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# Optimal control of an infinite-dimensional problem with a state constraint arising in the spatial economic growth theory\*

Raouf Boucekkine<sup>†</sup>

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Weihua Ruan<sup>§</sup>

## Abstract

We use Ekeland's variational principle together with Pontryagin's maximum principle to solve an optimal spatiotemporal economic growth model with a state constraint (no-negative capital stock) where capital law of motion follows a diffusion equation. We obtain the set of necessary optimal conditions for the solution to meet the state constraints for all time and locations. The maximum principle allows to reduce the infinite-horizon optimal control problem into a finite-horizon one ultimately leading to prove the uniqueness of the optimal solution with positive capital, and non-existence of the optimal solution with eventually strictly positive capital when the time discount rate is too large or too small.

**Keywords:** Diffusion and growth, Optimal Control, State constraint, Ekeland's variational principle, Pontryagin's maximum principle.

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# 1 Introduction

The optimal location of human activities is a core topic in economic theory. It's likely to be even more crucial as the environmental and sustainable development concerns will be more accurately accounted for.<sup>1</sup> Even though the New Economic Geography (see for example the excellent textbook of Fujita and Thisse, 2002) has considerably enriched the conceptual and methodological background of economic research of the field, the overwhelming part of the theoretical works is static. Clearly if sustainable development concerns have to be integrated into the framework, intertemporal considerations have to be embedded too. Therefore, the conception of spatiotemporal settings and inherent analysis methodologies is in principle a prerequisite.

The first attempt to incorporate spatiotemporal optimality in a seemingly spatiotemporal optimal control problem was due to Brito (2004), several other studies in the same vein followed (for example, Camacho *et al.*, 2008, or Brock and Xepapadeas, 2008). All these papers use transport (or diffusion–convection) equations (that's partial differential equations of the parabolic type typically meant to describe diffusion and convection/advection phenomena) to model spatiotemporal dynamics of goods and services, essentially capital goods, as a basic tool to address the issue of optimal location of economic activities and facilities. More recently, a quite active research line of the so-called *Spatial AK* (SAK) model has been taking place following an earlier work due to Boucekkine, Camacho and Fabbri (2013). SAK refers to the AK production, that's a production function linear in the stock of capital,  $K$ . We shall specialize in this model throughout this paper.<sup>2</sup>

A popular question around the SAK model implications is the convergence of the optimal spatiotemporal paths (if any) to a stationary spatial distributions and inherent characterization. A key result in Boucekkine *et al.* (2013) is that the optimal capital paths converge to a uniform distribution from any initial capital distribution under some parametric conditions involving the productivity of capital (that's the parameter  $A$  of the AK production function), the time discount rate or the spatial diffusion speed parameter of the inherent diffusion state equation describing the spatiotemporal dynamics of the capital stock. This is a striking result as the non-spatial AK model predicts just the contrary and rather enhances the role of initial spatial inequalities: a region (or a country) starting with a lower initial capital,  $K(0)$ , with respect to another place, will asymptotically remain lagged. Interestingly enough, Boucekkine *et al.* (2013) convergence conditions impose both a lower and an upper bound to the time discount rate. The former is standard in growth models, it assures that the value function is bounded. The second is much less standard. According

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<sup>1</sup>Early contributions to this economic theory area can be found in Arnott *et al.* (2008) and more recently in Boucekkine *et al.* (2022).

<sup>2</sup>The SAK model has suggested a number of extensions and discussions since 2013. Fabbri (2016) has extended the analysis of SAK to any Riemannian manifold, Boucekkine *et al.* (2019) have extended it to account for any (exogenous) spatiotemporal paths for population size and productivity  $A$ , Gozzi and Leocata (2023) have studied a stochastic version of SAK, and Ricci (2024) has provided with a preliminary discussion of the existence of non-negative optimal capital paths in the original SAK model.

to the authors, above this upper-bound, “...*everything can happen*” (p. 2724), which means that not only convergence to the uniform distribution is no longer granted but even existence of optimal solutions may not be ensured at all.

This “hole” in the analysis of optimal solution signals actually a more fundamental failure: the absence of a proper analysis of existence of optimal solutions in Boucekkine *et al* (2013). It turns out that the solution identified in the latter paper is the only possible solution if it does not lead to negative capital.<sup>3</sup> Indeed, the “optimal” solutions provided for example in Boucekkine *et al.* (2013, 2019) do not solve the original problems but sharper versions in which capital cannot be nonpositive. This is problematic because this means the existence of optimal solutions corresponding to the original optimal control problem is not yet settled. In this paper, we provide a solution to this problem. In the case where the Boucekkine *et al.*’s solution leads to negative capital, our new formulation allows us to reduce the infinite-horizon optimization problem to a two stage problem, a finite-horizon optimal control problem in the initial stage, followed by an infinite-horizon optimal control problem for which the solution in Boucekkine *et al.* (2013) is the only possible solution.<sup>4</sup>

We now describe some of the innovative ideas used in this paper to achieve these objectives. As alluded to above, Boucekkine *et al.* (2013) (and also Boucekkine *et al.*, (2019)) use a dynamic programming method adapted to infinite-dimensional problems, which requires rewriting the transport equations in appropriate functional Hilbert spaces. This method is effective to the SAK model without the state constraint, i.e., the capital remains nonnegative. In fact, it can be shown that the optimal solution derived this way is valid only if the capital is strictly positive in the spatial domain at all time. This excludes situations where the capital reaches zero at certain spots from time to time, which are possible in reality. In this paper, we use Ekeland’s variational principle and Pontryagin’s maximum principle to deal with such situations. The underlying mathematical problem is an infinite-dimensional optimal control problem in an infinite-time horizon with state constraints. Such problems have attracted enormous attentions for decades. See the classical textbooks by J.L. Lions (1971), H.O. Fattorini (1999) and A. Bensoussan *et al.* (2007). In particular, Pontryagin’s maximum principle is an important tool for handling state constraints, and it has long been formulated and widely applied in the area. See, e.g. X. Li and J. Yong (1991), H.O. Fattorini (1999), B. Hu and J. Yong (1995), J. P. Raymond and H. Zidani (1998, 1999), etc., and more recent work in E. Casa and K. Kunisch (2022, 2023).

On the other hand, most of the literature focuses on infinite-dimensional problems in a finite-time horizon. Only in the special case where the welfare functional is quadratic in the states and controls the infinite time horizon is considered, such as in A. Bensoussan *et al.* (2007) and E. Casa and K.

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<sup>3</sup>The latter problem is transparently documented in Boucekkine *et al.* (2013). In Theorem 3.1, page 2723, it’s explicitly written that the results hold “...*Provided the trajectory  $k^*(t, \theta)$ ...remains positive...*”.

<sup>4</sup>A valuable related paper in mathematical economics is Ballestra (2016) who has shown that an adequate choice of the transversality conditions enable a proper use of the Pontryagin principle to solve the SAK model, contrary to the claim in Boucekkine *et al* (2013), but the non-negativity of the stock of capital is not addressed.

Kunisch (2022, 2023). No similar results seem to exist in the case the welfare functional is not of the quadratic form. This is not surprising, as it has been observed that the forward-backward coupled diffusion equations of the states and costates are often ill-posed in the infinite horizon (cf. Boucekkine *et al.* (2009)). However, the SAK model under study is of this type.

In this paper we analyze the SAK model in an infinite time horizon using Pontryagin’s maximum principle in finite time horizons. This is possible because the parabolic equation for the SAK model is linear, and thus, the associated adjoint equation has a special structure that limits the possible forms of the solution. For this approach to succeed, we need first derive a set of necessary optimal conditions for the finite-horizon solutions to meet the state constraint, that is, the capital remaining nonnegative for all time and at all locations. The necessary conditions can be obtained using well-established techniques of H.O. Fattorini (1999), or more directly, of B. Hu and J. Yong (1995). The results allow us to eliminate most of the possible forms of the solutions to the adjoint equation in the case where the state constraint is not tight, i.e., when the capital is eventually positive. As a result, we are able to show the uniqueness of the optimal solution with positive capital, and non-existence of such an optimal solution when the time discount rate,  $\rho$ , is either too large or too small.

The paper is structured as follows. Section ?? presents the model. In Section ?? we derive the necessary condition for the optimal control in the finite horizon in a general form. Section ?? applies the necessary condition to the SAK model. Section ?? discusses the special case with spatially homogeneous technology advancement level and population distribution. We then present a few useful complementary numerical illustrations in Section ??. The proof of a technical lemma is left in Appendix.

## 2 The SAK model

We consider a closed economy, where both land and households are distributed over the unit circle on the plane,  $\mathcal{S} = \{(\sin \theta, \cos \theta) \in \mathbb{R}^2 : \theta \in [0, 2\pi]\}$ . Each location  $\theta \in [0, 2\pi]$  is populated by  $N(\theta)$  individuals. Local production  $Y(t, \theta)$  of the unique final good in the economy is a linear function of physical capital,  $K(t, \theta)$ :

$$Y(t, \theta) = A(\theta) K(t, \theta),$$

$A(\theta)$  stands for the exogenous location-dependent technological level. Capital  $K(t, \theta)$  evolves according to

$$\begin{cases} K_t - DK_{\theta\theta} = AK - Nc, & \text{for } t > 0, \quad \theta \in \mathcal{S}, \\ K(0, \theta) = K_0(\theta), & \text{for } \theta \in (0, 2\pi) \end{cases} \quad (1)$$

where  $D$  is a positive constant and  $c(t, \theta)$  is the per capita consumption. (To simplify notations, we use the slightly incorrect notation  $\theta \in \mathcal{S}$  throughout this paper to indicate both the point on

the unit circle and the corresponding value of  $\theta$ . Also, any function  $y$  defined on  $\mathcal{S}$  is assumed to satisfy the periodic condition

$$y(t, 0) = y(t, 2\pi), \quad y_\theta(t, 0) = y_\theta(t, 2\pi) \quad \text{for } t > 0,$$

and therefore the boundary conditions to the initial-boundary value problems in this paper will not be explicitly stated.) In addition, the capital should be nonnegative, i.e.,

$$K(t, \theta) \geq 0 \quad \text{for } t \geq 0, \quad \theta \in [0, 2\pi]. \quad (2)$$

The policy maker chooses  $c(\cdot, \cdot)$  to maximize overall welfare, defined as

$$J(K_0, c) := \int_0^\infty e^{-\rho t} \int_0^{2\pi} \frac{c(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt \quad (3)$$

where  $\rho > 0$  and  $\sigma \in (0, \infty) \setminus \{1\}$  are positive constants.

