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Negatives as fictions in 16th and 17th Century mathematics

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Abstract

The use of “negative” entities is a fact attested in various mathematical practices across various contexts since antiquity. Although we now have a better knowledge of this history in its complexity and its richness, many phenomena occurring in it are still blurred and oversimplified by the retrospective look we tend to posit on it under the heading of the history of negative *numbers*. Not only because of the difficulties attached to the use of a general category of “number” in history of mathematics, but also because our “numbers” are closely related to a specific kind of conceptualization in terms of “domain extension”. In this setting, the early modern period is often considered as playing a significant role, because it is the time when symbolic algebra emerged and because it is assumed that this symbolical approach forced actors to overcome various epistemological obstacles to the acceptance of new kind of “numbers”. In the first section of the paper, I detail several historical studies relying on this point of view and point to several of their shortcomings. In the second section, I propose a different point of view, based on the actor’s categories. I put particular emphasis on the fact that negative entities were conceived by many mathematicians from 16th and 17th as “fictitious” entities and I provide a contextual study of this general framework. In the third section, I show how this context allows a different reading of the various discussions and reflections on negatives occurring in the second half of the 17th century.

Key words : Negative quantities - Fictive numbers – False position - Michael Stifel - Gerolamo Cardano - René Descartes - John Wallis - Gottfried Wilhelm Leibniz.

Introduction

The use of “negative” entities¹ is a fact attested in various mathematical practices across various contexts since antiquity. They usually emerge as correlates to an operation (typically, but not necessarily, a form of “subtraction”), opposite to another one (typically, but not necessarily, a form of “addition”) – and also to the fact that this operation can be repeated “several times”. Whether or not the term used to denote these situations is related to the vocabulary of “negation” or is marked by a specific sign, one important feature in these contexts is the existence of opposite ways of manipulating mathematical objects and that this opposition obeys some rules, be they explicit or implicit. A typical example is given by the so-called “rule of signs” (which governs the relationship between what we now call “subtraction” and “multiplication”), but another example is given by the operations on what we now call “exponents”. More rarely, negative entities may appear as isolated results in the determination of sought quantities, in which case they are usually dubbed “impossible” or “absurd” solutions, since it is assumed that a number or a magnitude cannot be “less than nothing”. There is no necessity, however, in this (dis)qualification since there may exist ways of interpreting these results so that they lose their absurd character. A typical example is given by the interpretation of the sought quantity as a “debt”, a “lack” or a “default” of something.

Although we now have a better knowledge of this history in its complexity and its richness, many phenomena occurring in it are still blurred and oversimplified by the retrospective look we tend to posit on it under the heading of the history of negative *numbers*. Not only because of the difficulties attached to the use of a general category of “number” in history of mathematics, but also because our “numbers” are closely related to a specific kind of conceptualization. In a contemporary setting, the most common way to handle negative numbers is to see them as belonging to a domain (typically the “set” of integers), which extends that of another domain, that of “natural numbers”, and can in turn be extended into other domains: rational, real and complex numbers. In this setting, each domain can be said “part” of the other in the sense that there is a way to “enlarge” the initial domain by equipping it with new relations (typically by allowing the existence of “inverse”, without restriction, to some given operations) and introducing new entities which satisfy these new properties. This leads to the insistence put on the role of opposite relations, as in the picture we started with (although many usages provided in the historical sources does not testify for a symmetric treatment of “negative” and “positive” quantities)². Since these relations did not hold in the original domain, it is tempting to see these various “extensions” as the introduction of new “possibilities”. From the epistemological and psychological point of view, this process is thus often described in terms of the overcoming of “impossibilities”. Such an interpretation is, at first sight, consistent with the fact that negative entities appearing as isolated results have been called for a long time “impossible” or “absurd”. This framework, which I will designate as that of “domain

¹ Throughout this study, I will often use the expression negative “entities” to prevent talking too quickly of negative “numbers” or negative “quantities”. This deliberately informal expression should not be understood as having any ontological import and the realm of “entities” should be considered, depending on the authors, as possibly including pure “names”, “symbols” or “terms”.

² See papers in this special issue by Matthieu Husson, Agathe Keller and Karine Chemla.

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extension”³, tends to go along with a teleological reading in which one points to the apparent steps in the “overcoming” of epistemological “obstacles”. The progress of mathematical thought is thus represented as a way to destroy and conquer previous “absurdities” and extend the domain of expressive means and acceptable entities. The progressive acceptance of imaginary numbers or of non-Euclidean geometry are typical examples mentioned to foster this view. I will detail several studies relying on this point of view in the first section of this paper.

Relying on this retrospective account, it is natural (but, as I will show, misleading) to distinguish between the sporadic use of negative entities and their taking place as genuine mathematical objects *in a domain* (itself seen as belonging to a series of growing numerical domains). This is where early modern authors usually enter the picture. Indeed, when considering the origin of this logic of “domain extension”, a very common move is to relate it to the change of view occurring in the early modern period and, more specifically, to the birth of symbolic algebra at the end of the 16th century. Reasoning with symbols allowed, it is argued, to postpone the moment of interpretation and to manipulate expressions without meaning on a par with ones denoting real ‘ordinary’ entities – what Leibniz called a ‘blind’ or ‘symbolic’ knowledge⁴. Describing this new trend, Helena Pycior claims that it included, beyond new results (such as the famous solutions to cubic and quartic equations), “a new language (the symbolical), an *expanded algebraic universe (with negative and, tentatively, imaginary numbers)*, a new emphasis on analysis as the special method of algebra” (Pycior 1997, p. 1, my emphasis). At the beginning of his study on the history of numerical series, Giovanni Ferraro underlines in the same manner the crucial role played by what he calls the “abstract and symbolic concept of numbers” supposedly invented at the Renaissance. According to him, it “made it possible to introduce new species of numbers in addition to natural numbers. Irrational numbers, negative numbers, and imaginary numbers entered the realm of numbers, although (and this is a very important point) they were *epistemologically different* from natural numbers” (Ferraro 2008, p. 104). The emphasis put by the author on the “epistemologically different” nature of these “other” numbers is clearly related to the fact that they are supposed to extend through purely symbolical means and by symmetry the domains of numbers seen as ‘ordinary’ or ‘natural’ (like those occurring when counting a multiplicity or measuring a quantity, i.e. those which are supposed to occurred in more “concrete” situations).

Interestingly enough, negative numbers are precisely one of the key examples which were provided in support of the claim that there are ‘revolutions’ in mathematics (Dunmore 1992, p. 209-225). Taking into account the fact that negative quantities appeared in several contexts before early modern times, Dunmore argued that revolutions do not occur on the level of objects, but on a “meta-level”, that of beliefs, interpretations, values, standards attached to mathematical practice. This is attested, according to her account, by the resistances and discussions emerging around these entities in the early modern period – a fact which has clearly no equivalent in prior periods or in other cultural contexts. This early modern setting is thus presented as paving the way for a radical change in the general conception of what mathematics is about (which supposedly came to full completion in the 19th century). In this framework, the

³ Following Manders (1989).

⁴ *Meditationes de cognitione, veritate and ideis* (1684), Engl. transl. (Leibniz 1989), p. 23-27.

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question is not so much the appearance of negative numbers *per se* than, as was also argued by Pycior and Ferraro, the *epistemological* change occurring when they took place in the new “expanded algebraic universe”. The critical discussion of this so-called ‘epistemological’ change and the supposed ‘resistance’ it had to overcome will be the main target of my study.

Indeed, when considering these narratives in more detail, one is struck by the many grey areas they enclose. The most obvious one is that they usually have to concede that the full acceptance of this “expanded algebraic universe”, based on a conception of number seen as a symbolic expression, did not truly occur before the birth of more ‘formal’ approaches to mathematics *at the beginning of the 19th century* (see Pycior 1997, p. 39; Dunmore 1992, p. 217, 223). Accordingly, the previous periods, beginning with European Renaissance, since they were times of resistances and hesitations, appear as comparable to the so-called ‘crisis’ presumably induced by the discovery of incommensurable magnitudes in the Pythagorean world (see Dunmore 1992, p. 214)⁵. In the first section of this paper, I will go in more details into these historiographical considerations to indicate at the same time how widespread and how mysterious the interpretation of negative numbers in terms of “domain extension” and “epistemological obstacles” are. Then, I would like to argue in the opposite direction: what is striking in the period marking the birth of symbolic algebra – especially when considering the impressive increase in mathematical productions and discussions occurring at that time – is that one does *not* find in the sources anything resembling a ‘crisis’ related to negative entities. The reflections and discussions surrounding negative entities were very sparse and generally kept, as occurs in Cardano or Harriot, on the level of individual ruminations. More importantly, they were related to other issues (such as the appearance of roots of negative numbers in the resolution of equations or the computation of logarithms). Most of the prominent actors in the development of symbolic algebra, including Vieta, Stevin, Girard or Descartes, did not pay much attention to negative solutions to equations (they usually simply ignored them) or, when they mentioned them (for the “generality of the rule”, as Girard put it), did not seem to have much trouble with them.

One is struck, in particular, by the fact that the paradigm of the controversies mentioned in relation to negative numbers is given by debates such as the one occurring between Antoine Arnauld and Jean Prestet – a debate to which I will come back in section 3. Yet, this discussion began after the publication of the former’s *Nouveaux Elements de Géométrie* in 1667 and was really made public with the second edition of Prestet’s *Nouveaux Elements* in 1689. Indeed, Prestet (1689) contains the reproduction of an exchange of letters on negative numbers which occurred between Prestet and Arnauld around 1675 – that is to say almost one century *after* the beginning of symbolic algebra in the works of mathematicians such as Vieta or Stevin. In the second section, I would like to reverse the perspective and try to understand why there were no particular controversies accompanying the use of negative entities in 16th and first half of the 17th century, although the development of symbolic algebra certainly made them of a more widespread use. To this end, I will follow another strategy, based on actors’ categories and try to reconstruct the conceptual framework in which they took place. I would like to emphasize

⁵ On the fact that this ‘crisis’ appears as a convenient myth to recent historians of Greek mathematics, see Knorr (2001). My paper could be seen, *caeteris paribus*, as a similar enterprise with regard to the ‘crisis’ supposedly induced by the development of symbolical algebra in western mathematics.

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the role played by considering certain mathematical entities, such as ‘negative’ or ‘defective’ ones, as ‘fictions’. I will try to explain why this very common way to designate negative entities allowed mathematicians to incorporate these ‘absurd’ entities into their reasoning without much trouble – in a larger framework marked by the reappraisal of the ‘analytical’ mode of reasoning. For this reason, as I will show, the use of negative entities was considered by them in *continuity* with a mangle of existing practices and not as a revolutionary or a critical move. Finally, in a last section of a more epistemological flavour, I would like to assess in a different way the discussions occurring at the end of the 17th century. In particular, I would like to emphasize the fact that negative numbers appeared to some actors at the centre of a *network of problems*. I will conclude by sketching an alternative explanation to the fact that negative numbers were recognized as *bona fide* objects than the one given by the overcoming of “epistemological obstacles”.

1. Some preliminary historiographical considerations. On ‘revolutions’, ‘crises’, ‘epistemological obstacles’ and other similar analyst’s categories.

The general framework usually attached to a full acceptance of negative entities in mathematics is a transformation of the concept of “number” so radical that it has been presented as one of the rare examples of a genuine ‘revolution’ in mathematics (Dunmore 1992). The diagnosis of this radical transformation was certainly not new. It was already advocated in the first half of the 20th century by influential authors such as Jacob Klein⁶. Klein himself expanded interpretations dating from the 19th century in terms of the progressive introduction in history of mathematics of new ‘symbolic forms’ – or, more precisely, of a transformation of mathematics as dealing with such symbolic forms rather than with ‘quantities’⁷. As emphasized by G. Ferraro when mentioning this transformation, and in full accordance with Dunmore’s characterization, this change occurred not so much on the level of practices⁸, than on the meta-level of the “epistemological nature” of number (although this change surely prompted the development of new practices). The following passage, taken from the opening of a study on the change of the concept of number in early modern times, provides a summary of this general picture:

Jacob Klein's seminal work, *Greek Mathematical Thought and the Origin of Algebra*, suggests an explanation for this transformation of the number concept in the work of such scholars as Stevin and Viete. He proposes that a conceptual transformation occurred that permitted a symbolic understanding of number, allowing numbers to refer to things other than numbered objects rather than always being associated with a “definite number of definite objects.” An increased interest in “practical, i.e. applied, mathematical disciplines,” he implies, led to more elaborate techniques of calculation and the introduction of negative and irrational magnitudes. Additionally, he claims

⁶ Klein (1934-1936/1968).

⁷ This is the evolution to which Dunmore (1992) points when considering the change on what she calls the “meta-level”.

