline{Y_1^j} \sqrt{n} \geq E(\overline{Y_1^j} \sqrt{n})$ for $2 \leq i \leq 4$. The first probability can be obtained directly from the three-variate extension of Sheppard's Theorem and the second probability is $\Phi_4(\mathbf{R}_1)$. We must also account for the fact that there are three issues that could be the single issue that is not in agreement with the majority party and for the fact that there are two parties that could be the majority party. After algebraic reduction, we get a representation for $PA_3^2(a)$ as

$$PA_3^2(a) = \frac{3}{8} + \frac{3}{4\pi} \left\{ \text{Sin}^{-1}\left(\frac{2a+1}{3}\right) - \text{Sin}^{-1}\left(\frac{4a-1}{3}\right) \right\} - \frac{9}{2\pi^2} \int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}} \text{Sin}^{-1}\left(\frac{4a-1-3z^2}{4a+2-6z^2}\right) dz. \quad (10)$$

An exact integral solution of (10) can be found for the special case of $a = 1$ with $PA_3^2(1) = 0$. Computed values of $PA_3^2(a)$ are listed in Table 2 for each $a = .00(.10)1.00$ that were obtained from numerical integration of (10), along with the value for $a = .25$. A representation for $PA_3^1(a)$ is obtained from the identity relation $PA_3^1(a) = 1 - PA_3^3(a) - PA_3^2(a) - PA_3^0(a)$. After substituting the representations from (9) and (10) into this identity relation, algebraic reduction leads to

$$PA_3^1(a) = \frac{3}{8} - \frac{3}{4\pi} \left\{ \sin^{-1}\left(\frac{2a+1}{3}\right) + \sin^{-1}\left(\frac{4a-1}{3}\right) \right\} + \frac{9}{2\pi^2} \int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}} \sin^{-1}\left(\frac{4a-1-3z^2}{4a+2-6z^2}\right) dz. \quad (11)$$

An exact integral solution of (11) can be found for the special case of $a = 1$, with $PA_3^1(1) = 0$. Computed values of $PA_3^1(a)$ are listed in Table 2 for each $a = .00(.10)1.00$ that were obtained from numerical integration from (11), along with the value for $a = .25$.

4 The Impact of the Party Parity Assumption

Some discussion is in order regarding the impact of the party parity assumption that $q_1 - q_8 = a/2$ and $q_i = (1-a)/6$ for $2 \leq i \leq 7$. The possible voters' preferences on party positions on issues in Table 1 indicate that this assumption is equivalent to saying that the probability that any voter has a given set of preferences on issues is identical to the probability that the voter has preferences on issues with all of the R and L entries reversed. This leads to parity in voters' preferences for issues positions of Parties R and L, such that voters are equally likely to have an overall party alignment with either party and the majority party is equally likely to be either party. Situations of this nature with a complete balance of outcome possibilities will obviously tend to exaggerate the probability that paradoxical events are observed for large electorates, since the introduction of any consistent bias that favors the position of either party on the issues will typically lead to a very high probability of complete agreement with the majority party position on issues as $n \rightarrow \infty$. However, such a parity situation is not a completely implausible scenario, despite the fact that it does represent an extreme case.

Other more extreme theoretical models can be developed to obtain significantly greater probabilities that a Strict Ostrogorski's Paradox is observed. For example, consider a scenario in which the q_i probabilities are obtained with the following process. Randomly generate two variables, δ and ε , from some probability distribution on the interval $[0, 1/8]$, and let δ (ε) denote the propensity of voters to lean toward Party R (L) partisanship. Thus, voters are generally more disposed to favor the issue positions of Party R than Party L whenever $\delta > \varepsilon$. The q_i

probabilities for a *partisanship model* can then defined on the basis of δ and ε , as shown in Table 3.

Table 3 Feasible voter preferences with a partisanship model.

