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**On the (Ir)Relevance of Discount Factors for
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JEL Codes: D11, D90

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ON THE (IR)RELEVANCE OF DISCOUNT FACTORS FOR FUTURE ALLOCATIONS OF SCARCE RESOURCES *

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Abstract

This article is interested in future allocations of scarce resources in an environment where upper bounds and lower bounds are fixed on the stream of consumptions or extractions of the scarce resource. It is shown that we can compute the optimal planning of consumptions independently from an explicit sequence of discounting factors as soon as they are decreasing at a rate smaller than a bound linked to the concavity of the utility function and the choice of the sequences of lower and upper bounds. The optimal solution is unique and exhibits two regimes with a pivotal period in the middle. Therefore, one gets plans satisfying some kind of intergenerational fairness: while the highest effort is supported by the first generations, it then decreases for the remaining ones. The argument is then extended to partially renewable resources. Finally, we consider the role of the horizon and of a potential regret after a revision for the bounds.

KEY WORDS: intertemporal allocation, scarce or renewable resources, multiple regimes, discount rates, fairness

JEL CLASSIFICATION: D11, D90

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1. INTRODUCTION

Since Ramsey [14], Samuelson [15] and Koopmans [10] and Koopmans et al. [11], there is a large tradition to value infinite streams of payoffs using the discounted utilitarian model. Following some experimental evidence, the literature has also been interested in time-inconsistent preferences with a so-called present bias, the most popular representation being labelled as the “quasi-hyperbolic discounting model” (See, e.g. Laibson [12] and Montiel and Strzalecki [13]). Several related questions have been raised like the intergenerational equity (See, e.g., Svensson [16], Basu and Mitra [2] or Fleurbaey and Zuber [8, 9] and also [9] arguing for a negative discount rate).

But, as mentioned in the seminal paper of Chambers and Echenique [3], there are substantial disagreements among economists about the proper discount rate for discounting long-term streams. Several papers (see, e.g. Basili and Chateauneuf [1], Dong Xuan et al. [4]) have worked on the aggregation of experts’ opinion. Some others like [3] (see also among many others contributions Drugeon and Ha Huy [5, 6]) introduce axioms aiming at resolving conflicting discount rates via a social choice approach.

Deviating from these lines of research, we first follow the traditional route for evaluating infinite streams of consumptions by merely assuming the existence of a sequence of discounting factors, which in our framework need not to be explicit. The key difference with respect to the earlier literature comes from the fact that we impose a minimal and a maximal consumption for each year in order to allow a minimal surviving consumption with the lower bound and impose limitation to over consumption of the resource through the upper bound. It turns out that there exists a unique explicit optimal plan, if the discounting factors decrease at a rate smaller than a bound linked to the concavity of the utility function and the choice of the sequences of lower and upper bounds. So, under this condition, the discount factors become irrelevant and that one should rather pay attention to the technological bounds that define at every date the set of admissible extraction rates for the scarce resource.

This model is motivated by the optimal consumption or extraction of a scarce resource, which is indispensable for the survival of generations but detrimental for the environment. Lower bounds are here understood as minimal survival levels while upper bounds are driven by environmental concern aimed at reducing the environmental damages or mitigating global warming. The dynamics of these bounds is driven by the development of alternative clean technologies allowing to reduce the use of this polluting resource. The optimal solution exhibits two regimes with a pivotal period at the middle. We briefly study the sensibility of this pivotal period to the lower and upper bounds. In particular, we show that it is not affected by the asymptotic behaviour of the upper (resp. lower) bounds in the first (resp. second) case.

We complete the paper by discussing some related issues. First we show that considering a finite horizon or an infinite horizon leads to similar results. Then, we prove that the optimal solution is time consistent in the sense that the truncated stream

of consumptions after a given period is still the optimal path starting from the current stock of the resource for the truncated maximisation problem. We also imagine the situation where a future generation would like to revise the lower and upper bounds according to a better knowledge of the evolution of the technologies. Then, we show that this future generation will not blame the previous choice. Indeed if this new bounds would have been taken into account at the initial period, then the remaining available quantity of the resource for this future generation would be smaller or equal to the one decided initially. There is no regret against the initial decision since this future generation benefits from more possibilities.

The rest of this article is organized as follows. The problem under consideration, the main theorem, the properties of the optimal solution as well as the scope for the renewability of the resource are detailed in Section 2. Various issues are then described in Section 3, namely the role of the horizon length, the intertemporal consistency of the optimal solution and the scope for regret after a revision of the bounds. In a final section, we consider depreciation or renewability of the resource. While the solution looks surprisingly similar, this is an artefact because the conditions on the bounds for which a solution exists now depends on the depreciation (resp. renewability) coefficients.

2. EXTRACTION OF A NON-RENEWABLE RESOURCE

2.1 THE MAIN THEOREM

We consider the following problem of optimal extraction for a non-renewable resource depending on $q_0 > 0$, the initial quantity of the non-renewable resource, the lower and upper bounds of extraction per period, (α_n) and (β_n) , a sequence of strictly positive discount factors, $\mu = (\mu_n)$ and a utility function u from \mathbb{R}_+ to $\mathbb{R} \cup \{-\infty\}$:

$$(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{n=1}^{\infty} \mu_n u(x_n) \\ \forall n \geq 1, \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{\infty} x_n = q_0 \end{cases}$$

We posit the following assumption on the parameters. We denote by ℓ_1^+ the set of summable positive real sequences.