⁸ For a survey of the use of negative numbers in western culture before early modern time, see Sesiano (1985), which contains at its beginning a quick summary of some of the main occurrences of negative numbers before middle age, and Schubring (2005), chap. II section 2. See also (Wagner 2017, chap. 2, p. 46-58). For mathematics in Chinese, see Chemla & Guo (2004) and [Chemla’s paper in this issue](#). For mathematics in Sanskrit, see [Keller in this issue](#). For mathematics in Arabic, see Bellosta (2004) and [Sammarchi in this issue](#).

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that the practitioners were unaware of the “novelty” of their number concept until they came into contact with the *Arithmetic* of Diophantus. Moreover, he argues that treating numbers as symbols rather than as references to objects, was crucial to the emergence of modern mathematics (Neal 2002, p. 4)

Note that the emergence of “modern” mathematics is seen as following an epistemological transformation of practices, which was in previous periods “unaware” of itself. This is a key point in the argument (i.e. what would mark, in Dunmore’s vocabulary, the transition to the “meta-level”). In Klein’s terminology, the passage to the meta-level is mirrored on the level of objects by the introduction of what he called objects of “second intention”⁹. Another way to put it is that symbols appeared, with the development of symbolical algebra, not only as tools, ways of denoting ordinary mathematical objects, but as proper theoretical objects *per se*. Accordingly, the category of “number” transformed itself so as to become a purely symbolic form¹⁰.

A similar framework is recalled by Paolo Mancosu in his study on the history and philosophy of mathematical practice in seventeenth century. It occurs under the heading of the “algebraization of mathematics” precisely when introducing the debates over negative numbers in 17th century (“Perhaps one of the most interesting discussions”, he claims, “on the foundation of algebra”):

arithmetic develops more and more into a generalized arithmetic by relying on algebraic techniques and *extending the field of numbers* (...) The process is accompanied by a deep mutation in the *concept* of number, now seen as a symbolic entity as opposed to “pure” units whose mode of being may be subject to dispute. (Mancosu 1996, p. 86. My emphasis)¹¹

With or without reference to Klein, this picture has served as a background for many historical reconstructions of the history of “negative numbers”. We find, for example, the same idea of an extension of the field of numbers at the core of Helena Pycior’s study on the development of “impossible numbers” in early modern England (Pycior 1997). Opening her study with the first sentences of Cardano’s *Ars Magna*, she comments:

Thus the very advertisement to Girolamo Cardano's Great Art (*Ars magna*) suggested the dawning of a new epoch in the algebra of the Western world. Although less original than implied here, *The*

⁹ Here is how Klein describes this “decisive turn”: “Thus it appears that the concept of the species is for Vieta, its universality notwithstanding, irrevocably dependent on the concept of “arithmos.” *He preserves the character of the “arithmos” as a “number of...” in a peculiarly transformed manner.* While every *arithmos* intends *immediately the things or the units themselves* whose number it happens to be, his letter sign intends *directly the general character of being a number* which belongs to every possible number, that is to say it intends ‘number in general’ immediately, but the things or units which are at hand in each number only mediately. In the language of the schools: The letter sign designates the intentional object of a ‘second intention’ (*intentio secunda*), namely of a concept which itself directly intends another *concept* and not a *being*. Furthermore — and this is the truly decisive turn — this general character of number or, what amounts to the same thing, this ‘general number’ in all its indeterminateness, that is, in its merely possible determinateness, is accorded a certain independence which permits it to be the subject of ‘calculational’ operations.” (Klein 1968, p. 174).

¹⁰ On the “revolutionary” character of this extension of the concept of number as constitutive of our “modernity”, see also Page (1996).

¹¹ Without surprise Mancosu attaches the following note to this passage: “The reader is referred to Klein's classic *Greek Mathematical Thought and the Origin of Algebra* for a thorough analysis of these developments. Klein concentrated his attention in particular on Stevin, Descartes, and Wallis” (Mancosu 1996, p. 227, n. 53).

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Great Art – published in 1545, just two years after Copernicus's *De revolutionibus* and Vesalius's *De fabrica* – was an exciting scientific classic. Exciting in the mathematical way, the work announced the solution of two hitherto unsolved problems, finding the roots of cubic and quartic (or biquadratic) equations. As many other classics, it helped to redefine its field. In particular, *it fostered an expanding universe of algebraic objects through its consistent acknowledgment of negative roots* as well as its brush with imaginary roots. (Pycior 1997, p. 10. My emphasis)

Obviously, the reference to Copernicus and Vesalius is intended to relate Cardano's practice to a more general turn, reminiscent of the so-called "scientific revolution".

As should be clear from these various quotes, the framework which usually grounds this type of picture is that of the logic of "domain extension" – to the point of being sometimes even presented as a general framework for the description of historical developments in mathematics¹², or of human knowledge, if not both. For this reason, it also played a crucial role in Mathematics education under the influence of the notion of "epistemological obstacle", coming from Bachelardian history of science and of its use in the context of the "genetic epistemology" developed by Jean Piaget (Glaeser 1981). The basic insight is that human psychological development reproduces in small what historical developments show to us on a larger scale: the encounter and the overcoming of various epistemological obstacles¹³. Overcoming the apparent absurdity of some quantities being "less than nothing" is a typical example of this phenomenon. What appears in the classroom when students have to learn about this "negative numbers" is supposed to remind us in small of the difficulty that mathematicians had with these entities in the past. Accordingly, history of mathematics can provide a framework for studying these various difficulties and help the students distancing themselves from them. I mention this context because it has served, and still serves, as motivation for many historical studies, interested by a possible use of the history of negative numbers in the teaching of mathematics. Accordingly, some of the most important criticisms raised against this historical picture came from discussion in or with mathematics education (see Schubring 1986, 2005, Heffer 2008, 2011, Thomaidis 1993, Thomaidis and Tzanakis 2007).

Indeed, although the global picture drawn in the first section is very widespread, it is full of obscurities, if not inconsistencies. Let us start with the criterion proposed by Dunmore for characterizing the appearance of negative entities as a 'revolution'. One could certainly agree that what seems new is not so much the appearance of negative entities – a fact documented since Ancient Times in various contexts – than the fact that it became the object of reflections,

¹² On the logic of "domain extension", see Manders (1989). Commenting about the "ten laws" he diagnosed in 1975 (the tenth being "Revolutions never occur in mathematics"), Michael J. Crowe explains in this way how he was led to put some of them into question and in particular the last one: "I wish to note that my realization of the problematic character of these assumptions was intensified when I was invited around 1970 to present two lectures on the evolution of the notion of number at a mathematics summer institute at Pennsylvania State University. In preparing these lectures, I saw that *each new number type, from negatives to irrationals and the imaginary and hypercomplex numbers, rather than being welcomed by the mathematical community, met strong opposition*, which mathematicians set aside on grounds that can only be described as more pragmatic than principled" (Crowe 1992. My emphasis).

¹³ Depending on the author, this psychological equivalent of Haeckel's law ("ontogeny recapitulates phylogeny") can be considered as acting directly or, as is claimed by Piaget, in reverse order (the child first develops notions which came last in the historical development, typically, according to Piaget, topological notions before geometric ones).

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discussions and sometimes even controversies among scholars. Yet, a no less surprising fact when it comes to providing historical sources to this claim is that commentators usually mention as a testimony of this change the debate between Arnauld and Prestet, which occurs very late in our period (or worse, discussions from the 18th century)¹⁴. Moreover, this debate is pretty isolated, as are in fact the very rare reflections on negative numbers one finds in Cardano and in Harriot. By contrast, one sees immediately that there were *no* such discussions amongst the first proponents of symbolic algebra (and Klein's heroes): Vieta, Stevin or Descartes. In fact, as far as I can tell, there is not even a trace of *reflection* on this question in these authors¹⁵. When solving particular problems, they usually do not acknowledge negative solutions to equations, when there are some (the same is true in the vast majority of their predecessors), and when they consider negative solutions in general (for the general theory of equations or because they can be transformed into 'true' solutions by a change of variable), they do not comment on their specific nature.

The same kind of obscurity is attached to the image of an "expanded algebraic universe", which serves as a background for the supposed change of view. Although H. Pycior presented this feature as characteristic of the "new epoch" marked by the beginning of symbolic algebra, here is the conclusion she reached when closing her study on Oughtred and Harriot:

Oughtred, Harriot, and their disciples shared first and foremost a deep and irrevocable commitment to symbolical reasoning. As the two English algebraic pioneers fostered Viète's symbolical style, *they also perpetuated his hesitancy about the expanding algebraic universe*. They and most of their immediate successors reasoned symbolically, wrote algebraic equations for geometric problems, *but largely ignored negative and imaginary roots* or – in the most daring of cases, including Harriot's speculations in unpublished manuscripts – struggled to make some sense of them. In short, the algebra that was imported into England through Oughtred's and Harriot's textbooks was Viète's, not Cardano's. *In England, a deep appreciation of the symbolical style came first; openness to the expanding universe of algebra followed slowly* (Pycior 1997, p. 40).

We see here very clearly the many hesitations, to repeat Pycior's own term, attached to the diagnosis of a radical change of views ascribed to early modern times. Let us remember that on the very first page of her study, Pycior related this change to "a new language (the symbolical), an expanded algebraic universe (with negative and, tentatively, imaginary numbers), a new emphasis on analysis as the special method of algebra" (Pycior 1997, p. 1). As is clear from the passage quoted above, the first and third features are clearly attached to "Viète's symbolical style", but are *independent* of the second one (the expansions of the algebraic universe), supposedly attached to the legacy of Cardano. More than that, the first two features are presented as an apparent obstacle to the admission of the third (of which the acceptance "followed slowly"), in particular when it comes to the acceptance of negative entities. Indeed,

¹⁴ See Schrecker (1935), Schubring (2005), Mancosu (1996), Heffer (2011). The debate, on which I will come back in the last section, relies on the fact that the ratio of -1 to 1 should be equal to that of 1 to -1, although the usual rules for ratios should lead to the conclusion that $\frac{a}{b} < \frac{c}{d}$.

¹⁵ Accordingly, and in strong contrast to what is claimed by many commentators, negative entities are not central in Jacob Klein's picture (Klein's interpretation is based on the notion of "second intention", not on the logic of domain extension cf. [note ###](#)).

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what is presented here as Vieta's "hesitations" sounds like an euphemism compared to what was presented in the more detailed study conducted before in which it appeared as a "key difference" between two lines of development:

The statements and proofs of rules II and III of chapter 4 of *The Analytic Art* highlight key differences between Viète and Cardano, *differences that would separate Western algebraists through the early nineteenth century*. Cardano was basically an algebraic explorer. He was inclined to let problems dictate the development of his subject. Viète was a more traditionally philosophical thinker. He wanted to deal with the rules of the analytic art first, and only then move to its applications.

For Viète rules limited results; for Cardano results could lead to a broadening of rules. Viète's rules II and III set the boundaries for his use of "negative terms." *Rule II assured that impossible subtractions, generating negative numbers, would not slip into the analytic art*. Rule III sanctioned negative coefficients and even a rule of signs, but subject to the limitation that negative terms were terms of polynomials. Correspondingly, he almost completely confined the examples of *The Analytic Art* to equations with positive roots only, *and he avoided any discussion of negative roots*. (Pycior 1997, p. 38. My emphasis.)

Note that in this picture the acceptance of "impossible" objects goes in *opposite direction* to the "symbolic style" of Viète, followed by Harriot and Oughtred – in direct contradiction to the diagnosing of an "epistemological change" (the transformation of "number" into a purely symbolical entity), which was presented by so many commentators as a *condition* for the acceptance of negative numbers.

For this reason, the type of picture purported by the idea of an "expanding algebraic universe" has been criticized as sacrificing too much to a linear conception of the history of mathematics and as blending features which should be kept separated. The views of several actors going in divergent directions – as we just saw in the detail of the picture drawn by Pycior on the opposition between Viète's and Cardano's legacies – were conflated to form an apparent, but misleading, homogeneous and coherent landscape (Schubring 2005)¹⁶. Moreover, it has been recalled that describing this evolution in terms of the overcoming of "obstacles" just led to *select* some supposedly "progressive" aspects of the actors' thought and leave out more surprising ones¹⁷.

¹⁶ Schubring had already expressed serious doubts about the notion of "epistemological obstacles" and what he saw as a misleading teleological approach to the history of negative numbers in Schubring (1986). The idea that the "Kleinian" approach sacrifices much to a linear conception of history is also put forward by A. Malet, although without particular emphasis on negative numbers in Malet (2005).