It is natural to assume that consumption is nonnegative everywhere. In addition, optimal consumption should ensure that the stock of capital is nonnegative at all locations and at all times. The key then is to find such a solution for optimal consumption.

### 3 Pontryagin's Maximum Principle

In this section, we derive Pontryagin's maximum principle for the parabolic control problem with state constraint. We consider the initial-boundary value problem in a bounded time interval,

$$\begin{cases} y_t - [ay_\theta]_\theta = f(t, \theta, y, c) & \text{for } (t, \theta) \in (0, T) \times \mathcal{S}, \\ y(0, \theta) = y_0(\theta) & \text{for } \theta \in \mathcal{S} \end{cases} \quad (4)$$

where  $T > 0$  is a constant,  $y$  is the state variable,  $c$  is the control,  $a = a(\theta)$  is a strictly positive function, and  $f$  and  $y_0$  are given functions. The system dynamics is subject to the state constraint

$$y(t, \theta) \geq 0 \quad \text{for } (t, \theta) \in [0, T] \times \mathcal{S}.$$

In addition, the control,  $c(t, \theta)$ , is a nonnegative measurable function in  $[0, T] \times \mathcal{S}$ . Given the initial state  $y_0$ , the welfare functional to be maximized is

$$J(y_0, c) = \int_0^T \int_{\mathcal{S}} g(t, \theta, y(t, \theta), c(t, \theta)) d\theta dt + h(T, y(T, \cdot)), \quad (5)$$

where the functional  $h : \mathbb{R}_+ \times C(\mathcal{S}) \mapsto \mathbb{R}$  has a Fréchet derivative with respect to  $y$ .

We use the notations  $\mathcal{S}_T = [0, T] \times \mathcal{S}$  and

$$Q = \{y \in C(\mathcal{S}_T) : y(t, \theta) \geq 0 \text{ for all } (t, \theta) \in \mathcal{S}_T\}. \quad (6)$$

We make the following assumptions.

**Assumption 1.** 1.  $a : \mathcal{S} \mapsto \mathbb{R}$  is a  $C^2$  function and there exist positive constants  $\hat{a}$  and  $\tilde{a}$  such that  $\hat{a} \leq a(\theta) \leq \tilde{a}$  for all  $\theta \in \mathcal{S}$ .

2.  $f : \mathcal{S}_T \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$  and its derivative  $f_y$  are continuously differentiable, and there is a constant  $C$  such that

$$\begin{aligned} f(t, \theta, y, c) y &\leq C(y^2 + 1), \\ |f(t, \theta, y, c)| + |f_y(t, \theta, y, c)| &\leq C \end{aligned} \quad \text{for } (t, \theta, y, c) \in \mathcal{S}_T \times \mathbb{R}_+ \times \mathbb{R}_+.$$

3.  $g : \mathcal{S}_T \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$  its derivative  $g_y$  are continuously differentiable and there is  $C$  such that

$$|g(t, \theta, y, c)| + |g_y(t, \theta, y, c)| \leq C \quad \text{for } (t, \theta, y, c) \in \mathcal{S}_T \times \mathbb{R}_+ \times \mathbb{R}_+$$

and  $h : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuously differentiable.

The main result of this section is the maximum principle for the above optimal control problem given in Proposition ???. It is the basis of the analysis of the SAK model in the subsequent sections. Similar results to Proposition ??? exist in the literature, but none can be directly applied to the problem in this paper. For example, Theorem 10.3.1 in H.O. Fattorini (1999) gives the maximum principle for the general control system in an arbitrary Banach space with both a state constraint and a target condition, and Theorem 2.2 in B. Hu and J. Yong (1995) gives the maximum principle for the control system with a general semilinear parabolic equation with homogeneous Dirichlet boundary condition and without scrap value. Thus, we provide a version of the maximum principle that specifically applicable to the control system (??)–(??).

We first prepare a technical lemma. Let

$$\mathcal{C} = \{c : \mathcal{S}_T \mapsto \mathbb{R}_+ | c \text{ is measurable}\}$$

be the metric space with the Ekeland distance  $d(u, v)$  given by

$$d(u, v) = |\{(t, \theta) \in \mathcal{S}_T | u(t, \theta) \neq v(t, \theta)\}|. \quad (7)$$

(Recall that  $|\Omega|$  for a Lebesgue measurable set  $\Omega$  represents its measure.) It is well-known that  $(\mathcal{C}, d(\cdot, \cdot))$  is a complete metric space. Define the Hamiltonian

$$H(t, \theta, y, c, \mu, \psi) = \mu g(t, \theta, y, c) + \psi f(t, \theta, y, c) \quad \text{in } \mathcal{S}_T \times \mathbb{R}_+ \times \mathbb{R}_+$$



where  $\mu \geq 0$  is a constant. The following lemma gives the “linear approximations” of the state and welfare under spike perturbations.

**Lemma 1.** *Let Assumption ?? hold. Let  $\{\bar{c}, \bar{y}\}$  be a feasible pair and let  $c \in \mathcal{C}$  be fixed. Then, for any  $\delta \in (0, 1)$  there exists a measurable set  $E_\delta \subset \mathcal{S}_T$  and the control  $c_\delta$  defined by*

$$c_\delta(t, \theta) = \begin{cases} \bar{c}(t, \theta) & \text{if } (t, \theta) \in \mathcal{S}_T \setminus E_\delta, \\ c(t, \theta) & \text{if } (t, \theta) \in E_\delta, \end{cases} \quad (8)$$

such that  $|E_\delta| = \delta |\mathcal{S}_T|$  and the following hold:

$$y(\cdot, c_\delta) = \bar{y}(\cdot) + \delta z(\cdot) + o(\delta), \quad J(c_\delta) = J(\bar{c}) + \delta l + o(\delta) \quad (9)$$

(with the first  $o(\delta)$  is in space  $C^{\alpha, \alpha/2}(\mathcal{S}_T)$  for some  $\alpha \in (0, 1)$ ) where  $z$  and  $l$  satisfy

$$\begin{aligned} z_t - [az_\theta]_\theta &= f_y(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)) z + \phi(t, \theta), \\ z(0, \theta) &= 0 \end{aligned}$$

and

$$\begin{aligned} l &= \int_0^T \int_{\mathcal{S}} [g_y(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)) z(t, \theta) + \gamma(t, \theta)] d\theta dt \\ &\quad + \int_{\mathcal{S}} h_y(T, \bar{y}(T, \theta)) z(T, \theta) d\theta, \end{aligned}$$

respectively, with

$$\begin{aligned} \phi(t, \theta) &= f(t, \theta, \bar{y}(t, \theta), c(t, \theta)) - f(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)), \\ \gamma(t, \theta) &= g(t, \theta, \bar{y}(t, \theta), c(t, \theta)) - g(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)). \end{aligned}$$

A proof is included in Appendix.

Let  $d_0(y) : C(\mathcal{S}_T) \mapsto \mathbb{R}_+$  denote the functional

$$d_0(y) = |\min\{y, 0\}|_{C(\mathcal{S}_T)} \quad \text{for any } y \in C(\mathcal{S}_T). \quad (10)$$

It is clear that  $d_0(y) = 0$  if  $y$  is nonnegative in  $\mathcal{S}_T$ . In addition,  $d_0$  is Gâteaux differentiable at every  $y \in C(\mathcal{S}_T) \setminus Q$ , and its Gâteaux derivative  $\nabla d_0(y)$  is the same as the Clarke’s generalized gradient, which is convex and weak\*-compact. As a result,

$$|\nabla d_0(y)|_{\mathcal{M}(\mathcal{S}_T)} = 1 \quad \text{if } y \notin Q \quad (11)$$

and for any  $\xi \in \partial d_0(y)$ ,

$$\langle \xi, z - y \rangle + d_0(y) \leq d_0(z) \quad \text{for any } z \in C(\mathcal{S}_T) \quad (12)$$

where  $\mathcal{M}(\mathcal{S}_T)$  is the set of all Radon measures on  $\mathcal{S}_T$ .

The following is maximum principle for system (??)–(??).

**Proposition 1.** *Let Assumption ?? hold and let  $\{c^*, y^*\}$  be an optimal pair. Then there exists a constant  $\mu \geq 0$ , a function  $\psi \in L^q(0, T; W^{1,q}(\mathcal{S}))$  ( $1 < q < 3/2$ ) and a Radon measure  $\nu \in \{\nabla d_0(y^*)\}$  such that*

$$\mu + |\nu|_{\mathcal{M}(\mathcal{S}_T)} > 0,$$

$$\langle \nu, z - y^* \rangle \leq 0 \text{ for any } z \in Q,$$

$$\begin{aligned} \psi_t + [a\psi_\theta]_\theta &= -f_y(t, \theta, y^*(t, \theta), c^*(t, \theta)) \psi \\ &\quad - \mu g_y(t, \theta, y^*(t, \theta), c^*(t, \theta)) - \nu|_{(0,T) \times \mathcal{S}}, \\ \psi(T, \cdot) &= \mu h_y(T, y^*(T, \cdot)) + \nu|_{\{T\} \times \mathcal{S}}, \end{aligned} \quad (13)$$

and

$$H(t, \theta, y^*(t, \theta), c^*(t, \theta), \mu, \psi(t, \theta)) = \max_{c \geq 0} H(t, \theta, y^*(t, \theta), c, \mu, \psi(t, \theta)). \quad (14)$$

**Proof.** Note that the main difference between System (??)–(??) and that in B. Hu and J. Yong (1995) is the boundary condition and the presence of the scrap value  $h(T, y(T, \cdot))$ . Thus the proof follows closely the approach in that paper. The main idea is using Ekeland’s variational principle (Theorem 3.2.2 in H. Fattorini (1999)) together with a spike perturbation.

