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CASH-IN-ADVANCE CONSTRAINTS IN THE DIAMOND
OVERLAPPING GENERATIONS MODEL : NEUTRALITY
AND OPTIMALITY OF MONETARY POLICIES

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Cash-in-advance constraints in the Diamond overlapping generations model : neutrality and optimality of monetary policies

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Abstract

This paper is a study of money in overlapping generations models with cash-in-advances constraints *à la* Champ and Freeman and Hahn and Solow. We first offer a brief review of different features of the cash-in-advance constraint. Then we propose a general formulation. We show that neutrality of money prevails when the ratios of lump-sum monetary transfers and public expenditures to the money supply do not depend on the money growth rate. In order to implement the optimal economic growth solution, we show that one should follow the Chicago Rule *and* realize monetary transfers to young and old agents.

Keywords : Cash-in-advance constraints, overlapping generations model, neutrality of money, optimal monetary policy, Chicago Rule.

Résumé

Cet article est une étude des modèles à générations imbriquées dans lesquels sont introduites des contraintes de liquidités à la Champ-Freeman et Hahn et Solow. Nous faisons tout d'abord un bref panorama des différentes formes de contraintes de liquidités. Puis nous proposons une formulation générale. Nous montrons que la monnaie est neutre si les ratios des transferts monétaires forfaitaires et des dépenses publiques sur la masse monétaire ne dépendent pas du taux de croissance de celle-ci. La décentralisation de la solution optimale nécessite de respecter la règle de Chicago et d'effectuer des transferts monétaires à la fois aux agents jeunes et vieux.

Mots clés : Contraintes de liquidité, modèles à générations imbriquées, neutralité de la monnaie, politique monétaire, politique monétaire optimale, Règle de Chicago.

J.E.L Classification numbers : E31, E41, E52.

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1 Introduction¹

Overlapping generations models provide a very elegant framework to introduce money. Indeed, overlapping generations models exhibit a simple "friction" which inhibits people from directly buying goods in the absence of money (the seminal analysis is of course that of Samuelson [1958]). Thus money facilitates exchanges between agents who would otherwise be unable to trade. This confirms the intuition that the number of transactions in a monetary economy can be higher than in an economy with direct barter.

However, it is difficult to take overlapping generations models literally as models of money in terms of the intergenerational friction that makes room for currency (on this see Blanchard and Fischer [1989] page 163). For instance, it is well known that when the rate of return on non-monetary assets is greater than that of fiat money, the latter is simply not valued.

Researchers are then inclined to introduce further "frictions" in order to study monetary economies. It is often remarked that a major reason explaining why money is held is that it is more liquid than alternative assets. To capture the difference between liquid and illiquid assets it is useful to introduce a cash-in-advance constraint (surveys on this matter can be found in Sargent [1987] and in Villieu [1993])².

There is now a few studies devoted to analyze monetary matters in overlapping generations models with cash-in-advance constraints. The first study is that of Allais [1947] which focuses on the consequence of the mere existence of money on the level of the nominal interest rate. Woodford [1985]-[1987] studies the short and long-run consequences of credit expansion through inside money creation. Romer [1986] proposes a general equilibrium version of the Baumol-Tobin model. Kohn and Karacaoglu [1988] adapt the Woodford model to a small open economy to study the effects of devaluations. Champ and Freeman [1990] present a model in which unanticipated innovations in the stock of fiat money affect real variables (see also Champ and Freeman [1994]). An overlapping generations model with cash-in-advance constraint and with credit goods is analyzed by Schönfelder [1992] (pages 182-184). Hahn and

¹We would like to thank Herakles Polemarchakis, Jean-Marc Talon and Jean-Pierre Vidal for very helpful comments on a previous version of this paper.

²An alternative solution is to introduce money in the utility function (see Weiss [1980], Drazen [1981], Gale [1983], Abel [1987], McCallum [1987], Gahvari [1988], Yakita [1989]).

Solow [1995] also use an overlapping generations model with cash-in-advance constraints. They show that an economy with perfectly flexible wages can follow unmistakably pathological paths when an initial steady state is unexpectedly disturbed by an increase in the labor supply. Artus [1995] use a special feature of the cash-in-advance constraint to analyze the real effects of monetary policy.

The aim of this paper is to contribute to the preceding literature. We are particularly interested in the various forms of the cash-in-advance constraints. There exists several forms of cash-in-advance constraints when individual lives for two periods. Indeed, fiat money can be held to satisfy requirements that each person must hold fiat money worth (at least) a peculiar fraction of savings or of consumptions during youth and old age.

We first offer a brief review of the various forms of the cash-in advance constraints (section 2). Then we present a general form of the cash-in-advance constraint (section 3). Using this general form, we study the conditions under which money is or is not neutral. A necessary and sufficient condition for monetary neutrality is provided. Finally, we determine the monetary policy that will permit to implement optimal intertemporal allocations of resources (section 4). We show that optimal allocations are attainable if one controls the ratios of lump-sum monetary transfers and public expenditures to the money supply. We also show that a version of the Chicago Rule is implied by the optimal monetary policy. That is in order to reach the command optimum solution, one must follow the Chicago Rule and realize monetary transfers to young and old agents.

2 Overlapping generations models and cash-in-advance constraints

2.1 The fundamentals of the model

2.1.1 Agents

We make use of an overlapping generations model *à la* Allais (1947)-Diamond (1965). As usual, each individual lives for two periods. At each date t , N_t agents are born. The number of births grows at rate n so that $N_t = (1 + n)N_{t-1}$ (N_0 being given).

An agent born at date t has a life cycle utility function:

$$U(c_t, d_{t+1}) \tag{1}$$

where c_t and d_{t+1} are respectively the consumptions of good during youth and old age. $U(.,.)$ is increasing, differentiable and strictly quasi-concave. Each agent supplies inelastically one unit of labor during youth.

2.1.2 The firm

We assume that there is a single representative firm which produces Y_t quantity of good according to a production function with two inputs (capital: K ; labor: L_t):

$$Y_t = F(K_t, L_t) \tag{2}$$

It is assumed that $F(.,.)$ is increasing, differentiable, concave and exhibits constant returns to scale. Capital depreciates fully within a period.

Necessary conditions for profit maximizing are:

$$\omega_t = F_L(K_t, L_t) \tag{3}$$

$$1 + r_t = F_K(K_t, L_t) \tag{4}$$

where ω_t is the real wages, r_t the real rate of interest ($1 + r_t$ is the real price of capital at date t).