¹⁷ A typical example is analysed by Albrecht Heffer when turning to the so-called "epistemic difficulties" related to the introduction of negative numbers in the classroom (Heffer 2011). He notices that the use of the "number line" as an intuitive aid, whose paternity is ascribed to John Wallis, does not take into account the non-standard view Wallis held at the same time about what it means to divide a positive number by a negative one. Indeed, relying on the fact that the exponent $1/n$, for n positive, tends to infinity when n goes to zero before becoming negative, Wallis concluded in his *Arithmetica Infinitorum* (prop. 104) that the ratio of a positive to a negative number must therefore be something "greater than infinity" (*rationem plusquam infinitam*). (Thomaidis and Tzanakis 2007) have provided a similar argument by studying the confusion occurring in Descartes when dealing with the order relation amongst negative solutions to equation (which, for this reason, should not be coined "negative numbers" without a *caveat*).

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Indeed, even if we can clearly find in authors such as Wallis or Leibniz the general idea that mathematics allows for the progressive acceptance of entities deemed “impossible” by previous generations of mathematicians (more on this in the third section), one should not forget some intriguing and seemingly “archaic” features of their thought. Let me just mention, before coming back to it, the way in which Leibniz describes the appearance of negative solutions to problems for a project of handbook he wrote around 1700:

Negative quantities, when it comes to subtract a greater from a smaller, often occur in calculation. It is permissible to consider that they do not answer the question, but in truth they answer it most perfectly. Not only do they indicate that the question was badly conceived (even if it had been accepted with indulgence, since one could not foresee [the answer]), but they also indicate how it should have been conceived, and what should be answered if it was correctly conceived (Leibniz 2018 p. 149, my translation into English).¹⁸

Here, “negative quantities” are clearly seen not as genuine entities, but as expressions indicating how a certain problem was “badly conceived” (and has thus to be reformulated in order to receive a solution). They are certainly not *bona fide* objects, even of “second intention”, induced by a large acceptance of “symbolical” or “blind” thought in mathematics.

Before turning to a closer study of actors’ categories, let me address another interpretation, less preoccupied by the “meta-level”. Indeed, one could object, *contra* Dunmore, that some revolutions occur in silence and that they do not have to be manifested by discussions, reflections, controversies and progressive acceptance about this or that change. According to this view, the proper level in which a mathematical revolution occurs should be assessed at the level of practices and the fact that negative entities occurred in previous periods does not mean that a change of practice did not occur in relation to them in the early modern period. Indeed, a too narrow focus on the level of discourse would make us miss one of the most important changes occurring at the time through the development of symbolic algebra: the birth of Cartesian algebraic geometry (followed rapidly by the birth of differential calculus and an even broader expansion of the domain of geometry). In this context, negative numbers naturally occur from the variations of positions of points in the plane. Accordingly, the question should not be whether or not Descartes accepted explicitly negative entities in his mathematical ontology (he does not seem to have)¹⁹, than whether he acknowledges them in his practice (by the establishment of algebraic equations expressing a new type of mathematical generality which was inaccessible to his predecessors). This claim has been made forcefully by Ken Manders in his analysis of the logic of “domain extension” and in a series of papers in which he emphasizes:

¹⁸ *Quantitates negativae, cum a minor substrahi debet majus, saepe oriuntur in calculo, et licet non videantur respondere ad quaestionem, reapse tamen respondent perfectissime, non tantum enim indicant, quaestionem fuisse male conceptam (etsi venia danda sit, quia praevideri non poterat), sed etiam quomodo fuerit concipienda et quid ad eam recte conceptam sit respondendum* (GM VII, 70). The context is a problem in algebraic geometry in which one has to fix a parameter in order to obtain a positive solution: the appearance of negative solution indicate the bound for the variation of the parameter and allow to reformulate the problem so that no negative solutions occur.

¹⁹ In his theory of equations, Descartes explains in fact how to *get rid* of negative solutions by a translation of the data of the problem ([see quote note 12 and below](#)).

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Unification and clarification through model-completing domain extensions (or partial model completions and related constructions) are pervasive in the history of mathematics. Negative integers give unification in the theory of equations. Negative occurrences of line segments allow a uniform treatment of geometric solutions to equation. (Manders 1989)²⁰

In this setting, “domain extension” is not seen (or does not have to be seen) as the overcoming of “obstacles”. It is a dynamic of the practice itself which allows for more uniform and powerful techniques. The reason they succeed does not have to be the fact that actors change their concepts or beliefs, but simply the fact that they provided new solutions to problems in a more efficient way. But how appealing this view might appear, it must be treated with caution. Indeed, a closer study of Descartes’ geometrical practice shows that he does *not* allow an algebraic unknown to refer to a “negative” segment. This fact has very important and unfortunate consequences on the way in which he is lead to deal with equations in the treatment of geometrical problems²¹. In fact, not only has Descartes to devise equations which hold only locally and can, for this reason, describe *under different algebraic forms* one and the same curve (the exact opposite situation to what is claimed by Manders in the passage quoted above); but he also has to painfully devise a vast array of equations corresponding to the combinatorics of the variation of algebraic signs, whereas a reference to negative segments would have precisely allowed one and the same formula to refer to all of these possibilities at once²².

Although the interpretation of negative signs as indicating positions certainly provided a rather intuitive model and although taking into account negative solutions played a crucial role in the general theory of equations, one should be cautious in considering that this came with a change in the underlying ontology (be it on the level of practices or on the meta-level of foundational discussions). To interpret a negative sign as a way to refer to a positive entity taken in the opposite direction is an ancient strategy²³, and does not differ much from the usual interpretation in terms of lack or debt. As recalled by Heeffer, these reinterpretations could be seen more likely as a *refusal* of genuine negative entities occurring *on a par* with positive ones: they are intended to show that an apparently absurd expression, involving a negative sign, does in fact denote a “real” quantity²⁴. In the next section, we will see that this was the most common view on negatives until the end of the 17th century. Moreover, if negative expressions indicating positions brought uniformity in some situations, they also brought *disuniformity* in others. Let me dwell a little bit about this important feature.

Besides Descartes’ treatment of Pappus’ problem, an interesting case is given by the fact that the acceptance of negative as genuine entities would entail to give sense to basic algebraic and geometrical manipulations with them. An example of the first is of course the taking of

²⁰ The note attached to the first sentence refers to Cardano as a “transition” and the note attached to the second refers to Descartes as the climax of this generality of treatment.

²¹ See the analysis of the solutions to Pappus’ problem in *La Géométrie* given by (Maronne 2007, p. 49-94).

²² (Maronne 2007, p. 60). (Thomaidis 1993) also provides a nice example where Descartes draws a segment corresponding to the “false root” to equations, but simply ignore it when it comes to use this construction as a solution for the problem at stake (in that case, the trisection of an angle).

²³ See Husson’s paper in this issue.

²⁴ This claim is emphasized by Heeffer (2007) to point to the fact that studies of negative solutions to equations in medieval mathematics, such as Sesiano (1985), fail in their claim for the acceptance of negative numbers if they only mention documents in which these solutions are interpreted either as absurd or as debts.

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their square roots, as appeared with the solution to equations which Cardano called “sophistic” (and Descartes “imaginary”). This launched in Cardano a reflection on the uniformity of the rules governing the multiplication and, in particular, on the “rule of signs” (which should be contested if one allows a square to have a negative sign). I will come back to this reflection and its legacy in Commandino, Harriot and Arnauld in the last section of the paper. Another related issue was the possibility of inserting a proportional mean between a positive and a negative magnitude. It is striking that one of the first authors to posit the numerical line as an interpretation of negative quantities, Wallis, is also the one who intended to provide such a geometrical construction (which corresponds “arithmetically” to taking the square roots of negative quantities). By considering the usual diagram for the mean proportional between two segments AB (whose value is supposed to be $+b$) and BC (whose value is supposed to be $+c$), he then wonders what happens when B would be taken on the other side of A (with AB now equal to $-b$). His (wrongheaded) conclusion is that it amounts to taking the tangent in P instead of the perpendicular (see diagram below where Wallis describes the two cases, when B is inside the circle and when it is outside).

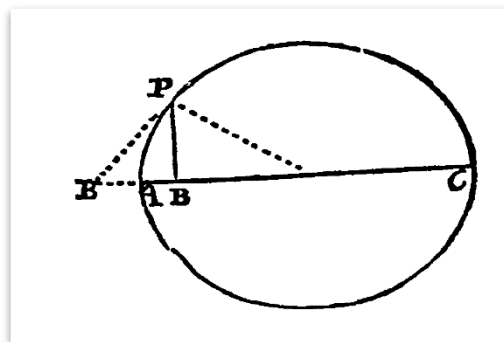


Fig. 1. Wallis 1685, p. 266

When dealing with negative segments, Wallis was henceforth led to *trouble* the uniformity of the construction of square roots, as Cardano, followed by Harriot, were led to trouble the uniformity of the rules of signs (by exploring if we could have “minus time minus gives minus”). Around the same period, Leibniz also worked (with Tschirnhaus) on this geometrical interpretation of square root of negatives, but he followed another path. He maintained the uniformity of construction (the taking of a proportional mean can always be performed, at least theoretically by the taking of a perpendicular between the two segments), but was led to the conclusion that it troubled the uniformity of operations. Indeed, in the resolution of this type of problem, one may have to form expressions such as $1 + \sqrt{-1}$. However, the two corresponding geometrical objects could not be juxtaposed (since $\sqrt{-1}$ denotes a perpendicular) whereas, according to Leibniz (following Euclid and Descartes), juxtaposition was *requested* in the very definition of what a geometrical addition should be (Ottaviani 2021).

The fact that we now mention *reflections* surrounding the appearance of negative entities might seem to go in Dunmore’s direction (a progressive change occurring at the “meta” level), but it should be noticed first that these discussions are still isolated (and sometimes even private, as is the case in Harriot’s reflection on the rule of signs or in the discussions between Tschirnhaus and Leibniz, of which no public traces ever transpired in their published works).

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terms entering multiplications and divisions between terms added or subtracted, Stifel reaches the following situations:

Regula multiplicationis & diuisionis, quantum attinet ad multiplicationem, quatuor uariationes habet huiusmodi.			
$\begin{array}{r} 0 + 6 \\ \times 4 \\ \hline 0 + 24 \end{array}$	$\begin{array}{r} 0 - 6 \\ - 4 \\ \hline 0 + 24 \end{array}$	$\begin{array}{r} 0 + 6 \\ \times 4 \\ \hline 0 - 24 \end{array}$	$\begin{array}{r} 0 - 6 \\ - 4 \\ \hline 0 + 24 \end{array}$
Quantum uero attinet ad diuisionem, quatuor similes uariationes habet huiusmodi:			
$\begin{array}{r} 0 + 24 \\ \div 6 \\ \hline 0 + 4 \end{array}$	$\begin{array}{r} 0 + 24 \\ \div 4 \\ \hline 0 - 6 \end{array}$	$\begin{array}{r} 0 - 24 \\ \div 6 \\ \hline 0 - 4 \end{array}$	$\begin{array}{r} 0 - 24 \\ \div 4 \\ \hline 0 + 6 \end{array}$

Fig. 2. Stifel 1544, p. 249.

As can be seen in the above table, Stifel does not write negative numbers in isolation, but keep the form “0 – 4”. However, he does not treat them merely as operands since they can appear as results of operation. In the third column above, for example, one multiplies “0 + 6” by “0 – 4” and obtains “0 – 24” (and the same for division in the line below: “0 – 24” divided by “0 + 6” gives “0 – 4”). Stifel then comments that although some of these computations might look at first as “the most futile nonsenses” (*vanissimae nugae*), they can lead to marvellous inventions (*mirificae inventiones*). For this reason, he says, he has to add a few words about what he designates as “fictive numbers” (*ficti numeri*). By that, he refers to the numbers “less than nothing”, which appears as results in the table under the form “0 – a number”. According to him, the introduction of these “fictions” should be considered in line with what was done when introducing other well-known entities such as roots for numbers having no roots (i.e. irrational or “surd” numbers)²⁷. This already indicates that a mathematical entity can be “fictive”, but nonetheless at most useful²⁸.

Stifel then recalls that the use of these “fictive numbers” is a widespread fact when considered as intermediary steps in calculations and provides a basic example by subtracting “10 – 5” from “8 – 2”. Since the first step in the calculation is to subtract 10 from 8, this leads to a number “less than nothing”. To justify this claim, Stifel indicates the possible variation of the *subtractendum*. When it is equal to the *subtractans* it gives zero and, when greater, a number greater than zero and when less and a number less than nothing. Accordingly, he claims, 0 should be considered as a kind of “middle term” between “true” and “absurd” numbers (*quod mediat inter numeros veros et numeros absurdos*). Note that “fictive” numbers are here presented not as a way of talking about “true” numbers (like when we interpret them as debts or lack)²⁹, but as a domain of its own, having with natural numbers some relations (as

²⁷ *Quemadmodum autem variae figuntur radices numerorum sub numeris non habentibus radices, fitque haec fictio summa utilitate pro rebus mathematicis : ita finguntur etiam non frustra, numeri infra 0, id est, infra nihil.* (Stifel 1544, p. 249).

