Informational Frameworks for Collective Decision Making: "A Suggested Compromise"

Bora Erdamar

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A DISSERTATION in Economics
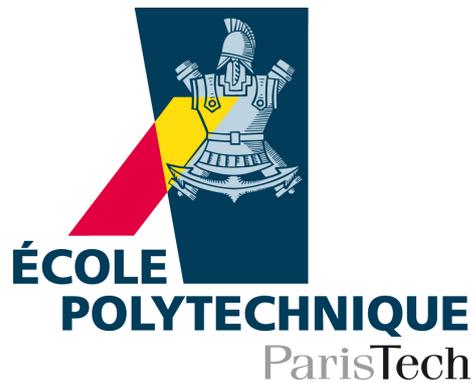
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To my wife *Hilal* and our son *Efe*, for the *preferred and approved* love...

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ABSTRACT

This thesis investigates the foundations of preference and utility theory used in Social Choice and Decision Theory.

The first chapter is the introduction.

The second chapter is composed of a survey of the existing results, motivations for a new framework that can combine many different approaches to aggregation of individual preferences and a proposal of a hybrid model, called preference-approval framework.

The third chapter asks the question of meaning of a consensus in such a framework. As an attempt to answer the question, this work follows a distance based approach, by a metric defined on the domain of preference-approvals and analyzes different ways of measuring homogeneity among the individual opinions. As a new modeling of these opinions, individuals are assumed to express themselves in terms of rankings over a set of options (alternatives) and threshold levels interpreted as the distinction between "approved" and "disapproved" alternatives.

The fourth chapter includes a manipulation analysis of aggregation rules over a voting profile composed of rankings and binary evaluations. Proposing a new notion of non-manipulability, this study provides a possibility result and some characterizations of impossibilities. Finally, further research problems for the art of designing new election systems and voting mechanisms are discussed with their potential implications for the society.

Keywords: social welfare functionals, interpersonal comparability, preference-approval, approval voting, Kemeny distance, consensus, nonmanipulability.

RÉSUMÉ

Cette thèse porte sur les fondations de la théorie des préférences et de l'utilité utilisée dans les domaines du choix social et de la théorie de la décision.

Le premier chapitre est l'introduction.

Le second chapitre composé d'une revue de la littérature et des résultats existants, d'une discussion des motivations pour envisager un nouveau cadre théorique permettant de combiner différentes approches de l'agrégation des préférences individuelles, et d'une proposition d'un modèle hybride appelé modèle de préférence-approbation.

Le troisième chapitre pose la question du sens que l'on peut donner au consensus dans un tel cadre théorique. Pour y répondre, ce travail fournit une approche basée sur la notion de distance, c'est-à-dire d'une métrique définie sur le domaine des préférence-approbations, et examine différentes faons de mesurer l'homogénéité au sein d'un ensemble d'opinions individuelles. Dans cette nouvelle modélisation des opinions, les individus s'expriment à la fois à travers un classement défini sur l'ensemble des alternatives et par un niveau de seuil, permettant de distinguer dans ce classement les alternatives "approuvées" de celles qui sont "désapprouvées".

Le quatrième chapitre comporte une analyse de la manipulabilité des règles d'agrégation définies sur un profil de votes composés de classements et d'évaluations binaires. En introduisant une nouvelle notion de non-manipulabilité, cette étude offre un résultat de possibilité, ainsi que certaines caractérisations d'impossibilités. La conclusion permet de discuter plusieurs questions de recherche future sur la manière de définir de nouveaux systèmes d'élections et mécanismes de votes, ainsi que leurs impacts potentiels sur la société.

Contents

Chapter 1

Introduction

Social Choice Theory investigates a wide range of collective decision making problems and deals with a variety of procedures. It covers the theories of elections on one side, and welfare economics on the other. In particular, designing voting rules, normative measurement issues like the evaluation of social welfare, national incomes or the measurement of inequality and poverty are among the concerns of this theory.

Although these problems are very different from each other, and require dissimilar approaches, they share a common feature. All of them includes the assessment of a social outcome to the preferences, choices or some other representations of the individual opinions under consideration, aggregation of which forms a profile of the society. On the other hand, differences between these problems lie mainly on the informational assumptions they use while processing the input of personal expressions or preferences in terms of measurability and comparability. For example, the well-known Arrovian framework

prohibits any cardinal information and interpersonal comparability. On the contrary, utilitarian approach demands for an interpersonally comparable cardinal framework.

In traditional voting theory, individuals in a group are represented solely by a ranking of the alternatives facing a group. However, this restriction may exclude some relevant information and many recent studies like *Approval Voting, Majority Judgment* and Range Voting have distinguished attention by the scholars of the field for the methods to include richer information.

In this dissertation, we analyze a new model on the foundations of the Social Choice Theory itself: representation of the individual opinions as *preference-approvals*. As its name suggests, in the *preference-approval* framework, agents facing an alternative set express themselves by an ordinal ranking over this alternative set and also by a threshold level interpreted as the distinction between their accepted and unaccepted alternatives. More explicitly, individuals provide the following information:

- Acceptable (approved) alternatives for them and a ranking of these acceptable alternatives.

- Unacceptable (disapproved) alternatives for them and a ranking of these alternatives with a natural requirement that all acceptable alternatives are above the unacceptable ones.

We use this framework for addressing three main issues of the Social Choice Theory:

1. What are the ways of representing individual opinions? In which methods those

opinions can be measured? What kind of interpersonal comparability assumptions can be used in collective decision making problems?

2. What is the meaning of a "consensus"? How can we measure the homogeneity among the individual opinions? In which ways measuring similarity of preferences can be used in collective decision making?

3. What are the ways of strategic misrepresentation of individual opinions? Up to which extent group choice procedures allow individuals to gain from expressing their views differently from the way they truly do?

In chapter 2, we analyze the informational bases of collective decision making problems. We provide a survey of the measurability and interpersonal comparability axioms used in the literature and give a taxonomy of the informational frameworks. We also briefly discuss *Utilitarianism* and *Rawlsian principles* and investigate the main difference in these approaches through the informational requirements. Finally, we analyze a new model of a hybrid platform, namely"preference-approval" structure. We show the following in this chapter:

- Measurability axioms for individual preferences can be partially ordered through "information sets".

- Axiomatically very similar rules can lead to very distinct outcomes simply because of the different informational frameworks they require.

- A unified framework of "preference-approval" can be used to analyze many recent models of voting theory, as well as the standard Arrovian rules. Moreover, this framework with its many simple extensions, calls for the design of new aggregation rules by incorporating various degrees of comparability and measurability assumptions.

In chapter 3, we consider the problem of measuring consensus level in a group of individuals who express themselves through rankings over an alternative set and also by a subset of approved (or disapproved) alternatives. This analysis requires the development of a distance concept between any given two *preference-approval*s. Intuitively, when all individuals have the same *preference-approvals* there is perfect consensus in the considered profile and any dissimilarity among them decreases the homogeneity, hence the consensus level. The difficulty is when distances between complex, so that multi-characteristic objects are considered, there is no "natural" scale for measuring the difference between these objects. For example, the difference between the wage payments from two job positions is easy to measure, as well as the difference between the number of working hours these positions require. But, what is the difference between these two job positions when their "wage & working hours" combinations are taken into account? Given any two preference-approvals, we propose measuring the dissimilarity between them by considering the convex combination of normalized Kemeny and Hamming metrics, for preference (ranking) and approval components respectively. By that way, we consider a metric which assigns a non-negative number to each pair of

preference-approvals. We show the following results in this chapter:

- A weighted sum of two distance measures that are not metrics can be a metric.

- For the neutral metric we propose, a consensus measure can be defined on the domain of *preference-approvals* which satisfies the standard properties of the literature like *unanimity, anonymity, neutrality*, as well as *maximum dissension* and *reciprocity*. [1]

- "Cloning does not always help to increase the homogeneity." More formally, given any two preference-approvals, increasing the population size by cloning these two preference-approvals will increase the consensus level only in a bounded way.

In Chapter 4, we focus on analyzing the strategic misrepresentation in the framework of preference-approvals. In the *preference-approval* framework, individuals may not only misrepresent their preferences but also what they approve of. Hence, we study a notion of manipulability where misrepresentation not only results in a more preferred alternative but also that the more preferred alternative is approved whereas the other one is not. We also consider a special domain, namely, *circular domain* for our analysis of possibility or impossibility for nonmanipulable rules. In a *circular domain*, the alternatives can be arranged on a circle so that for every alternative on the circle, there are two preferences in the domain in which top ranked and bottom ranked alternatives in of these preferences are respectively bottom ranked and top ranked in the other preference.

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[1] We invite the reader to check chapter 3 for the formal definitions.

In this chapter we establish the following results:

- The notion of non-manipulability we propose, which is defined in a new framework, coincides with the standard strategy-proofness when only *approval-invariant* rules are considered.

- Under some weak domain condition, there exists an *efficient* and *nonmanipulable* rule under which no agent is decisive.

- When the number of agents is even, we cannot have an *anonymous, efficient* and *nonmanipulable* rule on a circular domain.

- For the case of only two agents, there exists an *anonymous, unanimous* and *nonmanipulable* rule.

Chapter 2

Informational Frameworks in

Collective Decision Making

ABSTRACT

In this chapter, we provide a survey on measurability and comparability axioms that can be used in collective decision making problems and analyze a new informational basis, called *preference-approval* framework. At one extreme, pure ordinal framework can be modeled by utility functions which are equivalent under monotonic transformations. At the other extreme, the existence of an absolute scale for the individual utility leads to singleton information sets. In this context, *preference approval* framework corresponds to the case where utility functions are equivalent under monotonic transformations with one fixed point.

Keywords: social welfare functional, invariance transformations, interpersonal comparability, preference-approval.

2.1 Introduction

In collective decision making problems, the standard Arrovian framework rules out the use of any kind of cardinal information and interpersonal comparisons of utility. This restriction of information to ordinal preferences is one of the implicit assumptions that leads to the negative result of Arrow's impossibility theorem. In the literature, originally Sen [1] investigates some possible enrichments for the information bases by using utility functions instead of preferences. It is well-known that allowing for interpersonal comparability and richer informational requirements (for example, see Roemer [2]) can lead to positive results as *Utilitarianism* [3] and *Rawlsian Principles* [4].

Mainly two streams of studies exist for the analysis of informational frameworks. The first stream (Stevens [5], Krantz *et al.* [6], Roberts [7]) investigates *scales of measurement* which are identified by some admissible sets of transformations. For example, ordinal scales are defined by the set of all monotonic transformations, interval scales are defined by the set of increasing affine transformations and ratio scales are analyzed through the set of increasing linear transformations. Mandler [8] also investigates a variety of intermediate cases between cardinality and ordinality such as continuity and concavity by taking arbitrary sets of utility functions as the primitive of his analysis. The second stream (Sen [1, 9], d'Aspremont and Gevers [10, 11], Bossert and Weymark [12], Blackorby *et al.* [13]) formalizes *interpersonal comparability* assumptions in terms of equivalence relations over the set of utility functions. In this approach, a *social welfare functional* is required to be constant on informationally equivalent utility profiles.

As a result, richer informational assumptions imply fewer restrictions on social welfare functionals and some escapes from the impossibility theorem can be achieved.

In voting theory, many procedures (for example, see Brams and Fishburn [14]) and some recent models (e.g. Approval voting [15, 16, 17] , Majority judgment [18], Range voting [19]) use their own implicit assumptions on the informational bases for individuals. The analysis of voting rules with different informational requirements calls for a unified framework. Brams and Sanver [20] propose a way of combining the information of ranking and approval in a hybrid system which can be named as *preference approval* model. This approach leads to an extended Arrovian framework by incorporating two qualifications "good" and "bad" with a common meaning among individuals. [1]

In this chapter, we provide a survey on measurability and comparability axioms that can be used in collective decision making problems and analyze the *preference-approval* framework. At one extreme, pure ordinal framework can be modeled by utility functions which are equivalent under monotonic transformations. At the other extreme, the existence of an absolute scale for the individual utility leads to singleton information sets. In this context, *preference approval framework* corresponds to the case where utility functions are equivalent under monotonic transformations with one fixed point.

In Section 2, we introduce the basic notions. In section 3, we analyze *information sets* and equivalence relations on the set of utility functions for a single individual case. In Section 4, we extend our study to a finite set of individuals and analyze social welfare

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[1]One can see Kirman and Tesch [21] and also Davis [22] for a discussion on the representation of the identity of the economic agent from standard economic theory to the more recent mainstream approaches.

functionals and invariance transformations. Section 5 provides a brief discussion and comparison of Utilitarianism and Rawlsian Principles. Section 6 formalizes the informational requirement of the preference-approval framework. Finally, section 7 concludes by some remarks.