	$\frac{1}{8} + 3\delta$	$\frac{1}{8} - \delta$	$\frac{1}{8} - \delta$	$\frac{1}{8} - \varepsilon$	$\frac{1}{8} - \delta$	$\frac{1}{8} - \varepsilon$	$\frac{1}{8} - \varepsilon$	$\frac{1}{8} + 3\varepsilon$
Issue 1	R	R	R	R	L	L	L	L
Issue 2	R	R	L	L	R	R	L	L
Issue 3	R	L	R	L	R	L	R	L
Overall	R	R	R	L	R	L	L	L

The definitions from Table 2 lead to $E(Y_i^j) = 2(\delta - \varepsilon)$ for $i = 1, 2, 3$ and $E(Y_4^j) = 0$. If we suppose without a loss of generality that $\delta > \varepsilon$ as $n \rightarrow \infty$, then Party R will be the majority rule winner on all issues with probability approaching one. However, Party R will only be the majority party with probability .5 with this model since $E(Y_4^j) = 0$, so there is a very significant chance that a Strict Ostrogorski Paradox will be observed. However, the impact of this striking observation must be weighed against the relative degree of rationality that this partisanship model associates with the electorate.

Suppose that δ is significantly greater than ε , so that we have a population that has a strong bias toward adopting the issue positions that are taken by Party R. A randomly selected voter is very predictably most likely to have preferences that are in complete agreement with Party R on all issues, which is quite a rational outcome for this model. Unfortunately, Table 3 then tells us that a randomly selected voter is least likely to have agreement with Party R on two out of three issues, suggesting an electorate that displays very odd behavior for a group that is supposedly predisposed to be highly favorable toward the issue positions of Party R. So, while it is possible to define such a theoretical model, it falls out of the realm of plausibility.

In the same vein, it is possible to develop other models that give a significantly large probability of observing other forms of an Ostrogorski Paradox by making assumptions about different intensities of importance that parties might place on the passage of the various issues that are being considered. While these models can indeed fall into the realm of plausibility, they typically rely on the assumption that the majority party has some subset of issues for which it takes a position, but where it has a low intensity of concern about the ultimate vote outcome. However, it would not be particularly paradoxical or disconcerting for the majority party if the minority party position won on such issues that are considered to be of little importance.

The party parity model that we use in the current study attempts to give an upper bound on the estimate of the paradox probabilities with a *not* implausible model that assumes that the parties take issue positions with a real concern about the voting outcome on the issues, without making any *a priori* assumptions that are intentionally creating a specific situation that is tailored to produce the paradoxical outcome that is being studied.

5 The Case of Two Issues

When there are only two issues that are being considered, there are four possible sets of preferences that each voter might have on issues, as listed in Table 4. The possibility of ties in determining the party alignment of voters exists with even m , and no designation for party alignment is made for the voter in such cases.

Table 4 Feasible voter preferences for sequential elections on two issues

	t_1	t_2	t_3	t_4
Issue 1	R	R	L	L
Issue 2	R	L	R	L
Overall	R	-	-	L

Let b denote the probability that there is complete agreement between a randomly selected voter's preference and the position of the same party on each issue. The party parity assumption is then applied with $t_1 = t_4 = b/2$ and $t_2 = t_3 = (1-b)/2$, and the special case with $b = 1/2$ is equivalent to IC.

Following the same general logic that led to the development of the representation for $PA_3^3(a)$, the probability $PA_2^2(b)$ is given by $2\Phi_3(\mathbf{R}_3)$, where

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 2b-1 & \sqrt{b} \\ - & 1 & \sqrt{b} \\ - & - & 1 \end{bmatrix}. \quad (12)$$

As a result of Slepian (1962), $PA_2^2(b)$ does not decrease as b increases, and the three-variate extension of Sheppard's Theorem leads to

$$PA_2^2(b) = \frac{1}{4} + \frac{1}{2\pi} \text{Sin}^{-1}(2b-1) + \frac{1}{\pi} \text{Sin}^{-1}(\sqrt{b}). \quad (13)$$

We note from identities for special cases that $PA_2^2(0) = 0$, $PA_2^2(1) = 1$, and $PA_2^2(1/4) = 1/3$. Computed values of $PA_2^2(b)$ from (13) are listed in Table 5 for each value of $b = .00(.10)1.00$.