ASSUMPTION 2.1. The parameters of the problem (\mathcal{P}) satisfy:

- A) $(\alpha_n) \in \ell_1^+$ and $(\beta_n) \in \ell_1^+$, $\alpha_n < \beta_n$ for all n and $\sum_{n=1}^{\infty} \beta_n > q_0 > \sum_{n=1}^{\infty} \alpha_n$;
- B) u is a concave strictly increasing function on $]0, +\infty[$ and $u(0) = \lim_{x \rightarrow 0^+} u(x)$ ¹.

¹This limit could be either finite or $-\infty$.

For all $x > 0$, we denote by $\partial u(x)$ the superdifferential² of u at x , $u^-(x) = \min \partial u(x)$ and $u^+(x) = \max \partial u(x)$. Since u is concave and strictly increasing, for all (x, ξ) such that $0 < x < \xi$, $0 < u^-(\xi) \leq u^-(x) \leq u^+(x)$. The sequence of discount factors is restricted to:

ASSUMPTION 2.2. For all $n \in \mathbb{N}^*$, $\frac{\mu_{n+1}}{\mu_n} < \frac{u^-(\beta_n)}{u^+(\alpha_{n+1})}$.

Our approach provides a characterisation of the solution under the above assumption, which means that the discount factors are decreasing at a rate smaller than a bound link to the concavity of the function u and the choice of the sequences of lower and upper bounds α and β . This assumption is related to the optimality condition for a transfert of consumption between the period n and the next one $n + 1$. See the proof of Claim 1 in the proof of Theorem 1.

REMARK 2.1. Notice first that Assumption 2.2 is fully compatible with a configuration where u is differentiable over its domain and the superdifferential of u at x sums up to a singleton with, for every $x > 0$, $u^-(x) = u^+(x) = u'(x)$ but also with the so-called Inada assumption where $u'(0) = \lim_{x \rightarrow 0^+} \frac{u(x) - u(0)}{x}$ is infinite. As a matter of illustration, let us consider the quantity $q_0 = 1$, the bounds $\beta_n = b^n$ with $b \in]\frac{1}{2}, 1[$ and $\alpha_n = \lambda b^n$ with $\lambda < \frac{1-b}{b}$. Then, when $u(x) = \ln x$, Assumption 2 is satisfied if $\frac{\mu_{t+1}}{\mu_t} < \lambda b$. When $u(x) = x^\delta$ with $\delta \in]0, 1[$, Assumption 2 is satisfied if $\frac{\mu_{t+1}}{\mu_t} < (\lambda b)^{1-\delta}$. The range of discount factors for which Assumption 2 is fulfilled hence increases with δ since the utility function is closer to a linear function. If u is linear the bound $\frac{u'(\bar{\beta})}{u'(0)}$ is equal to 1 and the assumption just means that the discount factors are strictly decreasing.

Notice however that, if u has a finite slope at 0, that is $u(0)$ is finite and $u'(0) = \lim_{x \rightarrow 0^+} \frac{u(x) - u(0)}{x}$ is finite³, noticing that, for all t , $\beta_t \leq \bar{\beta}$ and $\alpha_{t+1} > 0$, one obtains that $u^-(\beta_t) \geq u^-(\bar{\beta})$ and $u^+(\alpha_{t+1}) \leq u'(0)$. This hence results in a simpler sufficient condition for Assumption 2 which only depends only on the upper bound of the sequence (β_n) and where, $u^-(\bar{\beta})$ being a decreasing⁴ function of $\bar{\beta}$, the smaller is $\bar{\beta}$ the weaker is the assumption:

PROPOSITION 2.1. Let us assume that u has a finite slope at 0. Let $\bar{\beta} = \sup\{\beta_n \mid n \in \mathbb{N}^*\}$. If, for all $n \in \mathbb{N}^*$, $\frac{\mu_{n+1}}{\mu_n} < \frac{u^-(\bar{\beta})}{u'(0)}$, then Assumption 2 is satisfied.

Our main result now states that, under Assumptions 1 and 2, the solution of Problem (\mathcal{P}) is unique and neither depends on the discounting factors (μ_n) nor on the utility function⁵ u . Furthermore, we provide an explicit formula to describe the solution.

² $\partial u(x) = \{y \in \mathbb{R} \mid u(x') - u(x) \leq y(x' - x), \forall x' \in \mathbb{R}_+\}$. Under Assumption 1, $\partial u(x)$ is a nonempty bounded interval. If u is differentiable, then $\partial u(x)$ is the singleton $\{u'(x)\}$.

³Note that $u'(0) > 0$ since u is concave and strictly increasing

⁴Not necessarily continuous if u is not differentiable.

⁵Actually, the same proof works if the instantaneous utility function u_n is period-dependent under the condition that Assumption 2 is fulfilled by each utility function u_n .

THEOREM 2.1. Let $\mathcal{S}_1 = \sum_{k=1}^{\infty} \alpha_k$, $\mathcal{S}_n = \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^{\infty} \alpha_k$ for $n \geq 2$ and $\bar{n} = \sup\{n \in \mathbb{N}^* \mid \mathcal{S}_n \leq q_0\}$.

- (I) Under Assumption 1, $\{n \in \mathbb{N}^* \mid \mathcal{S}_n \leq q_0\}$ is nonempty and \bar{n} is finite.
- (II) Under Assumptions 1 and 2, the problem (\mathcal{P}) has a unique solution x^* given by the following formula:

Case 1: if $\bar{n} > 1$, then x^* is defined by:

- A) $x_n^* = \beta_n$ for $1 \leq n < \bar{n}$, $x_{\bar{n}}^* = q_0 - \sum_{k=1}^{\bar{n}-1} \beta_k - \sum_{k=\bar{n}+1}^{\infty} \alpha_k \in [\alpha_{\bar{n}}, \beta_{\bar{n}}[$
- B) $x_n^* = \alpha_n$ for $n > \bar{n}$.

