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INDETERMINACY AND BANKRUPTCY : A NOTE

Jean-Marc TALLON

MAD

Indeterminacy and bankruptcy: a note

Jean-Marc Tallon*

October 1993

Preliminary

Abstract

It is shown in a model where agents can have different support for their expectations of states of the world, that the real indeterminacy result usual in the literature on incomplete markets holds true. This is shown on a simple example. Since in this type of models bankruptcy might occur, the result is further characterized. It is shown that equilibria without bankruptcy are indeterminate, whereas equilibrium allocations with bankruptcy are locally isolated although the rate of default is indeterminate.

1 Introduction

The formulation by Arrow [1953] and then Radner [1972] of a sequential economy was meant to capture in a more realistic way the intertemporal nature of economic and financial transactions, that were previously treated in a fundamentally static way through the notion of contingent goods (see Debreu [1959]). This intertemporal framework is, among other things, appropriate to deal with the notion of bankruptcy. Such a phenomenon could occur if agents do not agree *ex ante* on the set of possible states of the world, but are allowed to trade in assets that have non-zero returns in those states. Indeed, an “ignorant” agent can find himself with a debt superior to its wealth in the states he did not forecast. In that case, he defaults on (part of) his debt, and his creditor does not get the entire amount of her rights. Such a model is studied in Tallon [1993], where existence is shown under conditions on agents’ assets valuation, which cannot be “too different”.

The present note studies the issue of indeterminacy of equilibrium in a setup where agents have different views of the future. It is known since Cass [1985], Balasko and Cass [1989], and Geanakoplos and Mas-Colell [1989] that real indeterminacy of

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equilibrium is the rule in models of incomplete financial markets in which assets pay in units of account and agents place positive probability on the same set of states of the world. The question is then to know whether this property generalizes to a setting where agents' expectations do not have the same support. The issue of indeterminacy of such an equilibrium will be studied in a very simple case, *i.e.*, a two-period model with two agents, two states in the second period, one good *per spot* and inside money. One agent expects both states with positive probability, while the other places a zero probability on one of the state occurring.

In this simplified setup, it is shown that equilibrium is typically indeterminate. A further characterization is possible. There is a continuum of equilibria at which no bankruptcy occurs, while equilibrium allocations with default are locally unique. However, in the latter case, the level of bankruptcy is indeterminate. In other words, the same equilibrium allocation can be reached for a continuum of default rate, the price in the state where default occurs adjusting so that the good allocation does not change. An interesting outcome of the analysis is to show that there always exist equilibria with some default if the agent who is not expecting a state is a debtor. Intuitively, they are characterized by relatively low levels of spot prices in the state where default occurs.

Finally, still in parallel with the literature on incomplete markets (Cass [1989], Suda, Tallon and Villanacci [1992], Cass [1992], Pietra [1992]), one can wonder whether there exist sunspot equilibria, and how many there are. In the present setup, it is shown that sunspot equilibria exist and are indeterminate. Furthermore, as in the case of intrinsic uncertainty, equilibria at which bankruptcy occurs are locally unique, with still a nominal indeterminacy resulting in the default rate being indeterminate at this equilibria. This result points out that it is enough that one agent believes sunspots matter to actually obtain that the equilibrium allocation will depend upon the value of the sunspot.

Thus, the indeterminacy results obtained in the standard incomplete market framework seem to hold in this somewhat more general setting. Section 2 presents the model. Indeterminacy results are then stated and discussed in section 3, while concluding comments are gathered in section 4.

2 The model

The model studied here is a particular case of the one developed in Tallon [1993]. Two agents, $h = 1, 2$, exchange one good x on each spot market ($s = 0, \alpha, \beta$) and use inside money, b , to transfer wealth from one period to another.

Agent 1 is assumed to expect both states with strictly positive probability, while agent 2 expects only state α , putting a zero-probability on the occurrence of state β . The latter agent can therefore default in state β . In that case, he repays only his debt up to the amount of the value of his endowment, and his creditor only gets a fraction K of her rights.

Agent h 's consumption set is X_h . Her utility function is assumed to be separable across states and is denoted u_h^s , $s = \alpha, \beta$. Her endowment is $e_h = (e_h(0), e_h(\alpha), e_h(\beta))$. The following set of assumptions puts some structure on agents' preferences and endowments.

Assumption 1 :

- (i) $X_h = \mathbb{R}_+^C$,
- (ii) $e_h \in \mathbb{R}_{++}^3$,
- (iii) u_h^s is continuous, strictly increasing, and strictly concave.