Table 5 Computed values of $PA_2^2(b)$ and $PA_2^1(b)$.

b	$PA_2^2(b)$	$PA_2^1(b)$
.00	.0000	1.0000
.10	.2048	.7952
.20	.2952	.7048
.30	.3690	.6310
.40	.4359	.5641
.50	.5000	.5000
.60	.5641	.4359
.70	.6310	.3690
.80	.7048	.2952
.90	.7952	.2048
1.00	1.0000	.0000

The results in Table 5 clearly show that the probability of complete agreement increases dramatically as b increases for the case of two issues.

Using the logic from earlier arguments, we obtain a representation for the probability, $PA_2^0(b)$, that the Strict Ostrogorski Paradox occurs with two items

$$PA_2^0(b) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}(2b - 1) - \frac{1}{\pi} \sin^{-1}(\sqrt{b}). \quad (14)$$

Direct integration of (14) shows that $PA_2^0(0) = 0$, $PA_2^0(1) = 0$, and $PA_2^0(1/4) = 0$, which suggests that a Strict Ostrogorski Paradox cannot be observed when $m = 2$. This general result is proved rigorously in Corollary 1 of Deb and Kelsey (1980).

We can use this observation with (13) and (14) to obtain a simpler representation for $PA_2^2(b)$ as

$$PA_2^2(b) = \frac{1}{2} + \frac{1}{\pi} \text{Sin}^{-1}(2b - 1). \quad (15)$$

It then follows directly from earlier discussion that

$$PA_2^1(b) = \frac{1}{2} - \frac{1}{\pi} \text{Sin}^{-1}(2b - 1). \quad (16)$$

Then, (15) and (16) require that $PA_2^2(b) = PA_2^1(1 - b)$, as shown in Table 5.

6 The General Case of m Issues

The development of these types of representations becomes significantly more complex as the number of issues increases, since the number of possible combinations of voter agreements with party positions increases as 2^m . Attention is therefore restricted to the IC assumption when we consider $m \geq 4$. The initial analysis that was presented above for the two and three-issue cases is generalized by defining m binary variables to determine if the position of Party R is the winner by majority rule on each issue. Variable Z_i^j takes a value of +1 (-1) when the j^{th} voter is in agreement with the position of Party R (L) on the i^{th} issue. The m variables are formally defined as

$$\begin{aligned} Z_i^j = +1 & : \text{For } j^{\text{th}} \text{ voter agreement with the position of Party R on Issue } i & (17) \\ -1 & : \text{For } j^{\text{th}} \text{ voter agreement with the position of Party L on Issue } i. \end{aligned}$$

There are an equal number of possible voters' party position agreement combinations in both the +1 and -1 categories in the variable definitions in (17). Since each possible combination is equally likely with IC, it then follows directly that $E(Z_i^j) = 0$ and

$$E\left(Z_i^{j^2}\right) = 1 \text{ for all } 1 \leq i \leq m.$$

In order to determine $E(Z_i^j Z_k^j)$ for $1 \leq I < k \leq m$, we partition the set of all possible combinations of voter agreements with party positions on issues into 2^{m-2} subsets of cardinality four such that each of Z_i^j and Z_k^j can have values of +1 or -1 in each subset, while the party agreements on issues on the remaining $m-2$ issues are identical within each of the subsets. Obviously, $E(Z_i^j Z_k^j) = 0$ within each of these

subsets with IC, so it follows that $E(Z_i^j Z_k^j) = 0$ over the entire set of all possible combinations of voter agreements with party positions. Let $\omega_{i,k}$ denote the correlation between variables Z_i^j and Z_k^j , and the fact that $E(Z_i^j) = 0$ for all $1 \leq i \leq m$ coupled with earlier discussion leads to the observation that $\omega_{i,k} = 0$ for all $1 \leq i \leq k \leq m$.