In the standard version of the overlapping generations model, agents born at date t have two budget constraints (one for each period of life):

$$c_t + s_t = \omega_t \text{ and } d_{t+1} = (1 + r_{t+1})s_t \text{ where } s_t \text{ is savings.}$$

Let $s(\omega_t, r_{t+1})$ denotes the optimal savings function of the agent, then the dynamics of capital accumulation writes: $K_{t+1} = N_t s_t$; or $k_{t+1} = \frac{s(\omega_t, r_{t+1})}{1+n}$ in *per capita* variables.

If d_{t+1} is a normal good and savings is an increasing function of the interest rate (the substitution effect dominating the income effect) the dynamics of capital is well defined and monotone.

2.2 Money in the overlapping generations model with capital

We shall be interested in a monetary version of the Diamond's Model. Suppose that budget constraints write as:

$$P_t c_t + S_t + M_t = W_t + T_t^1 \quad (5)$$

$$P_{t+1} d_{t+1} = (1 + i_{t+1})S_t + (1 + \nu_{t+1})M_t + T_{t+1}^2 \quad (6)$$

The nominal variables are: the price of good (P_t), non monetary savings (S_t), wages (W_t), the nominal interest rate (i_{t+1}), lump-sum monetary transfers (T_t^1 and T_{t+1}^2) and proportional monetary transfers ($\nu_{t+1}M_t$).

In real terms, the budget constraints writes as:

$$c_t + s_t + m_t = \omega_t + \theta_t^1 \quad (7)$$

$$d_{t+1} = (1 + r_{t+1})s_t + (1 + \nu_{t+1})m_t + \theta_{t+1}^2 \quad (8)$$

where: $s_t = \frac{S_t}{P_t}$, $m_t = \frac{M_t}{P_t}$, $\omega_t = \frac{W_t}{P_t}$, $\theta_t^1 = \frac{T_t^1}{P_t}$, $1 + r_{t+1} = \frac{P_t(1+i_{t+1})}{P_{t+1}}$, $1 + \nu_{t+1} = \frac{P_t(1+\nu_{t+1})}{P_{t+1}}$, $\theta_{t+1}^2 = \frac{T_{t+1}^2}{P_{t+1}}$.

We can write the intertemporal budget constraint in real terms and one obtains:

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} = \omega_t + \theta_t^1 + \frac{\theta_{t+1}^2}{1 + r_{t+1}} + \left(\frac{1 + \nu_{t+1}}{1 + r_{t+1}} - 1\right)m_t \quad (9)$$

It is clear that if:

$$1 + \nu_{t+1} < 1 + r_{t+1} \quad (10)$$

the real return on monetary balances is lower than that on capital. Agents would then choose not to hold monetary balances in excess of what is implied by the cash-in-advance constraint. In any case, strictly positive savings implies $r_{t+1} \geq \nu_{t+1}$.

2.3 Usual cash-in-advance constraints

In the standard overlapping generations model, agents save in order to finance their consumption during old age. The usual cash-in-advance constraint implies that agents must hold money in order to finance their consumptions. In the standard overlapping generations model, this cash-in-advance constraint would lead agents to hold their savings entirely in the form of money. But this would cause $S_t = 0$; so that capital accumulation would be impossible.

Hahn and Solow [1995] suggest a constraint like $M_t \geq \mu P_{t+1} d_{t+1}$. In the case of proportional monetary transfer, one has:

$$(1 + v_{t+1})M_t \geq \mu P_{t+1} d_{t+1} \quad (11)$$

With such a constraint, a fraction of at least μ of consumption during old age is financed by money balances held at the beginning of old age (which supposes that (proportional) monetary transfers are realized at this moment).

We shall only study the case in which the cash-in-advance constraint is binding. The conditions under which this is satisfied will be explicitly analyzed in the examples studied later.

In real terms, when the cash-in-advance constraint is binding, one has:

$$m_t = \frac{\mu d_{t+1}}{1 + v_{t+1}} \quad (12)$$

The lifetime budget constraint can then be stated as:

$$c_t + \left(\frac{1 - \mu}{1 + r_{t+1}} + \frac{\mu}{1 + v_{t+1}} \right) d_{t+1} = \omega_t + \theta_t^1 + \frac{1}{1 + r_{t+1}} \theta_{t+1}^2 \quad (13)$$

The optimal choice of the consumer necessarily satisfies:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = \frac{1 + r_{t+1}}{1 + \mu \left(\frac{r_{t+1} - v_{t+1}}{1 + v_{t+1}} \right)} \quad (14)$$

The intuition of this condition is as follows. Suppose the agent is willing to increase his consumption during old age of Δd_{t+1} . Then, the increase in money balances in period $t + 1$ must be $\mu \Delta d_{t+1}$. But because of the proportional monetary transfers and inflation, increase in monetary balances in period t will be $\frac{\mu \Delta d_{t+1}}{1 + v_{t+1}}$. The remained portion of Δd_{t+1} is financed by increasing non-monetary savings. The amount required is then $\frac{(1 - \mu) \Delta d_{t+1}}{1 + r_{t+1}}$. To

sum up, in order to increase his or her consumption of Δd_{t+1} during old age an agent must lower his or her consumption during youth of $-\Delta c_t = \frac{\mu \Delta d_{t+1}}{1+\nu_{t+1}} + \frac{(1-\mu)\Delta d_{t+1}}{1+r_{t+1}}$. Optimal consumptions necessarily verify : $\Delta c_t U'_c + \Delta d_{t+1} U'_d = 0$, which gives condition (14).

2.4 Alternative forms of the cash-in-advance constraint

P. Artus [1995] studies a cash-in-advance constraint such as :

$$M_t \geq \mu P_t c_t \quad (15)$$

and assumes that the constraint is binding.

Budget constraints are unchanged:

$$\begin{aligned} P_t c_t + M_t + P_t s_t &= W_t + T_t^1 \\ P_{t+1} d_{t+1} &= (1 + i_{t+1}) s_t + (1 + \nu_{t+1}) M_t + T_{t+1}^2 \end{aligned}$$

When the cash-in-advance constraint is binding, the lifetime budget constraint expressed in real terms is now:

$$(1 + \mu - (\frac{1 + \nu_{t+1}}{1 + r_{t+1}})\mu) c_t + \frac{d_{t+1}}{1 + r_{t+1}} = \omega_t + \theta_t^1 + \frac{\theta_{t+1}^2}{1 + r_{t+1}} \quad (16)$$

The optimal choice of the consumer necessarily satisfies:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = (1 + \mu)(1 + r_{t+1}) - \mu(1 + \nu_{t+1}) \quad (17)$$

$$\Rightarrow \frac{U'_{c_t}}{U'_{d_{t+1}}} = (1 + r_{t+1})(1 + \mu \frac{r_{t+1} - \nu_{t+1}}{1 + r_{t+1}}) \quad (18)$$

Let us give an intuition of this condition. For the sake of variety, suppose that consumption during youth decreases by an amount Δc_t .