Case 2: if $\bar{n} = 1$, then x^* is defined by:

- A) $x_1^* = q_0 - \sum_{k=2}^{\infty} \alpha_k \in [\alpha_1, \beta_1[$
- B) $x_n^* = \alpha_n$ if $n \geq 2$.

REMARK 2.2. Note that, in Case 1, if $\mathcal{S}_{n_0} = q_0$, then $x_{n_0}^* = \alpha_{n_0}$ and similarly, in Case 2, if $\mathcal{S}_1 = q_0$, then $x_1^* = \alpha_1$. This formula means that the optimal extraction is obtained by choosing first the upper bound of the feasible interval at each date as long as it is possible without compelling the extraction to be below the lower bound afterwards. The lower bound is chosen after a pivoting date n_0 where the extraction level is between the lower and the upper bound in order to adjust the total extraction to q_0 . In all cases, the extraction is at the maximal level for the first periods and then at the minimal one, showing that, independently from the sequence of decreasing factors (μ_n) and the utility function u , the intertemporal maximisation of the present value leads the future generations to benefit from a better ecological environment.

REMARK 2.3. The pivotal period \bar{n} can be interpreted more generally as the first period where alternative clean technologies should be available allowing to stop using the scarce polluting resource at its upper acceptable regime and therefore to switch to its lower admissible regime. The pivotal period \bar{n} is the first one if $\beta_1 + \sum_{k=2}^{\infty} \alpha_k > q_0$, meaning that the lower bounds are quite large. Then, the only room to increase the extraction above the lower bound is at the first period without having the possibility to extract a quantity equal to the upper bound β_1 . Keeping the upper bounds fixed, if the lower bounds decrease, then the switching period \bar{n} increases but it never exceeds a finite upper bound \bar{n} , which is $\max\{n \in \mathbb{N} \mid \sum_{v=1}^n \beta_v \leq q_0\}$. More generally \bar{n} is weakly decreasing with respect to β and α . Note that if we change the upper bounds after period $\bar{n} - 1$ keeping them above the lower bounds, then the optimal solution remains the same as well as if we change the lower bounds before period $\bar{n} - 1$ keeping them below the upper bounds. It has been proved that, actually, the asymptotic behaviour of the upper bounds does not modify the result.

REMARK 2.4. Note that this same approach can be applied to a problem of minimising the intertemporal cost of cleanup a stock of a bad with discounting factors, lower and

upper bounds at each period and an instantaneous convex cost function. Under the same kind of assumptions, we get a unique solution depending only on the lower and upper bounds for each period but the shape is reverse in the sense that during the first periods, the level of cleaning is equal to the lower bound and then it switches to the upper bound.

Finally, a justification for the introduction of minimal consumption at each period springs from the protection of far future generations when the Inada Condition is not satisfied by u , that is $u'(0)$ is finite. Let us consider a sequence of strictly decreasing discounting factors (μ_n) converging to 0 and the following problem without extraction bounds:

$$(\mathcal{P}') \begin{cases} \text{Maximise } \sum_{n=1}^{\infty} \mu_n u(x_n) \\ \forall n \geq 1, 0 \leq x_n \\ \sum_{n=1}^{\infty} x_n = q_0 \end{cases}$$

PROPOSITION 2.2. Consider the problem without extraction bounds (\mathcal{P}') . The optimal solution x^* is such that $x_n^* = 0$ after some period \bar{n} : whatever the discount factors, the far enough future generations have nothing to consume which is unfair with respect to the intergenerational equity.

Note that if there are only upper bounds (β_n) such that $\sum_{n=1}^{\infty} \beta_n > 0$, the result still holds true.

Proof. We first show that $x_1^* > 0$. If $x_n^* > 0$ for some $n > 1$, then $x_{n-1}^* > 0$. Indeed, if $x_{n-1}^* = 0$, 0 is a solution of $\max\{\mu_{n-1}u(\xi) + \mu_n u(x_n^* - \xi) \mid \xi \in [0, x_n^*]\}$. There thus exists $\zeta_n \in \partial u(x_n^*)$ such that $\mu_{n-1}u'(0) - \mu_n \zeta_n \leq 0$. Hence, $\zeta_n \geq \frac{\mu_{n-1}}{\mu_n} u'(0)$. Since the sequence μ is strictly decreasing, $\zeta_n > u'(0)$, which contradicts the fact that $\zeta_n \leq u'^+(x_n^*) \leq u'(0)$. By backward induction, noticing that at least one consumption is positive, we can conclude that $x_1^* > 0$. Let $n > 1$ such that $x_n^* > 0$. Using the same argument as above, there exists $(\zeta_1, \zeta_n) \in \partial u(x_1^*) \times \partial u(x_n^*)$ such that $\mu_1 \zeta_1 - \mu_n \zeta_n = 0$. Hence $\zeta_n = \frac{\mu_1}{\mu_n} \zeta_1 \geq \frac{\mu_1}{\mu_n} u'^-(x_1^*)$. Since the sequence $(\frac{\mu_1}{\mu_n})$ tends to $+\infty$, $u'^-(x_1^*) > 0$ and $\zeta_n \leq u'^+(q_0)$, one concludes that this inequality is not satisfied if n is large enough. Finally, $x_n^* = 0$ for all n large enough. \square

2.2 PROOF OF THEOREM 2.1.

The argument of the proof is splitted between three steps. The first establishes the existence of \bar{n} , the second proves the existence of the solution and the third provides an explicit formula to describe the solution.