Variable Z_{m+1}^j is then defined to denote the contribution that the party alignment of the j^{th} voter makes toward Party R being the majority party, with

$$\begin{aligned} Z_{m+1}^j = +1 : & \text{ If } j^{\text{th}} \text{ voter is aligned with Party R} \\ & 0: \text{ If } j^{\text{th}} \text{ has no party alignment} \\ & -1: \text{ If } j^{\text{th}} \text{ voter is aligned with Party L.} \end{aligned} \quad (18)$$

Each of the possible combination of a voter's party agreement on issues can be paired with the equally likely combination in which the Party R and L positions are interchanged. Then, either both members of this pair do not have a party alignment or one is aligned with Party R while the other is aligned with Party L. It is therefore obvious that $E(Z_{m+1}^j) = 0$ with IC.

If m is odd, ties for party alignment of a voter cannot exist, so it must be true that

$$E\left(Z_{m+1}^j\right)^2 = 1, \text{ for odd } m. \quad (19)$$

When m is even, there are exactly $C_{m/2}^m$ different combinations of the 2^m possible voter agreements on the m issues for which a voter has no party alignment, with $Z_{m+1}^j = 0$. Since IC assigns an equally likely probability of $1/2^m$ to each possible combination,

$$E\left(Z_{m+1}^j\right)^2 = \frac{2^m - C_{m/2}^m}{2^m}, \text{ for even } m. \quad (20)$$

Consider the voters' party agreement on Issue h in the 2^m different possible combinations of voter agreements to obtain $E(Z_h^j Z_{m+1}^j)$. Half of these combinations have an alignment with Party R on Issue h , and this subset is denoted as S(R). There

are C_i^{m-1} combinations of voter agreements on the remaining $m-1$ party positions for issues in S(R) that will have exactly i issues in agreement with the position of Party L, and then Party R will then be the majority party if $0 \leq i \leq m/2$. The total number of combinations in S(R) for which Party R is the majority party is therefore given by #S(R), with

$$\# S(R) = \sum_{i=0}^{(m-1)/2} C_i^{m-1}, \text{ for odd } m \quad (21)$$

$$\# S(R) = \sum_{i=0}^{(m-2)/2} C_i^{m-1}, \text{ for even } m. \quad (22)$$

The remaining half of the possible combinations have agreement with Party L on Issue h , and we denote this subset as S(L). There are C_i^{m-1} combinations that have exactly i issues in agreement with the position of Party L in the remaining $m-1$ issues, and each such combination will have Party R as the majority party if $0 \leq i \leq (m-2)/2$. The total number of combinations in S(L) for which Party R is the majority party is therefore given by #S(L), with

$$\# S(L) = \sum_{i=0}^{(m-3)/2} C_i^{m-1}, \text{ for odd } m \quad (23)$$

$$\# S(L) = \sum_{i=0}^{(m-4)/2} C_i^{m-1}, \text{ for even } m. \quad (24)$$

The value of variable Z_h^j will then be +1 [-1] for each combination of possible voter agreements in S(R)[S(L)]. The expected value $E(Z_h^j Z_{m+1}^j)$ with IC is then obtained from

$$E(Z_h^j Z_{m+1}^j) = [(+1)\# S(R) + (-1)\# S(L)] / 2^m. \quad (25)$$

The correlation between Z_h^j and Z_{m+1}^j for all $1 \leq h \leq m$ follows from all of the above as

$$\omega_{h,m+1} = \frac{C_{(m-1)/2}^{m-1}}{2^{m-1}}, \text{ for all } 1 \leq h \leq m \text{ (odd)} \quad (26)$$

$$\omega_{h,m+1} = \frac{C_{(m-2)/2}^{m-1}}{\sqrt{2^{m-2} (2^m - C_{m/2}^m)}}, \text{ for all } 1 \leq h \leq m \text{ (even)}. \quad (27)$$

Let \mathbf{W}_{m+1} denote the correlation matrix for the $m+1$ variables that are defined in (17) and (18), with components $\omega_{i,j}$. The neutrality of IC toward the two parties gives the limiting probability of complete agreement with the majority party when $n \rightarrow \infty$ as $PA_m^m(IC) = 2\Phi_{m+1}(\mathbf{W}_{m+1})$.