This enables the agent to lower his or her money balances by $\mu \Delta c_t$. The increase in (real) savings is then $-\Delta c_t(1 + \mu) < 0$. This generates a direct increase in consumption during old age of $-(1 + r_{t+1})\Delta c_t(1 + \mu)$. However, as money balances are now lower, consumption during old age must decrease

by $\mu(1 + v_{t+1})\Delta c_t$. So, the total change in consumption during old age is: $\Delta d_{t+1} = -(1+r_{t+1})(1+\mu)\Delta c_t + \mu(1+v_{t+1})\Delta c_t$. Again, optimal consumptions necessarily verify : $\Delta c_t U'_c + \Delta d_{t+1} U'_d = 0$, which is condition (17).

Champ and Freeman [1990]-[1994] propose to consider the following cash-in-advance constraint:

$$M_t \geq \mu P_t \quad (19)$$

There is a legal constraint which requires agents to hold (at least) a constant amount of real monetary balances. Champ and Freeman suppose that the constraint is binding.

In this case, the lifetime budget constraint is:

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} = \omega_t + \theta_t^1 + \frac{\theta_{t+1}^2}{1 + r_{t+1}} + \mu \left(\frac{1 + v_{t+1}}{1 + r_{t+1}} - 1 \right)$$

The optimal choice of the consumer necessarily satisfies:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = 1 + r_{t+1}$$

Here, intuition is as usual.

Another form of cash-in-advance constraint is considered by Tirole [1985]. Suppose that (at least) a fraction of savings must be held in the form of money. This also implies that M_t is proportional to $(M_t + S_t)$, which writes:

$$M_t \geq \mu S_t \quad (20)$$

When the constraint is binding, $m_t = \mu s_t$, the intertemporal budget constraint is:

$$c_t + \frac{(1 + \mu)d_{t+1}}{1 + r_{t+1} + \mu(1 + v_{t+1})} = \omega_t + \theta_t^1 + \frac{(1 + \mu)\theta_{t+1}^2}{1 + r_{t+1} + \mu(1 + v_{t+1})} \quad (21)$$

The optimal choice of the consumer necessarily satisfies:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = \frac{1 + r_{t+1} + \mu(1 + v_{t+1})}{1 + \mu}$$

The economic rationale for this formula is as follows. Suppose that one wishes to decrease consumption during youth by an amount $\Delta c_t < 0$. The increase in monetary and real savings is $\Delta m_t + \Delta s_t = -\Delta c_t$. But the increase in real savings is only equal to $\frac{-\Delta c_t}{1+\mu}$. This increase in real savings permits to increase consumption during old age by $\frac{-(1+r_{t+1})\Delta c_t}{1+\mu}$. The increase in consumption due to the increase in monetary savings is: $\frac{-\mu(1+v_{t+1})\Delta c_t}{1+\mu}$. Hence, the total increase in consumption during old age is: $\Delta d_{t+1} = \frac{-(1+r_{t+1})\Delta c_t}{1+\mu} + \frac{-\mu(1+v_{t+1})\Delta c_t}{1+\mu}$. This leads to the arbitrage condition necessarily satisfied by the optimal consumptions.

3 Neutrality and monetary policies

This section will be devoted to the study of neutrality of monetary policies under the assumption of a general form of the cash-in-advance constraint:

$$m_t \geq m(c_t, s_t, e_{t+1}) \quad (22)$$

where $e_{t+1} = \frac{d_{t+1}}{1+v_{t+1}}$ is the real monetary balances - corrected of proportional transfers - which is needed to finance consumption during old age. We will suppose that the constraint is binding so that $m_t = m(c_t, s_t, e_{t+1})$.

$m(., ., .)$ is supposed to be a non-decreasing, differentiable function. We write the budget constraint for youth age as:

$$c_t + s_t + m_t \left(c_t, s_t, \frac{d_{t+1}}{1+v_{t+1}} \right) = \omega_t + \theta_t^1 \quad (23)$$

Substituting for real balances in old age budget constraint one obtains:

$$d_{t+1} = (1+r_{t+1})s_t + (1+v_{t+1})(\omega_t + \theta_t^1 - c_t - s_t) + \theta_{t+1}^2 \quad (24)$$

Differentiating both budget constraints with respect to s_t , one gets:

$$\left(1 + \frac{\partial m_t}{\partial c_t}\right) \frac{\partial c_t}{\partial s_t} + 1 + \frac{\partial m_t}{\partial s_t} + \frac{1}{1+v_{t+1}} \frac{\partial m_t}{\partial e_{t+1}} \frac{\partial d_{t+1}}{\partial s_t} = 0 \quad (25)$$

$$\frac{\partial d_{t+1}}{\partial s_t} + (1+v_{t+1}) \frac{\partial c_t}{\partial s_t} = r_{t+1} - v_{t+1} \quad (26)$$

From which one finds that:

$$\frac{\partial c_t}{\partial s_t} \left(1 + \frac{\partial m_t}{\partial c_t} - \frac{\partial m_t}{\partial e_{t+1}}\right) = -1 - \frac{\partial m_t}{\partial s_t} - \frac{r_{t+1} - v_{t+1}}{1 + v_{t+1}} \frac{\partial m_t}{\partial e_{t+1}} \quad (27)$$

$$\begin{aligned} \frac{\partial d_{t+1}}{\partial s_t} \left(1 + \frac{\partial m_t}{\partial c_t} - \frac{\partial m_t}{\partial e_{t+1}}\right) &= (1 + v_{t+1}) \left(1 + \frac{\partial m_t}{\partial s_t}\right) + \\ &\quad \left(1 + \frac{\partial m_t}{\partial c_t}\right) (r_{t+1} - v_{t+1}) \end{aligned} \quad (28)$$

The optimal choice of the consumer necessarily satisfies:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = \frac{(1 + v_{t+1})(1 + m'_{s_t}) + (r_{t+1} - v_{t+1})(1 + m'_{c_t})}{1 + m'_{s_t} + \frac{r_{t+1} - v_{t+1}}{1 + v_{t+1}} m'_{e_{t+1}}}$$

or:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = \frac{1 + r_{t+1} + m'_{s_t}(1 + v_{t+1}) + m'_{c_t}(r_{t+1} - v_{t+1})}{1 + m'_{s_t} + \frac{r_{t+1} - v_{t+1}}{1 + v_{t+1}} m'_{e_{t+1}}} \quad (29)$$

The various arbitrage conditions stated in section two can be seen are particular cases of the preceding expression.