2.2 Basic Notions

We consider an individual confronting a finite non-empty set A of alternatives. Letting $u : A \to \mathbb{R}$ be a "utility function", we write $\mathcal{U} = \mathbb{R}^A$ for the set of all real valued utility functions that can be defined on A. We examine the partitions of $\mathcal{U}$.[2]

Given a partition, which we denote by $\Pi(\mathcal{U})$, we write $\sim_\pi$ for the equivalence relation on $\mathcal{U}$ induced by $\Pi(\mathcal{U})$.[3] As a minimum requirement, we impose a condition on partitions that any two utility functions in the same cell of the partition imply the same orderings on A. First, for any u we define R_u as $x R_u y \Leftrightarrow u(x) \geq u(y) \ \forall x, y \in A$. Then, we write formally the following condition:

Condition O: $u \sim_\pi v \Rightarrow R_u = R_v$.

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[2]By a partition $\Pi(\mathcal{U})$ of a non-empty set $\mathcal{U}$, we mean a class of pairwise disjoint, nonempty subsets of $\mathcal{U}$ whose union is $\mathcal{U}$. That is, $\Pi(\mathcal{U})$ is a partition of $\mathcal{U}$ iff (i) $\emptyset \notin \Pi(\mathcal{U})$, (ii) $\bigcup_{\pi \in \Pi(\mathcal{U})} \pi = \mathcal{U}$, (iii) For any disjoint $\pi, \pi' \in \Pi(\mathcal{U}), \pi \cap \pi' = \emptyset$. The sets in $\Pi(\mathcal{U})$ are called the *cells* of the partition.

[3]For any given partition $\Pi(\mathcal{U})$, we can define an equivalence relation on $\mathcal{U}$ by setting $u \sim v$ precisely when u, v are in the same cell in $\Pi(\mathcal{U})$.

We refer to the partitions of $\mathcal{U}$ that satisfy the above condition as *admissible partitions*. We also say that $\Pi(\mathcal{U})$ is **admissible** if any u and v in the same cell of the partition are ordering equivalent.

In this context, a "finer than" relation can be defined on the set of admissible partitions. Let $\Pi(\mathcal{U})$ and $\Pi'(\mathcal{U})$ be any two admissible partitions. If every cell in $\Pi(\mathcal{U})$ is a subset of a cell in $\Pi'(\mathcal{U})$, we say $\Pi(\mathcal{U})$ is finer than $\Pi'(\mathcal{U})$. Formally, $\Pi(\mathcal{U})$ is **finer than** $\Pi'(\mathcal{U})$, denoted by $\Pi(\mathcal{U}) \sqsubseteq \Pi'(\mathcal{U})$, if for any $\pi \in \Pi(\mathcal{U})$, there exists $\pi' \in \Pi'(\mathcal{U})$ such that $\pi \subseteq \pi'$. For this relation, **the coarsest partition** satisfies $u \sim_\pi v \Leftrightarrow R_u = R_v$. On the other extreme, **the finest partition** is characterized by $u \sim_\pi v \Leftrightarrow u = v$. Note that in this case, every cell is a singleton. By definition, the coarsest and the finest partitions are admissible. [4]

2.3   Information Sets for an Individual

Let $\mathcal{F}$ be the set of all real valued functions. For any $\Phi \subseteq \mathcal{F}$, we define a binary relation $\sim_\Phi$ on $\mathcal{U}$ as follows: For any $u, v \in \mathcal{U}$, $u \sim_\Phi v \Leftrightarrow u(a) = \phi \circ v(a)$, for some $\phi \in \Phi$, for all $a \in A$. We refer ϕ as a *transformation*. Note that, Φ is generated by successively all pairs of $u, v \in \mathcal{U}$. The next lemma states the conditions for this binary relation to be an equivalence relation.

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[4] If $\Pi(\mathcal{U}) \sqsubseteq \Pi'(\mathcal{U})$ and $\Pi'(\mathcal{U}) \sqsubseteq \Pi(\mathcal{U})$ then we have $\Pi(\mathcal{U}) = \Pi'(\mathcal{U})$.

Lemma 2.3.1

$\sim_\Phi$ *is an equivalence relation if* Φ *satisfies the following conditions:*

(i) For any $\phi \in \Phi$, $\phi^{-1} \in \Phi$.

(ii) For any $\phi, \psi \in \Phi$, $\phi \circ \psi \in \Phi$.

Proof. Pick any $u, v \in \mathcal{U}$. let $u(A), v(A)$ be the images of A under u and v. . Consider

$\phi : v(A) \to u(A)$ such that $u(a) = \phi(v(a))$, for all $a \in A$. Since ϕ^{-1} is $\in \Phi$, we get

$v(a) = \phi^{-1}(u(a))$, for all $a \in A$. So, whenever $u \sim_\Phi v$, we also have $v \sim_\Phi u$. Hence,

$\sim_\Phi$ is symmetric.

Now, pick any $u, v, w \in \mathcal{U}$. Consider $\phi : v(A) \to u(A)$ such that $u(a) = \phi(v(a))$ and

$\psi : w(A) \to v(A)$ such that $v(a) = \psi(w(a))$, for all $a \in A$. Since $\phi \circ \psi \in \Phi$, we have

$u(a) = \phi \circ \psi(w(a))$, for all $a \in A$. So, whenever $u \sim_\Phi v$ and $v \sim_\Phi w$ we also get

$u \sim_\Phi w$. Hence, $\sim_\Phi$ is transitive. Finally, reflexivity of $\sim_\Phi$ follows from the above two

arguments. Therefore, $\sim_\Phi$ is an equivalence relation.

∎

From now on, we only consider Φ satisfying the above conditions in the lemma 2.3.1.

In this case, Φ induces a partition on $\mathcal{U}$, which we denote by $\Pi_\Phi(\mathcal{U})$. The relevant ques-

tion in this context is when the partitions induced by transformations are admissible. Note

that the orderings R_u, R_v implied by u, v are preserved under monotonic transformations.

So, given any u and v which are ordering equivalent, $\phi \circ u$ and $\phi' \circ v$ are also ordering

equivalent iff ϕ and ϕ' are monotonic transformations. We say that Φ is **admissible** if any

$\phi \in \Phi$ is a monotonic transformation. The next proposition characterizes the admissible

partitions.

Proposition 2.3.2

$\Pi_\Phi(\mathcal{U})$ *is admissible if and only if Φ is admissible.*

Proof.

"Only if" part is clear. So, to show the "if" part note that for Φ is admissible, every $\phi \in \Phi$ is monotonic. By Lemma 2.3.1, we can derive that $\sim_\Phi$ is an equivalence relation, and hence $\Pi_\Phi(\mathcal{U})$ is a partition. Furthermore, for any $u, v \in \mathcal{U}$, $u \sim_\Phi v$ iff u and v are ordering equivalent. So, $\Pi_\Phi(\mathcal{U})$ is admissible. ∎

Next theorem shows the relationship between the "finer than relation" on admissible partitions and the set of transformations that induce these partitions.

Theorem 2.3.3

For any admissible Φ and Φ', $\Phi \subseteq \Phi'$ if and only if $.\Pi_\Phi(\mathcal{U}) \sqsubseteq \Pi_{\Phi'}(\mathcal{U})$.

Proof.

For the "if" part, suppose $\Pi_\Phi(\mathcal{U}) \sqsubseteq \Pi_{\Phi'}(\mathcal{U})$. Consider Φ that induces $\Pi_\Phi(\mathcal{U})$ and take any $\phi \in \Phi$. Now, pick any u, v with $u = \phi \circ v$. Let $\pi \in \Pi_\Phi(\mathcal{U})$ be such that $u, v \in \pi$. For $\Pi_\Phi(\mathcal{U})$ is finer than $\Pi_{\Phi'}(\mathcal{U})$, there exists $\pi' \in \Pi_{\Phi'}(\mathcal{U})$ such that $\pi \subseteq \pi'$. So we have $u, v \in \pi'$ implying that $\phi \in \Phi'$. Thus, $\Phi \subseteq \Phi'$.

("only if" part) Take any admissible Φ, Φ' with $\Phi \subseteq \Phi'$. To see $\Pi_\Phi(\mathcal{U}) \sqsubseteq \Pi_{\Phi'}(\mathcal{U})$ pick any $\pi \in \Pi_\Phi(\mathcal{U})$. Take any $u, v \in \pi$. We have $u = \phi \circ v$ for some $\phi \in \Phi$. For

$\Phi \subseteq \Phi'$, we also have $u \sim_{\Phi'} v$ implying that $u, v \in \pi'$ for some $\pi' \in \Pi_{\Phi'}(\mathcal{U})$. Thus, $\Pi_\Phi(\mathcal{U}) \sqsubseteq \Pi_{\Phi'}(\mathcal{U})$.

■

Theorem 2.3.3 implies that a partial order can be defined on the information sets just by the investigation of the subsets of monotonic transformations.

2.4   Social Welfare Functionals and Invariance Transformations

In this section, we will investigate various degrees of interpersonal comparability though admissible set of transformations. In the Arrovian framework, there is no room for interpersonal comparability among individuals, or in other words it is completely ordinal. Given a utility profile, any monotonic transformation applied to this profile will be *informationally equivalent* to the first one in this framework. We will see that by different subsets of monotonic transformations, various degrees of comparability can be formalized. It is worthwhile to note that there is an inverse relationship with the size of these admissible subsets of transformations and the degree of comparable information.

For this purpose, first we will extend our analysis to a finite set N of individuals confronting a finite non-empty set A of alternatives. For the rest of this section, we will mainly use the analysis of D'Aspremont and Gevers [10].

For all $i \in N$ and for all $a \in A$, we denote by $u(a, i)$ the individual i's utiliy level for

the alternative a. By defining utility function on $A \times N$, we can now compare the welfare of an individual i at alternative a, to the welfare of an individual j at alternative b. We denote a profile of size n utility functions by $U = (u(\cdot, 1), \ldots, u(\cdot, n))$ and we write $\mathbf{U}$ for the set of all possible utility profiles.

Next, we define a **social welfare functional** as a mapping $F : \mathcal{D} \to \mathcal{O}$ where $\emptyset \neq \mathcal{D} \subseteq \mathbf{U}$ is the set of admissible profiles and $\mathcal{O}$ is the set of all orderings on A. Representation of the informational environment will be established through the partition of the set of admissible profiles into information sets, similar to our analysis in section 3.

Finally, for every $U^1, U^2 \in \mathbf{U}$, we write $R_U^1 = F(U^1)$ and $R_U^2 = F(U^2)$.

2.4.1 Measurability and Comparability Axioms

For each of the following cases, an *information-invariance* condition requires F to be constant on each of the information sets. In other words, two profiles U^1 and U^2 are informationally equivalent, if $F(U^1)$ and $F(U^2)$ are identical.

We define an *invariance transformation* as the following:

Definition 2.4.1

An ***invariance transformation*** *is a vector* $\phi = (\phi_1, \ldots, \phi_n)$ *of functions* $\phi_i : \mathbb{R} \to \mathbb{R}$ *for all* $i \in \{1, \ldots, n\}$ *whose application to a profile* U *results in an informationally equivalent profile.*

Similar to the one individual case, let Φ denote the set of invariance transformations

[5] used to generate the equivalence relation $\sim$. That is, for all $U^1, U^2 \in \mathcal{D}$, $U^1 \sim U^2$ if and only if there exists $\phi \in \Phi$ such that $U^2 = \phi \circ U^1$, where $\circ$ denotes component-by-component function composition.

First, we formalize the Arrovian world of ordinally measurable , non-comparable utilities.

Ordinally Measurable, Non-comparable utility levels (OMN)

For every $U^1, U^2 \in \mathcal{D}$, $R_U^1 = R_U^2$ if for every $i \in N$, ϕ_i is a strictly increasing transformation such that, for all $a \in A$, $u^2(a, i) = \phi_i(u^1(a, i))$ where $u^1(\cdot, \cdot), u^2(\cdot, \cdot)$ are the utility components of profiles U^1, U^2 respectively.

In the Arrovian framework, since the admissible monotonic transformations can be different for every individual, there is no possibility of comparing neither utility levels, nor utility gains and losses among them.

Next, we introduce interpersonal comparisons of utility levels while preserving ordinal measurability. This requirement leads to smaller set of admissible transformations, as follows:

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[5]As an important remark, for the rest of the chapter, elements of Φ are *vector* of functions, differently from the single individual case.