²⁸ *Vides certe, ut haec omnia vanissimis nugis videantur esse similima, et tamen Cossicae operationes, secundum ea factae, plane mirificas inventiones habent. Sed ut nihil obmittam eorum quae ad integritatem Arithmeticae pertinent, meamque facultatem non superant, dicendum mihi videtur hoc loco de Numeris fictis infra nihil.* (Stifel 1544, p. 249).

²⁹ Although Stifel also acknowledges this possibility. Interestingly enough, he emphasizes that we can have situation in which only fictive numbers occur in the determination of a true quantity. As an example he mentions

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manifested by the role of zero seen as a “middle term”). This is one reason why Stifel’s treatise is often considered as one the first in which “negative numbers” were considered as such (and not as isolated solution to equation or as operands). As we will see, this goes in Stifel with the fact of endowing “fictive numbers” with a structure of their own.

More generally, Stifel adds, algebra (“Cossa” in his vocabulary) allows one to deal on the same footing with entities which exist and with entities which we feign that they exist. In the same way that we can feign numbers “less than nothing”, we can also feign numbers “less than unity” (*minutae unitatis*), i.e. fractions of unity³⁰. Indeed, this is the way “fictions” were introduced in the treatise when dealing with the first absurdity one encounters when starting with the traditional (‘Euclidean’) definition of number as multiplicity of units, that of “parts” of unity (Stifel 1544, Lib. I, chap. 2, p. 8)³¹. One can even go further in this parallel as Stifel mentions with the example of the coupling of a geometrical progression with an arithmetical one. In this case, fractions of the units, the first kind of fiction, occur as *corresponding* to negative “exponents” (his term, which literally means “placed above”), the second kind of fiction. He then comments by showing on several examples that the operation on the first line of Figure 3 corresponds to the operations on the numbers in the second. For example, to divide the number with exponent -3 (i.e. $1/8$) with the number with the exponent 6 (i.e. 64) gives the number with exponent -9 (i.e. $1/512$).

-3	-2	-1	0	1	2	3	4	5	6
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64

Fig. 3. Stifel 1544, p. 250.

Two remarks are in order here. First, although the parallel between powers and fractions was already noticed by Arabic authors, it is here marked by the same sign than that of subtraction and explicitly related to *a specific kind of numbers (numeri ficti)*³². But more than that, these “fictive” numbers do not appear only as isolated results or intermediary steps of calculations. They manifest a structure of their own, which is interesting and useful in and of itself – since it allows to compute additively on powers, instead of multiplicatively on fractions³³.

the subtraction of $0 - 5$ from $0 - 3$ (p. 249). He wrongly states that this gives $0 + 3$, but it seems to be a *lapsus calami*.

³⁰ *Sic Cossa solet, pro immensa copia sua, iis uti quae sunt, et iis quae finguntur esse. Nam sicut supra unitatem ponuntur numeri integri, & infra unitatem finguntur minutiae unitatis, & sicut supra unum ponuntur integra, & infra unum ponuntur minuta seu fracta: si supra 0 ponitur unitas cum numeris, & infra 0 fingitur unitas cum numeris. Id quod pulchre repraesentari videtur in progressionem numerorum naturali, dum servit progressionem* (Stifel 1544, p. 250). This parallel is already mentioned in Lib. I, chap. 6, p. 48.

³¹ They also appear when Stifel introduces irrational numbers. Chapter 1 of Book II begins with the following question: are irrational numbers true or fictitious? (*Merito disputatur de numeris irrationalibus, an veri sint numeri, an ficti*, p. 103). Note that the question is presented as widespread amongst mathematicians. We will see it again appearing in Peletier’s treatise.

³² Bellosta (2004) & **Sammarchi in this issue**. Stifel’s idea was already developed by Chuquet in the preceding century, but left out of the published version of his treatise, see Spiesser (2005) and Kouteynikoff (2006).

³³ The autonomy of negative numbers is also marked by the fact that Stifel has a geometrical model for surfaces to be subtracted (which he indicates by crosshatched areas), see Rommevaux-Tani (2020).

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Interestingly enough, Cardano uses the very same term, *numeri ficti* (or *equationes fictae*), to qualify negative solutions to equations in his *Ars Magna* from 1545 – a more standard way of looking at negative entities at the time. What is striking, however, is that they do not occur in passing, in the resolution of specific problems, but in a more theoretical manner. This appears already in the very first chapter of the treatise when Cardano gives a first description of the solutions to cubic equations (dealt with in more detail in chapter 11 to 23). He first recalls that a square such as 9 can be obtained from two roots: 3 or -3 (in his notation “m: 3”). Henceforth, a quartic equation of the form $x^4 + 12 = 7x^2$, which can be solved in x^2 ($= 4$ or 3) has four solutions. The context is thus clearly that of the types of solutions to equations and a first hint at what will later be called the fundamental theorem of algebra. Cardano gives for the various kinds of cubics (which he separates into several cases since he does not allow negative coefficients to enter the equation itself) the number of positive (“true”) and/or negative (“fictive”) roots. At this occasion, he mentions the possible appearance of roots which are “neither true nor fictive” (*neque verae neque fictae*), i.e. what we would call imaginary roots.

Here, like in Stifel, the opposition is between “true” and “fictive” solutions, which Cardan also designates as debts – a vocabulary which will be explained later in the treatise and on which I will come back. He also sometimes calls these solutions “false” (*Ars Magna*, p. 4), a term which will be taken up by Descartes. But why consider these “fictive” or “false” solutions at all? One essential insight, which Cardano mentions in this first chapter and that he will later develop as such in chapter 6, is that an equation which could seem impossible to solve under a particular form can be *transformed* into another one which is solvable (by what we would call a change of variable)³⁴. Accordingly, as presented already at the end of the first chapter, it is possible to transform the “true” and “fictive” solutions one into the other by simply putting one or more terms on the other side of the sign of equality. Although it is a different use of the fiction, Cardano’s approach thus partakes with Stifel’s one the idea that these entities are taken in a general structure (here that of transformation between equations). As an example, Cardano takes the equation $x^3 + 11x^2 = 72$ and shows that its “fictive” solutions (-3 and $-4 - \sqrt{40}$) can be transformed by changing the order of the terms into the “true” solutions of another equation ($x^3 + 72 = 11x^2$). This is an interesting and original way of looking at the question since any fictive numbers can therefore be seen not as designating an entity directly, but as an indication on how to change the problem so that the solution becomes “true” – an idea we will find again in Stevin and later in Leibniz.

What it indicates is that the distinction between “true” and “fictive” is not so much in the sought quantity in and of itself, but in the formulation of the problem³⁵. The very same quantity which appears not to be “true” in one formulation could be “true” in another. This marks an important step in the passage from “absurd” to “fictitious” since “fictions” acts more like suppositions than like absurdities *stricto sensu*. They are entities or situation for which we simply do not know whether they exist or not (or, better, under which conditions they may exist). This will appear clearly by studying the broader use of the term “fiction” in mathematics at the time. But I will also come back in the last section on the role played by this interpretation

³⁴ On the importance of the role of substitution in Cardano’s approach, see Stedall (2011).

³⁵ See (Wagner 2017, p. 55).

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in the way champions of the “symbolic style” such as Wallis or Leibniz interpreted fictions such as negative numbers as taking place in the *continuity* of the ancient practice of “analysis” (and not at all as a revolutionary move), in which acts *as if* some solutions were given.

This general strategy is made fully explicit by Cardano in his famous chapter 37 in which he deals with the rules for supposing something false in a problem (*De regula falsum ponendi*). Cardano first explains that it is often possible to solve a problem by transforming the true solution into a fictive one (i.e., in our modern notations, to solve the problem in $-x$ instead of solving it in x). As an example, he gives a problem about the dowry of Francis’ wife, which is supposed to be 100 *aurei* more than Francis’ own property. In addition, we also know that the square of the dowry is 400 more than the square of the property. In order to determine the solution, Cardano proceeds by assuming that Francis possess “one thing m .” (*habere rem unam m.*), *res* designating the unknown and m : the minus sign³⁶. His wife’s dowry being 100 aurei more, the addition of her property with that of Francis can now be written in modern notations: $+100 - x$ (see image below for Cardano’s original rendering: *100 m: 1 pos* – where *pos* stands for the thing posited). By squaring these two expressions, Cardano obtains what we would note x^2 (in his notation *1 qd.*) on the one side and $10000 + x^2 - 200x$ on the other (*10000 p: 1 qd. m: 200 pos.*). According to the data of the problem, the difference between the two is equal to 400. One then easily concludes that 9600 equals $200x$, i.e. x equals 48.

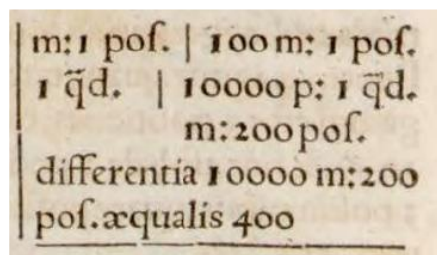


Fig. 4. Cardano 1545, p. 65³⁷

As can be seen, the equation is thus solved in x and Cardano can then conclude “so much he had in minus, i.e., in debt” (*tantum habuit m.; id est debiti*) – since our supposition was that Francis had this property “in minus”. Accordingly, “Francis has -48 *aurei*, without any capital or property, and the dowry of his wife is 52 *aurei*. By working this way, you can solve the most difficult and inextricable problems” (Heffer 2007). This is followed by similar examples of the same way of proceeding in elementary problems.

The last example treated in this chapter being that of imaginary solutions to algebraic equations (which Cardano calls *sophisticae*), the first series of examples has received few attention, although it is quite remarkable in its own way. Indeed it shows that there is no particular problem with negative solutions, which can always be reinterpreted as positive solutions to equations expressed in $-x$ (when the latter exists; if not, the problem would be truly “impossible”). The same idea appears in Stevin’s *Arithmétique* (1585), in a rare passage in

³⁶ It can be put before or after the sign for the quantity (be it a number or a *res*).

³⁷ This little *formularium* is inserted in a box next to the text which explains the operations in a mixture of natural language (with complete words such as *quadratum*, *positio*, etc.) and symbols (for numbers but also for operations such as p : and m :).

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which he discusses negative solutions explicitly (Stevin 1585, p. 332). He does not talk about “fictions”, but uses a similar (and rather strange) term in French by saying that solutions “by minus” are “solutions songées” (literally “dreamt”), as opposed to “true” solutions. They have, however, he claims, their own utility – the usual rhetoric for introducing fictions, as we have already seen in Stifel (and will encounter in many other authors from the period). Indeed, although Stevin does not acknowledge negative solutions to equations in isolation, they sometimes occur, as he recalls, as preliminary steps in a larger problem in which the sought quantity has then to be added to another given quantity. In this case, when the added quantity is greater than the sought one, the last result would be a positive quantity. It is henceforth useful to know how to solve an equation whose solutions are negative in the intermediary steps (we will encounter a similar situation, when one has to pass through fictive numbers in Peletier). Contrary to Stifel, Stevin writes them directly as -3 or -7 (*Arithmétique* p. 333, 334). In order to do so, he intends to explain on every example of equations for which he already gave a positive solution whether or not it admits a negative solution and how one can find it by simply changing x into $-x$ in the equation (so that one can recover one of the previous cases considered)³⁸.

As is well known, a similar strategy is to be found in Descartes’ *Géométrie* from 1637. Descartes did not talk about fictive solutions and called negative roots “false” – but recall that Cardano put “falsa” and “ficta” on a par³⁹. We should also keep in mind that the use of these fictions was taken in Cardano in a general setting, on which we shall soon return: “the positing of the false”. Like Cardano, Descartes had no problem to count negative roots amongst the solutions of equations (alongside “imaginary” ones), although he did not consider them in isolation when confronted with a specific problem. His strategy was, however, more general than that of his predecessors in the sense that it allowed transforming *all* the roots of an equation in order to obtain only true solutions⁴⁰. Accordingly, he did not proceed by substituting x by $-x$, but by adding to x a quantity sufficient to transform “false” roots into “true” ones. This technique also allowed to reinterpret the data of the problem in a way different from the traditional one (in terms of “debts” and “lack”). Indeed, the geometrical setting permits to see it as a question of position and the algebraic transformation can be seen as a way to change the point considered as marking the origin of the reference frame (although this is not Descartes’ terminology). This latter idea was already brought forward by Albert Girard in 1629, who provides us with a middle position between Cardano, Stevin and Descartes. Like the formers, he explained how a negative solution to an equation could be transformed into a positive one

³⁸ “Aucuns des precedens problèmes, de la proportion des nombres algebriques, reçoivent par dessus les solutions ci devant données, encore d'autre solution par $-$; Et combien les mesmes ne semblent que solutions songées, toutesfois elles sont utiles, pour venir par les mesmes aux vraies solutions des problèmes suivans par $+$; La cause est, qu'au valeur de 1 (1) trouvé par quelque des problemes precedens, il faudra aucunesfois encore aioufter quelque certain nombre, comme apparouiftra; d'ou s'enfuit que quand le nombre à aioufter sera maieur que ladicté folution par $-$, que leur différence fera vraie solution par $+$. Or lesdictes solutions par $-$ (lesquelles nous expliquerons par articles selon l'ordre des problèmes, & leurs différentes précédentes font telles » (Stevin 1585, p. 332, followed by the list of equations already encountered and reformulated in terms of $-x$. I note by (1) what Stevin indicates by putting the 1 into a circle).