Let $b_h^+ \geq 0$ denote purchases and $b_h^- \geq 0$ sales of inside money, h 's portfolio being $b_h = b_h^+ - b_h^-$. Let $p(s)$ be the price of the good in state s , and q the price of inside money. If agent 2 defaults in state β (in which case $b_2^+ = 0$ and $b_2^- > 0$), the fraction of his debt he reimburses is $K = \frac{p(\beta)e_2(\beta)}{b_2^-}$, i.e., it is the ratio of what the agent can pay (the value of his endowment) to the value of his debt.

Therefore, one can state the problems agents have to solve. Agent 1 faces the usual problem, except for the fact that she potentially does not get the full amount of what she is entitled to in state β .

$$\begin{aligned} & \max_{(x_1(0), x_1(\alpha), x_1(\beta), b_1)} \quad u_1^\alpha(x_1(0), x_1(\alpha)) + u_1^\beta(x_1(0), x_1(\beta)) \\ \text{s.t.} \quad & \begin{cases} p(0)(x_1(0) - e_1(0)) & = & -q(b_1^+ - b_1^-) \\ p(\alpha)(x_1(\alpha) - e_1(\alpha)) & = & b_1^+ - b_1^- \\ p(\beta)(x_1(\beta) - e_1(\beta)) & = & K b_1^+ - b_1^- \\ x_1(s) \in X_1 & \text{for } & s = 0, \alpha, \beta \\ b_1^+ \geq 0 & \text{and } & b_1^- \geq 0 \end{cases} \end{aligned} \quad (P_1)$$

Agent 2 has two problems to solve. The *ex ante* problem, which writes:

$$\begin{aligned} & \max_{(x_2(0), x_2(\alpha), b_2)} u_2^\alpha(x_2(0), x_2(\alpha)) \\ \text{s.t.} & \begin{cases} p(0)(x_2(0) - e_2(0)) = -q(b_2^+ - b_2^-) \\ p(\alpha)(x_2(\alpha) - e_2(\alpha)) = b_2^+ - b_2^- \\ x_2(s) \in X_2 \quad \text{for } s = 0, \alpha \\ b_2^+ \geq 0 \quad \text{and} \quad b_2^- \geq 0 \end{cases} \end{aligned} \quad (P_2)$$

while the *ex post* program, *i.e.* the problem agent 2 has to solve had state β occurred, is given by:

$$\begin{aligned} & \max_{x_2(\beta)} u_2^\beta(\bar{x}_2(0), x_2(\beta)) \\ \text{s.t.} & \begin{cases} p(\beta)(x_2(\beta) - e_h(\beta)) = \bar{b}_2^+ - \min(p(\beta)e_2(\beta) + \bar{b}_2^+, \bar{b}_2^-) \\ x_2(\beta) \in X_2 \end{cases} \end{aligned} \quad (\bar{P}_2)$$

In this program, the right hand side of the budget constraint is equal to \bar{b}_2^+ if it is positive, *i.e.* if the agent transfers positive wealth from period 0 to period 1, in which case no default occurs. It is equal to $\min(p(\beta)e_2(\beta), \bar{b}_2^-)$ if $\bar{b}_2^- > 0$. This term is the minimum of what agent 2 can repay and has to repay. Given the assumption of the model that an agent always repays its debt when he can, this minimum is equal to what will actually be reimbursed. If it is equal to $p(\beta)e_2(\beta)$, agent 2 is bankrupt and repays its debt up to the value of his endowments.

One can then define an equilibrium of this model.

Definition 1 : $((p, q, K), (x, b)) \in \mathbb{R}_+^3 \times \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+^6 \times \mathbb{R}^2$ is an equilibrium if:

- (i) given (p, q, K) , $(x_1(0), x_1(\alpha), x_1(\beta))$ solves (P_1) , and $(x_2(0), x_2(\alpha))$ solves (P_2) , and $x_2(\beta)$ solves (\bar{P}_2) ,
- (ii) $\sum_h (x_h - e_h) = 0$ and $\sum_h b_h = 0$ and,
- (iii) $K = \frac{\min(p(\beta)e_2(\beta), \bar{b}_2^-)}{\bar{b}_2^-}$ if $\bar{b}_2^- > 0$, and $K = 1$ otherwise.

The price of inside money will be normalized to 1 in the rest of the note without loss of generality. In the present setup there always exists an equilibrium (see Tallon [1993]). Whether this equilibrium is locally unique or indeterminate is now studied.