Ordinally Measurable, Comparable utility levels(OMCL)

For every $U^1, U^2 \in \mathcal{D}, R_U^1 = R_U^2$ if for every $i \in N, \phi_\circ$ is a strictly increasing transformation such that, for all $a \in A$, $u^2(a, i) = \phi_\circ(u^1(a, i))$ where $u^1(\cdot, \cdot), u^2(\cdot, \cdot)$ are the utility components of profiles U^1, U^2 respectively.

In this case, since individual utilities are transformed by only a *common* strictly increasing function, it is possible to compare utility levels interpersonally, whereas comparison of utility gains are losses are still not possible, due to ordinality. This framework is used by *Rawlsian approach.* for the comparison of individuals at minimum welfare level in different states. In section 5, we will have a closer look to the Rawlsian model.

We now introduce cardinally measurable informational framework without any interpersonal comparability, as the following:

Cardinally Measurable, Non-comparable utilities (CMN):

For every $U^1, U^2 \in \mathcal{D}, R_U^1 = R_U^2$ if there exist $2n$ numbers $\alpha_1, \ldots, \alpha_n, \beta_1 > 0, \ldots, \beta_n > 0$ such that, for all $i \in N$ and for all $a \in A$, $u^2(a, i) = \alpha_i + \beta_i \cdot u^1(a, i)$.

In this case, affine transformations are allowed for two utility profiles that are informationally equivalent. Although there is no restriction on the sign of origins (α_i), the scaling factors (β_i) should be strictly positive for all $i \in \{1, \ldots, n\}$, since otherwise

the transformation would not be monotonically increasing. Different origins and utility scales for individuals imply neither utility levels, nor utility gains and losses are interpersonally comparable in this cardinal framework. It is worthwhile to note that *bargaining solutions* have this informational requirement.

Cardinal unit comparability (CMCU):

For every $U^1, U^2 \in \mathcal{D}, R_U^1 = R_U^2$ if there exist $n + 1$ numbers $\alpha_1, \ldots, \alpha_n, \beta > 0$ such that, for all $i \in N$ and for all $a \in A$, $u^2(a, i) = \alpha_i + \beta \cdot u^1(a, i)$.

In this cardinal framework, due to the common scaling factor (b), now it is possible to compare utility gains and losses interpersonally. On the other hand, since the origins of the utility indices (α_i) can be different for all $i \in \{1, .., n\}$, utility levels are not comparable across individuals. This informational context is used in *utilitarian approach*, which we will be discussed in a more detailed way in Section 5.

Translation Scale Measurability

For every $U^1, U^2 \in \mathcal{D}, R_U^1 = R_U^2$ if there exist $n + 1$ numbers $\alpha_1, \ldots, \alpha_n$, such that for all $i \in N$ and for all $a \in A$, $u^2(a, i) = \alpha_i + u^1(a, i)$.

As a special case of the *Cardinal unit comparability*, in this framework, utility dif-

ferences are interpersonally comparable and in addition, numerical values of these differences are also meaningful. Since the origins of the utility scales may be different for each individual, utility levels are again, not interpersonally comparable.

Cardinal Full Comparability

For every $U^1, U^2 \in \mathcal{D}$, $R_U^1 = R_U^2$ if there exist two numbers $\alpha_\circ$ and $\beta_\circ > 0$ such that, for all $i \in N$ and for all $a \in A$, $u^2(a, i) = \alpha_\circ + \beta_\circ \cdot u^1(a, i)$.

In this fully comparable cardinal framework, since the origin and the scale factors are the same for everyone, both utility levels and differences can be compared across individuals. Note that the admissible affine transformation is quite restrictive, which allows for the richest amount of information that can be used for measurability and comparability, among the models we have analyzed until now.

It is worthwhile to note that even in the case of cardinal full comparability, exact numerical values of the utility levels are not important. By further restriction of the admissible set of transformations, we would reach the case of numerical full comparability in which every utility index is in absolute scale.

Numerical Full Comparability

For every $U^1, U^2 \in \mathcal{D}, R_U^1 = R_U^2$, for all $i \in N$ and for all $a \in A$, $u^2(a,i) = \mathcal{I} \cdot u^1(a,i)$.

In this final case, each utility level has a numerical meaning in addition to the all other properties that are satisfied by the previous frameworks. Note that, each utility function is equivalent to itself, in other words, the only admissible transformation is the identity function.

In the context of information sets, numerical full comparability is the other polar case of ordinal measurability. To be more explicit, numeral full comparability leads to the singleton information sets, which are the finest partitions, whereas in the case of ordinally measurable noncomparable utilities, partitions are the coarsest.

Until now, we have investigated *measurability* and *comparability* axioms that can be used in collective decision making problems. One can note that the assumptions on measurable information specifies the set of admissible transformations. On the other hand, up to which degree this measured information can be compared is determined by the common scale factors in these transformations.

2.5   Utilitarianism and Rawlsian Models

In this section we briefly discuss Utilitarianism and Rawlsian principles in the context of informational frameworks. We will provide their characterizations at the end of the sec-

tion to emphasize that the main difference between these two approaches arises from their requirements for the measurability and interpersonal comparability assumptions they use.

Utilitarianism [23, 24] suggests that among two states under consideration, the state with higher aggregate sum of utilities (or at least as high as the other), should be preferred (weakly, respectively) to the other one. It is worthwhile to note that the main factor for the social outcome is the sum of utilities, but the distribution of these utilities among the individuals has no effect in the outcome.

Formally, Utilitarianism can be defined as the following:

Utilitarianism: [6] *For all $U \in \mathbf{U}$ and for all $a, b \in A$, $a R_U b$ if and only if $\sum_i^n u(a, i) \geq \sum_i^n u(b, i)$.*

On the other hand, the Rawlsian rules [25, 4] focuses on the lowest levels of utility values in a given profile.[7] The main approach can be modeled as the following: Given any two alternatives $a, b \in A$, a should be strictly preferred to b. whenever $min_i U(a, i) > min_i U(b, i)$.

In the literature, two *social welfare functionals* are widely studied based on the above idea, namely *maximin* and *leximin* rules (for example, see Sen [27], Richardson *et al.* [28], Gaertner [29]).

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[6] More precisely, this rule is named as *pure utilitarianism* in the literature. For various other versions, d'Aspremont and Gevers [11]

[7] For a nice discussion on Rawlsian principle *the veil of ignorance*, one can see Baigent [26]

For a closer look to these rules, we proceed as follows: Given a utility profile $U \in \mathbf{U}$, and a state a, let $r_a(U)$ be the person who is the rth best-off under state a and has rank r. So, the best-off among all individuals at state a is $1_a(U)$ and the worst-off is $N_a(U)$, since there are N individuals. *Maximin rule* can be defined now as follows:

Maximin rule: For all $U \in \mathbf{U}$ and for all $a, b \in A$, $aR_U b$ if and only if $u(a, N_a(U)) \geq u(b, N_b(U))$.

The *maximin* rule implies that the Nth rank is a positional dictator. Hence, this rule can violate even the Pareto Principle [8]. The following lexicographic version of the *maximin* rule, on the other hand, satisfies the Pareto principle. For a detailed analysis of these rules one can check Hammond [30], d'Aspremont and Gevers [10] and Strasnick [31].

Leximin rule: For all $U \in \mathbf{U}$ and for all $a, b \in A$, $aP_U b$ if and only if there exists a rank $k \in \{1, \ldots, N\}$ such that $u(a, k_a(U)) > u(b, k_b(U))$ and $u(a, l_a(U)) > u(b, l_b(U))$ for all $l > k$ where $l \leq N$.

One can note that, *leximin rule* is very similar to the *maximin rule*. The main difference between these two rules lies on the treatment of the outcome when there is an indifference for the worst-off individuals. To be more explicit, if at the lowest level the

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[8]For the definition of *Strict Pareto* condition, please see below in this section.

utility values are the same for the states a and b, the *maximin rule* leads to an outcome of an indifference between a and b. On the other hand, if this is the case, *leximin rule* checks for the second-worst-off individuals, if they are also the same then the third-worst-off individuals and so forth.

We proceed by providing some widely-used axioms of the literature in the context of utility profiles, which will be used to characterize the above rules we discussed.

Anonymity (AN) Let σ be any permutation on the set of individuals N. For every $U^1, U^2 \in \mathbf{U}, R_U^1 = R_U^2$ if U^1 and U^2 are such that for all $i \in N$ and for all $a \in A, u^1(a, i) = u^2(a, \sigma(i))$.

Anonymity states that only the list of individual utility values should be important but the name tags of the individuals should not matter.

Strict Pareto (SP) For all $a, b \in A$ and for all $U \in \mathbf{U}, aR_U b$ if for all $i \in N$, $u(a, i) \geq u(b, i)$. If, moreover, for some $j \in N, i \in N, u(a, i) > u(b, i)$ then $aP_U b$.

Strict Pareto briefly says that if every individual favors a social state to another one, the social relation over these states should also reflect this accordingly.

Independence (I) For every $U^1, U^2 \in \mathbf{U}$, for all $a, b \in A$, R_{U^1} and R_{U^2} should coin-

cide on $\{a, b\}$ if $U^1 = U^2$ on $\{a, b\} \times N$.

Independence axiom states that if a and b obtain the same n-tuple of utilities in U^1 and U^2; the social relation between a and b for U^1 and U^2 should be the same as well, and should not be affected by any other third alternative.

Next, we discuss concerned and unconcerned voters axioms. Concerned voters are defined as the ones who are not indifferent between every pair of alternatives. On the other hand, unconcerned voters are defined as the ones who are indifferent between all given options. The relevant question in this context is up to which degree unconcerned individuals should have an influence on the collective choice. In particular, we can discuss this issue through utilitarianism: The voters who are unconcerned between two states a and b do not play any decisive role in the utilitarian collective decision for a and b, since these voters increase the sum of the utilities for these states in an equal way.

To formalize this property of "eliminating the influence of unconcerned voters on the decision rule", we have the following separability requirement:

Separability of unconcerned individuals (SE) For every $U^1, U^2 \in \mathbf{U}$, $R_{U^1} = R_{U^2}$ if there exists $M \subset N$ such that for all $i \in M$ and for all $a \in A$, $u^1(a, i) = u^2(a, i)$ while for all $h \in N \setminus M$ and for all $a, b \in A$, $u^1(a, h) = u^1(b, h)$ and $u^2(a, h) = u^2(b, h)$.

Separability axiom says that, if there are unconcerned voters ($h \in N \setminus M$) for the states a and b, and if all the rest of individuals have the same utility levels under the profiles U^1 and U^2 for all alternatives, then the social ranking should be the same for U^1 and U^2. As an implication, in the case of indifference between two states a and b for all individuals except two of them, the decision problem would be reduced to the disagreement on a and b of these two individuals only.

Now, we provide the following *Equity* axiom which requires that if every individual is indifferent between two states a and b except two agents, the one among these two agents which is in worse condition than the other determines the social relation between a and b. Formally we state the axiom of *Equity* as the following:

Equity For all $U \in \mathbf{U}$ for all $a, b \in A$ and for all $a, b \in N$. $aP_U b$ whenever for all $h \in (lN \setminus \{i, j\}), u(a, h) = u(b, h)$ and $u(b, i) < u(a, i) < u(a, j) < u(b, j)$.

Finally, just as the opposite of the *Equity* axiom, *Inequity* axiom requires that if every individual is indifferent between two states a and b except two agents, the one among these two agents which is in better condition than the other for these states, determines the social relation between a and b. Formally we state the axiom of *Inequity* as the following.

Inequity (INEQ) For all $U \in \mathbf{U}$ and for all $a, b \in A$ and for all $i, j \in N$, $bP_U a$ whenever for all $h \in (N \setminus \{i, j\}), u(a, h) = u(b, h)$ and $u(b, i) < u(a, i) < u(a, j) < u(b, j)$.

It is worthwhile to note that interpersonal comparisons of utility levels are required for applying the axioms of *Equity* and *Inequity*.

For the rest of this section we will provide a characterization of the *leximin rule* and *utilitarianism* through the above axioms we discussed. As a survey chapter, we omit the proofs of the following theorems and we invite the reader to see D'Aspremont and Gevers [10] and Sen [1, 9] for further discussion.

First, the next theorem shows a restriction on social welfare functionals by an informational basis.

Theorem 2.5.1

If F satisfies I, SP, AN, SE, and the informational requirement OMCL, it satisfies either EQ or INEQ.