For any  $\varepsilon > 0$  we define

$$F_\varepsilon(c) = \left\{ [J(c) - J(c^*) + \varepsilon]_+^2 + d_0(y(\cdot; c))^2 \right\}^{1/2} \quad (15)$$

where  $y(\cdot; c)$  is the solution to (??) corresponding to the control  $c$ . Then,

$$F_\varepsilon(c^*) = \varepsilon \leq \inf F_\varepsilon(c) + \varepsilon.$$

This means  $c^*$  is an  $\varepsilon$ -minimum of  $F_\varepsilon$ , which is bounded below and semi-lower continuous. Hence, by Ekeland’s variational principle, there exists  $c^\varepsilon$  such that

$$F_\varepsilon(c^\varepsilon) \leq F_\varepsilon(c^*), \quad d(c^\varepsilon, c^*) \leq \sqrt{\varepsilon} \quad (16)$$

and

$$F_\varepsilon(c^\varepsilon) - F_\varepsilon(c) \leq \sqrt{\varepsilon} d(c^\varepsilon, c) \quad \text{for any } c \in \mathcal{C}. \quad (17)$$

Let  $y^\varepsilon = y(\cdot; c^\varepsilon)$ . Fix a  $c \in \mathcal{C}$  and an  $\varepsilon > 0$ . For any  $\delta > 0$  and a measurable set  $E_\delta^\varepsilon \subset \mathcal{S}_T$  we construct the perturbation

$$c_\delta^\varepsilon(t, \theta) = \begin{cases} c^\varepsilon(t, \theta) & \text{for } (t, \theta) \in \mathcal{S}_T \setminus E_\delta^\varepsilon, \\ c(t, \theta) & \text{for } (t, \theta) \in E_\delta^\varepsilon. \end{cases} \quad (18)$$

Let  $y_\delta^\varepsilon = y(\cdot; c_\delta^\varepsilon)$  denote the state corresponding to the perturbation. An application of Lemma ?? leads to

$$y_\delta^\varepsilon = y^\varepsilon + \delta z^\varepsilon + o(\delta), \quad J(c_\delta^\varepsilon) = J(c^\varepsilon) + \delta l^\varepsilon + o(\delta), \quad (19)$$

where  $z^\varepsilon$  and  $z^{0, \varepsilon}$  satisfy equations

$$z_t^\varepsilon - [a z_\theta^\varepsilon]_\theta = f_y(t, \theta, y^\varepsilon(t, \theta), c^\varepsilon(t, \theta)) z^\varepsilon(t, \theta) + \phi^\varepsilon(t, \theta), \quad (20)$$

and

$$l^\varepsilon = \int_0^T \int_{\mathcal{S}} [g_y(t, \theta, y^\varepsilon(t, \theta), c^\varepsilon(t, \theta)) z^\varepsilon(t, \theta) + \gamma^\varepsilon(t, \theta)] d\theta dt + \int_{\mathcal{S}} h_y(T, y^\varepsilon(T, \theta)) z^\varepsilon(T, \theta) d\theta \quad (21)$$

respectively, with

$$\begin{aligned} \phi^\varepsilon(t, \theta) &= f(t, \theta, y^\varepsilon(t, \theta), c(t, \theta)) - f(t, \theta, y^\varepsilon(t, \theta), c^\varepsilon(t, \theta)), \\ \gamma^\varepsilon(t, \theta) &= g(t, \theta, y^\varepsilon(t, \theta), c(t, \theta)) - g(t, \theta, y^\varepsilon(t, \theta), c^\varepsilon(t, \theta)). \end{aligned}$$

We next choose a  $E_\delta^\varepsilon$  so that  $|E_\delta^\varepsilon| = \delta |\mathcal{S}_T|$ . By (??),  $d(c_\delta^\varepsilon, c^\varepsilon) = |E_\delta^\varepsilon|$ . Hence, by (??)

$$\begin{aligned} \sqrt{\varepsilon} |\mathcal{S}_T| &\geq \frac{F_\varepsilon(c^\varepsilon) - F_\varepsilon(c_\delta^\varepsilon)}{\delta} \\ &= \frac{1}{[F_\varepsilon(c^\varepsilon) + F_\varepsilon(c_\delta^\varepsilon)] \delta} \left\{ [J(c^\varepsilon) - J(c^*) + \varepsilon]_+^2 - [J(c_\delta^\varepsilon) - J(c^*) + \varepsilon]_+^2 \right. \\ &\quad \left. + [d_0(y^\varepsilon) - d_0(y_\delta^\varepsilon)] \right\}. \end{aligned}$$

Taking  $\delta \rightarrow 0$  and using (??), the right-hand side converges to

$$\frac{[J(c^\varepsilon) - J(c^*) + \varepsilon]_+}{F_\varepsilon(c^\varepsilon)} l^\varepsilon + \left\langle \frac{d_0(y^\varepsilon) \xi^\varepsilon}{F_\varepsilon(c^\varepsilon)}, z^\varepsilon \right\rangle \equiv \mu^\varepsilon l^\varepsilon + \langle \nu^\varepsilon, z^\varepsilon \rangle,$$

where

$$\mu^\varepsilon = \frac{[J(c^\varepsilon) - J(c^*) + \varepsilon]_+}{F_\varepsilon(c^\varepsilon)}, \quad \nu^\varepsilon = \frac{d_0(y^\varepsilon) \xi^\varepsilon}{F_\varepsilon(c^\varepsilon)}$$

and

$$\xi^\varepsilon(y^\varepsilon) = \begin{cases} \nabla d_0(y^\varepsilon), & \text{if } y^\varepsilon \notin Q, \\ 0 & \text{if } y^\varepsilon \in Q. \end{cases}$$

Hence, by (??),

$$\sqrt{\varepsilon} |\mathcal{S}_T| \geq \mu^\varepsilon l^\varepsilon + \langle \nu^\varepsilon, z^\varepsilon \rangle. \quad (22)$$

Note that by (??),

$$|\xi^\varepsilon(y^\varepsilon)|_{C(\mathcal{S}_T)^*} = 1 \quad \text{if } y^\varepsilon \notin Q.$$

It follows from (??) that  $\mu^\varepsilon \geq 0$  and

$$\mu^\varepsilon + |\nu^\varepsilon|_{C(\mathcal{S}_T)^*} = 1 \quad \text{for all } \varepsilon > 0.$$

Also, by (??),

$$\langle \nu^\varepsilon, z - y^\varepsilon \rangle \leq -d_0(y^\varepsilon) \leq 0 \quad \text{for any } z \in Q. \quad (23)$$

Next, by (??) and Lemma ??, it follows that

$$y^\varepsilon = y^* + \delta z^* + o(\delta), \quad J(c^\varepsilon) = J(c^*) + \delta l^* + o(\delta), \quad (24)$$

where  $z^*$  and  $l^*$  satisfy equations

$$\begin{aligned} z_t^* - [az_\theta^*]_\theta &= f_y(t, \theta, y^*(t, \theta), c^*(t, \theta)) z^*(t, \theta) + \phi^*(t, \theta), \\ z^*(0, \theta) &= 0 \end{aligned} \quad (25)$$

and

$$\begin{aligned} l^* &= \int_0^T \int_S [g_y(t, \theta, y^*(t, \theta), c^*(t, \theta)) z^*(t, \theta) + \gamma^*(t, \theta)] d\theta dt \\ &\quad + \int_S h_y(T, y^*(T, \theta)) z^*(T, \theta) d\theta, \end{aligned}$$

respectively, with

$$\begin{aligned} \phi^*(t, \theta) &= f(t, \theta, y^*(t, \theta), c(t, \theta)) - f(t, \theta, y^*(t, \theta), c^*(t, \theta)), \\ \gamma^*(t, \theta) &= g(t, \theta, y^*(t, \theta), c(t, \theta)) - g(t, \theta, y^*(t, \theta), c^*(t, \theta)). \end{aligned}$$

From (??) we see that  $y^\varepsilon \rightarrow y^*$  in  $C^{\alpha, \alpha/2}(\mathcal{S}_T)$  as  $\varepsilon \rightarrow 0$ . Thus, from (??) and (??) we find  $z^\varepsilon \rightarrow z^*$  in  $C^{\alpha, \alpha/2}$  and  $l^\varepsilon \rightarrow l^*$  in  $\mathbb{R}$  as  $\varepsilon \rightarrow 0$ . Since  $Q$  is finite codimensional in  $C(\mathcal{S}_T)$ , it follows from Lemma 3.2 of X. Li and J. Yong (1991) that the weakly-\* limit,  $(\mu, \nu)$ , of  $(\mu^\varepsilon, \nu^\varepsilon)$  as  $\varepsilon \rightarrow 0$  is positive. Taking  $\varepsilon \rightarrow 0$  in (??) we find  $\langle \nu, z - y^* \rangle \leq 0$  for any  $z \in Q$ . In addition, from (??) we find

$$\mu l^* + \langle \nu, z^* \rangle \leq 0 \quad \text{for any } c \in \mathcal{C}. \quad (26)$$