³⁹ See, for example, *Ars Magna*, p. 4: *falsa seu ficta*. Considering the title of chapter 37, *falsa* seems to be the largest category in which are included the negative (*ficta*) and the imaginary (*sophisticata*) roots.

⁴⁰ “Second, by increasing the roots by a quantity greater than any of the false roots we make all the roots true” (Descartes 1954, p. 168).

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by a change of sign⁴¹. But, like the latter, he interpreted also negatives as indicating segments going in opposite directions⁴². He even considered them as negative distances (something which is not found in Descartes). Commenting on the relative position of the points, he claims that one is then proceeding “as if” there were distances “less than nothing” and that this distance indicates what is in fact a “retrogradation”⁴³.

Girard also insists, more than Cardano and Stevin (whom he criticizes on this very point, alongside Vieta), on the fact that one should consider all the roots of equations, including negative (“solutions par moins”)⁴⁴ and “inexplicable” (i.e. imaginary) ones. Going a step further than Stevin, who allowed negative coefficients to enter the expressions in order to have a uniform treatment of equations, he insists on the fact that the introduction of these other entities, although “inept and absurd” (“inepte et absurde”) in the case of “indicible” roots, are necessary “for the generality of the rule”⁴⁵. This allowed him to formulate as a theorem, although without proof, one of the first explicit expressions of the fundamental theorem of algebra (or better of linear factorization)⁴⁶.

What I would like to emphasize is that in all of these authors, notwithstanding important nuances, we find the same global framework. Although for various reasons, they all accept the idea that the generality of algebra commands to consider the existence of roots being “fictive” or “false”, at least in certain circumstances. But these solutions pose no particular problem for a very simple reason: our actors usually look first for positive solutions. The interest in “other” solutions is therefore primitively driven by a desire to verify that a “true” solution has not been missed or, in Stifel, to exhibit a general structure of “exponents”. In case there is no positive solutions, but only negative ones, this is no particular problem again, since one can either declare the problem impossible or transform it so that it deals in fact with “true” numbers. We will come back to this general strategy in more detail in the last section of this paper, but it should be noticed right away that the recourse to the term “fiction” is hence very natural. One just introduces a solution “as if” it was ‘true’. In case it is not, one can always reinterpret the data of the problem so that it becomes so.

As I already mentioned, Cardano turns in *Ars Magna* chap. 37 to another famous case of “positing the false”, that concerning the square roots of negative numbers. I will postpone its study to concentrate on the first strategy relying on fictions⁴⁷, but it should be noticed that

⁴¹ However, unlike them, he insists that one should always give all the solutions and intends to show how Stevin and Vieta failed to do so in cases where there were negative solutions in addition to positive ones.

⁴² “Iufques icy nous n'avons encor expliqué à quoy servent les solutions par moins, quand il y en a. La solution par moins s'explique en Geometrie en retrogradant, & le moins recule, là où le + avance”. (Girard 1629, fol. F3v [the original edition does not contain page numbers]).

⁴³ “Assavoir monstrant lesdits points G & H, comme si les distances FG, FH estoient moins que rien, en retrogradant, prenant que FN, FD avancement, & FG, FH reculent en arrière”. (Girard 1629, fol. F3 r (the original edition does not contain page numbers, but indicates the folios by numbered letters)). The use of the term “retrogradation” is interesting as indicating a possible source in astronomy, [see Husson's paper in this issue](#).

⁴⁴ “Les solutions par – ne se doivent omettre” (Girard 1629, fol. D recto)

⁴⁵ “On pourrait dire : à quoi sert ces solutions qui sont impossibles ? Je répons à trois choses, pour la certitude de la règle générale & qu'il n'y a point d'autres solutions, & pour son utilité” (fol. D3 recto).

⁴⁶ Fol. E4 (not numbered) and Gilain (1991).

⁴⁷ According to Cardano, square roots of negative are not *fictae*, but *sophisticae*. This is made clear in the first chapter when he mentions a situation in which the solution is said neither fictive nor true (Cardano 1545, p. 4).

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Cardano was troubled enough by this case to launch a reflection on “plus and minus”. Yet this reflection was not a public debate and although it was published after his death, it did not have much influence on the majority of Cardano’s readers⁴⁸. Harriot was an exception, but he too did not publish his thoughts on the subject⁴⁹. At any rate it is important to keep in mind that these reflections were clearly attached not to negative numbers in and of themselves, but to imaginary ones and by the same token to the “rule of signs” (which prevents the multiplication of numbers with the same sign to have a negative output, and henceforth prevents the existence of negative squares). I will come back to these elements in the last section of this paper.

Another point I would like to emphasize right away by coming back to Stifel is the following one: the recourse to fictitious entities was clearly presented as related to a more general and widespread strategy in which it had *already* proven useful. As examples, Stifel mentions the introduction of “surd” and “broken” numbers. Indeed, according to the standard view inherited from Euclid, a “number” was defined *stricto sensu* as a multiplicity of (indivisible) units. In this sense, the other “numbers” were also “absurd” or “impossible”. Yet they were already of widespread use at the time. If symbolic algebra changed something in this setting, it is because it forced mathematicians to handle general expressions without knowing in advance whether or not they denoted ‘true’ quantities. When I take the root of an unknown, I have no way to know by this simple statement whether this expression denotes a ‘true’ number in the Euclidean sense or not. Moreover, as Cardano explained very well, this question cannot be solved by an inspection of the solution in isolation. Indeed, an expression such as “ -3 ”, which solves the equation $x^3 + 11x^2 = 72$ can easily be transformed into a “true” solution by changing the data of the problem ($x^3 + 72 = 11x^2$). In this case, the fact that *all* the real solutions are transformed into their opposite tend to indicate that these are the very same “quantities” considered under different expressions.

Now, if one focuses on the larger framework indicated by Stifel, one is struck to see that the recourse to fictions had in fact *nothing specific to negatives*. It was a common way to designate various entities entering into the mathematician’s reasoning. Let us take the example of Jacques Peletier du Mans’ *Algebre* from 1554. Like Stifel and Cardano, he calls negative solutions “fictive numbers” (“nombres feins”). Considering a problem leading to the equation $x^2 = 15x - 44$, he devises the two solutions 11 and 4, but in the second case, the conditions of the problem (which specify that we are looking for two numbers being 8 and 6 *below* the solution to the equation) involved accepting fictive numbers: “but it is by absurd numbers, which are fictive numbers, below nothing” (“mais c’est par nombre absurdes, qui sont nombres feins au-dessous de rien”). Peletier devises the two results (-4 and -2) and show that they satisfy the conditions of the problem. He then adds that they are useful in order to verify the general rules for the resolution of equations (Peletier 1554, p. 84). In the very same treatise, Peletier also uses this terminology to deal with irrational numbers. He even devotes a whole chapter of the second book, like Stifel, to determine whether or not they are fictitious entities⁵⁰. In his edition of Euclid from 1566, Candalle uses the same term to qualify incommensurable

⁴⁸ *Sermo de plus et minus*, in Cardano (1663).

⁴⁹ Tanner (1980a), see below section 3 ###

⁵⁰ Peletier (1554), p. 131: “De la nature des nombres irrationaux. S’ils sont vrez nombres ou feins”, Peletier’s conclusion to this very interesting development was that they were close enough to be ‘true’.

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magnitudes (and not numbers), because they involve an infinitary process in their determination⁵¹.

Now, Peletier also used this term in another context, to which Cardano might well have alluded in the title he gave for his chapter 37, that of the technique of “false position”. In his *Arithmetique* from 1549⁵², he rightly presents this technique as a way to “feign” a number as solution to a system of linear equations and then correct the result in order to obtain the “true” value⁵³. In the next chapter, he presents an expansion of this rule, which he attributes to Gemma Frisius (followed by Stifel), according to whom most of the algebraic questions could be solved by recourse to a kind of false position. This claim is crucial for two different reasons: first, we find here a confirmation that the introduction of “fictive numbers” was an existing strategy before Stifel, as he claimed. Indeed, Gemma Frisius talked already in 1540 about *numerus fictus* in the context of false position⁵⁴ and an exhaustive study remains to be done to see if this vocabulary already appears in earlier sources. In sources after Stifel and Peletier, the term appears in authors such as Trenchant (1558, p. 175) or Forcadel (1565, p. 125), or later Henrion (1621, p. 97) and Mersenne (1625, p. 639); the second, and more important reason, is to see that the rule of false position was considered by some of our actors as the very core of the art of “algebra”. Accordingly, we find authors considering that the handling of “fictive numbers” was the *very essence of algebra*. This is expressed most explicitly by Guillaume Gosselin, which talks about “nombres fains” when dealing with Diophantine problems and claims that Algebra is grounded on such fictions⁵⁵. This is also the case later in Alsted (1630)⁵⁶, Caramuel (1670, p. 109, quoting Alsted/Geysius) or Gottignies (1687, Lib. III, cap. 2, p. 12)⁵⁷. Notice that the association of algebra and the use of fictions is not related by our actors to the use of symbols. It is related to the hypothetical nature⁵⁸ of the entities introduced in the reasoning, a strategy which was already at play, according to them, in the context of false position (where no symbolic algebra is needed).

⁵¹ (Foix de Candalle 1566, p. 95).

⁵² The *Privilège du Roy* is dated from 1547.

⁵³ Peletier (1549), p. 86: “La pratique est telle. Feignons un nombre à plaisir...”; and 87: “Il faut noter qu’en la Règle de faux, soit d’une position soit de deux, quelque Nombre qu’on veuille feindre, tout revient toujours à un”.

⁵⁴ *Arithmeticae practicae methodus facilis*, p. xxi.

⁵⁵ *Il reste vne chose, c’est qu’il peut sembler à aucuns que ce calcul de nombres fains soit obscur & difficile, toutesfois on peut bien estimer qu’une règle si belle, qui est l’œuvre le plus difficile de toute l’Arithmetique simple (ie n’entens pas de l’Algebre, dont ceste cy est vne branche) ait peu estre produite de nature sans aucuns noeuz & difficultez, mais tout ainsi qu’elle est descenduë de nostre Algebre (qui est toute fondée sur telles fictions) aussi ne peut elle estre démontrée que par façon d’Algebre* (Gosselin, commenting on the rule of false position in his translation of Tartaglia (1578), I, f° 118-119). I thank Odile Kouteynikoff for having pointed this reference to me.

⁵⁶ Which contains in fact a treatise of Algebra (by J. Geysius), presented as dealing with fictitious numbers (Alsted 1630, 864 sq.).

⁵⁷ In Gottignies, this recourse to fictions in algebra is strongly related to the use of negative numbers.

⁵⁸ Caramuel explains that even if one feigns a sought quantity, it remains a “true” number and should not be called “false”, but “conditional” or “hypothetical” (“Adeoque non *Falsi*, sed *conditionales* & *hypothetici* nominari deberent”, Caramuel 1670, p.119). Gottignies compares the use of fictitious or imaginary quantities in algebra to the fictitious portrait of a person, which shares with the true person some features, but not all (Gottignies 1687, Lib. III, cap. 2, p. 12).

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This general framework will subsist until the end of the 17th century and even later⁵⁹. More importantly for us, the relation between the introduction of fictions and the method of false position confirms that actors themselves viewed this practice *in continuity* with an ancient practice in which one “feign” certain “false” solutions to problems in order to determine the “true” ones. No “crisis”, “epistemological obstacle” or “revolution” to be found here. In the case of Gosselin, this was taken in an even older framework since the introduction of “fictitious” entities and “fictitious” equations was something he found at play in Diophantine arithmetic⁶⁰. Once again, the proximity with the role played by supposition (of which we do not know in advance whether they are satisfied by the conditions of the problem or not) is obvious. But one has also to keep in mind that Diophantus was then considered, in line with Pappus, as one of the main sources for the Ancient art of “analysis”. Moreover, the “new algebra” was presented as a revival of this ancient method, which Greek mathematicians were supposed to have hidden⁶¹. As we will see in the last section of this paper, this reference to “analysis” in connection with fictions (and, more specifically with negative numbers) will be put *explicitly* to the fore by authors writing one century later.