The above theorem states that informational basis of ordinally measurable and interpersonally comparable utility levels with some axioms leads to a social welfare functional whose outcome is determined by either the worse-off or the better-off individuals in some specific states.

Next, we discuss the following characterization of the *leximin rule*.

Theorem 2.5.2

The leximin principle is characterized by I, SP, AN and EQ. [9]

Finally, the following theorem provides a characterization of the utilitarian rule.

Theorem 2.5.3

The utilitarian rule is characterized by conditions I, SP, AN and CMCU.

By the above characterizations, we note that the main difference between Utilitarianism and Rawlsian leximin rule arises from different informational bases. In other words, their axiomatic structures are almost the same, but the measurability requirements are quite different between Utilitarianism and Rawlsian rules.

2.6   Preference-Approval framework

So far we have seen various informational frameworks used in collective decision making problems and discussed their implications for some aggregation rules.

In recent literature of Social Choice Theory, there are many proposals of voting rules which call for new informational frameworks. (see, for example, Hillinger [32], Aleskerov et al. [33], Balinski and Laraki [18], Smith [19])

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[9]Many variations of these characterizations exist in the literature. For a simpler version, one can note that *Independence* and *Strict Pareto* implies the well-known *Neutrality* axiom, which briefly states that the labels of alternatives should not matter and all the relevant information for social welfare functional should be contained in the given utility values.

In particular Approval Voting, ([15, 16, 17]) requires a qualification profile of "approved" or "disapproved" alternatives and the ones which are approved by the highest number of individuals are the AV winners.[10] As a recent study (Sanver [35]) suggests, common meaning assumption for these two qualifications can be interpreted as the existence of a real number, say 0, whose meaning as a utility measure is common to all individuals. As an example, "being self-matched" in matching theory models (for example, see Roth and Sotomayor [36]) can be interpreted as the common zero of that framework.

In the context of this chapter, "existence of a zero for an individual" leads to the next invariance condition by using the the terminology of section 3. We denote by Φ^* the set of monotonic transformations with a fixed point, which is generated by successively all pairs of $u, v \in \mathcal{U}$ in the following condition:

Information sets of preference-approval framework for an individual:

For any $u, v \in \mathcal{U}$, $u \sim_{\Phi^*} v$ if and only if $v = \phi \circ u$, for some $\phi \in \Phi^*$, which is monotonically increasing and $\phi(0) = 0$.

The next lemma shows that preference-approval framework leads to finer partitions than the ones generated by the set of monotonic transformations, $\Phi^{\mathcal{M}}$, which is generated by successively all pairs of $u, v \in \mathcal{U}$.

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[10]One can see an application about choosing committees through approval ballots in Laffond, Gilbert and Lainé [34]

Lemma 2.6.1

$\Pi_{\Phi^*}(U)$ *is admissible and* $\Pi_{\Phi^*}(\mathcal{U}) \sqsubseteq \Pi_{\Phi^{\mathcal{M}}}(\mathcal{U})$.

Proof. First, note that $\sim_\Phi$ is an equivalence relation by Lemma 2.3.1. By definition, any $\phi \in \Phi^*$ is a monotonic transformation, therefore $\Pi_{\Phi^*}(U)$ is an admissible partition by Proposition 2.3.2. Finally, since $\Phi^* \subseteq \Phi^{\mathcal{M}}$ we get $\Pi_{\Phi^*}(\mathcal{U}) \sqsubseteq \Pi_{\Phi^{\mathcal{M}}}(\mathcal{U})$ by Theorem 2.3.3. ∎

We can generalize the above invariance condition to N many individuals case as follows:

Invariance under monotonic transformations with one fixed point:

For every $U^1, U^2 \in \mathcal{D}, R^1_U = R^2_U$ *if for every* $i \in N, \phi_i$ *is a strictly increasing transformation and* $\phi_i(0) = 0$ *such that, for all* $a \in A,\ u^2(a, i) = \phi_i(u^1(a, i))$ *where* $u^1(\cdot, \cdot), u^2(\cdot, \cdot)$ *are the utility components of profiles* U^1, U^2 *respectively.*

As the above analysis suggests, preference-approval framework provides a unified environment for studying standard and nonstandard aggregation rules, by incorporating a comparability at one fixed point. It is worthwhile to note that any aggregation function of the Arrovian model can be expressed in this framework, by simply introducing an "approval independence" condition. (Sanver [35] or Chapter 4 of this thesis). Moreover,

Preference-approval framework can be extended to analyze any kind of label voting, as *majority judgement* and *range voting* suggest. In the case of l labels, an extension of the model to the invariance under monotonic transformations with $l - 1$ fixed points would allow to study of these rules in a unified framework with standard Arrovian aggregation functions.

2.7 Concluding Remarks

Each rule that is used in a collective decision making problem has its own treatment for filtering the informational content provided by a society. In Arrow's world of ordinal utility, there is no possibility to compare anything except rankings provided by individuals. Utilitarianism asks for a cardinal framework which demands that utility differences are interpersonally comparable. Rawlsian rules require ordinal level of comparability to focus on the worst-group in the society. Approval voting asks for a binary qualification of the alternatives, whereas Majority Judgment Rule asks for a common language of seven labels for the evaluation of them. As these examples suggest, modifying the informational requirements can be one of the paths to follow for the design of new aggregation rules. However, as we try to illustrate in this chapter, some informational assumptions can be too demanding. Comparison of measurability properties in a common framework can avoid the confusion of axiomatic differences with the structural ones. Preference-approval framework provides a unified environment not only for a better examination of discrepancies and similarities between existing rules but also allows designing new rules

for collective decision making problems.

Chapter 3

Measuring Consensus in a Preference-Approval Context

ABSTRACT

We consider[1] measuring the degree of homogeneity for preference-approval profiles which include the approval information for the alternatives as well as the rankings of them. A distance-based approach is followed to measure the disagreement for any given two preference-approvals and we show that the proposed measure of consensus is robust to the extensions of the ordinal framework under the condition that a proper metric is used. This paper also shows that there exists a limit for increasing the homogeneity level in a group of individuals by simply replicating their preferences.

Keywords: Consensus, Approval voting, Preference-approval, Kemeny metric, Hamming metric.

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[1]**This chapter is based on a joint paper with the co-authors Garcìa Lapresta, Pérez-Román and Sanver [37]**

3.1 Introduction

In collective decision making problems, notion of consensus has been analyzed and interpreted in miscellaneous ways. Dictionary meaning of consensus is a general (unanimous) agreement within a group of people or agents. However, most of the decision making procedures (eg. elections, voting by committees, competitions) deal with a more realistic situation of partial agreement for the candidates or alternatives. Interpreting a partial agreement of individuals as a consensus up to some degree, the immediate question is how to measure that degree of agreement (Kacprzyk [38], Tastle and Wierman [39, 40]). Related questions include how to use this information to reach a final decision (Kemeny [41], Beliakov, Calvo and James [42]) and which procedures can be used to increase the level of consensus (Susskind and McKearnan [43], Strauss and Layton [44], Van Den Belt [45]). For an overview of different attributions of consensus, one can also see Martínez-Panero [46]. In this contribution, consensus is interpreted as the degree of homogeneity within a set of individuals and consensus measure is a scale for the similarity of preferences.

It is important to note that degree of consensus is dependent on the context of the preferences. Similarity of preferences when individuals submit linear orders over alternatives can be very different than the homogeneity of a profile composed of weak orders. In related literature, Kendall and Gibbons [47] considered measuring concordance among two linear orders. Then, Hays [48] and Alcalde-Unzu and Vorsatz [49] generalized the idea to any number of linear orders. Similarly, Bosch [50] proposed a measure of consensus

for any given profile of linear orders by a mapping which assigns a number between 0 and 1 according to the degree of homogeneity in that profile. Satisfying some desirable axioms such as *unanimity* (for every subgroup of agents, the highest degree of consensus is reached only if all agents have the same orderings), *anonymity* (permutation of agents does not lead to a change in the degree of consensus) and *neutrality* (permutation of alternatives does not lead to a change in the degree of consensus) Bosch's model has been investigated further for various domains. García-Lapresta and Pérez-Román [51] extended the consensus measure of Bosch [50] for weak orders and introduced new properties such as *maximum dissension* (in each subset of two agents, the minimum consensus is reached only if agents have linear orders which are inverses of each other) and *reciprocity* (replacing each order in the profile by their inverses does not lead to a change in the degree of consensus). Moreover, García-Lapresta and Pérez-Román [52] used the framework of Bosch [50] for weighted Kemeny distances, thereby dealing with the possibility of weighting discrepancies among weak orders.

Some recent models for collective decision making problems (eg. approval voting [16], majority judgment [18], range voting [19]) use nonstandard formulations of inputs in aggregation of preferences. These models assume that individuals adopt a common language when they evaluate alternatives. Therefore, instead of aggregating ordinal rankings these models deal with aggregating labels such as *approved* and *disapproved*. Brams and Sanver [53] suggests a framework that can be considered as a compromise between standard and nonstandard models by combining the information of ranking and

approval in a hybrid system which they call *preference-approval.* Individuals are assumed to submit a weak ordering on a given set of alternatives and a cut-off line to distinguish acceptable and unacceptable alternatives for them. An alternative which is ranked above (resp. below) the line is qualified as acceptable (resp. unacceptable). Preference-approval model extends the ordinal framework in a minimal way by incorporating two qualifications *good* and *bad* with a common meaning among individuals. It is worthwhile to note that the status-quo point in bargaining problems, the threshold level in public good problems and the alternative of being self-matched in matching problems can be interpreted as the cut-off lines when these models are translated to the preference-approvals. [2] In that sense, preference-approval model proposes a common framework in which nonstandard aggregation procedures and the standard ones in the literature can be analyzed in a natural way.

The problem of how to measure consensus for the extended ordinal frameworks is an open question in the literature. In this contribution, by following a distance-based approach we focus on measuring the degree of disagreement/agreement in preference-approval profiles. Since distance functions widely used in the literature are defined on various domains of ordinal rankings, the first difficulty is to derive a proper metric for extensions of weak orders. We propose a way of measuring distance separately for two types of informational content in preference-approvals and then we derive a metric defined by a convex combination of these distances. Technically speaking, for any given

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[2]For another interpretation of this idea, one can see Sertel and Yilmaz [54] and also Giritligil and Sertel [55].

pair of preference-approvals, first we use Kemeny metric [41] for weak orders to measure the distance regarding the ranking information. Secondly, we use Hamming metric [56] to measure the concordance with respect to acceptable or unacceptable alternatives. Proper aggregation of these two type of distances depends on the context of the particular problem. Noting that the choice of a particular convex combination of Kemeny and Hamming distances reflect the emphasis on the disagreement regarding approval or ranking, we briefly discuss various ways for aggregation. Then, we show that the proposed measure of consensus (based on García-Lapresta and Pérez-Román [51]) is robust to the extensions of the ordinal framework under the condition that a proper metric is used. By investigating the properties of the consensus measure for preference-approvals, we also show an unexpected result that the degree of homogeneity in a group of individuals cannot be increased by simply replicating their preferences.

This chapter is organized as follows. Section 2 introduces the basic notation and notions. Section 3 is devoted to the definition and some properties of consensus measures in general. Section 4 includes our proposal for measuring consensus in preference-approval context and some results. Finally, Section 5 concludes.

3.2   Preliminaries

Consider a set of agents $V = \{v_1, \ldots, v_m\}$ with $m \geq 2$ confronting a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ where $n \geq 2$. We assume that each agent ranks the alternatives in X by means of a weak order and additionally, evaluates each alternative as

either acceptable or unacceptable by partitioning the alternative set into approved (good) and disapproved (bad) alternatives. These two types of information exhibit the following consistency: Given two alternatives x and y, if x is approved and y is disapproved, then x is ranked above y.

Technically speaking, by a *weak order* (or *complete preorder*) on X we mean a complete[3] and transitive binary relation on X.

On the other hand a *linear order* on X is an antisymmetric weak order on X. We write $W(X)$ for the set of weak orders on X and $L(X)$ for the set of linear orders on X.

Given $R \in W(X)$, we let $\succ$ and $\sim$ stand for the asymmetric and the symmetric parts of R, respectively, i.e.,

$$x_i \succ x_j \quad \Leftrightarrow \quad \text{not } (x_j \, R \, x_i)$$

$$x_i \sim x_j \quad \Leftrightarrow \quad \left(x_i \, R \, x_j \ \text{ and } \ x_j \, R \, x_i \right).$$

By $\mathcal{P}(V)$ we denote the power set of V, i.e., $I \in \mathcal{P}(V) \Leftrightarrow I \subseteq V$; and by $\mathcal{P}_2(V)$ we mean the collection of subsets of V with at least two elements. That is, $\mathcal{P}_2(V) = \{ I \in \mathcal{P}(V) \mid \#I \geq 2 \}$, where $\#I$ is the cardinality of I. Analogously, we write $\mathcal{P}(X)$ for the power set of X.