STEP 1. Note that (\mathcal{S}_n) is strictly increasing since, for all $n \geq 1$, $\mathcal{S}_{n+1} - \mathcal{S}_n = \beta_n - \alpha_n > 0$. Note that $\mathcal{S}_1 = \sum_{k=1}^{\infty} \alpha_k < q_0$ by assumption. Since $\sum_{n=1}^{\infty} \beta_n > q_0$, there exists m_0 such that $\sum_{n=1}^{m_0} \beta_n > q_0$, hence $\mathcal{S}_n > q_0$ for all $n \geq m_0 + 1$. Therefore, the set $\{n \in \mathbb{N}^* \mid \mathcal{S}_n \leq q_0\}$ is nonempty since 1 belongs to it and it is bounded above by m_0 . Consequently, the upper bound \bar{n} exists and $\bar{n} \geq 1$.

STEP 2. The existence of a solution comes from the fact that the set $\{x \in \ell_1^+ \mid x_n \leq \beta_n \forall n \in \mathbb{N}^*\}$ is norm-compact in ℓ_1 and the function $x \rightarrow \sum_{n=1}^{\infty} \mu_n u(x_n)$ is continuous on this set since μ is bounded and u is continuous on $[0, \bar{\beta}]$. The problem (\mathcal{P}) hence has a solution x^* . We establish uniqueness at the end of the proof as a consequence of the explicit formula.

STEP 3. This third step is by far the more demanding and will be established by first proving a range of claims and then considering two separate cases for $\bar{n} > 1$ and $\bar{n} = 1$.

CLAIM 1. Let $n \geq 1$; if $x_n^* \in [\alpha_n, \beta_n[$, then $x_{n+1}^* = \alpha_{n+1}$.

Proof. Let $\bar{\xi} = \min\{\beta_n - x_n^*, x_{n+1}^* - \alpha_{n+1}\}$. By contraposition, if $x_{n+1}^* > \alpha_{n+1}$, then $\bar{\xi} > 0$. For all $\xi \in [0, \bar{\xi}]$, the sequence x defined by $x_{n'} = x_{n'}^*$ for all $n' \notin \{n, n+1\}$, $x_n = x_n^* + \xi$ and $x_{n+1} = x_{n+1}^* - \xi$ is feasible. From the optimality of x^* , one deduces that 0 maximises $\mu_n u(x_n^* + \xi) + \mu_{n+1} u(x_{n+1}^* - \xi)$ on the interval $[0, \bar{\xi}]$. Writing the first order necessary conditions, we get that there exists $(\zeta_n, \zeta_{n+1}) \in \partial u(x_n^*) \times \partial u(x_{n+1}^*)$ such that $\mu_n \zeta_n - \mu_{n+1} \zeta_{n+1} \leq 0$. Hence, from Assumption 2.2,

$$\frac{\zeta_n}{\zeta_{n+1}} \leq \frac{\mu_{n+1}}{\mu_n} < \frac{u'^-(\beta_n)}{u'^+(\alpha_{n+1})}$$

But we get a contradiction since $x_n^* \leq \beta_n$ implies that $\zeta_n \geq u'^-(\beta_n)$ and $x_{n+1}^* \geq \alpha_{n+1}$ implies $\zeta_{n+1} \leq u'^+(\alpha_{n+1})$. It derives that $x_{n+1}^* = \alpha_{n+1}$. \square

We complete the proof of Theorem 1 by considering the two cases $x_1^* = \beta_1$ and $x_1^* < \beta_1$.

CASE 1. $x_1^* = \beta_1$

We first remark that the set $F = \{n \in \mathbb{N}^* \mid \forall n' \leq n, x_{n'}^* = \beta_{n'}\}$ is nonempty and bounded above. Indeed, it is nonempty, since 1 belongs to F. As shown in Step 1, there exists m_0 such that $\sum_{n=1}^{m_0} \beta_n > q_0$. If there exists $n \geq m_0$ in F, then $q_0 = \sum_{\nu=1}^{\infty} x_{\nu}^* \geq \sum_{\nu=1}^n x_{\nu}^* = \sum_{\nu=1}^n \beta_{\nu} \geq \sum_{n=1}^{m_0} \beta_n > q_0$ a contradiction.

Let $n_0 = \max F + 1$. Clearly $x_{n_0}^* \in [\alpha_{n_0}, \beta_{n_0}[$. Indeed, if $x_{n_0}^* = \beta_{n_0}$, then n_0 belongs to F, which contradicts $n_0 = \max F + 1$. From Claim 1, for all $n \geq n_0 + 1$, $x_n^* = \alpha_n$. From the feasibility constraint, $q_0 = \sum_{n=1}^{n_0-1} \beta_n + x_{n_0}^* + \sum_{n=n_0+1}^{\infty} \alpha_n$ and $x_{n_0}^* \in [\alpha_{n_0}, \beta_{n_0}[$. Whence

$$\mathcal{S}_{n_0} = \sum_{n=1}^{n_0-1} \beta_n + \sum_{n=n_0}^{\infty} \alpha_n \leq q_0 < \sum_{n=1}^{n_0} \beta_n + \sum_{n=n_0+1}^{\infty} \alpha_n = \mathcal{S}_{n_0+1}$$

and $n_0 = \bar{n}$ and we recover the formula of the statement of the theorem.