The success of the designation of negative numbers as “fictive” in 16th century is impressive and should be taken into consideration by historians, as should be the larger framework I began to draw in the preceding paragraphs. Beside the above mentioned authors, it is to be found in many early modern treatises. Clavius, for example, devoted a complete chapter to these *numeri ficti*, very much in the spirit of Stifel, in his influential handbook of *Algebra* from 1608 (Clavius 1608). Amongst the German *Reichenmeister*, a similar development can be found around the same time in (Henisch 1609). This practice did not stop after Descartes (who favoured, as we have seen, the term “false” rather than “fictive”) and the terminology of fiction is still to be found in later and important handbooks such as that of Caramuel, Kersey, Millet de Chasles, Gottignies or Renaldini. The list of sources gets even longer, when one considers that the idea of *numeri ficti* could be rendered by synonymous terms. We already mentioned Stevin’s chapter dedicated to the solutions “songées”. Amongst the *Reichenmeister*, Roth (followed by Faulhaber) talked of negative solutions to equations as *gedicht*⁶². Jacques Hume, Vieta’s editor, called them “imaginaires”⁶³ (before the use of this term in a different meaning by Descartes) – a terminology which is also to be found in Wallis and in a number of texts from Leibniz, in synonymy with the term “fictive”.

We could multiply the references, but the above mentioned ones should suffice to establish the main claim of this section: when one adheres to the actors’ categories, one is struck to find quite the opposite of a crisis or a revolution surrounding the appearance of negative entities. To point to the fact that they were sometimes called “absurd” or “false” is not enough to allow

⁵⁹ See also Girard (1629), on algebraic questions: “on y procède le plus souvent comme aux fausses positions” (fol. C3 r). See also Schott (1661), Caramuel (1670) and until D’Alembert (see below [note ###](#)).

⁶⁰ Gosselin (2016). Indeed, in his translation of Diophantus, Xylander (1575) makes a very widespread use of the verb *finco* and the recourse to “fictitious” numbers will remain very present in authors from the tradition of Number theory such as Bachet, Fermat or Billy.

⁶¹ See, for example, Descartes AT X, 376.

⁶² Roth (1608); Faulhaber (1622). Roth also uses this term to designate imaginary roots to equations and even allows imaginary roots to enter the solution of concrete problems (see Schneider 2008 and Roth 1608, part III, problem 24, p. 175).

⁶³ (Hume 1635, p. 129).

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diagnosing something like an “epistemological obstacle” or any “resistance” here⁶⁴. In fact, using “absurd” or “false” entities was a very ancient strategy employed in analysis, in elementary arithmetic (beginning with the use of fractions of unities or surd numbers) and in the use of the *Regula falsi*. The most widespread way to consider negative numbers was henceforth a pragmatic one which considered them amongst other useful tools available to the mathematician in order to proceed in his computations without having to decide whether these expressions were denoting something “true” or not. On a larger scale, it was comparable to the use of hypotheses in sciences, in particular in astronomy where fictions were introduced in order to describe the trajectory of planets (typically by “imaginary” circles). The parallel with astronomical hypotheses was made explicitly by an author such as Ursus at the beginning of his *de Astronomicis hypothesibus, seu Systemate mundano, tractatus astronomicus et cosmographicus* (1597)⁶⁵. Although it was then harshly criticized by Kepler, who distinguished sharply between geometrical hypothesis and physical ones⁶⁶, what is interesting for us is the way Ursus depicts at this occasion astronomical hypotheses with what he presents as a *common practice* of the use of fictions in mathematics, be it in Algebra or in the use of the *Regula falsi*⁶⁷.

Beyond natural sciences, the use of “fictions” was also widespread in Law through the ancient practice of *factio juris*. The purpose was the same: in hypothetical situations, one could encounter incompatibilities between the situation imagined in the legal framework and the “real” situation to be taken into account. In that case, one just had to proceed “as if” the situation described by the law applied. Although all of these contexts might seem very distant, *they were not for the actors*. Leibniz, for example, makes the parallel between the use of fictions in the “casus perplexi” in Law and the practice of symbolical algebra as early as 1666⁶⁸. But the most striking example is given by Caramuel (1670), an author who was read by Leibniz. Indeed, a complete section in the introduction of his chapter on algebra is dedicated to the various ways of dealing with fictions. This development was motivated by the fact that Caramuel, like many other authors, thought of algebra as extending the technique of *Regula falsi*. This led him to study the broader question, inspired by a previous writing by Daniel Lipstorp: “whether Dialecticians, Jurists, Astronomers, Geometers and Arithmeticians, etc., posit the false and deduce the true from it?” (Caramuel 1670, p. 99). Caramuel thought that it was not the case,

⁶⁴ One could parallel with the numerous narratives originating in the fact that the Greek called some magnitudes « irrationals », cf. Knorr in note quoted above #####

⁶⁵ *Hypothesis, seu fictitia suppositio, est efficta Delineatio quorundam imaginariorum circularum imaginariae formae systematis Mundani, observandis motibus coelestibus accomodata, atque observandos salvandosque motus coelestium corporum, eorundemque calculum exprimendum, efficta, assumpta, introductaque* (Ursus 1597, pages not numbered). Montaigne already in his *Essais* II, 12, 537 establishes an intriguing parallel between the cosmetics as a way to feign some attributes of beauty and scientific fictions such as epicycles... The literature on the relationship between astronomy and fictions in renaissance and early modern periods is quite extensive. For an introduction to the various aspects of the topic, see Ait-Touati (2010).

⁶⁶ Kepler’s main argument is that the first ones, taken in and of themselves could lead to one and the same conclusion, whereas physical hypotheses, coupled with observations, were precisely meant to separate equivalent mathematical models.

⁶⁷ *Ut (gratia Exempli), in Arithmetice Illis Hypotheticis Regulis in Algebra posita unitate fictitia, inq: Regula falsi posito quovis falso seu ficto numero, verum quaesitumq: numerum invenire solemus : non tamen propterea peccamus, neq: contra sacras literas aliarumve Doctrinarum principia aliquid committimus, licet fictitii numeri, per se ut falsi fictiq: plerumq, mentiantur : quodq; idcirco mendacium peccatum dici nullo jure poterit, multo minus absurdum, licet ita videatur videri posit.*

⁶⁸ *Disputatio de casibus perplexis in jure*, November 1666 (A VI, 1, 235-236).

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but he had to face the objection that these various actors happily rely on “fictions”. By studying case by case these various uses, he then offers us a beautiful summary of all the various examples we encountered: the hypotheses of the astronomers, the *fictio juris*, the rule of false position and geometrical analysis (in particular *reductio ad absurdum*) – to which Caramuel adds an example to which I will return at the end of this paper: logarithms (and, in a note, infinitely small quantities). His conclusion was that in these various contexts, it was never a matter of concluding directly the true from the false, but of reasoning with hypothetical entities. This provides us with a nice way to conclude this journey with “fictions” by recalling how widespread this mode of thinking was not only in mathematics, but in various domains in which it was useful to reason with hypothetical entities.

3. *Discussions and reflections on negatives*

One aspect which might have reinforced the idea that enlarging the algebraic universe with fictitious entities could be interpreted in terms of a logic of “domain extension” is that this picture appeared as such in the actor’s categories. Moreover, it was presented as a way to *unify* retrospectively all the “impossible” entities which were proliferating with the development of symbolic algebra. But it appears, as far as I can tell, *in the second half of the 17th century*. Yet, even in this late context (compared to the beginning of symbolic algebra), I would like to argue that this practice was not seen as a rupture, a way to overcome “epistemological obstacles”, but, quite on the contrary, was conceived in the continuity of existing practices.

In the famous *scholium* to the prop. 190 of his *Arithmetica infinitorum* (1656), John Wallis presents what might appear as one of the first formulation of the logic of “domain extension”:

And therefore what arithmeticians usually do in other work, must also be done here; that is, where some ἀδύνατον is arrived at, which indeed must be assumed to be done, but nevertheless cannot actually be done, they consider some method of representing what is assumed to be done, though it may not be done in reality. And this indeed happens in all operations of arithmetic involving resolution, for example, in subtraction: if it is proposed that a larger number must be taken from a smaller, thus 3 from 2 or 2 from 1, since this can not be shown in reality, there are considered negative numbers, by means of which a supposed subtraction of this kind may be expressed, thus $2 - 3$, or $1 - 2$, or -1 . (Wallis 2004, p. 169).

He then pursues by indicating that the same process is at stake in other “resolutive” operations such as division, extraction of root and, more generally any insertion of proportional means. Accordingly, one might introduce a new kind of notation for the interpolation he just proposed to determine the quadrature of the circle, which cannot be designated “by a true number (not even by the usual said radicals, or surds)” (Wallis 2004, p. 162).

A similar claim is made in chapter 83 of his *Treatise of Algebra* (written for a large part in the 1670s, but published only in 1685), which had an even more widespread diffusion. The context is the same (the chapter deals with the impossibility to express the quadrature of the circle by a received notation involving numbers):

Nor is it strange, that such impossibility should arise; for the same happens in all the resolutive parts of arithmetick. And for most of them, we have already provided notations to express that

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impossibility. As for instance, addition is genetical, (or synthetical) and to any positive numbers, any positive numbers may be added, without coming to any impossibility. But subtraction, is analytical, or resolute: and here the case is sometimes possible; as if a lesser be to be subtracted from a greater ; ($3 - 2 = 1$.) But sometimes impossible; as if a greater to be taken from a lesser; ($2 - 3$.) in which case we are provided of a notation, to express that impossibility (and the measure of that impossibility) by a negative quantity, ($-1 = 2 - 3$.) imparting somewhat less than nothing. (Wallis 1685, p. 316)

In the following passage, Wallis deals, once again, with other “analytical” or “resolute”⁶⁹ operations such as division, extraction of roots, but adds the roots of algebraic equations and mentions imaginary solutions (which were not mentioned in the *Arithmetica infinitorum*). Another difference with the *Arithmetica infinitorum* is that here Wallis clearly refrains from talking about “negative numbers” (although he has no problem talking in the same paragraph about “positive numbers”). All of the notations introduced for impossible operations, he calls them “expedients”, useful to designate various kinds of “quantities” and when he talks of “number” in this case, it is to insist on the fact that they are “impossible” (as contrasted to “true” numbers).

The same idea is expressed by Leibniz in a contemporary text:

Sometimes also the operation which must be made in act is impossible, either for lack of time or in the absolute, even if it is possible to exhibit a construction of it, at least in our characters, or that it was already exhibited in the nature. Thus it is impossible to subtract when there is nothing, and yet this is represented in nature, for example when someone owes more than he has. In the same way it is impossible for a prime integer to be divided in act by another; hence the fraction arises, which represents a division to be made - the thing which is designated by this number being divided into parts more suitable for exhibiting this division. (Leibniz 2018, 107. My translation)⁷⁰

We find in these texts, for the first time it seems, the general picture in which mathematical thought is presented as the overcoming of various impossibilities related to a specific kind of operation (“analytical” or “resolute”⁷¹) and corresponding to the introduction of various kinds of new symbols. At first glance, these passages seem thus to testify for the acceptance of the “expanded algebraic universe” in which ‘number’ was progressively transformed into a symbolic expression (or “notation” or “character”). One century after the development of symbolic algebra, the obstacles and resistance to the expansion of the concept of “number” were, or so it seems, finally overcome. It is hence no surprise that Jacob Klein ended his study with Wallis, presented as the climax of the progressive transformation of the concept and a prelude to the full acceptance of “symbolic” knowledge accomplished by Leibniz.

⁶⁹ *Resolutio* and *compositio* were at the time another designation of Analysis and Synthesis. That this is what Wallis has in mind is made even clearer by the fact that he talks about “synthetical or compositive” operations. Interestingly enough, he also talks about “genetical” operations – a term which will be also used, in the same meaning, in 19th century mathematics.

⁷⁰ He then pursues by mentioning incommensurable or “surd” roots and imaginary roots.

⁷¹ In a text from the same period, but written after Wallis’s treatise, Leibniz takes up the same distinction between “analytic” and “synthetic” operations giving birth to the various categories of quantities (Leibniz 2018, p. 232).

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Yet, as we have seen, one can not find any such resistance or obstacles in previous authors. A closer look at our passage indicates, that *this is precisely what Wallis and Leibniz recall*. Arithmeticians, they claim, had in fact no problem introducing new notations for all kind of new entities before them. They do not even present this fact as related to the development of symbolic algebra (although symbolic algebra certainly helped indicating more clearly the general pattern here). It starts, for both of them as it was already the case with Stifel, with the introduction of fractions seen as numbers (*numeri fracti*) or irrationals (*numeri surdi*). As is obvious in Wallis' passages, the general framework is not given by a transformation of number into a symbolic entity, but by the much older technique of "analysis", in which one supposes a problem already solved in order to tackle it⁷².