Finally, we denote $\boldsymbol{a} = (a_1 \ldots, a_n)$ for the vectors in $\mathbb{R}^n$.

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[3]By completeness, for any given x_i and x_j in X, either x_i is at least as good as x_j or x_j is at least as good as x_i. Hence, any complete binary relation is also reflexive.

3.2.1 Preference-approval structures

For any given set of X of alternatives we define preference-approvals by partitioning X into A the set of *acceptable* (or *good*) alternatives and $U = X \setminus A$ the set of *unacceptable* (or *bad*) alternatives, where A and U can be empty sets.

Definition 3.2.1

A preference-approval on X is a pair $(R, A) \in W(X) \times \mathcal{P}(X)$ satisfying the following condition

$$\forall x_i, x_j \in X \ \big((x_i \, R \, x_j \text{ and } x_j \in A) \implies x_i \in A \big).$$

Note that if $x_i \, R \, x_j$ and $x_i \in U$, then we have $x_j \in U$.

We denote $\mathcal{R}(X)$ for the set of preference-approvals on X.

Given $R \in W(X)$, we let R^{-1} be the *inverse* of R such that

$$x_i \, R^{-1} \, x_j \iff x_j \, R \, x_i,$$

for all $x_i, x_j \in X$. Similarly, given a preference-approval $(R, A) \in \mathcal{R}(X)$, we write $(R, A)^{-1} = (R^{-1}, X \setminus A)$ for the preference-approval which is the *inverse* of (R, A).

Example 3.2.2

In order to illustrate preference-approval structures, consider the following example:

$$x_2 \; x_3 \; x_5$$

$$x_1$$

$$\overline{\hspace{3em}}$$

$$x_4 \; x_7$$

$$x_6$$

where alternatives in the same row are indifferent, alternatives in upper rows are preferred to those located in lower rows, alternatives above the dash line are acceptable (good) and those below the dash line are unacceptable (bad).

The inverse of the preference-approval above is the following:

$$x_6$$

$$x_4 \; x_7$$

$$\overline{\hspace{3em}}$$

$$x_1$$

$$x_2 \; x_3 \; x_5$$

We now introduce a system for codifying each preference-approval structure $(R, A) \in \mathcal{R}(X)$ by means of two vectors: $\boldsymbol{p}_R \in \mathbb{R}^n$ that represents the position of the alternatives, and $\boldsymbol{i}_A \in \{0, 1\}^n$ that represents acceptable alternatives.

It is worthwhile to note that there does not exist a unique system for codifying weak orders, since a weak order can be linearized in many different ways. We propose a codification based on a linearization of the weak order by assigning each alternative the

average of the positions of the alternatives within the same equivalence class.

Following García-Lapresta and Pérez-Román [51], for any given $R \in W(X)$ we assign the *position* of each alternative x_j in R through the mapping $P_R : X \to \mathbb{R}$ defined as

$$P_R(x_j) = n - \# \{x_i \in X \mid x_j \succ x_i\} - \frac{1}{2} \# \{x_i \in X \setminus \{x_j\} \mid x_i \sim x_j\},$$

where n is the number of alternatives.

The following table illustrates the codification of the preference-approval in Example 3.2.2.

$P_R(x_1) = 7 - 3 - \frac{1}{2} \cdot 0 = 4$
$P_R(x_2) = 7 - 4 - \frac{1}{2} \cdot 2 = 2$
$P_R(x_3) = 7 - 4 - \frac{1}{2} \cdot 2 = 2$
$P_R(x_4) = 7 - 1 - \frac{1}{2} \cdot 1 = 5.5$
$P_R(x_5) = 7 - 4 - \frac{1}{2} \cdot 2 = 2$
$P_R(x_6) = 7 - 0 - \frac{1}{2} \cdot 0 = 7$
$P_R(x_7) = 7 - 1 - \frac{1}{2} \cdot 1 = 5.5$

We denote $\boldsymbol{p}_R = (P_R(x_1), \dots, P_R(x_n))$ for the *position vector* of $R \in W(X)$. Note that the codification vector in Example 3.2.2 is $\boldsymbol{p}_R = (4, 2, 2, 5.5, 2, 7, 5.5)$.

On the other hand, given $A \subseteq X$, we define $I_A : X \longrightarrow \{0, 1\}$ the *indicator function*

(or *characteristic function*) of A:

$$I_A(x_j) = \begin{cases} 1, \text{ if } x_j \in A, \\ 0, \text{ if } x_j \in X \setminus A. \end{cases}$$

By $\boldsymbol{i}_A = (I_A(x_1), \dots, I_A(x_n))$ we denote the *indicator vector* of $A \subseteq X$.

Note that the preference-approval in Example 3.2.2 will be codified as $\boldsymbol{i}_A = (1, 1, 1, 0, 1, 0, 0)$

since x_1, x_2, x_3, x_5 are the accepted alternatives and x_4, x_6 and x_7 are the unaccepted

ones.

Given a preference-approval (R, A), we can completely characterize it by the ($\boldsymbol{p}_R$

, $\boldsymbol{i}_A$) tuple.

Remark 3.2.3

The condition appearing in Definition 4.2.1 can be written as:

$$\big(P_R(x_i) \geq P_R(x_j) \text{ and } I_A(x_j) = 1 \big) \implies I_A(x_i) = 1.$$

3.2.2 Distances and metrics

Usually, *distance* and *metric* are considered as synonymous. However, we follow the approach given by Deza and Deza [57], where distances and metrics are different concepts.

Definition 3.2.4

A distance *on a set* $D \neq \emptyset$ *is a mapping* $d : D \times D \to \mathbb{R}$ *satisfying the following conditions for all* $a, b \in A$:

1. $d(a, b) \geq 0$ *(non-negativity)*,

2. $d(a, b) = d(b, a)$ *(symmetry)*,

3. $d(a, a) = 0$ *(reflexivity)*.

If d satisfies the following additional conditions for all $a, b, c \in A$:

4. $d(a, b) = 0 \Leftrightarrow a = b$ *(identity of indiscernibles)*,

5. $d(a, b) \leq d(a, c) + d(c, b)$ *(triangle inequality)*,

then we say that d is a metric.

We now focus on Kemeny and Hamming metrics. Since any preference-approval has two components, an ordering and a partition on the set of alternatives, calculating the distance between any two preference-approvals requires to measure distances with respect to these components. We propose using Kemeny metric for weak orders and Hamming metric for the information regarding acceptable alternatives.

The Kemeny metric

The Kemeny metric was initially defined on linear orders by Kemeny [41], as the sum of pairs where the orders' preferences disagree. Subsequently, it has been generalized to the framework of weak orders (see Cook, Kress and Seiford [58] and Eckert and Klamler [59], among others).

Typically, the *Kemeny metric* on weak orders $d_K : W(X) \times W(X) \to \mathbb{R}$ is defined as the cardinality of the symmetric difference between the weak orders, i.e.,

$$d_K(R_1, R_2) = \#\big((R_1 \cup R_2) \setminus (R_1 \cap R_2)\big).$$

In this chapter, having a codification based approach we adopt the definition of Kemeny metric proposed by García-Lapresta and Pérez-Román [51] as the following:

$$d_K\big(R_1, R_2\big) = \sum_{\substack{i,j=1 \\ i<j}}^{n} \big| \operatorname{sgn}\big(P_{R_1}(x_i) - P_{R_1}(x_j)\big) - \operatorname{sgn}\big(P_{R_2}(x_i) - P_{R_2}(x_j)\big)\big| \,,$$

where sgn is the *sign function*:

$$\operatorname{sgn}(a) = \begin{cases} 1, & \text{if } a > 0\,, \\ 0, & \text{if } a = 0\,, \\ -1, & \text{if } a < 0\,. \end{cases}$$

It is worthwhile to remark that the Kemeny metric is a bounded metric in $W(X)$. That is, there exists some $M > 0$ such that $d_K(R_1, R_2) \leq M$ for all $R_1, R_2 \in W(X)$. One can immediately check that the maximum distance between orders with respect to Kemeny metric is $(\#X)^2 - \#X$.

The Hamming metric

The *Hamming metric* (Hamming [56]) $d_H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as[4]

$$d_H(\boldsymbol{a}, \boldsymbol{b}) = \#\{i \in \{1, \ldots, n\} \mid a_i \neq b_i\}.$$

We extend the Hamming metric from $\mathbb{R}^n$ to $\mathcal{P}(X)$ as the mapping $d_H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ defined by

$$d_H(A_1, A_2) = d_H(\boldsymbol{i}_{A_1}, \boldsymbol{i}_{A_2}).$$

Note that Hamming metric formulation above is equivalent to the following one.

$$d_H(A_1, A_2) = \#\big((A_1 \cup A_2) \setminus (A_1 \cap A_2)\big).$$

Clearly the Hamming metric on $\mathcal{P}(X)$ is a bounded metric as well and one can easily check that the maximum distance between any two subsets of X is $\#X$.

Mixing distances and metrics

In what follows, we state that d_K and d_H, although measuring distances regarding different kinds of information separately, cannot be aggregated as a total distance since d_K and d_H do not have the same codomains. Therefore, we first normalize these two metrics to the same codomain $[0, 1]$ via dividing by their maximum distances and we get d_R and d_A

---

[4]On binary vectors $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^n$, the Hamming metric and the l_1-metric (or Manhattan metric) coincide:

$$d_H(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^{n} |a_i - b_i|.$$

.

as distances regarding orderings and acceptable alternatives respectively.

Definition 3.2.5

1. *The mapping* $d_R : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow [0,1]$ *is defined as*

$$d_R\big((R_1, A_1), (R_2, A_2)\big) = \frac{d_K(R_1, R_2)}{(\#X)^2 - \#X} = \frac{\#\big((R_1 \cup R_2) \setminus (R_1 \cap R_2)\big)}{(\#X)^2 - \#X}.$$

2. *The mapping* $d_A : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow [0,1]$ *is defined as*

$$d_A\big((R_1, A_1), (R_2, A_2)\big) = \frac{d_H(A_1, A_2)}{\#X} = \frac{\#\big((A_1 \cup A_2) \setminus (A_1 \cap A_2)\big)}{\#X}.$$

Proposition 3.2.6

1. d_R *is a distance on* $\mathcal{R}(X)$ *and for all* $(R_1, A_1), (R_2, A_2) \in \mathcal{R}(X)$ *it holds*

   (a) $d_R\big((R_1, A_1), (R_2, A_2)\big) = 0 \Leftrightarrow R_1 = R_2.$

   (b) d_R *verifies triangle inequality.*

   (c) $d_R\big((R_1, A_1), (R_2, A_2)\big) = 1 \Leftrightarrow (R_1, R_2 \in L(X) \text{ and } R_2 = R_1^{-1}).$

2. d_A *is a distance on* $\mathcal{R}(X)$ *and for all* $(R_1, A_1), (R_2, A_2) \in \mathcal{R}(X)$ *it holds*

   (a) $d_A\big((R_1, A_1), (R_2, A_2)\big) = 0 \Leftrightarrow A_1 = A_2.$

   (b) d_A *verifies triangle inequality.*

   (c) $d_A\big((R_1, A_1), (R_2, A_2)\big) = 1 \Leftrightarrow A_2 = X \setminus A_1.$

3. Neither d_R nor d_A are metrics on $\mathcal{R}(X)$.

Proof. Let $(R_1, A_1), (R_2, A_2) \in \mathcal{R}(X)$.

1. Since d_R is the Kemeny metric normalized by a number, non-negativity, symmetry and reflexivity are obvious.

   (a) $d_R\big((R_1, A_1), (R_2, A_2)\big) = 0 \Leftrightarrow d_K(R_1, R_2) = 0 \Leftrightarrow R_1 = R_2$.

   (b) d_R inherits from Kemeny metric the property of triangle inequality.

   (c) In García-Lapresta and Pérez-Román [51] is proven that, for the Kemeny metric, the maximum distance between weak orders is not reached when one of them is not linear and, additionally, the maximum distance between linear orders is not reached when they are not inverses of each other.

