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The Computation of Risk Budgets under the Lévy Process Assumption

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The Computation of Risk Budgets under the Lévy Process Assumption

Abstract: This paper revisits the computation of Value-at-Risk and other risk indicators based on the use of Lévy processes. We first provide a new presentation of Variance Gamma Processes with Drift: we reconstruct them in an original way, starting from the exponential distribution. Then, we derive general Fourier formulas that allows to compute VaR quickly and efficiently, but also other typical indicators like Tail Conditional Expectation (TCE), TailVaR or Expected Shortfall. Based on this formula, we conduct a study of the term structure of VaR, and provide a discussion of the Basle 2 and Solvency II agreements.

Keywords: Value-at-Risk, TailVaR, Tail Conditional Expectation, Lévy processes, Variance Gamma process, Fourier transform, Basle 2, Solvency II.

Introduction

The Value-at-Risk (VaR) method is the most widely accepted method for risk measurement. It is used by all major financial institutions and is promoted by the Basle 2, Solvency II and UCITS 3 agreements. Choosing an appropriate VaR measure is an important and difficult task. There exists a wide range of studies about the development of VaR and risk measurement methods (see Jorion [2007]). The main purpose of this study is to put forward a new probabilistic framework for calculating VaR and other risk indicators. Instead of using the classical extreme value theory, which applies only to tails and is not able to render (or be calibrated on) the center of distributions, we develop a method based on the use of Lévy processes. The advantage of the complete distribution given by this alternative framework is to escape from the statistical estimation limitations due to the EVT framework. This enables to operate calculations with more confidence because of the ability to capture more accurately the true skewness and kurtosis of marginal distributions. Despite these advantages, it is difficult to calculate risk indicators with Lévy processes with usual numerical methods. An important contribution of the paper is the derivation of a general Fourier formula allowing to compute risk indicators quickly and efficiently.

As far as the choice of Lévy process is concerned, our study is grounded on the use of Variance Gamma processes. These dynamics, together with similar processes, have been used extensively at the turn of the century for the reconstruction of volatility surfaces. The reader can refer to the well-known papers of Carr and Madan [1998], Madan, Carr, and Chang [1998], or Carr, Geman, Madan, and Yor [2002] for the application of Variance Gamma and CGMY processes in option pricing. In this paper, on the contrary, we concentrate on risk management, and conduct a study on the computation of risk indicators when asset return dynamics are assumed to follow Variance Gamma Processes with Drift (hereafter VGPD). To do so, and based on the statistical work of Kotz, Kozubowski, and Podgorski [2001], we present a full and new reconstruction of the VGPD that relies on the exponential distribution.

The Basle 2 Capital Accord has proposed a multiplier superior to 3 on banks' internal 99% 10-day VaR calculated for market risk exposure. This *ad hoc* factor has not been fully explained and is poorly justified by arguing that the standard classical models of stock price dynamics do not adequately capture actual market risks. The objective of the Basle 2 agreement, with this recommendation of a protection equal to the maximum of the previous day's VaR and of 3 times the mean VaR observed during the past 60 days be

enforced, is clearly to protect institutions against both quick and slow market evolutions, with the usual caveat that the past may not always be a good indicator of the future. A simplified rule states that financial institutions should post capital equal to 3 times the 99%-10 days VaR. According to this rule, an institution investing in the CAC40 Index, or in a well-diversified portfolio that mimicks the index's tracker, should protect itself by an order of magnitude of 35% of its investment amount. As far as the QIS4 of Solvency II is concerned, it puts forward the 99.5%-1 year VaR, and promotes a 32% loading (for a duration of liabilities inferior to 3 years) for an investment in equities. We shall compare in this paper these percentages to those obtained when calibrating equity dynamics on VGPD processes and using the *ad hoc* VaR formula.