This consideration is crucial if one wants to pay attention to some important features of our texts. Let us read again how Wallis is describing this technique: "where some ἀδύνατον is arrived at, which indeed must be assumed to be done, but nevertheless cannot actually be done, they consider some method of representing what is assumed to be done, though it may not be done in reality". For readers of the time, this was a clear reference to one of the cases described by Pappus for "analysis" in the famous passage of Book 7 of the *Collectio*:

In the case of the problematic kind [scil. of analysis], we assume the proposition as something we know, then proceeding through its consequences, as if true, to something established, if the established thing is possible and obtainable, which is what mathematicians call 'given', the required thing will also be possible, and again the proof will be the reverse of the analysis; but should we meet with something established to be impossible (ἀδύνατον), then the problem too will be impossible (Pappus 1986, 82-83).

We find here a clear origin of the way in which we often time proceed "as if" (*pros ei*)⁷³ a result was obtained, although it may appear, at the end of the problem, as impossible. As indicated by Proclus, this was a natural way to obtain *reductio at absurdum* (by just reversing the analytical process into the production of a contradiction) – this would correspond to a particular case in which we know the fiction to be false at the beginning of the synthesis⁷⁴. In fact, way before the time of surd, broken or negative numbers, mathematicians were already accustomed to encounter, without much trouble, many "absurd" entities: parallel lines which meet (Eucl. I.27), circles that do intersect in more than two points (Eucl. III. 10), concentric circles that touch internally (Eucl. III.6) and other monsters already proliferating in Ancient

⁷² Notice also that when Leibniz describes what he calls "symbolical" or "blind" knowledge in his philosophical writings, the first kind of "symbols" he has in mind are, in fact, simply words of natural language, cf. *Meditationes de cognitione, veritate, et ideis, Acta Eruditorum*, November 1684, transl. (Leibniz 1989, p. 25).

⁷³ Latin translations were given by *tanquam* or *quasi*. Another expression used by author such as Cardano and Leibniz is *per analogiam*.

⁷⁴ "Every reduction to impossibility takes the contradictory of what it intends to prove and from this as a hypothesis proceeds until it encounters something admitted to be absurd and, by thus destroying its hypothesis, confirms the proposition it set out to establish.

In general, we must understand that all mathematical arguments proceed either from or to the starting-points, as Porphyry somewhere says. (...) Those that proceed to the starting-points are either affirmative of them or destructive. But those that affirm first principles are called 'analyses' (...); when they are destructive, they are called 'reductions to impossibility', for it is the function of this procedure to show that something generally accepted and self-evident is overthrown". (*In Eucl.* 255-256, transl. (Proclus 1992, p. 198-199)).

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Greek mathematics. That this was one of the received meaning of “fictions” in mathematics is confirmed by the following passage from Leibniz:

16. If space and time were anything absolute, that is, if they were anything else besides certain orders of things, then indeed my assertion would be a contradiction. But since it is not so, the hypothesis (that space and time are anything absolute) is contradictory, that is, it is an impossible fiction.

17. And the case is the same as in geometry, where by the very supposition that a figure is greater than it really is, we sometimes prove that it is not greater. This indeed is a contradiction, but it lies in the hypothesis, which appears to be false for that very reason.

(*Leibniz-Clarke Correspondence* IV, §16-17)⁷⁵

We also saw that this was one of the typical case of the use of “fictions” in mathematics as presented by Caramuel (on the same footing as *Regula falsi* or logarithms).

But there is more. Pappus’ passage on analysis does not stop where I cut it. It continues with the description of a very important practice related to these questions of possibility/impossibility: “Diorism is the preliminary distinction of when, how, and in how many ways the problem will be possible”. From the rare testimonies which we possess about this practice, we know that it was indeed a typical situation in which a *diorismos* could occur: confronted with an “impossible” result, it was sometimes possible to still provide a positive conclusion by identifying a condition to add in order to render the problem soluble. This means that to stumble upon an impossibility was not in and of itself an indicator of an absurdity, or, to be more precise, than some apparent impossibilities could be transformed into possibilities⁷⁶. No need to overcome any “epistemological obstacle”, to pass to the “meta-level” or to introduce “second intention” entities to do so.

This is very clearly the framework in which actors from the end of the 17th century interpreted the use of “fictions”, including the introduction of negative quantities. Just after the passage from the *Elementa matheseos universalis* quoted above, Leibniz explains that there are in fact *various kinds* of impossibility. The most straightforward one is the sheer contradiction, such as $3=4$. But, in other more intricate situations, the impossibility is only “by accident”⁷⁷. He then refers to the example of the intersection of a straight line and a circle when the line is at a distance of the centre exceeding the size of the radius. The apparition of imaginary solutions in the second-degree equation expressing this “intersection” is interpreted as an indication that a condition on the distance should be added in order to avoid this impossible situation to occur. This is a clear case of diorism. Another example he mentions is the use of infinitely large and infinitely small quantities in a projective context (where the apparition of the fiction of a point of intersection “at infinity” indicates that a certain sought angle should be seen as a right angle). In a similar text from 1699, which we already mentioned (*Mathesis universalis*), Leibniz explains on the same example of the intersection of the straight line and the circle that these are cases of *problema male formati*⁷⁸.

⁷⁵ As we have seen, Leibniz mentions the case of demonstration of impossibility in algebra in relation to the idea of a “blind” knowledge in the context of fictio juris as early as 1666, see [note #####](#).

⁷⁶ The term “apparent impossibility” is from Leibniz himself: (Leibniz 2008, p. 108).

⁷⁷ (Leibniz 2008, p. 108).

⁷⁸ (Leibniz 2008, p. 153).

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From this context, it follows that fictions should not be interpreted as denoting a specific kind of entities (living in a well identified “domain”). They belong to a conceptual strategy in which one can arrive at possible *or* at impossible solutions depending on the way in which the data are interpreted. But the most important fact to notice is that this strategy is never presented by our actors as something new. It takes place in a very ancient method attached to the framework of “analysis”. At any rate, it is not related by them to the extension of the concept of number, transformed into a purely symbolic expression by algebra. What symbolic algebra allows here, as was claimed by many other authors of the time, is more a way to uncover and revive an ancient art in the handling of suppositive entities in mathematics.

One may interject here, following Dunmore, that, notwithstanding the way actors portrayed their practice as taking place in a tradition, the very fact that they are publicly discussing about negative numbers and putting them in this larger context marks the beginning of a new era. The revolution would occur not so much in the use of negative numbers than in the way they are cast in an explicit reflection on the general framework in which they make sense. I am not sure that this can be convincingly argued since, as we saw, the idea of reflecting on a general practice, although certainly less developed, was already present in Stifel and in Cardano (and in many other authors) when introducing negative as fictions. As I recalled, chapters on “numeri ficti” are already to be found in Clavius’ or Henisch’ handbooks at the beginning of the 17th century. Moreover, we have encountered on the way many “meta” reflections on the fact that a practice (typically that of algebra) generalized another one (typically that of the *Regula falsi*) by extending the use of “fictitious” entities. But at any rate, the context in which Wallis and Leibniz discussed about negatives was *not* related to difficulties they would pose in and of themselves and it seems henceforth misleading, or at least ambiguous, to claim that they testify for discussions “about negative numbers”.

In Wallis, for example, the first motivation in the passages we came across is clearly that of the quadrature of the circle and the way we can introduce new “expedients” to refer to the sums of infinite series. When one looks more closely at the place where he introduced the “number line”, the context is also broader than negatives since the chapter deals in fact with the taking of their square roots. We also came across the passage in which he talks about the ratio of negative to positive as “greater than infinity” (*plusquam infinita*)⁷⁹ and, once again, the context was broader since it concerned the behaviour of $1/n$ as exponent (and the quadrature of associated curves). The same is true in Leibniz: in all the passages we encountered, the goal of the discussion was to arrive at imaginary and infinitely large or small quantities, where the real core of the discussion lied⁸⁰.

⁷⁹ Wallis’s expression is not easy to translate since it can be interpreted as saying that there is something greater than the infinite, or that there is something “more than infinite”. Parts of the debate between Varignon and Grandi at the beginning of the 18th century revolved around these two conflicting interpretations, which lead to many misunderstandings.

⁸⁰ One can also find in Leibniz, it is true, texts in which he introduces negative numbers in a purely axiomatic way (see Leibniz 2018, p. 174-175). In one of these texts, he introduces a list of axioms for this “calculus of magnitudes”, which corresponds nicely to what we now call the axioms for a field, and, in particular, relies on the two propositions “ $a - a = 0$ ” and “ $-1 \cdot a = -a$ ” as sufficient for the handling of subtraction on magnitudes (Leibniz 2021, p. 133-154). In this sense, one can argue that the end of our period marks the first attempts to introduce negatives as genuine objects of a theory in a rather formal way. But, besides the fact that these texts are very late

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To be sure, there were some reflections and discussion about negative entities, even in the preceding period. As an example of the first, one often mentions Cardano's inquiry into a possible alternative to the "rule of sign" exposed first in the *De Regula Aliza* (1570)⁸¹, then in his *Sermo de plus et minus* (published in Cardano 1663), followed by similar reflections in Commandino and Harriot (Tanner 1980a and 1980b). As an example of the second, one often mentions the debate between Arnauld and Prestet occurring in the middle of the 1670s. But several remarks are in order here. Firstly, although these examples look as a sample of a larger context, it should be emphasized that they exhaust almost all of the known corpus related to this issue before 1700⁸². Secondly, it is striking to see that the reflections of Cardano and Harriot are, in fact, rather isolated in their own work and almost of a private order; moreover, they are not on negative numbers per se, but about one of their use, namely what we now call the rule of signs and, behind it, the problem raised by the roots of negative squares (if minus by minus always gives a positive, there is no way a square can be negative). Finally, the public debate between Arnauld and Prestet is also about the rules of signs, which it transports in a larger setting, i.e. the way it perturbs the ordinary regime of "ratios". These various cases being already well studied in the literature, I will stay content to just quickly dwell a little bit upon the last remark by following the way in which Leibniz recalled this debate in 1712.

When Leibniz came back to this controversy in a text he wrote for the *Acta Eruditorum* in 1712⁸³, he reminded from the start that the issue at stake in the debate between Arnauld and Prestet was not negatives per se, but the question of what meaning could be ascribed to the insertion of a proportional mean between a positive and a negative. It was a matter of interpreting the fact that the ratio of -1 to 1 is equal to that of 1 to -1 , from an algebraic point of view, although the usual rules for manipulating ratios stipulate that if $a < c$ and $b > d$, we should have $\frac{a}{b} < \frac{c}{d}$. The problem of ratios between positive and negative was also, as we saw at the core of the difficulties agitating Tschirnhaus and Leibniz in their thinking about imaginary quantities in Paris (the square root of -1 can indeed be seen as the search for an x such that $\frac{-1}{x} = \frac{x}{1}$). Before that, it was also at play in Wallis' interpretation of negatives entering into ratios "greater than infinite" (*plusquam infinitae*). In fact, the latter constituted the immediate context of Leibniz' 1712 paper since he wrote it as a reaction to the quarrel between Grandi and Varignon on the status of Wallis's *plusquam infinita*.

The first argument Leibniz provides to support his own view goes as follow: since the foundation for having a proportion is, by definition, to dispose of a kind of "similitude" between ratios, one cannot define a true "proportion" in the case under study. Indeed, since the order relation occurring between the terms in both ratios are inverse one of each other, it does not

in Leibniz's work and remained completely unknown until very recent times, one should not forget that they take place in a corpus in which one *also* finds at the same period the other texts on "analysis", which we mentioned. At any rate, one would have to explain the coexistence of these two pictures, which do not match in the usual approach in terms of "epistemological obstacles".

⁸¹ On this treatise, published as an appendix to the reedition of the *Ars Magna* in 1570, see Confalonieri (2015). On predecessors to this alternative rule, see Wagner (2010).

⁸² To the corpus studied by Tanner, one may add reflections in Gottignies (see Mancosu 1996, p. 89-90).

⁸³ Leibniz (1712), French translation in (Leibniz 1995, p. 432-435).