2. Since d_A is the Hamming metric normalized by a number, non-negativity, symmetry and reflexivity are obvious.

   (a) $d_A\big((R_1, A_1), (R_2, A_2)\big) = 0 \Leftrightarrow d_H(A_1, A_2) = 0 \Leftrightarrow A_1 = A_2$.

   (b) d_A inherits from Hamming metric the property of triangle inequality.

   (c) $d_A\big((R_1, A_1), (R_2, A_2)\big) = 1 \Leftrightarrow (A_1 \cup A_2 = X \text{ and } A_1 \cap A_2 = \emptyset)$, i.e., $A_2 = X \setminus A_1$.

3. Let $(R_1, A_1), (R_2, A_2) \in \mathcal{R}(X)$ such that $R_1 \neq R_2$ and $A_1 \neq A_2$. Then, we have $d_R\big((R_1, A_1), (R_1, A_2)\big) = d_A\big((R_1, A_1), (R_2, A_1)\big) = 0$. Consequently, d_R and d_A do not verify identity of indiscernibles, hence they are not metrics.

The following example illustrates the calculation of distances for a given profile.

Example 3.2.7

Consider four agents confronting a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and having the following preference-approvals:

(R_1, A_1)	(R_2, A_2)	(R_3, A_3)	(R_4, A_4)
x_1	x_2	x_1	$\overline{}$
		$\overline{}$	x_3
x_2	x_1	x_2	x_2
$\overline{}$	$\overline{}$		
$x_3\ x_4$	$x_3\ x_4$	$x_3\ x_4$	$x_1 x_4$

These preference-approvals are codified as follows:

$$\boldsymbol{p}_{R_1} = (1, 2, 3.5, 3.5) \qquad \boldsymbol{i}_{A_1} = (1, 1, 0, 0)$$

$$\boldsymbol{p}_{R_2} = (2, 1, 3.5, 3.5) \qquad \boldsymbol{i}_{A_2} = (1, 1, 0, 0)$$

$$\boldsymbol{p}_{R_3} = (1, 2, 3.5, 3.5) \qquad \boldsymbol{i}_{A_3} = (1, 0, 0, 0)$$

$$\boldsymbol{p}_{R_4} = (3.5, 2, 1, 3.5) \qquad \boldsymbol{i}_{A_4} = (0, 0, 0, 0)$$

The following table shows the distances d_R and d_A between preference-approvals:

	d_R	d_A
$(R_1, A_1), (R_2, A_2)$	$\frac{2}{12}$	0
$(R_1, A_1), (R_3, A_3)$	0	$\frac{3}{12}$
$(R_1, A_1), (R_4, A_4)$	$\frac{8}{12}$	$\frac{6}{12}$
$(R_2, A_2), (R_3, A_3)$	$\frac{2}{12}$	$\frac{3}{12}$
$(R_2, A_2), (R_4, A_4)$	$\frac{6}{12}$	$\frac{6}{12}$
$(R_3, A_3), (R_4, A_4)$	$\frac{8}{12}$	$\frac{3}{12}$

Note that minimum distance regarding orderings is in between (R_1, A_1) and (R_2, A_2) since there is a disagreement only on the ranking of the first two alternatives. On the other hand, the maximum distance regarding orderings in this profile is attained by (R_4, A_4) and (R_1, A_1), which is also the distance between (R_4, A_4) and (R_3, A_3). Note that for these tuples, there is only one pair of alternatives (namely (x_2, x_4)) for which these preference-approvals agree on.

Similarly, the minimum distance regarding acceptability is in between (R_1, A_1) and (R_2, A_2) since there is a full agreement for the set of acceptable and unacceptable alternatives. On the other hand, the maximum distance regarding acceptability is attained by (R_4, A_4) by (R_1, A_1) which is also the distance between (R_4, A_4) and (R_2, A_2). Note that there is a disagreement on the acceptability of two alternatives (namely x_1 and x_2) for these preference-approvals.

For the rest of the section, first we define the neutrality of metrics and then we establish that a neutral metric can be deduced from the convex combinations of d_R and d_A.

Definition 3.2.8

A set $D \subseteq \mathbb{R}^n$ is stable under permutations *if for every permutation σ on $\{1, \dots, n\}$,*

it holds $(a_1^\sigma, \dots, a_n^\sigma) \in D$ for every $(a_1, \dots, a_n) \in D$.

Definition 3.2.9

Given a set $D \subseteq \mathbb{R}^n$ stable under permutations, a distance (or metric) $d : D \times D \longrightarrow \mathbb{R}$

is neutral *if for every permutation σ on $\{1, \dots, n\}$ it holds*

$$d\big((a_1^\sigma, \dots, a_n^\sigma), (b_1^\sigma, \dots, b_n^\sigma) \big) = d\big((a_1, \dots, a_n), (b_1, \dots, b_n) \big),$$

for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in D$.

Remark 3.2.10

The Kemeny metric is neutral (see García-Lapresta and Pérez-Román [51]). One can

easily check that the Hamming metric is neutral as well.

Remark 3.2.11

Given two distances $d_1, d_2 : D \times D \longrightarrow \mathbb{R}$, for every $\lambda \in [0, 1]$ the convex combination

$\lambda \, d_1 + (1 - \lambda) \, d_2$ is also a distance.

    In the next result we show that although d_R and d_A are not metrics, their convex

combinations are always metrics except for the degenerate values of $\lambda = 0$ and $\lambda = 1$.

Proposition 3.2.12

For every $\lambda \in (0,1)$ and all $(R_1, A_1), (R_2, A_2) \in \mathcal{R}(X)$, the following statements hold:

1. *$d_\lambda = \lambda \, d_R + (1 - \lambda) \, d_A$ is a neutral metric and $d_\lambda\big((R_1, A_1), (R_2, A_2)\big) \leq 1$.*

2. *$d_\lambda\big((R_1, A_1), (R_2, A_2)\big) = 1$ if and only if $R_1, R_2 \in L(X)$, $R_2 = R_1^{-1}$ and $A_2 = X \setminus A_1$.*

Proof.

1. By Remark 3.2.11, d_λ is a distance. By Proposition 3.2.6, d_λ verifies identity of indiscernibles property and triangle inequality. Then, d_λ is a metric. By Remark 3.2.10, the Kemeny and Hamming metrics are neutral and it is obvious that the convex combination $\lambda \, d_R + (1 - \lambda) \, d_A$ satisfies neutrality too.

2. By Proposition 3.2.6.

■

It is worthwhile to note that the aggregation of two distances for different kinds of information leads to two problems. The first one, which is technical, arises from the fact that d_K and d_H have different codomains for aggregation and a solution to this problem has been proposed by the proposition 3.2.12. The second one is deciding on the appropriate value of λ for the aggregation of these two distances. Since λ (resp. $1 - \lambda$) determines the weight of information regarding orderings (resp. acceptability), the value of λ should be decided before the implementation of the consensus measuring.

In practice, the selection of the lambda can be done in various ways. First, as in the case of voting in the committees, a moderator or a decision maker can decide on λ according to his principles. Although λ can take infinitely many values, the most important decision would be choosing the component of the preference (orderings or approval) that will have more weight than the other. Second, a separate aggregation rule can be applied and the outcome of that rule can be used as an optimal value of the λ. In particular, the mean or a trimmed mean of the submitted λ values can be used as the outcome. However, when an aggregation procedure is followed the issues regarding strategic behavior should be taken into consideration.

Example 3.2.13

The following table illustrates the changes in the total distances between preference-approvals in Example 3.2.7 with respect to the values of λ.

	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$
$d_\lambda\big((R_1, A_1), (R_2, A_2)\big)$	0.04167	0.08333	0.125
$d_\lambda\big((R_1, A_1), (R_3, A_3)\big)$	0.1875	0.125	0.0625
$d_\lambda\big((R_1, A_1), (R_4, A_4)\big)$	0.54267	0.58333	0.625
$d_\lambda\big((R_2, A_2), (R_3, A_3)\big)$	0.22917	0.20833	0.1875
$d_\lambda\big((R_2, A_2), (R_4, A_4)\big)$	0.5	0.5	0.5
$d_\lambda\big((R_3, A_3), (R_4, A_4)\big)$	0.35417	0.4583	0.5625

In these results, note that the preference-approval which has the minimum distance to (R_1, A_1) is (R_2, A_2) when we have $\lambda = 0.25$. However, when $\lambda = 0.75$ the result

changes to (R_3, A_3). To see why, note that when $\lambda = 0.25$ the distance d_A is weighted more than d_R implying that the disagreements on the set of accepted alternatives are more important than the disagreements on the orderings. This is reversed when $\lambda = 0.75$. For another illustration of a similar change in the distances with respect to λ, check that among the given preference-approvals (R_3, A_3) is the closest to (R_4, A_4) for $\lambda = 0.25$. However, for $\lambda = 0.75$ the previous result changes to (R_2, A_2).

3.3 Consensus measures

Consensus measures have been introduced and analyzed by Bosch [50] in the context of linear orders. Subsequently, García-Lapresta and Pérez-Román [51, 60] extended this notion to the context of weak orders by using distances. Although many nonstandard preferences are also analyzed for aggregation problems, the problem of measuring homogeneity for a set of these nonstandard preferences are not fully investigated in the literature. In this section, we focus on consensus measures for preference-approval structures and start with introducing basic notions for consensus measures in general.

3.3.1 Basic notions

First we include some pieces of notation.

Definition 3.3.1

A profile is a vector $\boldsymbol{R} = \big((R_1, A_1), \ldots, (R_m, A_m)\big) \in \mathcal{R}(X)^m$ *of preference-approvals,*

where (R_i, A_i) contains the preference-approval of the agent v_i, with $i = 1, \ldots, m$.

1. The inverse of $\boldsymbol{R}$ is $\boldsymbol{R}^{-1} = \big((R_1^{-1}, X \setminus A_1), \ldots, (R_m^{-1}, X \setminus A_m)\big)$.

2. Given a permutation π on $\{1, \ldots, m\}$ and $\emptyset \neq I \subseteq V$, we denote $\boldsymbol{R}_\pi = \big((R_{\pi(1)}, A_{\pi(1)}), \ldots, (R_{\pi(m)}, A_{\pi(m)})\big)$ and $I_\pi = \{v_{\pi^{-1}(i)} \mid v_i \in I\}$, i.e., $v_j \in I_\pi \Leftrightarrow v_{\pi(j)} \in I$.

3. Given a permutation σ on $\{1, \ldots, n\}$, we denote by
$\boldsymbol{R}^\sigma = \big((R_1^\sigma, A_1^\sigma), \ldots, (R_m^\sigma, A_m^\sigma)\big)$ the profile obtained from $\boldsymbol{R}$ by relabeling the alternatives according to σ, i.e., $x_i \, R_k \, x_j \Leftrightarrow x_{\sigma(i)} \, R_k^\sigma \, x_{\sigma(j)}$ and $x_i \in A_k^\sigma \Leftrightarrow x_{\sigma(i)} \in A_k$, for all $i, j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Definition 3.3.2

A consensus measure on $\mathcal{R}(X)^m$ is a mapping

$$\mathcal{M} : \mathcal{R}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0, 1]$$

that satisfies the following conditions:

1. Unanimity. For all $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds

$$\mathcal{M}(\boldsymbol{R}, I) = 1 \Leftrightarrow (R_i = R_j \text{ and } A_i = A_j, \text{ for all } v_i, v_j \in I).$$

2. Anonymity. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds

$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I).$$

3. Neutrality. *For all permutation* σ *on* $\{1, \ldots, n\}$, $\boldsymbol{R} \in \mathcal{R}(X)^m$ *and* $I \in \mathcal{P}_2(V)$, *it holds*

$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I).$$

Unanimity means that the maximum consensus in every subset of decision makers is only achieved when all opinions are the same. Anonymity requires symmetry with respect to decision makers, and neutrality means symmetry with respect to alternatives.

We now introduce additional properties that a consensus measure may satisfy.

Definition 3.3.3

Let $\mathcal{M} : \mathcal{R}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0, 1]$ *be a consensus measure.*

1. $\mathcal{M}$ *satisfies* maximum dissension *if for all* $\boldsymbol{R} \in \mathcal{R}(X)^m$ *and* $v_i, v_j \in V$ *such that* $i \neq j$, *it holds*

$$\mathcal{M}(\boldsymbol{R}, \{v_i, v_j\}) = 0 \iff (R_i, R_j \in L(X), \; R_j = R_i^{-1} \text{ and } A_j = X \setminus A_i).$$

2. $\mathcal{M}$ *is* reciprocal *if for all* $\boldsymbol{R} \in \mathcal{R}(X)^m$ *and* $I \in \mathcal{P}_2(V)$, *it holds*

$$\mathcal{M}(\boldsymbol{R}^{-1}, I) = \mathcal{M}(\boldsymbol{R}, I).$$

Maximum dissension means that in each subset of two agents[5], the minimum consensus level is only reached whenever preferences of agents are linear orders, each one the

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[5]It is clear that a society reaches the maximum level of consensus when all the opinions are the same. However, in a society with more than two members it is not an obvious issue to determine when there is the minimum level consensus (the maximum level of disagreement).

inverse of the other, and the good alternatives of each agent are the bad ones of the other. Reciprocity means that if all individual opinions are reversed, then the consensus does not change.