The outline of the paper is as follows. Section 1 provides a new presentation of VGPD, that are reconstructed in an original way starting from the exponential law. In section 2, we derive general Fourier formulas that allow to compute VaR, but also other types of risk indicators like Tail Conditional Expectation (hereafter TCE) quickly and efficiently. These formulas, that we use with VGPD, could in fact be used with any other type of Lévy process (or more generally with any process for which we have an analytic characteristic function, see for instance the seminal contribution of Carr, Geman, Madan and Yor [2003] that explains how to go beyond Lévy processes). Based on the first of these formulas, we conduct in section 3 a study of the term structure of VaR, and provide a discussion of the Basle 2 and Solvency II agreements.

1 A Lévy framework for asset returns

We provide a new presentation of Variance Gamma Processes with Drift, that are reconstructed in an original way starting from the exponential distribution. In particular, the goal of this section is to obtain the following VGPD characteristic function:

$$\Phi(u) = e^{i\theta tu} \left(\frac{1}{1 + \frac{1}{2}\sigma^2 vu^2 - i\delta vu} \right)^{\frac{t}{v}}$$

To do so, we rely on the description by Kotz, Kozubowski, and Podgorski [2001] of exponential, gamma and Laplace distributions and introduce an *ad hoc* homothety that allows us to conclude.

Let us start with the classical Laplace distribution whose probability dis-

tribution on \mathbb{R} is given by the density function:

$$f(x) = \frac{1}{2s} e^{-\frac{1}{s}|x-\theta|} \quad (1)$$

where $\theta \in \mathbb{R}$ and $s > 0$ are respectively location and scale parameters. Given the fact that the variance of (1) is $\sigma^2 = 2s^2$, f can also be seen as a function of θ and σ :

$$f(x) = \frac{1}{\sigma\sqrt{2}} e^{-\frac{\sqrt{2}}{\sigma}|x-\theta|} \quad (2)$$

The characteristic function associated with f is:

$$\phi(u) = \frac{e^{i\theta u}}{1 + \frac{1}{2}\sigma^2 u^2} \quad (3)$$

A skewed generalization of the classical Laplace law can be designed by introducing a new parameter β which controls the probability assigned to each side of θ :

$$g(x) = \frac{\beta}{1 + \beta^2} \frac{\sqrt{2}}{\sigma} \begin{cases} \exp\left(-\frac{\sqrt{2}\beta}{\sigma}(x - \theta)\right) & \text{if } x \geq \theta \\ \exp\left(-\frac{\sqrt{2}}{\sigma\beta}(-x + \theta)\right) & \text{if } x < \theta \end{cases} \quad (4)$$

Actually, when $\beta = 1$, the two probabilities are equal and the distribution is symmetric about θ : we obtain the classical Laplace law. An important property of β is that the parameter is scale invariant.

It can be shown that the characteristic function associated with (4) is:

$$\psi(u) = e^{i\theta u} \left(\frac{1}{1 - iu\frac{1}{\beta}\frac{\sigma}{\sqrt{2}}} \right) \left(\frac{1}{1 + iu\beta\frac{\sigma}{\sqrt{2}}} \right) = \frac{e^{i\theta u}}{1 + \frac{1}{2}\sigma^2 u^2 - i\delta u} \quad (5)$$

where δ is related to β as:

$$\delta = \left(\frac{1}{\beta} - \beta \right) \times \frac{\sigma}{\sqrt{2}} \quad (6)$$

Intuitively, δ represents the deviation from the mode θ , and is related to the asymmetry of jumps. Recall that the parameter β is scale invariant, whereas the scale of δ is given by the variance σ .

Based on the characteristic function (5), we can build the generalized asymmetric Laplace law as follows:

$$\Psi(u) = (\psi(u))^t = \left(\frac{e^{i\theta u}}{1 + \frac{1}{2}\sigma^2 u^2 - i\delta u} \right)^t \quad (7)$$

where the elevation of the characteristic function to the power of t is a consequence of the core i.i.d. assumption defining Lévy processes.