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seem possible to call these ratios “similar”⁸⁴. This is basically Arnauld’s initial claims for considering these ratios as “fictions”⁸⁵ (Leibniz calls them “imaginary” in his own paper)⁸⁶. However, Leibniz intends to establish this fact on a firmer ground by providing other arguments. He thus links it with another question, that of logarithms. Considering that a logarithm can be considered as a certain measure for ratios, one should be able to devise a logarithm for the ratio of -1 to 1 . Positing the logarithm of 1 to be 0 , this logarithm is the same as that of -1 . But Leibniz objects that -1 has no logarithm, since it could be neither positive (a positive logarithm corresponding to positive numbers greater than 1), nor negative (a negative logarithm corresponding to positive numbers lesser than 1). In the continuation of the passage, Leibniz provides a second argument by showing that the acceptance of logarithms of negative numbers would entail to accept logarithms of imaginary quantities too, a fact he considers even more absurd⁸⁷. He then refers to Wallis’ interpretation of the ratio to negatives as being *plusquam infinita*. Wallis’s mistake, according to Leibniz, was to neglect the preceding paradoxes and to have handled “imaginary” ratios in the same way as real ones. But he then pursues that it does not mean that one should reject quantities “less than zero”. According to him, and in perfect concordance to what we saw already in the preceding sections, this should be referred to a general strategy at play already in Ancient mathematics. His example is the angle of contingency, which was said by Euclid to be less than any rectilinear angle (although the meaning of “less” involved in the notion of magnitude in Euclid should not allow for such a statement to be true *stricto sensu*)⁸⁸. This is the strategy he claims to have followed, after many others, work when using “infinitely small” quantities⁸⁹.

I reconstrued the structure of Leibniz’s argument, because it provides us with a beautiful conclusion to our journey. Firstly, it indicates in a very vivid manner that negatives appeared as problem not so much when they emerged as possible solutions to equations, but in relation to *other* questions such as imaginary roots, logarithms, infinitely large and infinitely small quantities. This already helps explaining why the debates surrounding them occurred in the second half of the 17th century, as I have emphasized, and not with the development of symbolic algebra one century before. Second, it confirms that all of these examples were still considered by one of the champions of symbolic method, at the beginning of the 18th century, as belonging to a general strategy, the use of “fictions”, which he explicitly traced back to Greek Antiquity. More importantly, it provides us with an alternative picture than the usual one in terms of “domain extension”. We could propose to designate it as that of “crystallization”. Indeed, the process leading to the acceptance of a mathematical expression as denoting a genuine mathematical object is not related here to the overcoming of some “epistemological obstacles”, but to the fact that it appears at the crossroad of various independent contexts/problems which converge.

⁸⁴ (Leibniz 1995, p. 433).

⁸⁵ See Arnauld’s letter reproduced in (Prestet 1689, p. 367).

⁸⁶ For a quick description of the debate, one can consult Mancosu (1996), p. 88-89.

⁸⁷ If $\log(-1)$ were a real entity, it would be possible to divide it by 2 and thus obtain a logarithm for $\sqrt{-1}$.

⁸⁸ The fiction in this case is not the angle of contingency in and of itself, but the fact that Euclid said it was “smaller” than any rectilinear angle, although this relation of order/comparability contradicts his own definition of magnitudes as being comparable only when they are homogeneous.

⁸⁹ (Leibniz 1995, p. 435).

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Interestingly enough, a similar argument was made by Thomaidis, in the context of contemporary discussions on mathematics education, in order to counter a too simplistic view of the introduction of negative numbers in the classroom (Thomaidis 1993). Instead of relying on one miraculous representation, which was supposed to solve all the resistances experienced by students (typically the model of the number line), an alternative and much more fruitful strategy could be proposed, he claimed, in terms of a *plurality of approaches*. This also allowed Thomaidis to make a very simple remark, which is at the core of Leibniz's reflections (although Thomaidis makes no reference to Leibniz): there was indeed a context in which negative numbers were introduced very early not only *without any difficulty*, but *independently* of the algebraic setting to which the logic of "domain extension" points too exclusively, that of logarithms.

Accordingly, let me conclude this study by saying a quick word about this latter context, which is too often forgotten in the study of negative entities in early modern times. We already saw that Stifel introduced his *numeri ficti* in order to indicate an interesting parallel between numbers standing in an arithmetical and in a geometrical progression. The same development is to be found in many handbooks of algebra such as that of Clavius, who devoted the sixth chapter of his *Algebra* (1608) to these *numeri ficti*. This structure, extended to continuous domains, is at the core of the conception of logarithms in early modern times (and even sometimes serves to define them)⁹⁰. It is used as such by Napier as early as 1614, at the very beginning of his *Mirifici logarithmorum canonis descriptio*. By positing the logarithm of the whole sine (which he takes to be 1 000 000) as 0, he posits naturally the logarithms of numbers "greater than the whole sine" to be less than nothing:

For seeing it is manifest by the definition, that the sines decreasing from the whole sine, the Logarithmes increase from nothing: therefore contrariwise the numbers which yet we call Sines, increasing unto the whole sine, that is to 10000000, the Logarithmes must needs decrease to 0 or nothing: and by consequent the Logarithmes of numbers increasing above the whole sine 10000000, which we call Secants, or Tangents, and no more sines, shall be lesse then nothing.

Therefore we call the Logarithmes of the sines Abounding, because they are always greater then nothing, and set this marke + before them, or else none. But the Logarithmes which are lesse then nothing, we cal Defectiue, or wanting, setting this marke – before them. (Napier 1614, p. 4, Engl. transl. Napier 1616, p. 6).

Oughtred, in a chapter of his *Clavis mathematicae* (published posthumously in 1631), talked, in the case of the decimal logarithms, of "negative indices"⁹¹. As indicated by Napier in the continuation of the above quoted passage, these rules about negative and positive entities

⁹⁰ As Leibniz puts it: *si sint duae series sibi ordine respondententes fiantque numeri unius ex se invicem additione, ut numeri alterius multiplicatione, illa erit Logarithmorum, haec Numerorum* (GM V, 110). On the fact that one should be cautious not to project to quickly this meaning, which takes place on continuous domains, on Stifel himself, see Kouteynikoff (2006).

⁹¹ (Oughtred 1648, p. 157). This is one of the first occurrences of the term "negative", which I could find in the corpus. As emphasized by Wallis a few years later, in his *Mathesis Universalis* (Wallis 1657), the development of decimal fractions (of which Oughtred was a great proponent in England) was instrumental in the acceptance of exponents of the base noted -1 , -2 , -3 , etc., see (Wallis 1657, p. 65-66).

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are, for a large part, a matter of convention and of definition⁹². What is at stake is a certain structure conceived as a tool *for calculation*, not the handling of pre-existing objects/numbers. Interestingly enough, Napier and Mercator called for this reason logarithms *numeri artificiales*. Without surprise, another equivalent designation was... *numeri ficti*⁹³.

Be it as it may, what is interesting is to see negative numbers introduced as part of an artificial structure in which the question of their interpretation poses no particular problem. It is just a matter of designating two opposite directions (“contra”), through some explicit assumptions, with the great advantage that each direction corresponds nonetheless always to genuine quantities (numbers above or under the number chosen as reference, here for the “whole sine”). However, the choice of the very term “less than nothing”, which was employed for negative entities in an algebraic context, and of the sign of subtraction for designating them (in isolation) indicates that what is at stake is obviously the possibility of recuperating the whole additive structure underlying their use (as was already the case in Stifel and his followers). This is made clear by Napier in a following series of rules:

To adde Logarithmes of like signes, is to give the summe of them both, with their signe common to them both. [...]

To adde the Logarithmes of vnlike signes, is to giue the difference of them with the signe of the greatest number. [...]

Of two Logarithmes this is properly said to bee the Defectiue of that, and that the Abounding of this: when they haue both number and cyphers common, or the same, and all the signes + and – altogether contrary. [...]

To subtract an abounding Logarithme, is to adde his defective. [...]

To subtract a defective is to adde his abundant.

As emphasized by Thomaidis when mentioning this example (alongside the use of signs for indicating positions in Cartesian algebraized geometry): “negative numbers take their meaning through different classes of problems that cannot all be perceived at the same level” (Thomaidis 1993, p. 72). This is what was already indicated by Leibniz in his article from 1712. A too narrow focus on the history of the solution to algebraic equations and on the logic of “domain extension” has occulted part of the contexts in which negative entities occurred and the fact that they posed different kinds of questions, depending of the context of their use. Moreover, it has led to miss the role played by the category of “fiction”, which is precisely what allows for this plurality of meanings and their possible convergence.

⁹² “It was indeed left at libertie in the beginning, to attribute nothing, or 0. to any sine or quantitie for his Logarithme: but it was best to fit it to the whole sine, that the Addition or Substraction of that Logarithme which is most frequent in all Calculations, might never after be any troubel to us” (Napier 1616, p. 6).

⁹³ The same Caramuel who called the negative numbers “fictive” and devoted a whole introductory chapter on algebra dedicated to the use of fictions in mathematics, writes: *Logarithmi sunt ficti numeri* (Caramuel 1670, p. 102). In the paper dedicated to binary arithmetic, which Leibniz presented to the French Academy, we also find: “les Logarithmes, qui sont des espèces de nombres feints & supposés, sont un circuit que l'on prend pour arriver aux *Nombres Naturels*” (*Histoire de l'Academie des Sciences, avec les Mémoires, Année MDCCIII*, Paris, Jean Boudot, 1705, p. 62).

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Conclusion

It is not easy to determine when negative expressions entering the mathematician toolset became “negative numbers” seen as *bona fide* mathematical objects. It is not even clear that this question is meaningful. One turning point often mentioned on this issue is Euler (1770)⁹⁴. When one reads Euler, though, one is struck to see that he mentions from the start the fact that negative numbers are ways of talking about quantities such as money and can be conceived on the model of debts. Under this perspective, Euler’s treatment is in line with previous traditions. Yet there is another feature which is striking in Euler’s approach: not only does he introduce from the very beginning negative and positive numbers *on the same footing*, but he integrates both of them in a larger domain, that of “integers”. This presentation is certainly original in comparison with what we saw in the preceding century. It might have been prompted by the fact that problems of number theory, in which Euler worked a lot, can often admit of an infinite set of solutions in which negative and positive numbers enter on a par. But one might also consider the numerous efforts Euler made to give a proper meaning and autonomy to imaginary numbers (now seen as genuine numbers and not as particular type of roots to equations) or to the logarithm of negative numbers.

But that would not mean that a “revolution” would had occurred. In contrast to Euler’s approach, one could look for example at D’Alembert’s views, which is quite typical of the view expressed in the preceding period. In fact, the chapter on “negatives” in the *Encyclopédie* relates explicitly negative numbers to a kind of “false position”. D’Alembert opposed it very strongly to what he saw as useless metaphysical debates surrounding the supposed “absurdity” of quantities “less than nothing” (debates which were in fact, as we have seen, typical of the end of the 17th and beginning of 18th century): “Thus, negative quantities indicate truly positive quantities in computation, which were assumed in a false position”⁹⁵. This is the traditional view in terms of suppositive entities.

The contrast between D’Alembert and Euler’s positions indicate a crucial tension and debates on negative numbers are in fact numerous in 18th and 19th century (see Schubring 2005, Glière 2007). If our reconstruction is correct, these debates are not the continuation of struggles with “obstacles” related to the introduction of these objects with the development of symbolic algebra in the previous period. They are more likely related to their gaining the status of candidates as genuine objects (as opposed to “fictions”), an issue that originates in the second half of the 17th century, but really occurs at the beginning of the 18th. Considering our previous developments, it is natural to hypothesize – although this remains to be studied in more depth

⁹⁴ (Schubring 2005, p. 257-260).

⁹⁵ Here is the entry “négatifs” in the *Encyclopédie*: « quantités négatives, en Algebre, sont celles qui sont affectées du signe - , & qui sont regardées par plusieurs mathématiciens, comme plus petites que zéro. Cette dernière idée n'est cependant pas juste, comme on le verra dans un moment. (...) Imaginons, par exemple, qu'on cherche la valeur d'un nombre x, qui ajouté à 100 fasse 50, on aura par les regles de l'Algebre, $x + 100 = 50$, & $x = - 50$; ce qui fait voir que la quantité x est égale à 50, & qu'au lieu d'être ajoutée à 100, elle doit en être retranchée; de sorte qu'on auroit dû énoncer le problème ainsi: trouver une quantité x qui étant retranchée de 100, il reste 50; en énonçant le problème ainsi, on auroit $100 - x = 50$, & $x = 50$; & la forme négative de x ne subsisteroit plus. Ainsi les quantités négatives indiquent réellement dans le calcul des quantités positives, mais qu'on a supposées dans une fausse position. Le signe - que l'on trouve avant une quantité sert à redresser & à corriger une erreur que l'on a faite dans l'hypothese, comme l'exemple ci - dessus le fait voir très clairement (*Encyclopédie*, vol. 11 (1765), p. 72).

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– that the way these objects gained their meaning in a multilayered historical setting is also a way in which they reached a form of autonomy. Indeed, the passage from the status of tool to that of object is certainly favoured by the fact that one and the same expression appears as a useful tool *in different unrelated contexts*. More precisely, it makes connections between these previously unrelated contexts appear in a way that cannot be explained by the specific meaning these expressions had in each of these contexts and this suggest a kind of “self-subsistence”. Instead of being considered as ways of talking of other (“true”) objects, as is the case in the model of “fictions”, they appear as the objects we are talking about.

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