3.4   Measuring consensus for preference-approvals

We now introduce our proposal for measuring consensus in the context of preference-approvals. As we have discussed previously, richer information content of the preference-approval structures can be desirable to distinguish some threshold levels as well as the rankings in collective decision problems. On the other hand, analyzing the homogeneity level in such profiles asks for an extension of the standard measures of consensus in the literature. We show that the consensus measure introduced by García-Lapresta and Pérez-Román [51, 60] for weak orders is robust to the additional approval information for ordinal preferences when the metric proposed by Proposition 3.2.6 is used as an input.

Definition 3.4.1

Given a metric $d : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathbb{R}$, *the mapping*

$$\mathcal{M}_d : \mathcal{R}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0, 1]$$

is defined by

$$\mathcal{M}_d\left(\boldsymbol{R}, I\right) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in I \\ i < j}} d\big((R_i, A_i), (R_j, A_j)\big)}{\dbinom{\#I}{2} \cdot \Delta_n},$$

where

$$\Delta_n = \max \left\{ d\big((R_i, A_i), (R_j, A_j)\big) \mid (R_i, A_i), (R_j, A_j) \in \mathcal{R}(X) \right\}.$$

Note that the numerator of the quotient appearing in the above expression is the sum of all the distances between the preference-approvals of the profile, and the denominator is the number of terms in the numerator's sum multiplied by the maximum distance between preference-approvals. Consequently, that quotient belongs to the unit interval and it measures the disagreement in the profile.

3.4.1 Some results
Proposition 3.4.2

For every metric $d : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathbb{R}$, $\mathcal{M}_d$ *satisfies unanimity and anonymity.*

Proof. Let $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$.

1. Unanimity.

$$\mathcal{M}_d(\boldsymbol{R}, I) = 1 \quad \Leftrightarrow \quad \sum_{\substack{v_i, v_j \in I \\ i < j}} d\big((R_i, A_i), (R_j, A_j)\big) = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \forall \, v_i, v_j \in I \;\; d\big((R_i, A_i), (R_j, A_j)\big) = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \forall \, v_i, v_j \in I \;\; (R_i, A_i) = (R_j, A_j) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \forall \, v_i, v_j \in I \;\; (R_i = R_j \;\text{ and }\; A_i = A_j).$$

2. Anonymity. Let π be a permutation on $\{1, \ldots, m\}$.

$$\sum_{\substack{v_i,v_j \in I_\pi \\ i<j}} d\big((R_{\pi(i)}, A_{\pi(i)}), (R_{\pi(j)}, A_{\pi(j)})\big) =$$

$$= \sum_{\substack{v_{\pi(i)},v_{\pi(j)} \in I \\ \pi(i)<\pi(j)}} d\big((R_{\pi(i)}, A_{\pi(i)}), (R_{\pi(j)}, A_{\pi(j)})\big) =$$

$$= \sum_{\substack{v_i,v_j \in I \\ i<j}} d\big((R_i, A_i), (R_j, A_j)\big) \ .$$

Thus, $\mathcal{M}_d(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}_d(\boldsymbol{R}, I)$. $\blacksquare$

If $\mathcal{M}_d$ verifies neutrality, then we say that $\mathcal{M}_d$ is the *consensus measure associated with d*.

Proposition 3.4.3

If $d : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathbb{R}$ *is a neutral metric, then* $\mathcal{M}_d$ *is a consensus measure.*

Proof. By Proposition 3.4.2, $\mathcal{M}_d$ satisfies unanimity and anonymity. Obviously, if d is neutral, then $\mathcal{M}_d$ verifies neutrality and thus $\mathcal{M}_d$ is a consensus measure.

$\blacksquare$

Theorem 3.4.4

For every $\lambda \in (0,1)$, $\mathcal{M}_{d_\lambda}$ *is a consensus measure that satisfies maximum dissension and reciprocity.*

Proof. By Proposition 3.4.3, $\mathcal{M}_{d_\lambda}$ is a consensus measure.

1. *Maximum dissension.* First of all, notice that $\mathcal{M}_{d_\lambda}(\boldsymbol{R}, \{v_i, v_j\}) = 0$ if and only if $d_\lambda\big((R_i, A_i), (R_j, A_j)\big)$ is maximum. This is equivalent to $d_R\big((R_i, A_i), (R_j, A_j)\big)$

and $d_A\big((R_i, A_i), (R_j, A_j)\big)$ are maximum. By Proposition 3.2.6, $d_\lambda\big((R_i, A_i), (R_j, A_j)\big)$

is maximum if and only if $(R_1, R_2) \in L(X)$, $R_2 = R_1^{-1}$ and $A_2 = X \setminus A_1$.

2. *Reciprocity.* Given $(R_1, A_1), (R_2, A_2) \in \mathcal{R}(X)$, we only need to prove:

(a) $d_R(R_1, R_2) = d_R(R_1^{-1}, R_2^{-1})$ (see García-Lapresta and Pérez-Román [51]).

(b) $d_H(A_1, A_2) = d_H(A_1^{-1}, A_2^{-1})$:

$$d_H(A_1, A_2) =$$

$$= \#\big((A_1 \cup A_2) \setminus (A_1 \cap A_2)\big) =$$

$$= \#\big((A_1 \cup A_2) \cap (A_1 \cap A_2)^{-1}\big) =$$

$$= \#\big((A_1 \cup A_2) \cap (A_1^{-1} \cup A_2^{-1})\big) =$$

$$= \#\big((A_1^{-1} \cup A_2^{-1}) \cap (A_1^{-1} \cap A_2^{-1})^{-1}\big) =$$

$$= \#\big((A_1^{-1} \cup A_2^{-1}) \setminus (A_1^{-1} \cap A_2^{-1})\big) =$$

$$= d_H(A_1^{-1}, A_2^{-1}).$$

Taking into account (a) and (b), we have

$$d_\lambda\big((R_1, A_1), (R_2, A_2)\big) = d_\lambda\big((R_1, A_1)^{-1}, (R_2, A_2)^{-1}\big).$$

Thus, $\mathcal{M}_{d_\lambda}(\boldsymbol{R}^{-1}, I) = \mathcal{M}_{d_\lambda}(\boldsymbol{R}, I)$.

∎

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3.4.2 An Illustrative example

Example 3.4.5

Consider again the preference-approvals in Example 3.2.7:

(R_1, A_1)	(R_2, A_2)	(R_3, A_3)	(R_4, A_4)
x_1	x_2	x_1	$\overline{}$
x_2	x_1	$\overline{}$	x_3
$\overline{}$	$\overline{}$	x_2	x_2
$x_3\ x_4$	$x_3\ x_4$	$x_3\ x_4$	$x_1 x_4$

In the following table we illustrate the level of consensus reached in some representative subsets of agents for three values of λ:

	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$
$\mathcal{M}_{d_\lambda}(\boldsymbol{R}, \{v_1, v_2\})$	0.95833	0.91666	0.875
$\mathcal{M}_{d_\lambda}(\boldsymbol{R}, \{v_1, v_3\})$	0.8125	0.875	0.9375
$\mathcal{M}_{d_\lambda}(\boldsymbol{R}, \{v_3, v_4\})$	0.64583	0.54167	0.4375
$\mathcal{M}_{d_\lambda}(\boldsymbol{R}, \{v_1, v_2, v_3, v_4\})$	0.69097	0.67361	0.65625

In the first row, the level of consensus is measured for the first two agents. Recall that λ is the coefficient for d_R . Since these agents only disagree on the orderings, an increase in λ puts more emphasis for that disagreement and leads to a decrease in the level of consensus. On the contrary, the first and the third agents totally agree on orderings but they disagree on the set of acceptable alternatives. Hence, we see in the second row that

the level of consensus increases when λ increases as the importance of that agreement increases (simultaneously, disagreement on the set of acceptable alternatives becomes less important since $1 - \lambda$ decreases). In the third row we focus on the third and fourth agents. These agents have disagreement on orderings and on acceptable alternatives at the same time, so they reach the minimum level of consensus for all considered cases so far. Note that the level of consensus decreases for these two agents when the weight of d_R increases. Finally in the last row, consensus level is measured for the full profile. We see that as λ increases, the level of consensus in the profile decreases. According to these results we conclude that for this profile, individuals have more agreement on which alternatives are socially acceptable than the ordering of those alternatives.

3.4.3   Replications

In some collective decision procedures, especially for the multi-rounded voting systems, analyzing the preference-updating schemes can be useful for the moderator to see the changes in the level of consensus. In particular, coalition formations can lead to the occurrence of the same preference as many as the number of the agents in a coalition. Hence, analyzing homogeneity for a given set of preferences when there are some replications of preferences has its own interest. Having this motivation, we consider a metric $d : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathbb{R}$ and the associated consensus measure $\mathcal{M}_d : \mathcal{R}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0, 1]$. For each $t \in \mathbb{N}$, it is possible to extend $\mathcal{M}_d$ to t

replicas of profiles of $\mathcal{R}(X)^m$ and subsets of V:

$$\mathcal{M}_d^t : \mathcal{R}(X)^{tm} \times \mathcal{P}_2(t\,V) \longrightarrow [0,1].$$

Thus, $\mathcal{M}_d^t(t\,\boldsymbol{R}, t\,I) \in [0,1]$ measures the consensus in the multiset of agents[6] $t\,I = I \uplus \overset{t}{\cdots} \uplus I$ generated by t replicas of I for the profile generated by t replicas of $\boldsymbol{R} \in \mathcal{R}(X)^m$, $t\boldsymbol{R} = (\boldsymbol{R}, \overset{t}{\ldots}, \boldsymbol{R}) \in \mathcal{R}(X)^{tm}$.

Proposition 3.4.6

Let $d : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathbb{R}$ be a metric. For each profile of two preference-approvals $\boldsymbol{R} = \big((R_1, A_1), (R_2, A_2)\big) \in \mathcal{R}(X)^2$ such that $d\big((R_1, A_1), (R_2, A_2)\big) = \delta$ and every $t \in \mathbb{N}$, it holds:

$$\mathcal{M}_d^t(t\,\boldsymbol{R}, t\,I) = 1 - \frac{t \cdot \delta}{(2t-1) \cdot \Delta_n},$$

where $\Delta_n = \max\big\{d\big((R_i, A_i), (R_j, A_j)\big) \mid (R_i, A_i), (R_j, A_j) \in \mathcal{R}(X)\big\}$.

Proof. Consider $\boldsymbol{R} = \big((R_1, A_1), (R_2, A_2)\big) \in \mathcal{R}(X)^2$ with $d\big((R_1, A_1), (R_2, A_2)\big) = \delta$ and $I = \{v_1, v_2\}$. Given $t \in \mathbb{N}$, $t\boldsymbol{R} = \big((R_1, A_1), \ldots, (R_{2t}, A_{2t})\big)$, where $(R_{2k-1}, A_{2k-1}) = (R_1, A_1)$ and $(R_{2k}, A_{2k}) = (R_2, A_2)$, for every $k \in \{1, 2, \ldots, t\}$.

$$\mathcal{M}_d^t(t\,\boldsymbol{R}, t\,I) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in tI \\ i < j}} d\big((R_i, A_i), (R_j, A_j)\big)}{\dbinom{\#(t\,I)}{2} \cdot \Delta_n}.$$

---

[6]List of agents where each agent occurs as many times as the multiplicity. For instance, $2\{v_1, v_2\} = \{v_1, v_2\} \uplus \{v_1, v_2\} = \{v_1, v_2, v_1, v_2\}$.