We show that the above inequality is equivalent to

$$\begin{aligned}
0 &\leq \int_0^T \int_{\mathcal{S}} \{ \mu [g(t, \theta, y^*(t, \theta), c^*(t, \theta)) - g(t, \theta, y^*(t, \theta), c(t, \theta))] \\
&\quad + \psi(t, \theta) [f(t, \theta, y^*(t, \theta), c^*(t, \theta)) - f(t, \theta, y^*(t, \theta), c(t, \theta))] \} d\theta dt \\
&= \int_0^T \int_{\mathcal{S}} [H(t, \theta, y^*(t, \theta), c^*(t, \theta), \mu, \psi(t, \theta)) \\
&\quad - H(t, \theta, y^*(t, \theta), c(t, \theta), \mu, \psi(t, \theta))] d\theta dt
\end{aligned}$$

for any  $c \in \mathcal{C}$ . Using (??) and (??), we find

$$\begin{aligned}
\int_0^T \int_{\mathcal{S}} [z^* \psi]_t d\theta dt &= \int_0^T \int_{\mathcal{S}} [z_t^* \psi + z^* \psi_t] d\theta dt \\
&= \int_0^T \int_{\mathcal{S}} [\psi \phi^* - \mu z^* g_y^*] d\theta dt + \langle \nu, z^* \rangle_{(0, T) \times \mathcal{S}}
\end{aligned}$$

where

$$g_y^* = g_y(t, \theta, y^*(t, \theta), c^*(t, \theta)).$$

On the other hand, since

$$z^*(0, \theta) = 0, \quad \psi(T, \cdot) = \mu h_y(T, y^*(T, \cdot)) + \nu|_{\{T\} \times \mathcal{S}},$$

it follows that

$$\begin{aligned}
\int_0^T \int_{\mathcal{S}} [z^* \psi]_t d\theta dt &= \int_{\mathcal{S}} z^*(T, \theta) \psi(T, \theta) d\theta \\
&= \mu \int_{\mathcal{S}} h_y(T, y^*(T, \theta)) z^*(T, \theta) d\theta + \langle \nu, z^*(T, \cdot) \rangle_{\mathcal{S}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\mu \int_{\mathcal{S}} h_y(T, y^*(T, \theta)) z^*(T, \theta) d\theta + \langle \nu, z^*(T, \cdot) \rangle_{\mathcal{S}} + \mu \int_0^T \int_{\mathcal{S}} z^* g_y^* d\theta dt \\
&= \int_0^T \int_{\mathcal{S}} \psi(t, \theta) \phi^*(t, \theta) d\theta dt - \langle \nu, z^* \rangle_{(0, T) \times \mathcal{S}}.
\end{aligned} \tag{27}$$

By (??),

$$\mu \int_0^T \int_{\mathcal{S}} [z^* g_y^* + \gamma^*(t, \theta)] d\theta dt + \mu \int_{\mathcal{S}} h_y(T, y^*(T, \theta)) z^*(T, \theta) d\theta + \langle \nu, z^* \rangle_{\mathcal{S}_T} \leq 0.$$

As a result, by (??),

$$\int_0^T \int_{\mathcal{S}} [\mu \gamma^*(t, \theta) + \psi(t, \theta) \phi^*(t, \theta)] dt d\theta \leq 0.$$

This is equivalent to

$$0 \leq \int_0^T \int_{\mathcal{S}} \{ \mu [g(t, \theta, y^*(t, \theta), c^*(t, \theta)) - g(t, \theta, y^*(t, \theta), c(t, \theta))] + \psi(t, \theta) [f(t, \theta, y^*(t, \theta), c^*(t, \theta)) - f(t, \theta, y^*(t, \theta), c(t, \theta))] \} d\theta dt$$

for any  $c \in \mathcal{C}$ . Since  $c(t, \theta)$  is arbitrary, (??) follows.

The proof is complete.

The following observation plays an important role in the next section. It distinguishes the optimal solution with positive state from others.

**Remark 1.** Let  $\nu \in \mathcal{M}(\mathcal{S}_T)$  be the Radon measure in Proposition ???. It is easy to see that

$$\text{supp } \nu \subset \{(t, \theta) \in \mathcal{S}_T : y^*(t, \theta) = 0\}. \quad (28)$$

Indeed, for any  $\eta \in C(\mathcal{S}_T)$  with  $\text{supp } \eta \subset \mathcal{S}_T \setminus \text{supp } y^*$ , there is  $\varepsilon > 0$  such that  $z^\pm := y^* \pm \varepsilon \eta \in Q$ . Hence,

$$\pm \varepsilon \langle \nu, \eta \rangle = \langle \nu, z^\pm - y^* \rangle \leq 0.$$

This implies that  $\langle \nu, \eta \rangle = 0$ .

## 4 Optimal solution for the SAK model

In Boucekine *et al.* (2013) a value function  $\bar{V}(K_0)$  is derived with the control

$$\bar{c}(t, \theta) = (\lambda_0 - g) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} [\varphi_0(\theta)]^{-1/\sigma} e^{gt}, \quad (29)$$

where  $\lambda_0$  is the principal eigenvalue of the differential operator  $\mathcal{L}$  defined by

$$\mathcal{L}u(\theta) := Du''(\theta) + A(\theta)u(\theta) \quad \text{for } \theta \in \mathcal{S} \quad (30)$$

and  $\varphi_0$  is the normalized (in  $L^2(\mathcal{S})$ ) positive eigenfunction corresponding to  $\lambda_0$ ,

$$f(\theta) = [\varphi_0(\theta)]^{-1/\sigma} N(\theta), \quad (31)$$

and

$$g = \frac{\lambda_0 - \rho}{\sigma}. \quad (32)$$

With this control, the value function is

$$\bar{V}(K_0) = \frac{\langle K_0, \varphi_0 \rangle^{1-\sigma}}{1-\sigma} \left[ \frac{\langle f, \varphi_0 \rangle}{\lambda_0 - g} \right]^\sigma. \quad (33)$$

This value function is valid only if the associated trajectory,  $\bar{K}(t, \theta)$ , which satisfies (??) with  $c$  replaced by  $\bar{c}$ , is nonnegative for all  $t \geq 0$ ,  $\theta \in \mathcal{S}$ . In the case where  $\bar{K}(t, \theta)$  takes negative value for some  $(t, \theta)$ , we call the pair  $\{\bar{c}, \bar{K}\}$  *not feasible*.

In this section, we explore the feasibility of  $\{\bar{c}, \bar{K}\}$  and show that it is the only possible optimal pair with positive state if it is feasible. In the case where it is not feasible, we show how an optimal solution can be obtained by solving a finite-horizon optimal problem, provided that an optimal pair with eventually positive state exists. We finally determine asymptotic behavior of such a state.

Let us first recall a well-known property of the eigenvalues of  $\mathcal{L}$ . For reference, see, e.g., Theorems 2.4.2 and 2.5.1 in Brown *et al.* (2013):

**Proposition 2.** *Suppose  $D > 0$  is a constant and  $A \in C(\mathcal{S}; \mathbb{R}^+)$ . Then, the eigenvalues of the operator  $\mathcal{L}$  on  $W^{2,2}(\mathcal{S}; \mathbb{R})$  are countable and can be ordered as a decreasing sequence  $\{\lambda_n\}_{n=0}^\infty$  such that  $\lambda_n \rightarrow -\infty$ . Furthermore, the multiplicity of the principal eigenvalue,  $\lambda_0$ , is one, and there exists a strictly positive eigenfunction  $\varphi_0$  corresponding to  $\lambda_0$ .*

#### 4.1 Small time discount

In the case where  $\rho \leq (1 - \sigma)\lambda_0$ , we show that  $\{\bar{c}, \bar{K}\}$  is never feasible.

**Proposition 3.** *Suppose  $\rho \leq (1 - \sigma)\lambda_0$ . Then  $\{\bar{c}, \bar{K}\}$  is not feasible. In addition, if an optimal solution  $\{c^*, K^*\}$  exists, then for any  $T > 0$  there exists  $(t, \theta)$  such that  $t > T$ ,  $\theta \in [0, 2\pi]$  and  $K^*(t, \theta) = 0$ .*

**Proof.** We first show that if  $\{c^*, K^*\}$  is an optimal pair, then there is no  $T > 0$  such that  $K^*(t, \theta) > 0$  for all  $t > T$ ,  $\theta \in \mathcal{S}$ .

Assume for contradiction that such a  $T$  exists. Using a time translation if necessary, we can assume that  $T = 0$ . Hence, in view of Remark ??, the Radon measure  $\nu$  in Proposition ?? vanishes. As a result,  $\mu > 0$  and we can choose  $\mu = 1$ . Therefore, the adjoint variable  $\psi$  and  $c$  are related by (??) which takes the form

$$e^{-\rho t} \frac{c^*(t, \theta)^{1-\sigma}}{1-\sigma} - \psi(t, \theta) c^*(t, \theta) = \arg \max_{c \geq 0} \left\{ e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} - \psi(t, \theta) c \right\}.$$

It follows that

$$\psi(t, \theta) = e^{-\rho t} c^*(t, \theta)^{-\sigma}, \quad (34)$$

then  $\psi$  satisfies the adjoint problem (??) which takes the form

$$\begin{cases} \psi_t + D\psi_{\theta\theta} + A\psi = 0, & \text{if } 0 < t < \tau, \theta \in \mathcal{S}, \\ \psi(\tau, \theta) = e^{-\rho\tau} c^*(\tau, \theta)^{-\sigma}, & \text{for } \theta \in \mathcal{S}. \end{cases} \quad (35)$$

for any  $\tau > 0$ . Since  $\tau$  is arbitrary, we see that the first equation in (??) holds for any  $t > 0$ ,  $\theta \in (0, 2\pi)$ . Using Fourier expansion, we can write

$$\psi(t, \theta) = \sum_{n \geq 0} \sum_i e^{-\lambda_n t} \langle \psi(t, \cdot), \varphi_{n,i} \rangle \varphi_{n,i}(\theta)$$

where  $\{\varphi_{n,i}\}$  is the set of normalized eigenfunctions of  $\mathcal{L}$  corresponding to the  $n$ th eigenvalue,  $\lambda_n$ . Note that for  $\{c^*, K^*\}$  to be feasible, it is necessary that  $c^*(t, \theta) > 0$  for all  $(t, \theta)$ . Thus  $\psi(t, \theta) > 0$  for all  $(t, \theta)$ . Since  $\lambda_n < \lambda_0$  and any eigenfunction corresponding to  $\lambda_n$  changes sign in  $\mathcal{S}$  for any  $n > 0$ , it follows that  $\langle \psi(t, \cdot), \varphi_n \rangle = 0$  for all  $t > 0$  for any  $n > 0$ , because, otherwise,  $\psi$  cannot remain positive. Thus there is  $a > 0$  such that  $\psi(t, \theta) = a e^{-\lambda_0 t} \varphi_0(\theta)$ . By (??),

$$c^*(t, \theta) = \alpha \varphi_0(\theta)^{-1/\sigma} e^{gt}, \quad (36)$$

where  $\alpha = a^{-1/\sigma}$  and  $g$  is given by (??).