3 Indeterminacy of equilibrium

In order to show that there exists a continuum of equilibria in this economy, one has to define an auxiliary economy. In this auxiliary economy agent 1 solves the following program:

$$\begin{aligned} \max_{x_1(0), x_1(\alpha)} \quad & u_1^\alpha(x_1(0), x_1(\alpha)) + u_1^\beta(x_1(0), e_1(\beta) + \frac{p(\alpha)}{p(\beta)}K(x_1(\alpha) - e_1(\alpha))) \\ \text{s.t.} \quad & p(0)(x_1(0) - e_1(0)) + p(\alpha)(x_1(\alpha) - e_1(\alpha)) = 0 \end{aligned}$$

while agent 2 solves:

$$\begin{aligned} \max_{x_2(0), x_2(\alpha)} \quad & u_2^\alpha(x_2(0), x_2(\alpha)) \\ \text{s.t.} \quad & p(0)(x_2(0) - e_2(0)) + p(\alpha)(x_2(\alpha) - e_2(\alpha)) = 0 \end{aligned}$$

Thus, this economy has complete markets, and consequently an equilibrium exists. Observe that it is parametrized by K and $\theta = \frac{p(\alpha)}{p(\beta)}$. The following propositions state that an equilibrium of the auxiliary economy is an equilibrium of the original economy, and that the equilibrium allocation varies with the parameter θ .

Proposition 1 *Let $K = 1$ and $(p(0), p(\alpha))$ be an equilibrium price vector of the auxiliary economy θ . Then, $(p(0), p(\alpha), \frac{p(\alpha)}{\theta})$, together with $K = 1$, is an equilibrium price vector of the original economy if $\theta \in (0, \min_h \frac{e_h(\beta)}{e_h(\alpha)})$.*

Proof : Define $x_h(\beta) = \theta(x_h(\alpha) - e_h(\alpha)) + e_h(\beta)$, $b_h^+ = \max(0, p(\alpha)(x_h(\alpha) - e_h(\alpha)))$ and $b_h^- = \max(0, p(\alpha)(e_h(\alpha) - x_h(\alpha)))$. Observe that if $\theta \in (0, \min_h \frac{e_h(\beta)}{e_h(\alpha)})$, then $x_h(\beta) > 0$, or in other terms, $\max(0, p(\alpha)(e_h(\alpha) - x_h(\alpha))) < p(\beta)e_h(\beta)$.

Now, by construction of the optimization problems in the auxiliary economy, if $(x_h(0), x_h(\alpha))$ is a solution to h 's program, $(x_h(0), x_h(\alpha), x_h(\beta))$, where $x_h(\beta)$ is defined above, is a solution to the programs in the original economy. Observe also that market clearing conditions in the auxiliary economy imply that markets clear in state β as well. Finally, there is no bankruptcy since $\frac{\min((p(\alpha)/\theta)e_2(\beta), b_2^-)}{b_2^-} = 1$ if $\theta \in (0, \min_h \frac{e_h(\beta)}{e_h(\alpha)})$. \diamond

The following proposition shows through the study of the properties of equilibria of the auxiliary economy, that equilibria of the original economy are indeterminate.

Proposition 2 *Let $K = 1$ and $\theta \in (0, \min_h \frac{e_h(\beta)}{e_h(\alpha)})$. Then, there exists an equilibrium of the auxiliary economy. If the endowment vector is not Pareto optimal, the equilibrium allocation varies with θ .*

Proof : Existence of an equilibrium is standard. Observe that if e is not Pareto-optimal there exists at most one θ such that e is an equilibrium of that economy (see e.g. Portier and Tallon [1993]). Now, let $\hat{\theta}, \tilde{\theta} \in (0, \min_h \frac{e_h(\beta)}{e_h(\alpha)})$. If $\widehat{x}_h(\alpha) \neq \widetilde{x}_h(\alpha)$, then the result is shown. If $\widehat{x}_h(\alpha) = \widetilde{x}_h(\alpha)$, then $\widehat{x}_h(\alpha) \neq e_h(\alpha)$ or $\widetilde{x}_h(\alpha) \neq e_h(\alpha)$, since otherwise e would be an equilibrium of both economies, contradicting the fact that there is at most one θ such that e is an equilibrium of the economy parametrized by it. Therefore, one has $\widehat{x}_h(\alpha) - e_h(\alpha) = \widetilde{x}_h(\alpha) - e_h(\alpha) \neq 0$. This in turn implies that $\widehat{x}_h(\beta) \neq \widetilde{x}_h(\beta)$. Consequently, the equilibrium allocation is different in both economies. \diamond

Combining the two propositions, one gets that, if the endowment is not Pareto-optimal, there is a continuum of equilibria in the original economy. Therefore, there is one degree of real indeterminacy in this model, as it would be the case in the model where both agents expect both states with strictly positive probability. It is worth noticing that there is no bankruptcy at any of the equilibria of the continuum exhibited. Indeed, the bounds on θ ensure that agent 2 will not default if state β occurs. Intuitively, it is clear that for small θ , $p(\beta)$ is relatively high, meaning that the real value of agent 2's debt is low in state β . Therefore he can reimburse the integrality of it. One can then wonder whether there is also a continuum of equilibria with bankruptcy. The following proposition shows that this is not the case in this example.