The characteristic function (7) can be expressed in the following manner:

$$\Psi(u) = e^{i\theta tu} \left(\frac{1}{1 - iu\frac{1}{\beta}\frac{\sigma}{\sqrt{2}}} \right)^t \left(\frac{1}{1 + iu\beta\frac{\sigma}{\sqrt{2}}} \right)^t \quad (8)$$

which leads to the very useful representation:

$$Y_t \stackrel{d}{=} \theta t + (G_1 - G_2) \quad (9)$$

where G_1 and G_2 are two i.i.d. Gamma $\mathcal{G}(t, \lambda_1)$ and $\mathcal{G}(t, \lambda_2)$ random variables with:

$$\lambda_1 = \frac{\beta\sqrt{2}}{\sigma} \quad \text{and} \quad \lambda_2 = \frac{\sqrt{2}}{\sigma\beta}$$

and where a Gamma random variable $\mathcal{G}(t, \lambda)$ is defined by the density:

$$f_{\mathcal{G}(t,\lambda)}(x) = \frac{\lambda^t}{\Gamma(t)} x^{t-1} e^{-\lambda x}$$

and spectrum:

$$\Phi_{\mathcal{G}(t,\lambda)}(u) = \left(\frac{1}{1 - \frac{i u}{\lambda}} \right)^t \quad (10)$$

Also note that, solving for (6), we have:

$$\beta = \beta(\sigma, \delta) = -\frac{\delta}{\sigma\sqrt{2}} + \sqrt{1 + \frac{\delta^2}{2\sigma^2}}$$

The process defined by (8) and (9), where t is a time, is underparametrized: θ , σ , and β are not sufficient to permit any behavior of the four first moments of financial dynamics. We need to introduce a fourth parameter v , and we do this by performing a homothety on σ , δ and t as follows:

$$\sigma \rightarrow \sigma\sqrt{v}$$

and:

$$\delta \rightarrow \delta v$$

and:

$$\theta \rightarrow \theta v$$

and:

$$t \rightarrow \frac{t}{v}$$

thus defining $G'_1 \rightsquigarrow \mathcal{G}(t, \frac{\hat{\beta}\sqrt{2}}{\sigma\sqrt{v}})$ and $G'_2 \rightsquigarrow \mathcal{G}(t, \frac{\sqrt{2}}{\hat{\beta}\sigma\sqrt{v}})$ where $\hat{\beta} = \beta(\sigma\sqrt{v}, \delta v)$, and yielding finally the new random variable:

$$X_t \stackrel{d}{=} \theta t + (G'_1 - G'_2) \quad (11)$$

which is the marginal r.v. at time t of a variance gamma process with drift.

The characteristic function associated with (11) is:

$$\Phi(u) = e^{i\theta tu} \left(\frac{1}{1 - iu\frac{1}{\hat{\beta}}\frac{\sigma\sqrt{v}}{\sqrt{2}}} \right)^{\frac{t}{v}} \left(\frac{1}{1 + iu\hat{\beta}\frac{\sigma\sqrt{v}}{\sqrt{2}}} \right)^{\frac{t}{v}} \quad (12)$$

Replacing $\hat{\beta}$ by its value and developing, we finally obtain the searched characteristic function:

$$\Phi(u) = e^{i\theta tu} \left(\frac{1}{1 + \frac{1}{2}\sigma^2 v u^2 - i\delta v u} \right)^{\frac{t}{v}} \quad (13)$$

where the right part of the right member of this equation is the well-known characteristic function of a variance gamma random variable, and the left part of this member accounts for an added pure drift¹.

¹Based on the characteristic function (13), we can compute the moments of the VGPD. These moments are interesting to compute for two reasons. First, they can be used as a tool to calibrate distribution parameters (although we take a different route – maximum likelihood – in the third section of this article for performing calibrations). Second, they are interesting *per se*, because they give a clear meaning to model parameters. We obtain for the four first moments of processes whose marginal variables X_t are characterized as above in formula (13):

1. Mean

$$E(X_t) = (\delta + \theta) t$$

The mean depends both on the drift θ (mode of the Laplace distribution) and on the deviation due to asymmetrical jumps.