7 The General Case of m Issues – Partial Agreement

Suppose that we are interested in the probability that there is nearly complete agreement, in the sense that only Issue 1 has a majority rule outcome for a party position that is in disagreement with the majority party. This would be determined by finding the resulting correlation matrix \mathbf{W}_{m+1}^1 where the signs of the variable values for Z_1^j are reversed, which would reverse the sign on all correlation terms in \mathbf{W}_{m+1} that involve Z_1^j . As a result, it is still true that $\omega_{h,k}^1 = \omega_{h,k} = 0$ for all $1 \leq h < k \leq m$ and $\omega_{g,m+1}^1 = \omega_{g,m+1}$ for all $1 < g \leq m$. The only difference between \mathbf{W}_{m+1}^1 and \mathbf{W}_{m+1} is that $\omega_{1,m+1}^1 = -\omega_{1,m+1}$. With the neutrality of IC toward the two parties and the symmetry of IC with respect to the m possible issues that could be the single issue in disagreement with the majority party position, it follows that $PA_m^{m-1}(IC) = 2m\Phi_{m+1}(\mathbf{V}_{m+1}^1)$.

This logic can easily be extended to the general case in which exactly k issues have majority rule agreement with party positions that are in disagreement with the majority party positions. The correlation matrix that is applicable to the associated probability is \mathbf{W}_{m+1}^k , which comes from \mathbf{W}_{m+1} simply by negating the $\omega_{i,m+1}$ correlation values for $1 \leq i \leq k$. The same probability value will be obtained, regardless of which specific set of k issues are selected to have their $\omega_{i,m+1}$ terms

negated in order to obtain the $\omega_{i,m+1}^1$ values. There are C_k^m different sets of k issues and there are two parties that could be the majority party, so $PA_m^{m-k}(IC) = 2C_k^m \Phi_{m+1}(\mathbf{W}_{m+1}^k)$.

This observation can be extended to produce some interesting results.

Theorem 1. $PA_m^{m-k}(IC) \geq PA_m^k(IC)$, for $0 \leq k \leq m/2$.

Proof. Given the definition of \mathbf{W}_{m+1}^k , $\omega_{i,j}^k \geq \omega_{i,j}^{k^*}$ for all $1 \leq i < j \leq m+1$ when $k < k^*$. This observation is contingent upon the requirement that $\omega_{i,m+1} > 0$, which is true from (26) and (27). It then follows from Slepian (1962) that $\Phi(\mathbf{W}_{m+1}^k) \geq \Phi(\mathbf{W}_{m+1}^{k^*})$. Given that $C_k^m = C_{m-k}^m$, $2C_k^m \Phi(\mathbf{W}_{m+1}^k) \geq 2C_{m-k}^m \Phi(\mathbf{W}_{m+1}^{m-k})$ if $k \leq m/2$. **QED**

Theorem 2. $PA_m^{m-k}(IC) + PA_m^k(IC) = C_k^m \left(\frac{1}{2}\right)^{m-1}$ for $m \geq 2$ with $0 \leq k \leq m$.

Proof. The limit probability $PA_m^{m-k}(IC)$ is obtained from the positive orthant probability $\Phi_{m+1}(\mathbf{W}_{m+1}^k)$, which is the probability that Party R is the majority party and that there are exactly k majority rule outcomes on issues that are in disagreement with the position of Party R. This orthant probability can alternatively be obtained as the difference in two probabilities. The first of these probabilities represents the situation in which there are exactly k majority rule outcomes on issues that are in disagreement with the position of Party R. This situation makes no determination of the majority party, and the associated correlation matrix \mathbf{Z}_m on this joint distribution is obtained from \mathbf{W}_{m+1}^k by removing all correlation terms that are related to Z_{m+1}^j .

Given the definition of \mathbf{W}_{m+1}^k , all correlations in \mathbf{Z}_m are equal to zero, which gives

$$\Phi_m(\mathbf{Z}_m) = \left(\frac{1}{2}\right)^m.$$

We then subtract the second probability that there are exactly k majority rule outcomes on issues that are in disagreement with the position of Party R, when Party L is the majority party. This second probability is obtained by using the assumptions

that led to the development of $\Phi_{m+1}(\mathbf{W}_{m+1}^k)$, except that the signs on variable Z_{m+1}^j are reversed. This reverses the signs on all correlation terms that involve Z_{m+1}^j and leads to an associated positive orthant probability that is equivalent to $\Phi_{m+1}(\mathbf{W}_{m+1}^{m-k})$.