3.1 Definitions of monetary policies

Let us suppose that total money supply (\bar{M}_t) grows at rate ε_t :

$$\bar{M}_t = (1 + \varepsilon_t) \bar{M}_{t-1} \quad (30)$$

We then introduce the following parameters:

$$\alpha_t \bar{M}_t = N_t T_t^1 \quad (31)$$

$$\beta_t \bar{M}_t = N_{t-1} T_t^2 \quad (32)$$

$$\gamma_t \bar{M}_t = (1 + \nu_t) \bar{M}_{t-1} \quad (33)$$

$$\phi_t \bar{M}_t = G_t \quad (34)$$

where G_t denotes nominal public expenditures.

We shall assume that the government controls the value of lump-sum monetary transfers (T_t^1 and T_t^2), the nominal public expenditures G_t and the

money growth rate ε_t . The choices of these variables determines the values of α_t , β_t and ϕ_t which can then be considered as policy parameters. From the budget constraint of the government one obtains the value of γ_t . Indeed, without public debt, the government is faced with a budget constraint such as:

$$\varepsilon_t \bar{M}_{t-1} = \bar{M}_t - \bar{M}_{t-1} = G_t + N_t T_t^1 + N_{t-1} T_t^2 + \nu_t \bar{M}_{t-1} \quad (35)$$

Using the parameters introduced before, one obtains :

$$\gamma_t = 1 - \alpha_t - \beta_t - \phi_t \quad (36)$$

Equation (33): $(1 + \nu_t) \bar{M}_{t-1} = \gamma_t \bar{M}_t = \gamma_t (1 + \varepsilon_t) \bar{M}_{t-1}$ implies that:

$$\gamma_t = \frac{1 + \nu_t}{1 + \varepsilon_t} \quad (37)$$

We note that when the growth rate of the money supply and the value of γ_t are fixed, one determines necessarily the proportional monetary transfers (i.e. the value of ν_t).

Let us now concentrate on the equilibrium of the market for money. The aggregate demand of monetary balances is: $N_t P_t m_t$. The equilibrium in the market for money implies that:

$$\bar{M}_t = N_t P_t m_t \quad (38)$$

from which, one then gets:

$$\theta_t^1 = \alpha_t \frac{\bar{M}_t}{N_t P_t} = \alpha_t m_t \quad (39)$$

$$\theta_t^2 = \beta_t \frac{\bar{M}_t}{N_{t-1} P_t} = (1 + n) \beta_t m_t \quad (40)$$

$$(1 + \nu_t) m_{t-1} = \frac{P_{t-1}}{P_t} (1 + \nu_t) \frac{\bar{M}_{t-1}}{N_{t-1} P_{t-1}} = \frac{\gamma_t \bar{M}_t}{N_{t-1} P_t} = (1 + n) \gamma_t m_t \quad (41)$$

3.2 The real equilibrium

We now study the real part of the general equilibrium of our monetary economy.

3.2.1 The agent

At a general equilibrium, every budget constraints are satisfied and choices of consumptions and savings are optimal.

So, taking into account (31), (32) and (33), it follows that:

$$m_t = m \left(c_t, s_t, \frac{d_{t+1}m_t}{(1+n)(1-\alpha_{t+1}-\beta_{t+1}-\phi_{t+1})m_{t+1}} \right) \quad (42)$$

$$c_t + s_t + m_t = \omega_t + \alpha_t m_t \quad (43)$$

$$d_{t+1} = (1+r_{t+1})s_t + (1+n)m_{t+1}(1-\alpha_{t+1}-\phi_{t+1}) \quad (44)$$

When it is possible to solve this system for c_t , s_t , d_{t+1} , equations (42), (43) and (44) allow to compute c_t , s_t and d_{t+1} as functions of m_t , m_{t+1} , ω_t , r_{t+1} , α_t , β_t , and ϕ_t ³. Then, using the arbitrage equation:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = \frac{(1+n)(1-\alpha_{t+1}-\beta_{t+1}-\phi_{t+1})\frac{m_{t+1}}{m_t}(m'_{s_t} - m'_{c_t}) + (1+r_{t+1})(1+m'_{c_t})}{1+m'_{s_t} - m'_{e_{t+1}} + \frac{(1+r_{t+1})}{1+n} \frac{m_t}{m_{t+1}} \frac{m'_{e_{t+1}}}{1-\alpha_{t+1}-\beta_{t+1}-\phi_{t+1}}} \quad (45)$$

one obtains a first-order difference equation in m_t parametrized by ω_t , r_{t+1} , and the policy instruments α_t , β_t , and ϕ_t .

Three variants of this equation are stated in the appendix for the case of Cobb-Douglas life-cycle utility and production functions. In each case, we provide a condition sufficient to insure that the cash-in-advance constraint is binding.

3.2.2 The firm

Defining $k_t = \frac{K_t}{N_t}$, one readily checks that:

$$\omega_t = F'_L(k_t, 1) \quad (46)$$

$$1+r_t = F'_K(k_t, 1) \quad (47)$$

³The case $m_t = m$ is considered separately.

3.2.3 Dynamics

The law of motion of capital is as usual:

$K_{t+1} = N_t s_t$ where $s_t = s(\omega_t, r_{t+1}, \alpha_t, \beta_{t+1}, \phi_{t+1})$. Using per capita variables one also gets:

$$k_{t+1} = \frac{s_t}{1+n} \quad (48)$$

The values of $\alpha_t, \beta_t, \phi_t$ being given, (45) and (48) define a first-order system of difference equations in m_t and k_t . We have only one predetermined variable (k_0), m_0 being a free variable. We then have the following theorem.

Theorem 1. *A necessary and sufficient condition for monetary neutrality is that equations (42)-(48) are independent of the money growth rate. This condition is satisfied if α_t does not depend on ε_t , if $\alpha_t + \phi_t$ does not depend on ε_t , and if either β_t does not depend on ε_t or $m'_e = 0$ and $m'_s = m'_c = 0$.*

Proof : The theorem follows readily from a close inspection of equations (42)-(48). ■

Neutrality of money prevails if all monetary transfers are realized in a way which does not depend on the money growth rate. When only some monetary transfers do not depend on the money growth rate, monetary neutrality is warranted if the cash-in-advance function satisfies some specific conditions. This conditions are satisfied in the Champ and Freeman version of the cash-in-advance constraint.

The special case of no proportional monetary transfers (case $\nu_t = 0$) is worth of interest. Then $\gamma_t = \frac{1}{1+\varepsilon_t}$ and except for the case where $m'_e = 0$ and $m'_s = m'_c = 0$, changes in the money growth rate modify the real equilibrium : monetary policy is no more neutral. With the Champ and Freeman constraint, $m_t = \mu$ and the term β_t vanishes in equation (45). Considering (44), neutrality obtains if and only if $\beta_t + \gamma_t$ is independant of ε_t . This is always the case when there are no public expenditures. These properties hinge crucially on the way increases in money supply are distributed to agents.