2. Variance

$$E\left([X_t - E(X_t)]^2\right) = (\sigma^2 + \delta^2 v) t$$

3. Asymmetry

$$E\left([X_t - E(X_t)]^3\right) = (2\delta^3 v^2 + 3\delta\sigma^2 v) t$$

Note that the sign of this expression is the same as the sign of δ , motivating the

2 Computing Risk Indicators

We now come to the computation of risk indicators when asset return dynamics are represented by Lévy processes.

2.1 Computing VaR

We derive below a formula allowing to compute directly the cumulative distribution associated with any Lévy process' marginals as a Fourier transform of a well behaved function of the characteristic function. Recall that VaR is defined by:

$$F(\text{VaR}_\alpha) = \alpha \quad (14)$$

where α is the no-ruin probability and F is the c.d.f. of the random variable under study.

The following formula is well-known and gives a relationship between F and the characteristic function Φ of the examined random variable:

$$F(x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \frac{\Phi(u)}{iu} du$$

The use of the above formula can be problematic, because it is not stable around $u = 0$. This is the reason why we design a new formula, as presented in the following proposition (formulas of this type were developed independently by Lee [2004] in the context of option pricing):

Proposition 2.1 *The following relationship holds for any random variable that is sufficiently well-behaved (admitting a density that is integrable with respect to the Lebesgue product measure):*

$$F(x) = \frac{e^{ax}}{2\pi} \int_{-\infty}^{+\infty} e^{iux} \frac{\Phi(ia - u)}{a + iu} du \quad (15)$$

where a is an arbitrary positive real number.

introduction of θ to permit means and skewnesses of opposite signs.

4. Leptokurticity

$$E\left([X_t - E(X_t)]^4\right) = (3\sigma^4 v + 12\sigma^2 \delta^2 v^2 + 6\delta^4 v^3)t + (3\sigma^4 + 6\sigma^2 \delta^2 v + 3\delta^4 v^2)t^2$$

Note that when $v = 0$, $E\left([X_1 - E(X_1)]^4\right) = 3\sigma^4 t^2$ yielding a kurtosis coefficient $K = 3$.

Proof. Formula (15) can be obtained as follows. Let f be the density of the infinitely divisible distribution X under study, and Φ its characteristic function which can be expressed as:

$$\Phi(u) = \int_{-\infty}^{+\infty} e^{iux} f(x) dx$$

First, a being any strictly positive real number, we introduce the new function:

$$g(x) = e^{-ax} F(x) = e^{-ax} \int_{-\infty}^x f(s) ds$$

The Fourier transform of this function is given by:

$$\Lambda(u) = \int_{-\infty}^{+\infty} e^{iux} g(x) dx = \int_{-\infty}^{+\infty} e^{iux} \left(e^{-ax} \int_{-\infty}^x f(s) ds \right) dx$$

so that:

$$\Lambda(u) = \int_{-\infty}^{+\infty} \int_{-\infty}^x e^{iux} e^{-ax} f(s) ds dx$$

Noting that $-\infty < s < x < +\infty$, we can swap the integrals as:

$$\Lambda(u) = \int_{-\infty}^{+\infty} \int_s^{+\infty} e^{iux} e^{-ax} f(s) dx ds$$

Therefore:

$$\Lambda(u) = \int_{-\infty}^{+\infty} f(s) \left(\int_s^{+\infty} e^{iux} e^{-ax} dx \right) ds = \int_{-\infty}^{+\infty} f(s) \left[\frac{e^{-(a-iu)x}}{-(a-iu)} \right]_s^{+\infty} ds$$

and, because $|e^{-(a-iu)x}| = e^{-ax}$ tends to zero when x becomes infinite:

$$\Lambda(u) = \int_{-\infty}^{+\infty} f(s) \left(-\frac{e^{-(a-iu)s}}{-(a-iu)} \right) ds = \frac{1}{a-iu} \int_{-\infty}^{+\infty} f(s) e^{-(a-iu)s} ds$$

Finally:

$$\Lambda(u) = \frac{1}{a-iu} \int_{-\infty}^{+\infty} f(s) e^{i(u+ia)s} ds$$

permits to write readily:

$$\Lambda(u) = \frac{\Phi(u+ia)}{a-iu}$$

The function g being the inverse Fourier transform of Λ :