As a result, we find that

$$\begin{aligned}\Phi_{m+1}(\mathbf{W}_{m+1}^k) &= \left(\frac{1}{2}\right)^m - \Phi_{m+1}(\mathbf{W}_{m+1}^{m-k}) \\ 2C_k^m \Phi_{m+1}(\mathbf{W}_{m+1}^k) &= C_k^m \left(\frac{1}{2}\right)^{m-1} - 2C_k^m \Phi_{m+1}(\mathbf{W}_{m+1}^{m-k}) \\ 2C_k^m \Phi_{m+1}(\mathbf{W}_{m+1}^k) + 2C_{m-k}^m \Phi_{m+1}(\mathbf{W}_{m+1}^{m-k}) &= C_k^m \left(\frac{1}{2}\right)^{m-1}.\end{aligned}\quad \text{QED}$$

A related observation then follows directly from the proof of Theorem 2.

Corollary 1. $PA_{m/2}^{m/2}(IC) = C_{m/2}^m \left(\frac{1}{2}\right)^m$ for all even $m \geq 2$.

8 The Case of Four Issues with IC

Some results can be obtained for $P_m^k(IC)$ in the special case of four issues.

Corollary 1 directly leads to $P_4^2(IC) = 3/8$. A representation can be obtained for limit probability $PA_4^4(IC)$ from the identity $PA_4^4(IC) = 2\Phi_5(\mathbf{W}_5)$. Representations for multivariate normal positive orthant probabilities generally become extremely complex in cases with more than four variables, except for special cases in which very restrictive conditions can be placed on the associated correlation matrix for the distribution.

A reasonable representation is obtained for $\Phi_5(\mathbf{W}_5)$ by appealing to Boole's Equation (see Johnson and Kotz (1972), pg. 52), which describes a procedure that can be used to express positive orthant probabilities with an odd number of dimensions in terms of a linear combination of positive orthant probabilities with fewer dimensions. With the correlation matrix \mathbf{W}_5 , Boole's Equation results in

$$\Phi_5(\mathbf{W}_5) = \frac{1}{2} \left[1 - 5 \left(\frac{1}{2} \right) + \{6\Phi_2(\mathbf{Z}_2) + 4\Phi_2(\mathbf{U}_2)\} - \{4\Phi_3(\mathbf{Z}_3) + 6\Phi_3(\mathbf{U}_3)\} + \{\Phi_4(\mathbf{Z}_4) + 4\Phi_4(\mathbf{U}_4)\} \right] \quad (28)$$

Here, \mathbf{Z}_j denotes a correlation matrix for a distribution on j variables with all correlation terms are equal to zero, as above. The correlation matrix \mathbf{U}_j is defined on j variables with terms $u_{i,h} = 0$ for all $1 \leq i < h < j$ and $u_{i,j} = \sqrt{\frac{9}{40}}$ for all $1 \leq i \leq j-1$. The term $\sqrt{\frac{9}{40}}$ is obtained from (27).

Sheppard's Theorem can be used to obtain simple representations for $\Phi_2(\mathbf{U}_2)$ and $\Phi_3(\mathbf{U}_3)$, while $\Phi_4(\mathbf{U}_4)$ is a special case of a representation that is considered in Gehrlein (1979). After substitution and algebraic reduction, (28) reduces to

$$\Phi_5(\mathbf{W}_5) = \frac{1}{32} + \frac{1}{4\pi} \text{Sin}^{-1} \left(\sqrt{\frac{9}{40}} \right) + \frac{3}{2\pi^2} \int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}} \text{Sin}^{-1} \left(\frac{-z^2}{1-2z^2} \right) dz \quad (29)$$

Using the fact that $PA_4^4(IC) = 2\Phi_5(\mathbf{W}_5)$ with (29) yields

$$PA_4^4(IC) = \frac{1}{16} + \frac{1}{2\pi} \text{Sin}^{-1} \left(\sqrt{\frac{9}{40}} \right) - \frac{3}{\pi^2} \int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}} \text{Sin}^{-1} \left(\frac{z^2}{1-2z^2} \right) dz \approx .1245 \quad (30)$$

Numerical integration is used to obtain the value of .1245 for $PA_4^4(IC)$ from (30).