We show that  $K^*$  becomes negative for any  $a > 0$ . Let  $u(t, \theta) = K^*(t, \theta) e^{-gt}$ . Then

$$u_t = (\mathcal{L} - g)u - \alpha f$$

where  $f$  is given by (??). The equation is linear, and its solution can be written as

$$u(t, \theta) = \tilde{u}(t, \theta) - \alpha \hat{u}(t, \theta)$$

where  $\tilde{u}$  and  $\hat{u}$  are solutions to the initial-boundary value problems

$$\begin{cases} \tilde{u}_t - D\tilde{u}_{\theta\theta} = (A - g)\tilde{u} & \text{for } t > 0, \theta \in \mathcal{S}, \\ \tilde{u}(0, \theta) = K_0(\theta) & \text{for } \theta \in \mathcal{S}, \end{cases} \quad (37)$$

and

$$\begin{cases} \hat{u}_t - D\hat{u}_{\theta\theta} = (A - g)\hat{u} + f & \text{for } t > 0, \theta \in \mathcal{S}, \\ \hat{u}(0, \theta) = 0 & \text{for } \theta \in \mathcal{S}, \end{cases} \quad (38)$$

respectively. Clearly,  $u(t, \theta)$  and  $K^*(t, \theta)$  have the same sign. We show that  $\alpha \hat{u}(t, \theta) > \tilde{u}(t, \theta)$  if  $t$  is large. It suffices to show that

$$\alpha \langle \hat{u}(t, \cdot), \varphi_0 \rangle > \langle \tilde{u}(t, \cdot), \varphi_0 \rangle$$



for large  $t$ . Note that  $\rho \leq (1 - \sigma) \lambda_0$  is equivalent to  $\lambda_0 \leq g$ . We denote

$$\hat{u}_0(t) = \langle \hat{u}(t, \cdot), \varphi_0 \rangle, \quad \tilde{u}_0(t) = \langle \tilde{u}(t, \cdot), \varphi_0 \rangle.$$

Multiplying both sides of the differential equations in (??) and (??) by  $\varphi_0$ , and integrating over  $[0, 2\pi]$ , we find

$$\begin{aligned} \hat{u}'_0(t) &= (\lambda_0 - g) \hat{u}_0(t) + \langle f, \varphi_0 \rangle, & \hat{u}_0(0) &= 0, \\ \tilde{u}'_0(t) &= (\lambda_0 - g) \tilde{u}_0(t), & \tilde{u}_0(0) &= \langle K_0, \varphi_0 \rangle. \end{aligned}$$

The solutions are

$$\hat{u}_0(t) = \frac{\langle f, \varphi_0 \rangle}{\lambda_0 - g} \left[ e^{(\lambda_0 - g)t} - 1 \right], \quad \tilde{u}_0(t) = \langle K_0, \varphi_0 \rangle e^{(\lambda_0 - g)t} \quad (39)$$

if  $\lambda_0 \neq g$  and

$$\hat{u}_0(t) = \langle f, \varphi_0 \rangle t, \quad \tilde{u}_0(t) = \langle K_0, \varphi_0 \rangle \quad (40)$$

if  $\lambda_0 = g$ . Since  $\langle f, \varphi_0 \rangle > 0$ , in both cases  $\alpha \hat{u}_0(t) > \tilde{u}_0(t)$  for large  $t$ . This proves that there is no  $T > 0$  such that  $K^*(t, \theta) > 0$  for all  $t > T$  and  $\theta \in \mathcal{S}$ .

Recall that  $\bar{c}$  is given by (??) which is in the form (??) with

$$\alpha = (\lambda_0 - g) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle}, \quad (41)$$

it follows that  $\{\bar{c}, \bar{K}\}$  is not feasible.

This completes the proof.

## 4.2 Larger time discount

In the case where  $\rho > (1 - \sigma) \lambda_0$ , the pair  $\{\bar{c}, \bar{K}\}$  may or may not be feasible. We first show that whenever  $\{\bar{c}, \bar{K}\}$  is feasible, it is the only possible optimal pair with positive state.

**Proposition 4.** *Suppose that  $\rho > (1 - \sigma) \lambda_0$  and that  $\{\bar{c}, \bar{K}\}$  is feasible. If  $\{c^*, K^*\}$  is an optimal pair such that  $K^*(t, \theta) > 0$  for all  $t > 0$  and  $\theta \in \mathcal{S}$ , then  $\{c^*, K^*\} = \{\bar{c}, \bar{K}\}$ .*

**Proof.** Let  $\psi(t, \theta)$  be defined by (??). As in the proof of Proposition ??,

$$\psi(t, \theta) = a e^{-\lambda_0 t} \varphi_0(\theta)$$

for some  $a > 0$ . As a result,  $c^*(t, \theta)$  satisfies (??) with  $\alpha = a^{-1/\sigma}$ . It remains to show that  $\alpha$  is given by (??).

Let  $\bar{\alpha}$  denote the right-hand side of (??). We show that  $\alpha \leq \bar{\alpha}$ . Let  $\tilde{u}$  and  $\hat{u}$  be solutions of (??)

and (??), respectively. Then, as in the proof of Proposition ??,

$$\tilde{u}(t, \theta) - \alpha \hat{u}(t, \theta) = K^*(t, \theta) e^{-gt}.$$

Since  $K^*(t, \theta) > 0$  for all  $t > 0$ ,  $\theta \in \mathcal{S}$ , it follows that

$$\langle \tilde{u}(t, \theta), \varphi_0 \rangle > \alpha \langle \hat{u}(t, \theta), \varphi_0 \rangle \quad \text{for all } t > 0. \quad (42)$$

Note that  $\rho > (1 - \sigma)\lambda_0$  is equivalent to  $\lambda_0 > g$ . It follows from (??) that

$$\langle \tilde{u}(t, \theta), \varphi_0 \rangle = \langle K_0, \varphi_0 \rangle e^{(\lambda_0 - g)t}, \quad \langle \hat{u}(t, \theta), \varphi_0 \rangle = \frac{\langle f, \varphi_0 \rangle}{\lambda_0 - g} \left[ e^{(\lambda_0 - g)t} - 1 \right].$$

Hence, (??) implies

$$\alpha < (\lambda_0 - g) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} \frac{e^{(\lambda_0 - g)t}}{e^{(\lambda_0 - g)t} - 1} \quad \text{for all } t > 0.$$

This leads to

$$\alpha \leq (\lambda_0 - g) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} \equiv \bar{\alpha}_0. \quad (43)$$

Now, since  $\alpha \leq \bar{\alpha}_0$  and the welfare function  $J(K_0, c)$  given by (??) is increasing in  $c$ , and since  $\{\bar{c}, \bar{K}\}$  is feasible, it follows that  $\alpha = \bar{\alpha}_0$ .

This completes the proof.

### 4.3 Initial period

In the case where  $\{\bar{c}, \bar{K}\}$  is not feasible, an optimal pair  $\{c^*, K^*\}$  can still exist. In this case either  $K^*$  is eventually positive in the sense that there is  $T > 0$  such that  $K^*(t, \theta) > 0$  for all  $t > T$  and  $\theta \in \mathcal{S}$ , or there exist a sequence  $\{(t_k, \theta_k)\}_{k=1}^\infty$  in  $(0, \infty) \times \mathcal{S}$  such that

$$0 < t_1 < \dots < t_k < \dots \rightarrow \infty,$$

and  $K^*(t_k, \theta_k) = 0$ . In the former case, by a time-translation we see from Proposition ?? that  $\{c^*, K^*\} = \{\bar{c}, \bar{K}\}$  for  $t > T$  provided that  $\{\bar{c}, \bar{K}\}$  for  $t \geq T$  is feasible, where  $\bar{c}$  is given by

$$\bar{c}(t, \theta) = (\lambda_0 - g) \frac{\langle K^*(T, \cdot), \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} [\varphi_0(\theta)]^{-1/\sigma} e^{g(t-T)} \quad \text{for } t \geq T, \quad (44)$$

and  $\bar{K}$  is the solution to the initial-boundary value problem

$$\begin{cases} K_t - DK_{\theta\theta} = AK - Nc, & \text{for } t > T, \quad \theta \in \mathcal{S}, \\ K(T, \theta) = K^*(T, \theta), & \text{for } \theta \in \mathcal{S} \end{cases} \quad (45)$$

During the initial period,  $0 \leq t < T$ , we can determine  $c^*(t, \theta)$  by solving a finite-horizon optimal control problem

$$J(K_0, c^*) = \sup_{c \geq 0} \int_0^T \int_0^{2\pi} e^{-\rho t} \frac{c(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt + e^{-\rho T} \bar{V}(K^*(T, \cdot)). \quad (46)$$

We prove

**Proposition 5.** *Suppose that  $\rho > (1 - \sigma)\lambda_0$  and that  $\{c^*, K^*\}$  is an optimal pair such that  $K^*(t, \theta) > 0$  for all  $(t, \theta) \in (T, \infty) \times \mathcal{S}$ . Suppose  $\{\bar{c}, \bar{K}\}$  with  $\bar{c}$  given by (??) and  $\bar{K}$  solving (??) is also feasible for  $t \geq T$ , then, the costate  $\psi$  given by (??) is a solution to the terminal-boundary value problem*

$$\begin{cases} \psi_t + D\psi_{\theta\theta} + A\psi = -\nu|_{(0,T) \times \mathcal{S}}, & \text{if } 0 < t < T, \quad \theta \in \mathcal{S}, \\ \psi(T, \theta) = \nu|_{\{T\} \times \mathcal{S}} + e^{-\rho T} \bar{c}(T, \theta)^{-\sigma}, & \text{for } \theta \in \mathcal{S}. \end{cases} \quad (47)$$

where

$$\bar{c}(T, \theta) = (\lambda_0 - g) \frac{\langle K^*(T, \cdot), \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} [\varphi_0(\theta)]^{-1/\sigma}.$$

**Proof.** Proposition ?? with

$$\begin{aligned} f(t, \theta, K, c) &= A(\theta)K - N(\theta)c, & g(t, \theta, K, c) &= e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} N(\theta), \\ h(t, K) &= e^{-\rho t} \bar{V}(K) \end{aligned}$$

implies that there exists a constant  $\mu$ , a function  $\psi$ , and a Radon measure  $\nu$  such that  $\mu + |\nu|_{\mathcal{M}(\mathcal{S}_T)} > 0$  and

$$\begin{cases} \psi_t + D\psi_{\theta\theta} + A(\theta)\psi = -\nu|_{(0,T) \times \mathcal{S}}, & \text{if } 0 < t < T, \quad \theta \in \mathcal{S}, \\ \psi(T, \theta) = \nu|_{\{T\} \times \mathcal{S}} + \mu e^{-\rho T} \nabla_K \bar{V}(K^*(T, \cdot)), & \text{for } \theta \in \mathcal{S}. \end{cases} \quad (48)$$