3.3 Two different dynamics

What are the implications of the choice of one or another form of the cash-in-advance constraint for the equilibrium dynamics ?

What will be shown, is that assuming a cash-in-advance constraint such as “ μd ” or “ μ ”⁴, it is possible to formulate the equilibrium dynamics as a first-order difference equation in k_t .

So, in that case, incorporating a monetary side into the original version of the Diamond model does not add a dimension to the dynamics.

This is in contrast with what obtains if one introduces a cash-in-advance constraint such as “ μc ” or “ μs ”. Indeed, one gets a second-order system of difference equation (one with a forward looking equation and a backward looking equation). In that case, the dynamical equilibrium is determinate and converges to a steady-state equilibrium if the latter is a saddle-point.

3.3.1 The “ μd ” model

From (42) one gets:

$$d_{t+1} = (1+n)\gamma_{t+1}\mu m_{t+1}$$

Together with equations (44) and (48), one has:

$$m_{t+1} \left(\frac{1-\mu}{\mu} \gamma_{t+1} - \beta_{t+1} \right) = R_{t+1} k_{t+1} \quad (49)$$

In the latter equation, one can clearly see that there is a - static - link between m_{t+1} and k_{t+1} . The corresponding dynamics is studied in appendix 1 under the assumption of Cobb-Douglas life-cycle utility and production functions. We find that when μ is small enough, there exists a unique non trivial steady-state equilibrium and the sequence of capital stocks converges to it monotonically.

3.3.2 The “ μ ” model

With this special feature of the cash-in-advance constraint, the real balances are time invariant. k_t is then the only dynamical variable of the model. The dynamics is as in the Diamond’s model. A complete study of this dynamics under Cobb-Douglas life-cycle utility and production functions can be found in appendix 2. We find that there are generally two steady-state equilibria one which is locally stable, the other being locally unstable.

⁴In what follows, “ μd ” refers to the cash-in-advance constraint (12), whereas “ μ ” refers to the cash-in-advance (19). Similarly, “ μs ” will refer to the cash-in-advance (20) whereas “ μc ” will refer to (15).

3.3.3 The “ μs ” model

The choice of a cash-in-advance constraint of the “ μs ” form has implications on the dynamics.

Using (20) and the equilibrium condition for the market of capital (48), one has:

$$m_t = \mu(1 + n)k_{t+1} \quad (50)$$

We see that there is a very simple relation between money and capital which simplifies greatly the study of the dynamics.

But, in contrast with the “ μd ” model, this link concerns real money balances at date t and capital a date $t + 1$ (and not capital at date t). This is a forward looking equation. Rearranging the latter one can arrive at a second-order difference equation in k_t . The case of Cobb-Douglas life-cycle utility and production functions is studied in appendix 3. We find that when μ is small enough, one gets a steady-state which is a saddle-point.

3.3.4 The “ μc ” model

With this form of cash-in-advance one can show that the equilibrium dynamics is also given by a second-order difference equation in k_t .

Using (15), (43) and (48), yields:

$$(1 + n)k_{t+1} = \left[\omega_t - \left(\frac{\mu + 1}{\mu} - \alpha_t \right) m_t \right] \quad (51)$$

The real balance m_t is a function of k_t and k_{t+1} . This is a forward looking equation. The dynamics is studied in appendix 4 under the assumption of Cobb-Douglas life-cycle utility and production functions. Again, we find that when μ is small enough, there exists a non-trivial steady-state which is a saddle-point.

4 The optimal monetary policy

By definition, the optimal monetary policy is that policy which decentralizes the command optimum solution, *i.e.* the desired intergenerational redistribution of resources..

In this section we first state the command optimum and show how to decentralize it using only monetary instruments.

4.1 The command optimum solution

The optimal intergenerational redistribution of resources - the command optimum solution - is obtained as the solution of the following problem:

$$\max_{\{c_t, d_t, K_{t+1}\}_{t=0}^{+\infty}} \sum_{t=-1}^{+\infty} \delta^t U(c_t, d_{t+1}) \quad (52)$$

$$Y_t = F(K_t, N_t) = K_{t+1} + N_t c_t + N_{t-1} d_t \quad (53)$$

where c_{-1} and K_0 are given.

This problem is studied in Marchand, Michel and Pestieau [1990]. The necessary and sufficient conditions for optimality include the intragenerational and intergenerational trade-off conditions for every periods t :

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = F_K(k_{t+1}, 1) \quad (54)$$

$$\delta U'_{c_t} = (1+n)U'_{d_t} \quad (55)$$

We shall note $\{c_t^*, d_t^*, k_{t+1}^*\}_{t=0}^{+\infty}$ the solution of the preceding problem.

The decentralization of the command optimal solution implies to replicate the complete trajectory.

4.2 Optimal monetary policy

We now show that the command optimum solution can be decentralized by a monetary policy.

To achieve this aim, it is necessary and sufficient for monetary policy to be designed in order to give to each agent the amount of resources he needs to:

- 1) buy the optimal consumptions
- 2) choose the optimal rate of savings.

Given the optimal trajectory $\{c_t^*, d_t^*, k_{t+1}^*\}_{t=0}^{+\infty}$, let us define the following variables $\forall t \geq 0$:

$$s_t^* = (1+n)k_{t+1}^* \quad (56)$$

$$v_t^* = r_t^* = F'_K(k_t^*, 1) - 1 \quad (57)$$

$$m_t^* = m\left(c_t^*, s_t^*, \frac{d_{t+1}^*}{1 + r_{t+1}^*}\right) \quad (58)$$

At date $t = -1$, we set:

$$m_{-1}^* = m\left(c_{-1}, (1+n)k_0, \frac{d_0^*}{F'_K(k_0, 1)}\right)$$

Finally, $\forall t \geq 0$, one defines⁵:

$$\gamma_t^* = 1 - \alpha_t^* - \beta_t^* = \frac{(1 + r_t^*)m_{t-1}^*}{(1+n)m_t^*} \quad (59)$$

$$\theta_t^{1*} = c_t^* + (1+n)k_{t+1}^* + m_t^* - F_L(k_t^*, 1) \quad (60)$$

$$\theta_{t+1}^{2*} = d_t^* - (1 + r_t^*)(1+n)k_t^* - (1 + r_t^*)m_{t-1}^* \quad (61)$$

These two equations allow us to compute respectively α_t^* and β_t^* :

$$\alpha_t^* = \frac{\theta_t^{1*}}{m_t^*}$$

$$\beta_t^* = \frac{\theta_t^{2*}}{(1+n)m_t^*}$$

If (57), (59), (60) and (61) are satisfied at all dates, it is readily checked that an agent born at date t will in effect choose a rate of savings equal to s_t^* . Indeed, the last two conditions ensure that this agent can afford to buy c_t^* and d_{t+1}^* . Moreover, with (57), the individual arbitrage equation:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = \frac{1 + r_{t+1} + m'_{s_t}(1 + v_{t+1}) + m'_{c_t}(r_{t+1} - v_{t+1})}{1 + m'_{s_t} + \frac{(r_{t+1} - v_{t+1})}{1 + v_{t+1}}m'_{e_{t+1}}}$$

reduces to:

$$\frac{U'_{c_t}}{U'_{d_{t+1}}} = 1 + r_{t+1}^*$$

⁵ Assuming no public expenditures.