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \Lambda(u) du$$

we obtain:

$$e^{-ax}F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \frac{\Phi(u+ia)}{a-iu} du$$

so that finally:

$$F(x) = \frac{e^{ax}}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \frac{\Phi(u+ia)}{a-iu} du$$

A simple change of variable ($u = -u$) yields the searched formula:

$$F(x) = \frac{e^{ax}}{2\pi} \int_{-\infty}^{+\infty} e^{iux} \frac{\Phi(-u+ia)}{a+iu} du$$

■

One can use formula (15), in which a choice of characteristic function Φ has to be made, and perform an Fast Fourier Transform (FFT) to obtain F , before achieving a root search on F to compute VaR. With Matlab, the result comes quasi-instantaneously because an FFT code applied to a vector of values of Φ yields quickly a vector of values of F , to which it remains to apply a simple search function.

2.2 Computing Other Risk Indicators

Most risk indicators are based on the use of the cumulative distribution function of values exceeding a given x_0 :

$$F(x|X > x_0) = \Pr(X < x | X > x_0) \quad (16)$$

and on the mean conditional on this value x_0 being surpassed:

$$M(x_0) = E[X | X > x_0] \quad (17)$$

When X is the loss incurred by a financial asset and when $x_0 = \text{VaR}_c(X)$, where $\text{VaR}_c(X)$ is the VaR at the confidence threshold c , $M(x_0)$ represents the average loss conditional on VaR_c being surpassed. This is the well-known Tail Conditional Expectation (TCE), or Conditional Tail Expectation (CTE). Indeed:

$$\text{TCE}_c(X) = E[X | X > \text{VaR}_c(X)] \quad (18)$$

As for the cumulative distribution function of excesses, it is defined by:

$$F_{x_0}(x) = \Pr(X - x_0 < x | X > x_0) \quad (19)$$

To this expression, corresponds the average value of excesses beyond x_0 , which is given by:

$$e(x_0) = E[X - x_0 | X > x_0] \quad (20)$$

If $x_0 = \text{VaR}_c(X)$, $e(x_0)$ represents the average value of losses beyond $\text{VaR}_c(X)$, conditional on this threshold surpassed. This is the well-known Conditional VaR or CVaR defined by:

$$\text{CV}_c(X) = e(\text{VaR}_c(X)) \quad (21)$$

or as:

$$\text{CV}_c(X) = E[X - \text{VaR}_c(X) | X > \text{VaR}_c(X)] \quad (22)$$

Finally, the Expected Shortfall is defined in the literature either as in formula (22), or as:

$$\text{ES}_c(X) = E[(X - \text{VaR}_c(X)) \mathbf{1}_{X > \text{VaR}_c}] \quad (23)$$

which is equivalent to:

$$\begin{aligned} \text{ES}_c &= E[X - \text{VaR}_c | X > \text{VaR}_c] \Pr(X > \text{VaR}_c) \\ &= \text{CV}_c(X) \Pr(X > \text{VaR}_c) \end{aligned}$$

For the sake of illustration, we shall compute here the following indicator:

$$H(x) = E(X | X \leq x) = \frac{E(X \mathbf{1}_{X \leq x})}{P(X \leq x)} = \frac{G(x)}{F(x)}$$

where in this instance X is the opposite of a loss and x is typically negative. We explain below how it is possible to expand the computation of F to the one of G (and therefore of H), based on the knowledge of Φ . All other risk indicators can be computed in the same manner.

Proposition 2.2 *The following relationship holds for any random variable that is sufficiently well-behaved (admitting a density that is integrable with respect to the Lebesgue product measure):*

$$G(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} \frac{\Phi'(v + i\alpha)}{v + i\alpha} dv \quad (24)$$

where a is an arbitrary positive real number.