Theorem 2 can then be used in conjunction with (30) to obtain a representation for $PA_4^0(IC)$, with

$$PA_4^0(IC) = \frac{1}{16} - \frac{1}{2\pi} \text{Sin}^{-1} \left(\sqrt{\frac{9}{40}} \right) + \frac{3}{\pi^2} \int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}} \text{Sin}^{-1} \left(\frac{z^2}{1-2z^2} \right) dz \approx .0005 \quad (31)$$

Attention is turned to the situation in which there is only partial agreement with four issues, with the development of a representation for $PA_4^3(IC)$. Issue 1 will have the only majority rule outcome that is in disagreement with the issue position of

the majority party, Party R, when both $\overline{Z}_1^j \sqrt{n} \leq E\left(\overline{Z}_1^j \sqrt{n}\right)$ and $\overline{Z}_i^j \sqrt{n} \geq E\left(\overline{Z}_i^j \sqrt{n}\right)$ for each $2 \leq i \leq 5$. This probability is equivalent to the probability that $\overline{Z}_i^j \sqrt{n} \geq E\left(\overline{Z}_i^j \sqrt{n}\right)$ for each $2 \leq i \leq 5$ minus the probability that $\overline{Z}_i^j \sqrt{n} \geq E\left(\overline{Z}_i^j \sqrt{n}\right)$ for each $1 \leq i \leq 5$, which is $\Phi_4(U_4) - \Phi_5(W_5)$. There are four issues that could be the single issue that is in disagreement with the issue position of the majority party, and there are two parties that could be the majority party. The symmetry of IC with respect to issues and parties leads to the conclusion that $PA_4^3(IC) = 8\{\Phi_4(U_4) - \Phi_5(W_5)\}$. After performing all necessary substitution and algebraic reduction,

$$PA_4^3(IC) = \frac{1}{4} + \frac{1}{\pi} \text{Sin}^{-1}\left(\sqrt{\frac{9}{40}}\right) + \frac{6}{\pi^2} \int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}} \text{Sin}^{-1}\left(\frac{z^2}{1-2z^2}\right) dz \approx .4406 . \quad (32)$$

A representation for the remaining probability $PA_4^1(IC)$ can be obtained from the identity relationship $\sum_{i=0}^4 PA_4^i(IC) = 1$, which leads to

$$PA_4^1(IC) = \frac{1}{4} - \frac{1}{\pi} \text{Sin}^{-1}\left(\sqrt{\frac{9}{40}}\right) - \frac{6}{\pi^2} \int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}} \text{Sin}^{-1}\left(\frac{z^2}{1-2z^2}\right) dz \approx .0594 . \quad (33)$$

9 Conclusion

The possibility of the existence of a Strict Ostrogorski Paradox presents a very interesting phenomenon that could lead to very unsettling outcomes in group decision-making situations. However, this phenomenon cannot exist in two-issue voting situations for any n . When three-issue situations are considered, the results of Table 2 indicate that the probability of such an outcome never reaches as much as a two percent for large electorates, regardless of the propensity of voters to align their

views with the standards of political parties. The results of (31) indicate that the probability of observing a Strict Ostrogorski Paradox in four-issue situations is nearly zero with IC for large electorates.

Given our discussion regarding the propensity of models with the assumption of part parity to exaggerate the likelihood that such paradoxical outcomes will be observed, we can conclude that it is very unlikely that a Strict Ostrogorski Paradox, or any other extreme form of Ostrogorski's Paradox, would ever be observed in any real situation with large electorates over the range of the number of issues that we have been considering.

An alternative approach to the problem of deriving these probability representations in the limit as $n \rightarrow \infty$ that was mentioned previously has been used to obtain alternative forms to verify numerical results from all of the representations that are given above for $m = 2, 3, 4$. Details of these derivations are available from the authors upon request.

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