Note that the Hamiltonian is given by

$$H(t, \theta, K, c, \mu, \psi) = \mu e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} N(\theta) + \psi(t, \theta) [A(\theta)K - N(\theta)c].$$

Thus, the optimal pair  $\{c^*, K^*\}$  satisfies

$$\begin{aligned} & \mu e^{-\rho t} \frac{c^*(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) + \psi(t, \theta) [A(\theta) K^*(t, \theta) - N(\theta) c^*(t, \theta)] \\ = & \arg \max_{c \geq 0} \left\{ \mu e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} N(\theta) + \psi(t, \theta) [A(\theta) K^*(t, \theta) - N(\theta) c] \right\} \end{aligned}$$

for  $(t, \theta) \in (0, T) \times \mathcal{S}$ . It follows that

$$\mu e^{-\rho t} c^*(t, \theta)^{-\sigma} = \psi(t, \theta) \quad \text{for } (t, \theta) \in (0, T) \times \mathcal{S}. \quad (49)$$

We first show that  $\mu > 0$ . If  $\mu = 0$ , then  $|\nu|_{\mathcal{M}(\mathcal{S}_T)} > 0$ . Therefore, (??) is non-homogeneous. As a result,  $\psi(t, \theta)$  is nontrivial. This contradicts (??). This proves that  $\mu \neq 0$ . Since  $\mu$  is nonnegative by Proposition (??), it follows that  $\mu > 0$ . Using scaling if necessary, we can choose  $\mu = 1$ .

It remains to show that

$$\nabla_K \bar{V}(K^*(T, \cdot)) = \bar{c}(T, \theta)^{-\sigma}. \quad (50)$$

Let  $T' > T$ . Since  $\{\bar{c}, \bar{K}\}$  with  $\bar{c}$  given by (??) and  $\bar{K}$  solving (??) is feasible and optimal, by dynamic programming

$$J(K^*(T, \cdot), \bar{c}) = \sup_{c \geq 0} \int_T^{T'} \int_0^{2\pi} e^{-\rho(t-T)} \frac{c(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt + e^{-\rho(T'-T)} \bar{V}(\bar{K}(T', \cdot)).$$

As a result, since  $\bar{K}(t, \theta) > 0$  for  $(t, \theta) \in (T, T') \times \mathcal{S}_T$ , by Proposition ??, the costate  $\bar{\psi} =: e^{-\rho(t-T)} \bar{c}(t, \theta)^{-\sigma}$  satisfies

$$\begin{cases} \bar{\psi}_t + D\bar{\psi}_{\theta\theta} + A\bar{\psi} = 0, & \text{if } T < t < T', \quad \theta \in \mathcal{S}, \\ \bar{\psi}(T', \theta) = e^{-\rho(T'-T)} \nabla_K \bar{V}(\bar{K}(T', \cdot)), & \text{for } \theta \in \mathcal{S}. \end{cases}$$

This leads to

$$e^{-\rho(T'-T)} \nabla_K \bar{V}(\bar{K}(T', \cdot)) = \bar{\psi}(T', \theta) = e^{-\rho(T'-T)} \bar{c}(T', \theta)^{-\sigma}.$$

Taking  $T' \rightarrow T$  on both sides and using continuity, we obtain (??).

The proof is complete.

#### 4.4 Asymptotic behavior

Finally, in the case where an optimal pair  $\{c^*, K^*\}$  has an eventually positive state  $K^*$ , we can determine its asymptotic behavior by deriving the limit of the detrended capital,  $K^*(t, \theta) e^{-gt}$ .

**Proposition 6.** *Suppose that  $\rho > (1 - \sigma) \lambda_0$  and that  $\{c^*, K^*\}$  is an optimal pair. Also suppose that  $K^* > 0$  for all  $t > T$ ,  $\theta \in \mathcal{S}$ , and that  $\{\bar{c}, \bar{K}\}$  defined by (??) and (??) is feasible for  $t > T$ .*

Let  $n$  be the smallest positive integer such that  $\langle K^*(T, \cdot), \varphi_{n,i} \rangle \neq 0$  for some eigenfunction  $\varphi_{n,i}$  corresponding to eigenvalue  $\lambda_n$ . Then,

$$\rho \leq \lambda_0 - \sigma \lambda_n. \quad (51)$$

Furthermore,

$$\lim_{t \rightarrow \infty} K^*(t, \theta) e^{-gt} = e^{-gT} \left\{ \langle K^*(T, \cdot), \varphi_0 \rangle \varphi_0(\theta) + \sum_{k \geq n} \sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta) \right\} \quad (52)$$

if (??) holds strictly, where

$$\bar{\alpha}_T = (\lambda_0 - g) \frac{\langle K^*(T, \cdot), \varphi_0 \rangle}{\langle f, \varphi_0 \rangle}. \quad (53)$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} K^*(t, \theta) e^{-gt} = e^{-gT} & \left\{ \langle K^*(T, \cdot), \varphi_0 \rangle \varphi_0(\theta) + \sum_i \langle K^*(T, \cdot), \varphi_{n,i} \rangle \varphi_{n,i}(\theta) \right. \\ & \left. + \sum_{k > n} \sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta) \right\} \end{aligned} \quad (54)$$

if (??) holds equally.

**Proof.** By Proposition ??,

$$c^*(t, \theta) = \bar{c}(t, \theta) \equiv \bar{\alpha}_T [\varphi_0(\theta)]^{-1/\sigma} e^{g(t-T)} \quad \text{for } t \geq T \quad (55)$$

Substituting the right-hand side of (??) into (??) and using Fourier series expansion, we obtain

$$\begin{aligned} K^*(t, \theta) = e^{g(t-T)} & \left\{ \langle K^*(T, \cdot), \varphi_0 \rangle \varphi_0(\theta) + \sum_{k \geq n} \sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta) \right\} \\ & + \sum_{k \geq n} \sum_i \left[ \langle K^*(T, \cdot), \varphi_{k,i} \rangle - \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \right] e^{\lambda_k(t-T)} \varphi_{k,i}(\theta). \end{aligned}$$

Since  $K^*(t, \theta) > 0$  for  $t > T$ , and since  $\varphi_{k,i}(\theta)$  changes the sign for  $k \geq 1$ , it follows that  $g \geq \lambda_n$ , which is equivalent to (??).

In the case where  $g > \lambda_n$ ,

$$\begin{aligned} K^*(t, \theta) e^{-gt} &= e^{-gT} \left\{ \langle K^*(T, \cdot), \varphi_0 \rangle \varphi_0(\theta) + \sum_{k \geq n} \sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta) \right\} \\ &\quad + \sum_{k \geq n} \sum_i \left[ \langle K^*(T, \cdot), \varphi_{k,i} \rangle - \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \right] e^{(\lambda_k - g)t - \lambda_k T} \varphi_{k,i}(\theta) \end{aligned}$$

converges to the right-hand side of (??).

In the case where  $g = \lambda_n$ , by Proposition ??,  $g > \lambda_k$  for  $k > n$ . Hence, the above relation leads to

$$\begin{aligned} \lim_{t \rightarrow \infty} K^*(t, \theta) e^{-gt} &= e^{-gT} \left\{ \langle K^*(T, \cdot), \varphi_0 \rangle \varphi_0(\theta) + \sum_{k \geq n} \sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta) \right. \\ &\quad \left. + \sum_i \left[ \langle K^*(T, \cdot), \varphi_{n,i} \rangle - \frac{\bar{\alpha}_T \langle f, \varphi_{n,i} \rangle}{\lambda_n - g} \right] \varphi_{n,i}(\theta) \right\}. \end{aligned}$$

This is the same as (??).

This completes the proof.

Clearly, in the case where  $\{\bar{c}, \bar{K}\}$  is feasible, the results of Proposition ?? hold with  $T = 0$ . Conversely, since any eigenfunctions corresponding to eigenvalues  $\lambda_n$  with  $n \geq 1$  change sign in  $\mathcal{S}$ , it is possible that the right-hand sides of (??) and (??) with  $T = 0$  reaches negative values in  $\mathcal{S}$ . In this case  $\{\bar{c}, \bar{K}\}$  cannot be feasible. This observation leads to the following corollary.

**Corollary 1.** *Suppose  $\rho > (1 - \sigma) \lambda_0$ . Let  $\bar{\alpha}_0$  be defined by (??) and let  $n$  be the smallest positive integer such that  $\langle K_0, \varphi_{n,i} \rangle \neq 0$  for some eigenfunction  $\varphi_{n,i}$  corresponding to eigenvalue  $\lambda_n$ . Then,  $\{\bar{c}, \bar{K}\}$  is not feasible in the following cases:*

1.  $\rho > \lambda_0 - \sigma \lambda_n$ ;
2.  $\rho < \lambda_0 - \sigma \lambda_n$  and the function

$$\theta \mapsto \langle K_0, \varphi_0 \rangle \varphi_0(\theta) + \sum_{k \geq n} \sum_i \frac{\bar{\alpha}_0 \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta)$$

is not positive for all  $\theta \in \mathcal{S}$ ;

3.  $\rho = \lambda_0 - \sigma \lambda_n$  and the function

$$\theta \mapsto \langle K_0, \varphi_0 \rangle \varphi_0(\theta) + \sum_i \langle K_0, \varphi_{n,i} \rangle \varphi_{n,i}(\theta) + \sum_{k \geq n} \sum_i \frac{\bar{\alpha}_0 \langle f, \varphi_{k,i} \rangle}{\lambda_k - g} \varphi_{k,i}(\theta)$$

is not positive for all  $\theta \in \mathcal{S}$ .

The proof follows immediately from Proposition ??.

We note from Item 1 in the above corollary that for any initial capital which is not propotional to the principal eigenfunction  $\varphi_0$ ,  $\{\bar{c}, \bar{K}\}$  is not feasible if  $\rho$  is sufficiently large. Therefore, high time discount rate will lead to depletion of the capital in a future time at certain locations.