Proof. By analogy with the approach used for F , we write, with α strictly positive:

$$\zeta(x) = e^{-\alpha x} G(x) = e^{-\alpha x} \int_{-\infty}^x s f(s) ds$$

We have the Fourier transform:

$$\hat{\zeta}(v) = \int_{-\infty}^{+\infty} e^{ivx} \zeta(x) dx$$

so that:

$$\hat{\zeta}(v) = \int_{-\infty}^{+\infty} e^{ivx} \left(e^{-\alpha x} \int_{-\infty}^x s f(s) ds \right) dx$$

and:

$$\hat{\zeta}(v) = \int_{-\infty}^{+\infty} \int_{-\infty}^x e^{ivx} e^{-\alpha x} s f(s) ds dx$$

Noting that:

$$-\infty < s < x < +\infty$$

we can swap integrals as follows:

$$\hat{\zeta}(v) = \int_{-\infty}^{+\infty} \int_s^{+\infty} e^{ivx} e^{-\alpha x} s f(s) dx ds$$

so that:

$$\hat{\zeta}(v) = \int_{-\infty}^{+\infty} s f(s) \left(\int_s^{+\infty} e^{ivx} e^{-\alpha x} dx \right) ds = \int_{-\infty}^{+\infty} f(s) \left[\frac{e^{-(\alpha-iv)x}}{-(\alpha-iv)} \right]_s^{+\infty} ds$$

and:

$$\hat{\zeta}(v) = \int_{-\infty}^{+\infty} s f(s) \left(-\frac{e^{-(\alpha-iv)s}}{-(\alpha-iv)} \right) ds = \frac{1}{\alpha-iv} \int_{-\infty}^{+\infty} s f(s) e^{-(\alpha-iv)s} ds$$

Finally:

$$\hat{\zeta}(v) = \frac{1}{\alpha-iv} \int_{-\infty}^{+\infty} s f(s) e^{i(v+i\alpha)s} ds$$

Recall that:

$$\Phi(z) = \int_{-\infty}^{+\infty} f(s)e^{izs} ds$$

for $z \in \mathbb{C}$.

Differentiating, and assuming we are in a context where it is possible to swap the integral and differentiation signs, we obtain:

$$\begin{aligned} \frac{\partial \Phi(z)}{\partial z} &= \frac{\partial \left(\int_{-\infty}^{+\infty} f(s)e^{izs} ds \right)}{\partial z} \\ &= \int_{-\infty}^{+\infty} f(s) \frac{\partial e^{izs}}{\partial z} ds \\ &= \int_{-\infty}^{+\infty} f(s) i s e^{izs} ds \\ &= i \int_{-\infty}^{+\infty} s f(s) e^{izs} ds \end{aligned}$$

Therefore:

$$\hat{\zeta}(v) = \frac{1}{\alpha - iv} \frac{\Phi'(v + i\alpha)}{i}$$

or equivalently:

$$\hat{\zeta}(v) = \frac{\Phi'(v + i\alpha)}{v + i\alpha}$$

We only need to invert this expression:

$$\zeta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} \hat{\zeta}(v) dv$$

so that:

$$e^{-\alpha x} G(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} \frac{\Phi'(v + i\alpha)}{v + i\alpha} dv$$

and finally:

$$G(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} \frac{\Phi'(v + i\alpha)}{v + i\alpha} dv$$

■

3 Illustration

We illustrate the formulas obtained above by using the first of them (which is valid for all types of Lévy processes) in the VGPD context to compute VaRs at various confidence levels and time horizons. Finally, we go the opposite way, computing implicit process parameters based on given VaR levels.

3.1 Computing VaR with VGPD

We compute the VaR attached to an investment in a portfolio replicating the French CAC40 index. We calibrate a VGPD on the index return process for the data covering the period (01/03/01 - 04/15/09), using a maximum likelihood estimation (the density of the VGPD process is available in closed-form). Based on this calibrated VGPD dynamics, we can compute VaR at various confidence intervals and protection time windows. To do so, we incorporate the characteristic function (13) of the VGPD process displayed in the first inside equation (15) of proposition 2.1, and finally we compute VaR using equation (14).