The optimality condition (54) ensures that (c_t^*, d_{t+1}^*) is the solution to the preceding equation.

Let us now show that the system of transfers is well balanced, *i.e.* that: $\alpha_t^* + \beta_t^* + \gamma_t^* = 1$.

To do so, let us add the monetary transfers (expressed in per capita variable) distributed at date t . This yields:

$$\theta_t^{1*} + \frac{\theta_t^{2*}}{1+n} = (\alpha_t^* + \beta_t^*)m_t^*$$

$$(\alpha_t^* + \beta_t^*)m_t^* = c_t^* + \frac{d_t^*}{1+n} + m_t^* - (1+r_t^*)\frac{m_{t-1}^*}{1+n} - F(k_t^*, 1) + (1+n)k_{t+1}^* \quad (62)$$

Using the fact that: $v_t^* = r_t^*$, it follows that:

$$\alpha_t^* + \beta_t^* + \gamma_t^* = 1 \quad (63)$$

Then one can choose independently α_t^* and β_t^* .

We then have shown that:

Theorem 2: *With the following appropriate monetary policy: $\forall t \geq 0$,*

$$\begin{aligned} \alpha_t^* m_t^* &= c_t^* + (1+n)k_{t+1}^* + m_t^* - F_L(k_t^*, 1) \\ (1+n)\beta_t^* m_t^* &= d_t^* - (1+r_t^*)(1+n)k_t^* - (1+r_t^*)m_{t-1}^* \end{aligned}$$

competitive market forces achieve the command solution $\{c_t^, d_t^*, k_{t+1}^*\}_{t=0}^{+\infty}$.*

Clearly, any growth rate of the nominal money supply can be chosen. In fact, only redistribution effects matter. The latter can be designed by the appropriate choice of the ratios of monetary transfers to the money supply.

The condition $v_t^* = r_t^*$ is a version of the Chicago Rule when there are proportional transfers. According to this rule, an optimal monetary policy should be designed so that the real return on money holdings is equal to that on capital. Generally, this involves a steady contraction of money supply at a rate sufficient to bring the nominal interest rate down to zero. Such a policy obtains when there are no proportional transfers. Indeed, suppose that $\nu_t = 0$. Then the condition $v_t = r_t$ implies $1 + r_t = \frac{p_{t-1}}{p_t}$, so that by definition, $i_t = 0$. However, in such a case, it is no longer true that the

optimal money growth rate is indeterminate. Indeed, as $\gamma_t^* = \frac{1}{1+\varepsilon_t}$, one has:

$$\varepsilon_t^* = \frac{(1+n)m_t^*}{(1+r_t^*)m_{t-1}^*} - 1.$$

The Chicago rule is generally found in existing analyses of the optimal inflation rates (see e.g. Weiss [1980], Drazen [1981], Woodford [1985]). Exceptions are provided by McCallum [1983] and [1987]. McCallum suggests that the Chicago Rule prevails if models with finite-lived agents take account of the transaction-facilitating services of money. The present analysis which takes the services of money into account, is clearly in line with this statement.

Remark: Recall that if $\gamma_t < \frac{1}{1+\varepsilon_t}$, then $\nu_t < 0$, and there is a proportional tax on money balances which requires the government to implement a "monetary" reform. If this proves impossible, then one has to choose ε_t such as $\varepsilon_t \leq \frac{1}{\gamma_t^*} - 1$. Finally, monetary transfers are positive only in the case $\theta_t^{1*} \geq 0$, $\theta_t^{2*} \geq 0$ and $\gamma_t^* \geq \frac{1}{1+\varepsilon_t}$.

5 Conclusion

In this paper we have proposed a general form of the cash-in-advance constraint which encompasses several formulations.

We showed that money is neutral when the ratios of monetary transfers to money supply do not depend on the growth rate of the latter. When only some ratios do not depend on the the money growth rate, monetary neutrality obtains if the cash-in-advance constraint satisfies some properties.

We also showed that the formulations of the cash-in-advance constraints of Champ and Freeman and Hahn and Solows yield very simple dynamics similar to the dynamics of the Diamond's model. These formulations provide then a useful framework for monetary analysis.

We also proved that optimal intergenerational resources sharing can be decentralized by choosing adequate ratios of monetary transfers to the money supply. In that case the choice of the money supply growth rate is a matter of indifference. We showed that the optimal monetary policy does imply the Chicago Rule.

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Appendix 1

Study of the dynamics: case “ μd ”

We suppose that the life-cycle utility function is as follows:

$$U(c_t, d_{t+1}) = c_t d_{t+1}^a$$

and that the production function is:

$$F(K_t, L_t) = AK_t^\lambda L_t^{1-\lambda}$$

We have a static link between real money balances and the stock of capital (64):

$$k_{t+1} = \frac{m_{t+1}}{R_{t+1}} \left(\frac{1-\mu}{\mu} \gamma - \beta \right) \quad (64)$$

Under the Cobb-Douglas assumption, equation (45) writes:

$$(1+n) \frac{m_{t+1}}{R_{t+1}} \left(\frac{1-\mu}{\mu} \gamma - \frac{a}{1+a} \beta \right) = \frac{a}{1+a} \omega_t + m_t \left(\frac{\alpha a}{1+a} - 1 \right) \quad (65)$$

Using (64) and (65), one obtains a first-order difference equation in k_t :

$$k_{t+1} = \frac{A}{1+n} \left[\frac{a}{1+a} \frac{Y}{X} (1-\lambda) - \left(1 - \frac{\alpha a}{1+a} \right) \frac{\lambda}{X} \right] k_t^\lambda \quad (66)$$

where:

$$X = \left(\frac{1-\mu}{\mu} \gamma - \frac{a}{1+a} \beta \right)$$

and:

$$Y = \left(\frac{1-\mu}{\mu} \gamma - \beta \right)$$

Suppose that k_t is fixed, and let us look at the effects of parameters α , β and γ on capital accumulation. One readily checks that:

$$\begin{aligned} \frac{\partial k_{t+1}}{\partial \alpha} &> 0 \\ \frac{\partial k_{t+1}}{\partial \beta} &< 0 \\ \frac{\partial k_{t+1}}{\partial \gamma} &> 0 \end{aligned}$$

In order to get the usual dynamics of the Diamond's model, it is sufficient that the term into brackets in equation (66) be positive. This is the case if μ is small enough. If this is the case, there does exist a unique non trivial steady-state equilibrium and the sequence of capital stocks converges to it monotonically. Using (64) and the dynamics of k_t , the dynamics of m_t obtains.