## 5 Constant $A$ and $N$

We consider the case where  $A$  and  $N$  are constants. In this case the eigenvalues of  $\mathcal{L}$  are,

$$\lambda_n = A - Dn^2 \quad \text{for } n = 0, 1, 2, \dots \quad (56)$$

and the corresponding normalized eigenfunctions are

$$\varphi_0(\theta) = \frac{1}{\sqrt{2\pi}}, \quad \varphi_{n,1}(\theta) = \frac{\sin n\theta}{\sqrt{\pi}}, \quad \varphi_{n,2}(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}.$$

In addition,  $\bar{c}$  is independent of  $\theta$  and

$$\bar{c}(t) = \frac{A - g}{2\pi N} e^{gt} \int_0^{2\pi} K_0(\theta) d\theta. \quad (57)$$

We first show that  $\{\bar{c}, \bar{K}\}$  is feasible if  $K_0$  is constant.

**Proposition 7.** *Suppose that  $A$  and  $N$  are constants, that  $\rho > (1 - \sigma)A$ , and that  $K_0$  is a constant. Then  $\{\bar{c}, \bar{K}\}$  given by*

$$\bar{c}(t) = \frac{K_0}{N} (A - g) e^{gt}, \quad \bar{K}(t) = K_0 e^{gt} \quad \text{for all } t \geq 0$$

*is feasible and optimal.*

**Proof.** Since  $A$ ,  $N$ , and  $K_0$  are constant, by (??),

$$\bar{c}(t) = \frac{K_0}{N} (A - g) e^{gt}.$$

Substituting it into (??) and noting that  $\bar{K}$  depends only on  $t$ , we see that

$$\frac{d\bar{K}}{dt} = A\bar{K} - K_0 (A - g) e^{gt}, \quad \bar{K}(0) = K_0.$$

It is easy to see that the solution is  $\bar{K}(t) = K_0 e^{gt}$ . This completes the proof.

In the case where  $K_0(\theta)$  is not constant, we obtain from Propositions ?? and ?? that

**Corollary 2.** Suppose  $\rho > (1 - \sigma)A$  and that  $\{\bar{c}, \bar{K}\}$  is feasible. Let  $n$  be the smallest positive integer such that

$$\int_0^{2\pi} K_0(\theta) \cos n\theta d\theta \neq 0 \text{ or } \int_0^{2\pi} K_0(\theta) \sin n\theta d\theta \neq 0. \quad (58)$$

Then

$$\rho \leq (1 - \sigma)A + \sigma Dn^2. \quad (59)$$

Furthermore,

$$\lim_{t \rightarrow \infty} K^*(t, \theta) e^{-gt} = \frac{1}{2\pi} \int_0^{2\pi} K_0(\theta) d\theta$$

if (??) holds strictly and

$$\begin{aligned} \lim_{t \rightarrow \infty} K^*(t, \theta) e^{-gt} &= \frac{1}{2\pi} \int_0^{2\pi} K_0(\theta) d\theta + \frac{1}{\pi} \int_0^{2\pi} K_0(\theta) \sin n\theta d\theta \sin n\theta \\ &\quad + \frac{1}{\pi} \int_0^{2\pi} K_0(\theta) \cos n\theta d\theta \cos n\theta \end{aligned}$$

if (??) holds equally.

To prove, it suffices to observe that the function  $f$  in (??) is constant. Therefore,  $\langle f, \varphi_{k,i} \rangle = 0$  for any  $k \geq 1$ . Also, notice that since  $\{\bar{c}, \bar{K}\}$  is feasible,  $T = 0$ .

The above result also shows that for any non-constant initial capital distribution,  $K_0(\theta)$ ,  $K(t, \theta)$  will reach zero at some future time and location if  $\rho$  is sufficiently large (equivalently, if  $D$  is sufficiently small).

**Corollary 3.** Let  $\{c^*, K^*\}$  is an optimal pair, and let  $n$  be the smallest positive integer such that (??) is true. If either

$$\rho > (1 - \sigma)A + \sigma Dn^2 \quad (60)$$

or

$$\rho = (1 - \sigma)A + \sigma Dn^2$$

and the function

$$\begin{aligned} K_\infty(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} K_0(\theta) d\theta + \frac{1}{\pi} \int_0^{2\pi} K_0(\theta) \sin n\theta d\theta \sin n\theta \\ &\quad + \frac{1}{\pi} \int_0^{2\pi} K_0(\theta) \cos n\theta d\theta \cos n\theta \end{aligned}$$

is not nonnegative everywhere in  $\mathcal{S}$ , then  $\{\bar{c}, \bar{K}\}$  is not feasible. That is, there exists  $(t, \theta) \in (0, \infty) \times \mathcal{S}$  at which  $K^*(t, \theta) = 0$ .

This result follows immediately from Corollary ???. Since (??) does not hold,  $\{\bar{c}, \bar{K}\}$  is not feasible.



Hence,  $c^* \neq \bar{c}$  for some  $(t, \theta) \in (0, \infty) \times \mathcal{S}$ . Therefore, the costate  $\psi$  does not satisfy the first equation in (??) for all  $(t, \theta)$ . As a result,  $\nu \neq 0$ . In view of Remark ??, the set  $\{(t, \theta) : K^*(t, \theta) = 0\}$  is not empty.

**Remark 2.** *In general, a non-constant function has non-zero inner products with eigenfunctions corresponding to  $\lambda_1$ . Therefore, from Corollary ?? we see that if*

$$\rho > (1 - \sigma)A + \sigma D,$$

*then  $\{\bar{c}, \bar{K}\}$  is generally not feasible. This means the optimal trajectory of the state,  $K^*(t, \theta)$ , is generally not eventually positive. I.e., there exists a sequence  $(t_k, \theta_k)$  with  $t_k \rightarrow \infty$  and  $\theta_k \in \mathcal{S}$  such that  $K^*(t_k, \theta_k) = 0$  for each  $k$ .*

## 6 Numerical examples

In this section we illustrate results from the previous sections. In a first subsection, we reconsider the numerical examples in Boucekkine et al. (2013) in which the pair  $\{\bar{c}, \bar{K}\}$  is feasible. Our results confirms theirs in the case where the time discount is low, and extend theirs when the time discount is high. In the second subsection, we examine a case where the pair  $\{\bar{c}, \bar{K}\}$  is not feasible, but there exists an optimal pair  $\{c^*, K^*\}$  with  $K^*$  eventually positive. Therefore, there is an initial period in which the capital trajectory along the optimal control reaches zero.

In both subsections we use the same parameter values as in Boucekkine *et al.* (2013),

$$A = 1/3, \quad D = 1, \quad N = 1, \quad \sigma = 0.8$$

### 6.1 Cases with $\{\bar{c}, \bar{K}\}$ feasible

We use the same initial capital as in Boucekkine *et al.* (2013),

$$K_0(\theta) = \begin{cases} 20, & \text{if } 0 \leq \theta < \pi, \\ 10, & \text{if } \pi \leq \theta < 2\pi. \end{cases} \quad (61)$$

In this case,  $\lambda_0 = 1/3$  is the only positive eigenvalue of the operator  $\mathcal{L}$ ,  $(1 - \sigma)\lambda_0 = 0.0667$ , and

$$\langle K_0, \varphi_0 \rangle = 15\sqrt{2\pi}, \quad \langle f, \varphi_0 \rangle = (2\pi)^{(1+\sigma)/2\sigma}.$$

As in Boucekkine *et al.* (2013), we run two exercises. One with  $\rho = 0.07$  which is slightly greater than the lower bound  $(1 - \sigma)A$ . The other with  $\rho$  at the upper bound  $(1 - \sigma)A + \sigma D = 0.8667$ .

**Case 1: Low discount rate.** With  $\rho = 0.07$ , we find  $g = 0.3292$  and  $\bar{\alpha} = 0.0198$ . Computations show that  $\{\bar{c}, \bar{K}\}$  is feasible detrended capital  $K(t, \theta) e^{-gt}$  converges to

$$\frac{1}{2\pi} \int_0^{2\pi} K_0(\theta) = 15 \quad \text{for } \theta \in \mathcal{S}$$

as time tends to infinite. Figure ?? shows the evolution of detrended capital. Note that very fast, initial disparities vanish and detrended capital reaches a constant value.

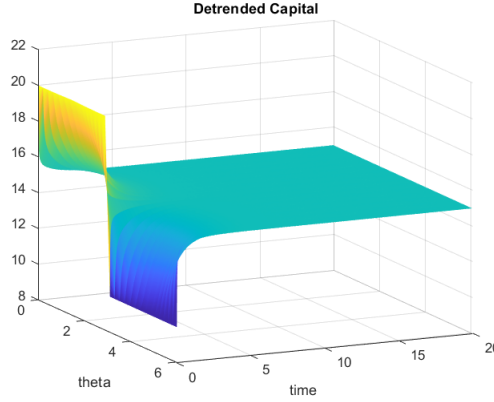


Figure 1: Low discount rate,  $\rho = 0.07$ .

**Case 2: High discount rate.** With  $\rho = 0.8667$ , we find  $g = -2/3$  and  $\bar{\alpha} = 4.7559$ . Computations shows that  $\{\bar{c}, \bar{K}\}$  is again feasible. The detrended capital converges to

$$K_D(\theta) = 15 + \frac{20}{\pi} \sin \theta$$

as predicted by Corollary ??. The evolution of the detrended capital is showing in Fig. ??