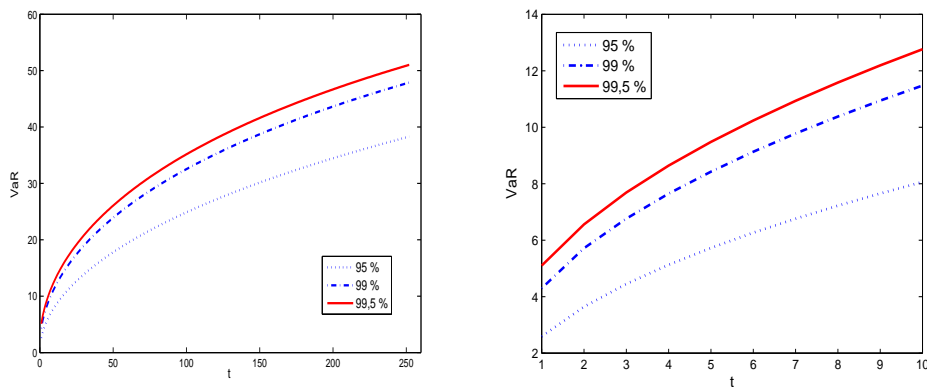


Figure 1: VaR w.r.t. investment horizon (in days)

In Figure 1 (left), we display VaR computed respectively at the 95, 99 and 99.5% confidence intervals, and for investment horizons ranging up to 252 open days, so up to one year. The following features can be observed. First, and obviously, the longer the investment horizon, the higher VaR, or equivalently the amount of required protection. Second, and this is also obvious, the higher the confidence level, the higher VaR. Third, when the confidence level is high, increasing it further does not change very much the amount of required protection. Fourth, a change in the confidence interval has a small impact on VaR compared to a change of the investment horizon. This is very important, because this sheds some light on many debates about whether a VaR at the 99 or the 99.5% confidence interval should be chosen. Clearly, this is not the most important point, and the real difficulty lies in the choice of investment horizon.

Consider for instance a very illiquid corporate bond, held by a bank, not by an insurance company, so that the holding investment time is supposed (by the regulator and by the bank itself) short, typically of the order of magnitude of a couple of days. With these instruments, it is not clear how many days are necessary for computing VaR, and the 30 open days VaR will be twice as big as the 10 open days one. In this instance also, the confidence level will not be critical for the determination of the appropriate protection.

In Figure 1 (right), we plot a zoom and display VaR computed respectively at the 95, 99 and 99.5% confidence intervals, and for investment horizons ranging up to 10 open days. At short horizons also, we observe that the choice of investment horizon is far as important as the choice of confidence level. As far as the 99%-10 days VaR is concerned, it is important in finance because it is the one promoted by the Basle 2 Agreement. In the case of the CAC40 Index, it is equal to approximately 11.5% of the total investment.

As far as medium term horizons are concerned, the VGPD model predicts a 99.5%-1 year VaR equal to about 51% of the total investment. The difference with the 32% level promoted by the QIS4 of Solvency II is huge. This is a clear indication that a 32% loading does not correspond to a 99.5%-1 year VaR. Our conclusion stops here, and we do not go as far as to recommend a 51% loading. Why? Because, and as examined above, the sensitivity to the investment horizon is huge, and it remains to be shown that setting a one year horizon is the best and most relevant choice in the interest of insurance companies. Note finally that the 32% loading of QIS4 corresponds to a 99%-4 months VaR or to a 95%-9 months VaR on the French market (based on the VGPD model).

3.2 Implicit Distributions

In the previous subsection (see Figure 1), we compute the 99%-10 days VaR associated with the CAC40 Index. It is equal to 11.4763%. Assume now that VaR has been corrected by the risk manager, or computed based on another model, or just given as a rule of thumb and as the result of a consensus. Assume for instance that the 99%-10 days VaR has been estimated at various levels ranging between 11 and 13%. The question that arises then is to what Lévy process marginals do these VaRs correspond to. To simplify matters, we assume all the parameters of the Lévy process describing the CAC40 Index fixed (the calibration procedure yields $\delta = -0.0011398$, $\sigma = 0.01537$ and $\theta = 0.0008247$), except the kurtosis parameter ν .