CASE 2. $x_1^* < \beta_1$

Applying Claim 1, we conclude that $x_n^* = \alpha_n$ for all $n \geq 2$. From the feasibility condition $x_1^* + \sum_{n=2}^{\infty} \alpha_n = q_0$, one deduces that $x_1^* = q_0 - \sum_{n=2}^{\infty} \alpha_n$. From Assumption 1,

$\mathcal{S}_1 = \sum_{n=1}^{\infty} \alpha_n < q_0$ and $\mathcal{S}_2 = \beta_1 + \sum_{n=2}^{\infty} \alpha_n > x_1^* + \sum_{n=2}^{\infty} \alpha_n = q_0$. It is obtained that $n_0 = 1$ and we recover the formula of the statement of the theorem.

Since the solution is fully characterized by the formula provided in the statement of the theorem, we conclude that the solution is unique and depends only on the pivotal period \bar{n} , which is determined by the lower and upper bounds of extraction (α_n) and (β_n) .

3. SOME FACETS OF THE SOLUTION

3.1 THE FINITE HORIZON CASE

If we consider the same problem as the one in Section 2 with a finite horizon N , then we get the same functional forms for the solutions. Indeed, all the proofs work identically replacing the infinite sums by finite ones. The only difference appears when \bar{n} is equal to N , so that the lower bound α_n is never reached at the optimal solution.

Let (α_n) and (β_n) in \mathbb{R}_+^N such that $\alpha_n < \beta_n$ for all n and $\sum_{n=1}^N \beta_n > q_0 > \sum_{n=1}^N \alpha_n$, let $\mu = (\mu_n)$ be a sequence of discounting factors in \mathbb{R}_+^N and u be a concave strictly increasing utility function from \mathbb{R}_+ to $\mathbb{R} \cup \{-\infty\}$. Let us now consider the following maximisation problem of the present value of an extraction plan $(x_n) \in \mathbb{R}_+^N$:

$$(\mathcal{P}_N) \begin{cases} \text{Maximise } \sum_{n=1}^N \mu_n u(x_n) \\ \forall n \in [[1, N]], \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^N x_n = 1 \end{cases}$$

COROLLARY 3.1. Let $\mathcal{S}_1^N = \sum_{k=1}^N \alpha_k$, $\mathcal{S}_n^N = \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^N \alpha_k$ for $n = 2, \dots, N$, and $\bar{n} = \sup\{n \in \{1, \dots, N\} \mid \mathcal{S}_n^N \leq q_0\}$. If, for all $n \in [[1, N-1]]$, $\frac{\mu_{n+1}}{\mu_n} < \frac{u'(\beta_n)}{u'(\alpha_{n+1})}$, the problem (\mathcal{P}_N) has a unique solution (\hat{x}_n) given by the explicit formula:

Case 1: $N > \bar{n} > 1$, then \hat{x} satisfies:

- A) $\hat{x}_n = \beta_n$ for $1 \leq n < \bar{n}$, $\hat{x}_{\bar{n}} = q_0 - \sum_{k=1}^{\bar{n}-1} \beta_k - \sum_{k=\bar{n}+1}^N \alpha_k \in [\alpha_{\bar{n}}, \beta_{\bar{n}}[$
- B) $\hat{x}_n = \alpha_n$ for $N \geq n > \bar{n}$.

Case 2: $\bar{n} = 1$, then \hat{x} satisfies:

- A) $\hat{x}_1 = q_0 - \sum_{k=2}^N \alpha_k \in [\alpha_1, \beta_1[$
- B) $\hat{x}_n = \alpha_n$ if $n \geq 2$.

Case 3: $\bar{n} = N$, then \hat{x} satisfies $\hat{x}_n = \beta_n$ for $1 \leq n < N$, $\hat{x}_N = q_0 - \sum_{k=1}^{N-1} \beta_k \in [\alpha_N, \beta_N[$.

The above result shows that considering a finite or an infinite horizon model does not change the optimal solution for the first periods if the bounds are the same for these first periods. More precisely, let us consider the initial problem (\mathcal{P}) with an

infinite horizon with the discounting factors, lower and upper bounds and the utility function satisfying Assumption 1 and 2. Let us choose a finite horizon N and let us assume that the lower and upper bounds are the same for the periods 1 to $N - 1$ and, at the period N , $\hat{\alpha}_N = \sum_{n=N}^{\infty} \alpha_n$ and $\hat{\beta}_N = \sum_{n=N}^{\infty} \beta_n$. Then one easily checks that for the problem (\mathcal{P}) and for the problem (\mathcal{P}_N) , the solutions are the same for the periods up to $N - 1$. Indeed, we remark that $S_n = S_n^N$ for $n \leq N$ and, if $\bar{n} < N$, $\hat{x}_{\bar{n}} = 1 - \sum_{k=1}^{\bar{n}-1} \beta_k - \sum_{k=\bar{n}+1}^{\infty} \alpha_k = 1 - \sum_{k=1}^{\bar{n}-1} \beta_k - \sum_{k=\bar{n}+1}^{N-1} \alpha_k - \hat{\alpha}_N$.

The same result works when we start with a finite horizon N and we push it forward to $\tilde{N} > N$ or $+\infty$. If the lower and upper bounds α_n and β_n are not modified for the periods 1 to $N - 1$ and the final bounds α_N and β_N are shared over the remaining periods, that is, $\alpha_N = \sum_{n=N}^{\tilde{N}} \tilde{\alpha}_n$ and $\beta_N = \sum_{n=N}^{\tilde{N}} \tilde{\beta}_n$ or $\alpha_N = \sum_{n=N}^{\infty} \tilde{\alpha}_n$ and $\beta_N = \sum_{n=N}^{\infty} \tilde{\beta}_n$, then the optimal plans for the problem (\mathcal{P}_N) and for the extended horizon problem $(\mathcal{P}_{\tilde{N}})$ or (\mathcal{P}) coincide up to the period $N - 1$ if the discounting factors for the periods after N are satisfying Assumption 2.