Let us now give a condition insuring that the cash-in-advance constraint is binding. We must verify that: $1 + v_t = \frac{(1+n)\gamma_t m_t}{m_{t-1}} \leq 1 + r_t$.

We have: $M_t = \frac{1+\varepsilon_t}{1+n} M_{t-1} = \frac{1+\varepsilon_t}{1+n} \mu p_t d_t$. But we also have:

$$p_t d_t = M_{t-1} + (1 + i_{t-1}) p_{t-1} s_{t-1} \text{ so that } p_t d_t = \frac{(1+i_t) p_{t-1} s_{t-1}}{1-\mu}.$$

From this, one has: $m_t = (1 + \varepsilon_t) \frac{\mu}{1-\mu} R_t k_t$. Assuming a constant money growth rate, we have: $R_t \geq \frac{\gamma(1+n) R_t k_t}{R_{t-1} k_{t-1}}$. This condition writes also after a few computations :

$$\left[\frac{a}{1+a} \frac{Y}{X} (1-\lambda) - \left(1 - \frac{\alpha a}{1+a}\right) \frac{\lambda}{X} \right] < \frac{\lambda}{\gamma}$$

Appendix 2

Study of the dynamics: case “ μ ”

We saw that in this case, the dynamics is very close to that of the real version of the Diamond's model. In particular, one notice that the individual arbitrage condition only depends on the rate of interest.

Using (45), (44) (43) and (48), one gets a first-order difference equation in k_t :

$$k_{t+1} = \frac{1}{1+n} \left[\frac{a\omega_t}{1+a} - \frac{\mu}{1+a} \left((1-\alpha)a + \frac{(\beta+\gamma)(1+n)}{R_{t+1}} \right) \right] \quad (67)$$

Using the Cobb-Douglas specification for the production function yields:

$$k_{t+1} = \phi k_t^\lambda - \eta k_{t+1}^{1-\lambda} - \rho \quad (68)$$

where,

$$\begin{aligned} \phi &= \frac{Aa(1-\lambda)}{(1+n)(1+a)} \\ \eta &= \frac{\mu(\beta+\gamma)(1+n)}{(1+n)(1+a)A\lambda} \\ \rho &= \frac{\mu(1-\alpha)a}{(1+n)(1+a)} \end{aligned}$$

when a steady-state equilibrium exists, it necessarily verifies:

$$k + \rho = \phi k^\lambda - \eta k^{1-\lambda}$$

Let us study the function $g(k) = \phi k^\lambda - \eta k^{1-\lambda}$. Under the assumption $\lambda < 1/2$, this function is first increasing with respect to k - in $k = 0$, the slope is infinite - and reaches a maximum at:

$$k^* = \left(\frac{\epsilon \lambda}{\eta(1-\lambda)} \right)^{\frac{1}{1-2\lambda}}$$

This enables us to conclude that there may be zero, one or two steady-states. The case unique steady-state obtains when $g(k)$ and $k + \rho$ are tangent. The higher ρ , the less likely is the existence of a steady-state.

Now, let us define by k_0 the value of k lower than k^* such that the slope of $g(\cdot)$ is equal to one.

k_0 is such that:

$$\phi \lambda k_0^{\lambda-1} - \eta(1-\lambda)k_0^{-\lambda} = 1$$

We now assume that there is two steady-states, \bar{k}_1 and \bar{k}_2 , with $\bar{k}_1 < \bar{k}_2$. We have:

$$\bar{k}_1 < k_0 < k_2$$

The slope of $g(\cdot)$ at a steady-state is:

$$\frac{dk_{t+1}}{dk_t} = \frac{\phi \lambda k^{\lambda-1}}{1 + \eta(1-\lambda)k^{-\lambda}}$$

The value of this slope will be higher or lower than one if the following expression is positive or negative:

$$\phi \lambda k^{\lambda-1} - 1 - \eta(1-\lambda)k^{-\lambda}$$

We deduce from this, that \bar{k}_1 is locally unstable, whereas \bar{k}_2 is locally stable.

We now give a condition insuring that the cash-in-advance constraint is binding. We must verify that: $1 + v_t = \frac{(1+n)\gamma_t m_t}{m_{t-1}} \leq 1 + r_t$. From the cash-in-advance constraint we have: $m_t = \mu$. From this, we must verify that: $R_t \geq \gamma(1+n)$. This implies that $k_t \leq \left(\frac{A\lambda}{\gamma(1+n)} \right)^{\frac{1}{1-\lambda}}$. We suppose that $\bar{k}_2 \leq \left(\frac{A\lambda}{\gamma(1+n)} \right)^{\frac{1}{1-\lambda}}$.

Appendix 3

Study of the dynamics: case "μs"

When the production function is Cobb-Douglas, using (42), (43), (44) and (45) one obtains:

$$\begin{aligned} & (1 + \mu) \left[\frac{R_{t+1}}{\mu} m_t + (1 + n)(\gamma + \beta)m_{t+1} \right] \\ = & a \left[\omega_t - m_t \left(1 - \alpha + \frac{1}{\mu} \right) \right] \left[(1 + n)\gamma \frac{m_{t+1}}{m_t} \mu + R_{t+1} \right] \end{aligned}$$

Using (50) one gets a second-order difference equation in k_t :

$$\begin{aligned} & (1 + \mu) \left[A\lambda k_{t+1}^\lambda (1 + n) + (1 + n)^2 \mu (\gamma + \beta) k_{t+2} \right] \\ = & a \left[A(1 - \lambda)k_t - \mu(1 + n)k_{t+1} \left(1 - \alpha + \frac{1}{\mu} \right) \right] \\ & \left[(1 + n)\gamma \frac{k_{t+2}}{k_{t+1}} \mu + \lambda A k_{t+1}^{\lambda-1} \right] \end{aligned}$$