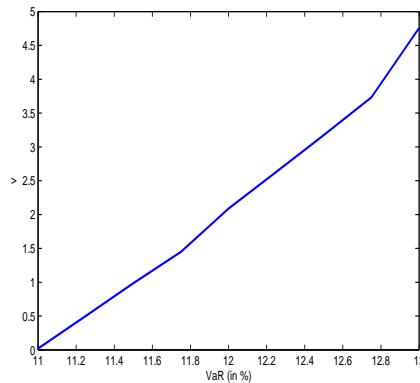


Figure 2: Implicit ν as a function of the 99% – 10D VaR

In Figure 2, we plot the implicit ν corresponding to the chosen VaR level (performing a root search on the function given VaR, which is itself a root search on the function F given in this article). Not surprisingly, the higher the VaR selected by the risk manager, the higher the kurtosis parameter of the VGPD process.

This is an example of a mapping between VaR and kurtosis, when the literature is more accustomed to direct matches between VaR and variance. This is also an example where we do the contrary of what is presented in the previous subsection. Here, we are not interested in VaR *per se*, but in implying the regulators or risk managers' view on the market from the VaR levels that they promote.

In Figure 3, we plot two implicit distributions for two values of ν . We chose these two values because they represent extreme reasonable limits ob-

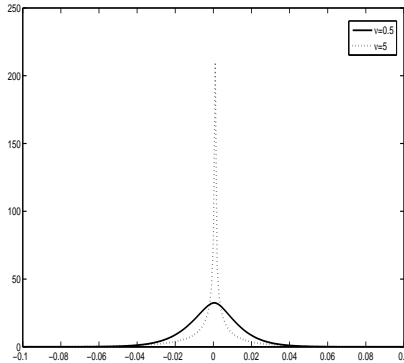


Figure 3: Implicit distributions for $\nu = 0.5$ and $\nu = 5$

served in calibrations. The case $\nu = 0.5$ yields a Gaussian-like density, while the case $\nu = 5$ corresponds to a very leptokurtic distribution. Obtaining an implicit ν for regulators is therefore a way to understand how heavy-tailed they perceive markets.

Another interesting experience consists in assuming that the Basle 2 agreement's spirit is the incentive to post capital equal to three times the 99%-10 days VaR, so to three times 11.4763% in the case of the CAC40. The question is then to what type of VaR (on which a factor 1 and not 3 is imposed) does this loading correspond to? In Figure 4, we plot the implicit (confidence level, investment horizon) couples yielding a VaR equal to three times 11.4763%, so to 34.4289%. We see that with such a loading, an investment in the CAC40 is protected a hundred open days at the 99.5% confidence level. A larger investment horizon will correspond of course to a smaller confidence level. For instance, if the investment horizon is one year, then this investment can be considered protected only at a confidence level of about 92%.

Conclusion

This article rebuilds the Variance Gamma Process with Drift from the beginning, namely starting with the first law of Laplace - the exponential law. We derive formulas that allow to compute VaR and other risk indicators instantaneously when asset returns are modeled by Lévy processes. We also discuss the VaR levels promoted by the Basle 2 and Solvency II agreements

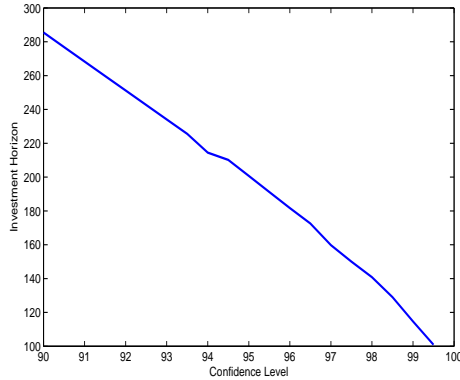


Figure 4: Implicit Basle 2 Curve

and compare them to those obtained with the Lévy method. The proposed algorithm is calibration efficient and stable, because it uses all the points of empirical distributions, contrary to the EVT method which relies on the calibration and modeling of a unique distribution tail.

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