Proposition 3 *There is a finite number of equilibrium allocations with bankruptcy.*

Proof : An allocation (x_1, x_2) is an equilibrium allocation with bankruptcy associated to $(p(0), p(\alpha), p(\beta), K)$ if and only if it satisfies the following:

(i) $x_1(\beta) = e_1(\beta) + e_2(\beta)$ and $x_2(\beta) = 0$,

(ii) $(x_1(0), x_1(\alpha))$ is a solution to:

$$\max_{x_1(0), x_1(\alpha)} u_1^\alpha(x_1(0), x_1(\alpha)) + u_1^\beta(x_1(0), e_1(\beta) + e_2(\beta))$$

$$\text{s.t. } p(0)(x_1(0) - e_1(0)) + p(\alpha)(x_1(\alpha) - e_1(\alpha)) = 0$$

and $(x_2(0), x_2(\alpha))$ is a solution to:

$$\max_{x_2(0), x_2(\alpha)} u_2^\alpha(x_2(0), x_2(\alpha))$$

$$\text{s.t. } p(0)(x_2(0) - e_2(0)) + p(\alpha)(x_2(\alpha) - e_2(\alpha)) = 0$$

$$(iii) \quad p(\beta)e_2(\beta) \leq p(\alpha)(e_2(\alpha) - x_2(\alpha)) \text{ and } K = \frac{p(\beta)e_2(\beta)}{p(\alpha)(e_2(\alpha) - x_2(\alpha))} ,$$

$$(iv) \quad \sum_h x_h = \sum_h e_h.$$

Therefore the problem is in some sense dichotomic. On the one hand, the allocation at spot 0 and α is determined through the solution of a complete market equilibrium, while, on the other hand, the assumption of bankruptcy gives the equilibrium allocation at spot β . Consequently, since equilibrium allocations in a complete market economy are in finite number (see e.g. Debreu [1970]), there are a finite number of equilibrium allocations with bankruptcy. \diamond

This result could seem surprising. It is due to the particular structure of the problem, for once the equilibrium problem on spot β is solved (and it can be solved independently since there are only two agents and one bond), the problem reduces to a problem with complete markets. This is what permits the proof. Were there a third state expected by both households, there still would be some indeterminacy, even though one would get two degrees of real indeterminacy for equilibria with no default but only one degree for equilibria involving bankruptcy. Furthermore, in a model with more than two agents, bankruptcy of one of the agent would not be enough to determine the equilibrium allocation at spot β .

Another fact comes from the proof of local uniqueness. Observe that both $p(\beta)$ and K are determined on the spot β market. In condition (iii), $x_2(\alpha)$, the equilibrium consumption of agent 2 in state α , does not depend on K or $p(\beta)$. It can be rewritten:

$$\theta K = \frac{e_2(\beta)}{e_2(\alpha) - x_2(\alpha)}$$

Consequently, only the product θK is determined in equilibrium. There is therefore some nominal indeterminacy, that translates in an indeterminacy of the default rate. In other words, there is a continuum of equilibrium default rate $K < 1$, all leading to the same real allocation, the price $p(\beta)$ adjusting itself to the level of K .

Rewriting the above equilibrium condition in still another way,

$$p(\beta) = \frac{Kp(\alpha)(e_2(\alpha) - x_2(\alpha))}{e_2(\beta)}$$

one can see that the higher $e_2(\beta)$ at K given, the lower $p(\beta)$. This says that for large endowment in state β , the price in that state has to be very low if bankruptcy is observed, for it makes the real value of the debt very large. Similarly, $p(\beta)$ and

K move in the same direction, meaning that relatively high inflation prevents large amount of default.

Finally, it is worth noticing that if $b_2^- > 0$, there always exists an equilibrium at which bankruptcy occurs. This comes from the fact that if $b_2^- > 0$, that is to say if $p(\alpha)(e_2(\alpha) - x_2(\alpha)) > 0$, then one can always find a (possibly very small) value of $p(\beta)$ and of $K < 1$ such that an equilibrium exists.

To conclude on that issue, if there is an equilibrium at which agent 2 defaults, then there is a continuum of default rates giving rise to the same equilibrium allocation, the latter being locally isolated.