To sum up, the initial extraction plan until a given period N does not depend on the lower and upper bounds for each period after $N + 1$ but just on the remaining aggregate bounds at $N + 1$, $\sum_{n=N+1}^{\infty} \alpha_n$ and $\sum_{n=N+1}^{\infty} \beta_n$. Otherwise stated, the choice of the lower and upper bounds, which are the key parameters of the model, is really important for the first periods together with the lower and the upper bounds for the total extraction defined as $\sum_{n=1}^{\infty} \alpha_n$ and $\sum_{n=1}^{\infty} \beta_n$. Hence, if we have in mind that these bounds are determined by experts using forecasts over the evolution of productivity of the current technologies and the one of alternative technologies, we just need a finite (even short) horizon expectations to take the right decisions for the first periods.

3.2 NO REGRET AFTER A REVISION OF THE BOUNDS

Let us now consider that the initial decision gets revised at a given period N . Starting from Problem (\mathcal{P}) and the optimal solution (\hat{x}_n) given by Theorem 1, let $N > 1$ and $q_N = q_0 - \sum_{n=1}^N \hat{x}_n$, for q_N the remaining stock of the non renewable resource at the end of Period N if the optimal extraction plan is completed until Period N . Keeping unchanged the lower and upper bounds, we get a new maximisation problem for the forthcoming periods:

$$(\mathbb{Q}^N) \left\{ \begin{array}{l} \text{Maximise } \sum_{n=N+1}^{\infty} \mu_n u(x_n) \\ \forall n \geq N+1, \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=N+1}^{\infty} x_n = q_N \end{array} \right.$$

We remark that the set defined by the constraints is nonempty since the truncation of the optimal plan $(\hat{x}_n)_{n=N+1}^{\infty}$ satisfies the constraints. A direct application of the Bellman principle leads to the following result:

PROPOSITION 3.1. The optimal solution $(\tilde{x}_n)_{n=N+1}^{\infty}$ of Problem (\mathbb{Q}^N) is just the truncation of the optimal solution (\hat{x}_n) that is $\tilde{x}_n = \hat{x}_n$ for $n \geq N + 1$.

We now assume that the lower and upper bounds after period N are revised thanks to a new analysis of the experts or a better knowledge of the technology transition. We still assume that they are consistent with the available stock $q_N = 1 - \sum_{n=1}^N \hat{x}_n$ and that Assumption 2 is satisfied.

Let us denote $(\tilde{\alpha}_n)_{n=N+1}^\infty$ and $(\tilde{\beta}_n)_{n=N+1}^\infty$ the new bounds satisfying $\tilde{\alpha}_n < \tilde{\beta}_n$ for all $n \geq N+1$ and $\sum_{n=N+1}^\infty \tilde{\alpha}_n < q_N < \sum_{n=N+1}^\infty \tilde{\beta}_n$. One may now wonder whether such a change could have modified the optimal path before Period N had it been anticipated at Period 0. The answer is clearly yes but the next proposition shows that the new optimal plan leads to a larger extraction level with the new bounds at each period before period N . Otherwise stated, at Period N , there is no regret about the previous extractions levels because there remains a greater quantity of the resources for the next generations. In other words, with the initial bounds, the optimal plan is more moderate in the sense that the quantities extracted during the first periods are lower than the optimal ones with the new revised bounds. In that sense, the initial plan is more favorable to the future generations.

Let us denote by $(\dot{\alpha}_n)_{n \geq 1}$ and $(\dot{\beta}_n)_{n \geq 1}$ the sequences defined by $\dot{\alpha}_n = \alpha_n$ and $\dot{\beta}_n = \beta_n$ for $n \leq N$ and $\dot{\alpha}_n = \tilde{\alpha}_n$ and $\dot{\beta}_n = \tilde{\beta}_n$ for $n > N$. We first remark that the modified lower and upper bounds satisfy the feasibility condition. Indeed:

$$\begin{aligned} \sum_{n=1}^{\infty} \dot{\beta}_n &= \sum_{n=1}^N \beta_n + \sum_{n=N+1}^{\infty} \tilde{\beta}_n \\ &> \sum_{n=1}^N \beta_n + q_N \\ &= \sum_{n=1}^N \beta_n + q_0 - \sum_{n=1}^N \hat{x}_n \\ &= \sum_{n=1}^N (\beta_n - \hat{x}_n) + q_0 \geq q_0 \end{aligned}$$

and the same type of computation shows that $\sum_{n=1}^{\infty} \dot{\alpha}_n < q_0$.

PROPOSITION 3.2. Let (\hat{x}_n) denote the optimal solution associated to $(\dot{\alpha}_n)_{n \geq 1}$ and $(\dot{\beta}_n)_{n \geq 1}$. (\hat{x}_n) satisfies the following inequalities: $\hat{x}_n \leq \dot{x}_n$ for all $n \in \llbracket 1, N \rrbracket$.

PROOF. Two configurations will be successively considered according to the position of \bar{n} with respect to N .