Let us now define:

$$z_t = \frac{(1 + n)\mu}{Aa(1 - \lambda)} k_{t+1} k_t^{-\lambda}$$

The difference equation can then be re-written as:

$$z_{t+1} = \Phi(z_t) = D \frac{1 - Bz_t}{-1 + Cz_t}$$

where:

$$\begin{aligned} B &= (1 + a)\left(1 + \frac{1}{\mu}\right) - a\alpha \\ C &= \left(1 + a + \frac{\beta}{\gamma}\right)\left(1 + \frac{1}{\mu}\right) - a\alpha \\ D &= \frac{\lambda}{a(1 - \lambda)\gamma} \end{aligned}$$

Φ is an hyperbola. It can be checked that $B < C$. This implied the existence of a unique steady-state \bar{z} such that:

$$1/C < \bar{z} < 1/B$$

If this steady-state is unstable, as z_t is a free variable it will jump to its steady-state value. Instability of the steady-state obtains if the absolute value of the slope of $\Phi(\cdot)$ at $1/A$ is strictly higher than one. This implies that:

$$\frac{DB^2}{C-B} > 1$$

or:

$$\frac{\lambda}{a(1-\lambda)} \frac{\left[(1+a) \left(1 + \frac{1}{\mu} \right) - a\alpha \right]^2}{\beta \left(1 + \frac{1}{\mu} \right)} > 1$$

This condition is satisfied if μ is small enough. In that case, z_t jumps to \bar{z} . The dynamics of the capital is then given by the difference equation:

$$\bar{z} = \frac{(1+n)\mu}{Aa(1-\lambda)} k_{t+1} k_t^{-\lambda}$$

Let us now give a condition insuring that the cash-in-advance constraint is binding. We must verify that: $1 + v_t = \frac{(1+n)\gamma_t m_t}{m_{t-1}} \leq 1 + r_t$.

Given the definition of \bar{z} , the inequality reduces to:

$$\bar{z} \leq \frac{\lambda}{a\gamma(1-\lambda)}.$$

Appendix 4

Study of the dynamics: case “ μc ”.

Let us recall the expression of the difference equation (69):

$$k_{t+1} = \frac{1}{1+n} \left[\omega_t - \left(\frac{\mu+1}{\mu} - \alpha \right) m_t \right] \quad (69)$$

Under the assumption of a Cobb-Douglas life-cycle utility function, (45) is now:

$$(1+n) \frac{m_{t+1}}{R_{t+1}} \left(\gamma + \frac{1}{1+a} \beta \right) = m_t \left(\frac{\mu+1}{\mu} - \frac{\alpha}{1+a} \right) - \frac{1}{1+a} \omega_t \quad (70)$$

Using (69) and (70) one obtains a second-order difference equation in k_t .

We now concentrate on the existence of a steady-state. Combining the two dynamic equations at steady-state, one gets an equation in k :

$$\frac{1}{1+a}\omega = (\omega - (1+n)k) \frac{1+n}{\frac{\mu+1}{\mu} - \alpha} \left[\frac{1+\mu}{\mu} - \frac{\alpha}{1+a} - \frac{1+n}{R} \left(\frac{\beta}{1+a} + \gamma \right) \right]$$

A trivial solution to the preceding equation is $k = 0$. Another solution is:

$$\frac{\frac{\mu+1}{\mu} - \alpha}{(1+n)(1+a)} = \left[1 - \frac{1+n}{A(1-\lambda)} k^{1-\lambda} \right] \left[\frac{\mu+1}{\mu} - \frac{\alpha}{1+a} - (1+n) \left(\frac{\beta}{1+a} + \gamma \right) \frac{k^{1-\lambda}}{A\lambda} \right]$$

The right-hand side of the later equation is a function of k which is positive at 0, reaches a minimum (where the value of the function is negative), and tends to infinity as k increases. One cannot exclude that there exists two non trivial steady-state values for k . However, the first term into bracket is necessarily non negative otherwise savings would be higher than the wage income of an individual. Let us denote \tilde{k} the value of k such that:

$$\tilde{k}^{1-\lambda} = \frac{A(1-\lambda)}{1+n}$$

There will exist a non trivial steady-state between 0 and \tilde{k} if the following condition is met:

$$\frac{\frac{\mu+1}{\mu} - \alpha}{(1+n)(1+a)} < \frac{\mu+1}{\mu} - \frac{\alpha}{1+a}$$

But this is always true. Let then be \bar{k} the non trivial steady-state value.

In order to study the dynamics, let us introduce the variable $u_t = m_t k_t^{-\lambda}$. Let us take the ratio of (69) to (70); this yields a first-order difference equation in u_t :

$$\frac{u_{t+1}}{A\lambda} \left(\gamma + \frac{\beta}{1+a} \right) = \frac{u_t \left(\frac{\mu+1}{\mu} - \frac{\alpha}{1+a} \right) - \frac{A(1-\lambda)}{1+a}}{A(1-\lambda) - \left(\frac{\mu+1}{\mu} - \alpha \right) u_t}$$

We re-write this equation like:

$$u_{t+1} = \Psi(u_t) = D \frac{Bu_t - 1}{1 - Cu_t}$$

where:

$$\begin{aligned} D &= \frac{\lambda A}{\gamma(1+a) + \beta} \\ B &= \left(\frac{\mu+1}{\mu} - \frac{\alpha}{1+a} \right) \frac{1+a}{A(1-\lambda)} \\ C &= \left(\frac{\mu+1}{\mu} - \alpha \right) \frac{1}{A(1-\lambda)} \end{aligned}$$

One checks that: $B > C$. A study of Ψ allow us to conclude that there is a unique steady-state value \bar{u} such that:

$$1/B < \bar{u} < 1/C$$

A sufficient condition of instability of this steady-state is:

$$\Psi'(1/B) > 1$$

so that:

$$D \frac{B^2}{B-C} > 1$$

or:

$$\frac{\left[\left(\frac{\mu+1}{\mu} - \frac{\alpha}{1+a} \right) \frac{1+a}{1-\lambda} \right]^2 \frac{\lambda}{\gamma(1+a)+\beta}}{\frac{a(\mu+1)}{\mu(1-\lambda)}} > 1$$

This inequality holds if μ is small enough.

u_t then jumps to its steady-state value \bar{u} . We now have a static relation between m_t and k_t :

$$m_t = \bar{u} k_t^\lambda$$

Substituting for m_t in equation (69), one obtains a first-order difference equation in k_t . It is easy to see that the dynamics converges monotonically to a steady-state \bar{k} .

We now give a condition insuring that the cash-in-advance constraint is binding. We must verify that: $1 + v_t = \frac{(1+n)\gamma_t m_t}{m_{t-1}} \leq 1 + r_t$. From $m_t = \bar{u} k_t^\lambda$, and the dynamic equation of k_t the inequality reduces to: $\bar{u} \geq \frac{\mu(A(1-\lambda) - \frac{\lambda A}{\gamma})}{1+\mu(1-\alpha)}$.