Another natural issue that arises is the one of sunspot equilibrium. It might have been noticed that use was never made of the assumption that endowments are different across states. Actually, all the results previously obtained hold true when the economy is of the sunspot type. In particular, there is a continuum of equilibria at which sunspot matters. Furthermore, if agent 2 is a debtor there exists an equilibrium where default occurs. This shows that, not very surprisingly, bankruptcy may be caused by factors that are not related to the fundamentals of the economy. It is easy to see that such cases occur for values of θ greater than one, implying a higher inflation rate in state α than in state β . The real value of agent 2's debt is, in that case, higher in state β , which explains why, even though he has the same endowment as in state α , he can not repay his debt. This result points out that it is not necessary for the sunspot to be commonly expected to have real effects. Indeed, in the present framework, it is enough that one agent believes in sunspots to get that the equilibrium allocation is different according to the value of the sunspot. There is therefore no need of any "social recognition" of the sunspot to ensure it has some real effect.

4 Concluding remarks

The example studied in this note is admittedly rather particular. Some of the assumptions could be relaxed at no cost, such as the assumption of a unique good. However, some of the results are very dependent on this particular structure.

It has been shown, on this simple example, that the real indeterminacy result, usual in the literature on incomplete financial markets with nominal assets, carries over in the case where agents can have different support for their expectations of exogenous variables, *i.e.* states of the world. However, the real indeterminacy exhibited

concerned only equilibria at which no default is observed.

Nevertheless, there is, in this simplistic example, some nominal indeterminacy at equilibria where bankruptcy occurs, in the sense that the rate of default is indeterminate, although the equilibrium allocation is locally isolated. This result is very dependent on the structure of the example, which reduces to a complete market economy when default occurs. A plausible conjecture is that in a more general setting, the real indeterminacy result holds true (both at equilibrium with and without bankruptcy) for a “large” class of economies (an economy being characterized by utility functions, endowments, and a description of the states expected by each agent). However, existence of equilibria with bankruptcy is not ensured, and it is not clear what conditions on the fundamentals of the economy might imply the existence of equilibrium with bankruptcy. Accordingly, indeterminacy of equilibrium with bankruptcy should hold for a smaller class of economies than the one usually considered (loosely speaking, economies in which endowment is not Pareto optimal).

Finally, the results obtained for sunspot economies are similar to the ones obtained in economies with intrinsic uncertainty. This captures in a nice way the idea that optimism plays a role in determining the equilibrium that arises, even though it is not necessary that this “optimism” be shared by all agents. It is indeed the case that in an economy where one agent believes in the effect of a sunspot, both equilibrium with and without bankruptcy can exist.

References

- K. ARROW. Le rôle des valeurs boursières pour la répartition la meilleure des risques. *Econometrie C.N.R.S.*, 11:41–47, 1953.
- Y. BALASKO AND D. CASS. The structure of financial equilibrium : I. exogenous yields and unrestricted participation. *Econometrica*, 57:135–162, 1989.
- D. CASS. *On the “Number” of equilibrium allocations with incomplete financial markets*. Working paper 85-16, CARESS University of Pennsylvania, 1985.
- D. CASS. Sunspots and incomplete financial markets : the leading example. In G. Feiwel, editor, *The economics of imperfect competition : Joan Robinson and beyond*, MacMillan, London, 1989.
- D. CASS. Sunspots and incomplete financial markets : the general case. *Economic Theory*, 2:339–361, 1992.
- G. DEBREU. *Theory of value*. Wiley, New York, 1959.

G. DEBREU. Economies with a finite set of equilibria. *Econometrica*, 38:387–392, 1970.

J. GEANAKOPOLOS AND A. MAS-COLELL. Real indeterminacy with financial assets. *Journal of Economic Theory*, 47:22–38, 1989.

T. PIETRA. The structure of the set of sunspot equilibria in economies with incomplete financial markets. *Economic Theory*, 2:320–338, 1992.

F. PORTIER AND J.M. TALLON. *Incomplete financial markets and the role of monetary policy: A macroeconomic presentation*. mimeo, M.A.D. Paris I, 1993.

R. RADNER. Existence of equilibrium of plans, prices, and price expectations in a sequence of markets. *Econometrica*, 40:289–303, 1972.

S. SUDA, J.M. TALLON AND A. VILLANACCI. Real indeterminacy of equilibria in a sunspot economy with inside money. *Economic Theory*, 2:309–319, 1992.

J.M. TALLON. *Bankruptcy in a general equilibrium model where agents have different views of the future*. Document de travail, M.A.D., Université Paris I, 1993.