CASE 1. $N \geq \bar{n}$. Then $\hat{x}_n = \alpha_n$ for all $n > \bar{n}$, so $q_N = \sum_{n=N+1}^{\infty} \alpha_n$. Consequently, for all $n \leq N$,

$$\begin{aligned}
\mathcal{S}_n &= \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^{\infty} \alpha_k \\
&= \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^N \alpha_k + \sum_{n=N+1}^{\infty} \alpha_k \\
&= \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^N \alpha_k + q_N \\
&> \sum_{k=1}^{n-1} \beta_k + \sum_{k=n}^N \alpha_k + \sum_{n=N+1}^{\infty} \tilde{\alpha}_k = \mathring{\mathcal{S}}_n
\end{aligned}$$

One deduces that $\mathring{\mathcal{S}}_{\bar{n}} \leq \mathcal{S}_{\bar{n}} \leq q_0$, which implies that the new pivotal period \mathring{n} is greater or equal to the initial one \bar{n} . If $N < \mathring{n}$, then for all $1 \leq n \leq N$, $\mathring{x}_n = \mathring{\beta}_n = \beta_n \geq \hat{x}_n$. If $\mathring{n} \leq N$, we distinguish two cases. If $\bar{n} < \mathring{n}$, then for all $1 \leq n < \mathring{n}$, $\mathring{x}_n = \mathring{\beta}_n = \beta_n \geq \hat{x}_n$, and for all $\mathring{n} \leq n \leq N$, $\mathring{x}_n \geq \mathring{\alpha}_n = \alpha_n = \hat{x}_n$.

If $\bar{n} = \mathring{n}$, then for all $1 \leq n < \mathring{n}$, $\mathring{x}_n = \mathring{\beta}_n = \beta_n = \hat{x}_n$, for all $\mathring{n} < n \leq N$, $\mathring{x}_n = \mathring{\alpha}_n = \alpha_n = \hat{x}_n$, and

$$\begin{aligned}
\mathring{x}_{\mathring{n}} &= q_0 - \sum_{k=1}^{\mathring{n}-1} \mathring{\beta}_k - \sum_{k=\mathring{n}+1}^{\infty} \mathring{\alpha}_k \\
&= q_0 - \sum_{k=1}^{\mathring{n}-1} \beta_k - \sum_{k=\mathring{n}+1}^{\infty} \mathring{\alpha}_k \\
&> q_0 - \sum_{k=1}^{\mathring{n}-1} \beta_k - \sum_{k=\mathring{n}+1}^{\infty} \alpha_k = \hat{x}_{\mathring{n}}
\end{aligned}$$

The last inequality comes from the facts that $\mathring{n} + 1 \leq N + 1$ and $q_N = \sum_{n=N+1}^{\infty} \alpha_n > \sum_{n=N+1}^{\infty} \tilde{\alpha}_n = \sum_{n=N+1}^{\infty} \mathring{\alpha}_n$.

Gathering the previous results, in all cases, we get that $\mathring{x}_n \geq \hat{x}_n$ for $1 \leq n \leq N$.

CASE 2. $\bar{n} > N$. Then $\hat{x}_n = \beta_n$ for $1 \leq n \leq N$, so $q_N = q_0 - \sum_{k=1}^N \beta_k$ and $\sum_{k=N+1}^{\infty} \tilde{\alpha}_k < q_0 - \sum_{k=1}^N \beta_k$. Whence

$$\begin{aligned}
\hat{s}_{N+1} &= \sum_{k=1}^N \hat{\beta}_k + \sum_{k=N+1}^{\infty} \hat{\alpha}_k \\
&= \sum_{k=1}^N \beta_k + \sum_{k=N+1}^{\infty} \tilde{\alpha}_k \\
&< q_0
\end{aligned}$$

Hence $\hat{n} \geq N + 1$ and $\hat{x}_n = \hat{\beta}_n = \beta_n = \hat{x}_n$ for $1 \leq n \leq N$. \square

3.3 DEPRECIATION OR RENEWABILITY OF THE RESOURCE

In this section, we consider the limit case where the instantaneous utility function is linear, say $u(x) = x$. At each period, we introduce in the model a depreciation or renewability coefficient $\gamma_n > 0$. It means that if the stock of the resource at the end of the period $n - 1$ is q_{n-1} , then the quantity available for the generation n is $\gamma_n q_{n-1}$. If γ_n is less than 1, then it is a rate of depreciation of the resource whereas if $\gamma_n > 1$, it is a rate of renewability. Both cases are treated below in the same way.

The stream of consumptions $(x_n)_{n=1}^{\infty}$ is feasible if for all $n \geq 1$, $0 \leq x_n \leq q_{n-1}$ and $q_n = \gamma_n(q_{n-1} - x_n)$. Let Γ_n defined by $\Gamma_1 = 1$ and $\Gamma_n = \prod_{k=1}^{n-1} \gamma_k$. Γ_n is the quantity of resource available at period n if we have one unit available at date 0 and no consumption during the $n - 1$ first periods. Then the non-negative stream of consumptions $(x_n)_{n=1}^{\infty}$ is feasible if:

$$\sum_{n=1}^{\infty} \frac{1}{\Gamma_n} x_n \leq q_0$$

The optimal allocation problem becomes:

$$(\mathcal{P}(\gamma)) \begin{cases} \text{Maximise } \sum_{n=1}^{\infty} \mu_n x_n \\ \forall n \geq 1, \alpha_n \leq x_n \leq \beta_n \\ \sum_{n=1}^{\infty} \frac{1}{\Gamma_n} x_n = q_0 \end{cases}$$

If we consider a change of variable with ξ defined by $\xi_n = \frac{1}{\Gamma_n} x_n$, the problem has the same structure as Problem (\mathcal{P}) :

$$\begin{cases} \text{Maximise } \sum_{n=1}^{\infty} \mu_n \Gamma_n \xi_n \\ \forall n \geq 1, \frac{1}{\Gamma_n} \alpha_n \leq \xi_n \leq \frac{1}{\Gamma_n} \beta_n \\ \sum_{t=1}^{\infty} \xi_t = q_0 \end{cases}$$

The new discounting factors are $\bar{\mu}_n = \mu_n \Gamma_n$ and the new bounds are $\frac{1}{\Gamma_n} \alpha_n$ and $\frac{1}{\Gamma_n} \beta_n$.