## 6.2 A case with $\{\bar{c}, \bar{K}\}$ not feasible

We use the initial capital

$$K_0(\theta) = \begin{cases} 10, & \text{if } 0 \leq \theta < \pi, \\ 0, & \text{if } \pi \leq \theta < 2\pi, \end{cases}$$

and choose  $\rho = 0.3$ . Computation shows that  $\{\bar{c}, \bar{K}\}$  is not feasible with  $\bar{K}$  taking negative values in the period  $0 \leq t \leq 0.3$ . An optimal pair  $\{c^*, K^*\}$  satisfies

$$\begin{cases} K_t^* - DK_{\theta\theta}^* = AK^* - Nc^*, & \text{for } 0 < t < T, \quad \theta \in \mathcal{S}, \\ K^*(0, \theta) = K_0(\theta) & \text{for } \theta \in \mathcal{S} \end{cases} \quad (62)$$

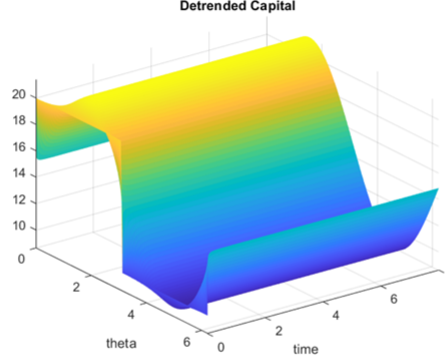


Figure 2: High discount rate  $\rho = 0.8667$ .

where  $T > 0$  such that  $K^*(t, \theta) > 0$  for  $t \geq T$ . The costate

$$\psi^*(t, \theta) = c^*(t, \theta)^{-\sigma} e^{-\rho t} \quad (63)$$

is a solution to the terminal-value problem

$$\begin{cases} \psi_t^* + D\psi_{\theta\theta}^* + A\psi^* = -\nu|_{(0,T) \times \mathcal{S}}, & \text{if } 0 < t < T, \quad \theta \in \mathcal{S}, \\ \psi^*(T, \theta) = \psi_T(\theta), & \text{for } \theta \in \mathcal{S}. \end{cases} \quad (64)$$

where, by Proposition ??,

$$\psi_T(\theta) = e^{-\rho T} \left[ (\lambda_0 - g) \int_0^{2\pi} K^*(T, \theta) d\theta \right]^{-\sigma} \quad (65)$$

and  $\nu$  is the Gâteaux derivative of the functional

$$d_0(K^*) = |\min\{K^*, 0\}|_{C(\mathcal{S}_T)}.$$

(Note that  $\nu$  does not show up in the terminal condition because of the relation (??) and  $K^*(T, \theta) > 0$  in  $\mathcal{S}$ .) System (??)–(??) form a couple system that can be represented by operators

$$K^* = \mathcal{P}[\psi^*], \quad \psi^* = \mathcal{Q}[K^*] \quad (66)$$

where  $\mathcal{P}[\psi]$  is the solution of (??) with  $c^* = [\psi e^{\rho t}]^{-1/\sigma}$ , and  $\mathcal{Q}[K]$  is the solution of (??) with  $\nu$  the Gâteaux derivative of  $d_0(K)$  and  $\psi_T$  given by (??) with  $K^*$  replaced by  $K$ . Thus,  $(K^*, \psi^*)$  is a fixed point of the operator  $(K, \psi) \mapsto (\mathcal{P}[\psi], \mathcal{Q}[K])$ .

We obtain a numerical approximation of the solution  $(K^*, \psi^*)$ . Computation shows that  $K^*(t, \theta) > 0$  for  $t \geq T \approx 0.26$ . Graphs of  $K^*$  and  $c^*$  for  $0 \leq t \leq 0.4$  are showing in Fig.??.

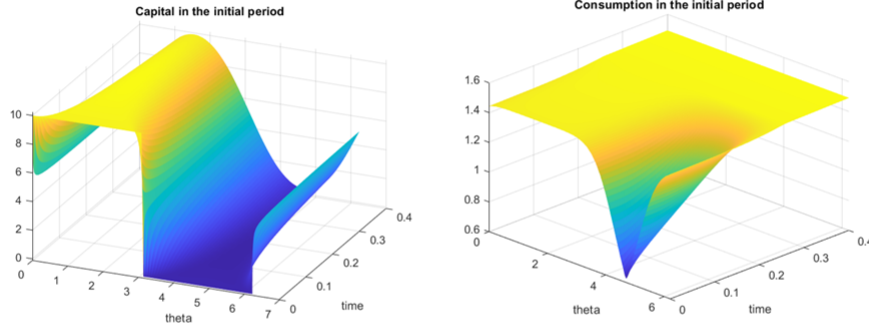


Figure 3: A case with  $\{\bar{c}, \bar{K}\}$  not feasible. Initial period is roughly  $0 \leq t \leq 0.26$ . Left: the capital in the initial stage. Right: the consumption in the initial stage.

As can be seen, unlike the case where  $\{\bar{c}, \bar{K}\}$  is feasible, the consumption  $c^*$  in the initial period is not spatially uniform. Specifically  $c^*$  is lower near locations where  $K^*$  is near zero.

## Appendices

### Proof of Lemma ??

By the definition of  $c_\delta$  in (??),  $d(c_\delta, \bar{c}) \leq |E_\delta|$ . Let

$$z_\delta(t, \theta) = \frac{1}{\delta} [y(t, \theta; c_\delta) - \bar{y}(t, \theta)] \quad \text{in } \mathcal{S}_T.$$

Then,  $z_\delta$  satisfies

$$\begin{aligned} (z_\delta)_t - [a(z_\delta)_\theta]_\theta &= b_\delta(t, \theta) z_\delta(t, \theta) + \frac{1}{\delta} \chi_{E_\delta}(t, \theta) \phi(t, \theta) & \text{if } t \in (0, T), \theta \in \mathcal{S}, \\ z_\delta(0, \theta) &= 0 & \text{if } \theta \in \mathcal{S}, \end{aligned}$$

where

$$b_\delta(t, \theta) = \int_0^1 f_y(t, \theta, \bar{y}(t, \theta) + s(y(t, \theta, c_\delta(t, \theta)) - \bar{y}(t, \theta)), c_\delta(t, \theta)) ds,$$

and  $\chi_{E_\delta}$  is the characteristic function of  $E_\delta$ . From the regularity of the parabolic equation (??) and Assumption ??, we see that  $b_\delta$  and  $\phi$  are uniformly bounded. Hence, by the Hölder's estimate there is  $\alpha \in (0, 1)$  such that

$$|y(\cdot, c_\delta) - \bar{y}|_{C^{\alpha, \alpha/2}(\mathcal{S}_T)} \leq C |\chi_{E_\delta}|_{L^p(\mathcal{S}_T)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0,$$

where the constant  $C$  is independent of  $E_\delta$ . As a result,

$$b_\delta(t, \theta) \rightarrow f_y(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)) \quad \text{in } L^p(\mathcal{S}_T) \text{ as } \delta \rightarrow 0$$

for any  $p$  such that  $1 \leq p < \infty$ . Comparing equations for  $z_\delta$  and  $z$  we derive

$$\begin{aligned} (z_\delta - z)_t - [a(z_\delta - z)_\theta]_\theta &= b_\delta(t, \theta)(z_\delta - z) + (b_\delta(t, \theta) - f_y(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)))z \\ &\quad - \left(1 - \frac{1}{\delta}\chi_{E_\delta}(t, \theta)\right)\phi(t, \theta), \\ (z_\delta - z)(0, \theta) &= 0. \end{aligned}$$

Using Lemma 3.2 in B. Hu and J. Yong (1995), we see that  $|z_\delta - z|_{C^{\alpha, \alpha/2}(\mathcal{S}_T)} \rightarrow 0$  as  $\delta \rightarrow 0$ . This proves the first relation in (??).

To prove the second relation in (??), we let

$$\begin{aligned} l_\delta &= \frac{1}{\delta} [J(c_\delta) - J(\bar{c})] \\ &= \frac{1}{\delta} \left\{ \int_0^T \int_{\mathcal{S}} [g(t, \theta, y(t, \theta, c_\delta(t, \theta)), c_\delta(t, \theta)) - g(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta))] d\theta dt \right. \\ &\quad \left. + [h(T, y(T, \cdot, c_\delta(T, \cdot))) - h(T, \bar{y}(T, \cdot))] \right\}. \end{aligned}$$

By the definition of  $c_\delta$  and (??), it follows that

$$\begin{aligned} l_\delta &= \int_0^T \int_{\mathcal{S}} \left[ \beta_\delta(t, \theta) z_\delta(t, \theta) + \frac{1}{\delta} \chi_{E_\delta}(t, \theta) \gamma(t, \theta) \right] d\theta dt \\ &\quad + \int_{\mathcal{S}} \eta_\delta(\theta) z_\delta(T, \theta) d\theta + \frac{1}{\delta} \int_{\mathcal{S}} \chi_{E_\delta}(t, \theta) \gamma(t, \theta) d\theta, \end{aligned}$$

where

$$\begin{aligned} \beta_\delta(t, \theta) &= \int_0^1 g_y(t, \theta, \bar{y}(t, \theta) + s(y(t, \theta, c_\delta(t, \theta)) - \bar{y}(t, \theta)), c_\delta(t, \theta)) ds, \\ \eta_\delta(\theta) &= \int_0^1 h_y(T, \bar{y}(T, \theta) + s(y(T, \theta, c_\delta(T, \theta)) - \bar{y}(T, \theta))) ds. \end{aligned}$$

It is clear that

$$\beta_\delta(t, \theta) \rightarrow g_y(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)), \quad \eta_\delta(\theta) \rightarrow h_y(T, \bar{y}(T, \theta))$$

as  $\delta \rightarrow 0$ .

Comparing equations for  $l_\delta$  and  $l$ , we find

$$\begin{aligned} l_\delta - l &= \int_0^T \int_{\mathcal{S}} [\beta_\delta(t, \theta) z_\delta(t, \theta) - g_y(t, \theta, \bar{y}(t, \theta), \bar{c}(t, \theta)) z(t, \theta)] d\theta dt \\ &\quad + \int_{\mathcal{S}} [\eta_\delta(\theta) z_\delta(T, \theta) - h_y(T, \bar{y}(T, \theta)) z(T, \theta)] d\theta \\ &\quad - \int_0^T \int_{\mathcal{S}} \left(1 - \frac{1}{\delta} \chi_{E_\delta}(t, \theta)\right) \gamma(t, \theta) d\theta dt. \end{aligned}$$

Using again Lemma 3.2 in B. Hu and J. Yong (1995) and the convergence  $z_\delta \rightarrow z$  in  $C^{\alpha, \alpha/2}(\mathcal{S}_T)$ , we find  $l_\delta - l \rightarrow 0$  as  $\delta \rightarrow 0$ .

This completes the proof of the lemma.

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