Let us now have a look on the assumptions. Since the utility function is linear, Assumption 2 just means that the discounting factors are strictly decreasing. If the resource faces a depreciation between two periods, this means that $\gamma_n < 1$, then if μ_n is decreasing so is $\mu_n \Gamma_n$ since $\frac{\bar{\mu}_{n+1}}{\bar{\mu}_n} = \gamma_n \frac{\mu_{n+1}}{\mu_n} < 1$.

If the resource is partially renewable, $\gamma_n > 1$, then the new discount factors are decreasing if $\gamma_{t+1} \mu_{t+1} < \mu_t$. If this condition is not satisfied, it would mean that the utility increases if one saves one unit of the resource at the period t to consume γ_{t+1} units at the next period. This means that the renewal coefficient is high enough so that it is better to postpone the consumption, which means that the problem has no solution if this happens at all periods. It could be reasonable from an economic point of view to assume that the new discount factors $\bar{\mu}_n$ are decreasing.

The new problem has a feasible solution if and only if $\sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \alpha_n \leq q_0 \leq \sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \beta_n$. If these conditions are not fulfilled, the lower and upper bounds are not well adjusted to the economic environment. Indeed, if $\sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \alpha_n > q_0$, the endowments in initial resources is not sufficient even if each generation consumes only the minimal quantity α_n and if $\sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \beta_n < q_0$, the endowments in initial resources is too large so that each generation can consume the maximal quantity β_n without exhausting the resource.

Applying the previous characterisation of the solution for the non-renewable case, we get the following results.

COROLLARY 3.2. Under the following assumptions:

- (I) $(\alpha_n) \in \ell_1^+$ and $(\beta_n) \in \ell_1^+$, $\alpha_n < \beta_n$ for all n and $\sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \beta_n > q_0 > \sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \alpha_n$;
- (II) $(\mu_n \Gamma_n)$ is a strictly decreasing sequence;

the unique solution x^* of the problem $(\mathcal{P}(\gamma))$ is characterised, letting $\mathcal{S}_1^\gamma = \sum_{k=1}^{\infty} \frac{1}{\Gamma_k} \alpha_k$, $\mathcal{S}_n^\gamma = \sum_{k=1}^{n-1} \frac{1}{\Gamma_k} \beta_k + \sum_{k=n}^{\infty} \frac{1}{\Gamma_k} \alpha_k$ for $n \geq 2$ and $\bar{n} = \sup\{n \in \mathbb{N}^* \mid \mathcal{S}_n^\gamma \leq q_0\}$, as follows:

Case 1: if $\bar{n} > 1$, then x^* is defined by:

- A) $x_n^* = \beta_n$ for $1 \leq n < \bar{n}$, $x_{\bar{n}}^* = \gamma_{\bar{n}} \left(q_0 - \sum_{k=1}^{\bar{n}-1} \frac{1}{\Gamma_k} \beta_k - \sum_{k=\bar{n}+1}^{\infty} \frac{1}{\Gamma_k} \alpha_k \right) \in [\alpha_{\bar{n}}, \beta_{\bar{n}}[$
- B) $x_n^* = \alpha_n$ for $n > \bar{n}$.

Case 2: if $\bar{n} = 1$, then x^* is defined by:

- A) $x_1^* = q_0 - \sum_{k=2}^{\infty} \frac{1}{\Gamma_k} \alpha_k \in [\alpha_1, \beta_1[$
- B) $x_n^* = \alpha_n$ if $n \geq 2$.

REMARK 3.1. Surprisingly, the solution looks similar with depreciation or with partially renewable resources. But, this is an artefact. Indeed, the bounds (α_n) and (β_n) for which a solution exists are not the same since the condition $\sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \alpha_n \leq q_0 \leq \sum_{n=1}^{\infty} \frac{1}{\Gamma_n} \beta_n$ depends on the depreciation (resp. renewability) coefficients. Furthermore,

the turning period \bar{n} depends on the new discount factors, which are depending on these coefficients. Otherwise stated, while the qualitative structure of the solutions remains identical, the level of consumptions at each date are not comparable. Let us consider an example to illustrate this issue.

EXAMPLE. Let us assume that $q_0 = 1$, the renewal coefficient is constant equal to γ , the bounds are geometric sequences $\beta_n = b^n$ with $b \in]\frac{1}{2}, 1[$ and $\alpha_n = \lambda b^n$ with $\lambda < \frac{1-b}{b}$. The depreciation (resp. renewability) coefficient is compatible with the feasibility constraint if $\sum_{k=1}^{\infty} \lambda \frac{b^k}{\gamma^{k-1}} < 1 < \sum_{k=1}^{\infty} \frac{b^k}{\gamma^{k-1}}$, that is if $\lambda \gamma^2 b < \gamma - b < \gamma^2 b$, which determines an interval around 1. Note that $\mathcal{S}_n^\gamma = \sum_{k=1}^{n-1} \frac{b^k}{\gamma^{k-1}} + \lambda \sum_{k=n}^{\infty} \frac{b^k}{\gamma^{k-1}}$. It is clearly decreasing with respect to γ . Hence, noticing that the pivotal period is determined by $\bar{n} = \sup\{n \in \mathbb{N}^* \mid \mathcal{S}_n^\gamma \leq 1\}$, one deduces that the transition period is increasing with γ . This means that if the renewal coefficient increases, then the optimal solution changes in the sense that the maximal consumption β_n is chosen for a longer period before the switch to the minimal consumption α_n after the pivotal period \bar{n} .

□

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