Since

$$
d\big((R_i, A_i), (R_j, A_j)\big) =
\begin{cases}
0, & \text{if } i, j \text{ are both even,} \\[1em]
0, & \text{if } i, j \text{ are both odd,} \\[1em]
\delta, & \text{otherwise,}
\end{cases}
$$

we obtain

$$
\sum_{\substack{v_i, v_j \in tI \\ i < j}} d\big((R_i, A_i), (R_j, A_j)\big) = \sum_{i=1}^{2t-1} \sum_{j=i+1}^{2t} d\big((R_i, A_i), (R_j, A_j)\big) =
$$

$$
= \left(\sum_{i=1}^{t} i + \sum_{j=1}^{t-1} j \right) \delta = t^2 \cdot \delta \, .
$$

On the other hand, we have

$$
\binom{\#(t\,I)}{2} = \binom{2t}{2} = 2t^2 - t \, .
$$

Consequently,

$$
\mathcal{M}_d^t \,(t\,\boldsymbol{R}, t\,I) = 1 - \frac{t \cdot \delta}{(2t - 1) \cdot \Delta_n} \, .
$$

∎

Remark 3.4.7

Under the assumptions of Proposition 3.4.6, it holds:

$$
\lim_{t \to \infty} \mathcal{M}_d^t \,(t\,\boldsymbol{R},\, t\,I) = 1 - \frac{\delta}{2\Delta_n} .
$$

Particulary, if $R_1 \in L(X)$ and $(R_2, A_2) = (R_1^{-1}, A_1^{-1})$, then:

$$
\lim_{t \to \infty} \mathcal{M}_d^t \,(t\,\boldsymbol{R},\, t\,I) = \frac{1}{2} .
$$

Note that Remark 3.4.7 illustrates a surprising result that the level of consensus (or homogeneity) in a group of individuals cannot be increased by simply replicating their preferences. In fact, as the particular case of a polarized profile suggests, increasing the number of inverse preferences can only lead to a consensus level of $\frac{1}{2}$.

Example 3.4.8

Consider $I = \{v_1, v_4\}$ in Example 3.2.7. Their preference-approvals over four alternatives are:

$$
\begin{array}{cc}
\underline{(R_1, A_1)} & \underline{(R_4, A_4)} \\[4pt]
x_1 & \underline{} \\[4pt]
& x_3 \\[4pt]
x_2 & \\[4pt]
\underline{} & x_2 \\[4pt]
x_3\ x_4 & x_1 x_4
\end{array}
$$

In the following table we illustrate the changes in the level of consensus when we replicate the agents $\{v_1, v_4\}$ for three values of λ:

	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$
$\delta = d_\lambda\big((R_1, A_1), (R_4, A_4)\big)$	0.54167	0.58333	0.66250
$\mathcal{M}_{d_\lambda}(\boldsymbol{R}, I)$	0.458333	0.41666	0.375
$\mathcal{M}_{d_\lambda}(2\,\boldsymbol{R}, 2\,I)$	0.63889	0.61111	0.58333
$\mathcal{M}_{d_\lambda}(5\,\boldsymbol{R}, 5\,I)$	0.69907	0.67593	0.65278
$\mathcal{M}_{d_\lambda}(15\,\boldsymbol{R}, 15\,I)$	0.71983	0.69828	0.67672
$\mathcal{M}_{d_\lambda}(30\,\boldsymbol{R}, 30\,I)$	0.72457	0.70339	0.68220
$\lim\limits_{t\to\infty} \mathcal{M}_d^t(t\,\boldsymbol{R}, t\,I)$	0.72917	0.70833	0.68750

The first row shows the distances between these two preference-approvals with respect to three different values of λ. Consensus levels are illustrated in the second row. Note that when the size of the profile is doubled by cloning the preferences of each agent, as it is shown in the third row, consensus levels are increased for each values of λ. According to the results in table, we see that as the number of replications are increased the level of consensus also increases as it might be expected. However, our results also shows that there exists a limit for increasing the homogeneity level in a group of individuals by simply replicating their preferences.

3.5 Concluding remarks

Many collective decision making problems of voting, matching, bargaining and public goods implicitly use some threshold levels which are naturally described in the preference-

approval framework. We explore the problem of measuring consensus in this hybrid informational system by following a distance-based approach. Measuring homogeneity in terms of distances raises the question of how to evaluate the similarity between any two preferences. Although this question has been answered for various types of ordinal rankings like linear or weak orders over alternatives, we are not aware of a formal treatment of this problem for nonstandard preferences. Enriching the informational content by approval notion asks for a more sophisticated evaluation of similarity of preferences.

Given any two preference-approvals we propose measuring the concordance of them by convex combinations of normalized Kemeny and Hamming metrics. At this stage, our proposal depends on the ex-ante selection of the coefficients (λ and $1 - \lambda$) for Kemeny and Hamming metrics depending on the context of the relevant problem. Due to the relative importance of the disagreements with respect to the orderings of the alternatives or the approval of them, λ can be chosen by a moderator or by an aggregation rule. Experimental studies related to the optimal selection of λ for different contexts would give more insight for the implementation of this procedure, but would be the subject of a separate paper.

By deriving a proper metric that takes into account of two pieces of information, next we deal with measuring homogeneity according to these two components. We see that the measure of consensus introduced by García-Lapresta and Pérez-Román [51] can be extended for preference-approvals when that measure is based on a metric that satisfies some desirable axioms. Among the interesting results, we show that the degree of

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homogeneity in a group of individuals cannot be increased by simply replicating their preference-approvals.

For further research, analyzing metrics which can identify correlation between rankings and accepted alternatives in preference-approvals invites interesting questions. Additionally, using weighted distances to measure the discrepancies with respect to the position of the alternatives in the rankings would be a complement of this chapter. How to apply our model for truncated preferences on the subsets of a given alternative set arises another appealing question especially when there is large number of alternatives under consideration.

Chapter 4

On Manipulabilition from an Unacceptable Social Choice to an Acceptable one

ABSTRACT

Non-manipulability in collective decision making problems has been analyzed mainly through the axiom of strategy-proofness. In this chapter[1], we propose a new concept of non-manipulability. We postulate that each agent misreports his preferences if and only if the misrepresentation leads to a change of the social outcome from an unacceptable one for this agent to an acceptable one. For the formulation of this idea the preference-approval framework is used. Possibility and impossibility results for the existence of a non-manipulable rule are provided.

Keywords: non-manipulability, strategy-proofness, preference-approval

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[1]**This chapter is based on a joint working paper with the co-authors Sanver and Sato [61]**

4.1 Introduction

In collective decision making problems, non-manipulability of the aggregation rules has been widely studied in the literature.[2] *Strategy-proofness*, one of the central axioms in the theory of social choice, leads to impossibility results in most environments.

For example, by the Gibbard–Satterthwaite theorem (Gibbard [63] and Satterthwaite [64]) we know that on the universal domain of preferences, each *strategy-proof* social choice function whose range contains more than two alternatives is dictatorial.[3]Therefore, the standard notion of *strategy-proofness* does not lead to any normatively appealing rule which is robust to misrepresentation. In other words, *strategy-proofness* does not serve as a practically useful criterion of the robustness to misrepresentation.

We propose a new non-manipulability condition by using approval notion rather than rankings. We postulate that each agent manipulates the social outcome if and only if the social outcome changes from an unacceptable one for himself to an acceptable one. Since the standard model of social choice does not contain this type of binary evaluation, we use a framework which has richer informational content than the standard Arrovian framework.

In the recent literature of Voting Theory, there are some new models of aggregation mechanisms which use different formulations of inputs as individuals' messages in bal-

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[2]For an eloquent survey for the related literature, one can check Barbera [62].

[3]This result extends to many other frameworks. For example, in the context of choice aggregations, Baigent [65] shows that *Unanimity, Neutrality* and a condition called the *Bilateral Dropping Property* are sufficient for a choice aggregation to be emphmanipulable.

loting procedures. For example, *Approval voting* ([15], [16]) allows voters to express themselves through two labels as approved or not approved. *Majority judgement* [18] method extends this freedom of expression to seven labels as *Excellent, Very Good, Good, Good, Passable, Inadequate, Mediocre, Bad*. Finally, *Range Voting* [19] asks voters to provide a numerical score for the candidates within a fixed interval such as 0-100. Non-manipulability of these rules are also investigated in the literature as in [66], [18]. However, the difference between the standard and nonstandard settings in the formulation of inputs asks for further analysis of issue.

Brams and Sanver [53] suggest a model by combining the standard ordinal world of rankings with evaluation through approval in a hybrid system called *preference-approval*. Each agent is assumed to have an ordering on a given set of alternatives and a cut-off line to distinguish acceptable and unacceptable alternatives for them. An alternative which is ranked above (resp. below) the line is qualified as acceptable (resp. unacceptable).

We use the preference-approval model as a basis for our study and we investigate the existence of of non-manipulable aggregation rules in this framework.

Our results contribute to two lines of research. One is about non-manipulability of social choice rules and the other is about non-standard formulations of agents' characteristics. In the literature, various paths can be noticed for the non-manipulability analysis. One of them, as in Sato [67, 68], analyses different notions of non-manipulability in Arrovian framework and provides further examination of *strategy-proofness.*. Another stream, investigates the manipulability under some domain conditions, such as *dichoto-*

mous preferences and for specific rules as in Vorsatz.[69]. In this study, we attempt to investigate a new notion of non-manipulability, with a domain condition in preference-approval framework.

This chapter is organized as follows. Section 2 introduces the basic notion and definitions. Section 3 is devoted to the analysis of non-manipulability of preference-approvals. Finally, Section 4 includes concluding remarks.

4.2 Basic notions and definitions

We consider a set of agents $N = \{1, \ldots, n\}$ with $n \geq 2$ confronting a finite set of alternatives X where $|X| = m \geq 3$. By 2^X we denote the set of all subsets of X.

We write $\mathcal{L}$ for the set of linear orders [4] on X. For each $R \in \mathcal{L}$, we denote the strict part of R by P. Finally, for each $R \in \mathcal{L}$ and each $k \in \{1, \ldots, m\}$, we write $r_k(R)$ for the kth ranked alternative in R.

Now, we introduce the primitive of our model, namely *preference-approval* which incorporates hybrid information of ordinal rankings and the approval notion.

4.2.1 Preference-approval framework

We consider a framework in which each agent not only ranks the alternatives in X by means of a linear order but also evaluates each alternative as either acceptable or unac-

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[4]A linear order is a complete, transitive, and antisymmetric binary relation.

ceptable. [5]

We provide the formal definition of a preference-approval, as the following.

Definition 4.2.1

A preference-approval is a pair $p = (R, A) \in \mathcal{L} \times 2^X$ *satisfying the following condition*

$$\forall x, y \in X \ \big((x \mathrel{R} y \text{ and } y \in A) \ \Rightarrow \ x \in A\big).$$

Let $U = X \setminus A$.

We interpret A as the set of *acceptable* alternatives and U as the set of *unacceptable* alternatives. So, the above condition says that if an alternative is approved, all alternatives preferred to this alternative should be approved as well. Similarly we have if $x \mathrel{R} y$ and $x \in U$, then $y \in U$.

One can note that, with this definition, we embed the notion of approval to the primitives of the individual preferences. Therefore, strategic behavior of "approving an alternative in an approval ballot" (or in any other aggregation rule which uses this notion) becomes a different issue from the evaluation of the alternatives during formation of the "preference-approval"of an individual. An analogy would be the difference between the strategic behavior of providing a ranking of alternatives in an election method and the primitives as standard preferences (linear or weak orders over alternatives).

We denote a profile of preference-approvals by $\boldsymbol{p} = (p_1, \ldots, p_n)$ where $p_i = (R_i, A_i)$ is a preference-approval of agent i. $\mathcal{A}$ denotes the set of all preference-approvals.

---

[5]We interchangeably use the terms "approved","acceptable", "eligible", "appropriate", and so on.

Considering misrepresentation, when $p_i \in \mathcal{A}$ in $\boldsymbol{p} \in \mathcal{A}^N$ is replaced by $p_i' \in \mathcal{A}$, we write $(p_i', \boldsymbol{p}_{-i})$ for the new profile.

$\mathcal{D} \subseteq \mathcal{L}$ denotes the set of *admissible preferences* and we write $\mathcal{P}(\mathcal{D}) = \{(R_i, A_i) \in \mathcal{A} \mid R_i \in \mathcal{D}\}$ for the set of admissible preference-approvals. Interchangeably, we write $\mathcal{P}$ for $\mathcal{P}(\mathcal{D})$ when the meaning is clear.

Providing the basic model, next we discuss aggregation rules defined for "preference-approval" profiles and we propose our notion of non-manipulability in the following part.

4.2.2   Preference-approval aggregation

We consider single-valued functions defined over preference-approval profiles. For each $\mathcal{D} \subseteq \mathcal{L}$, a *rule* is a mapping f from $\mathcal{P}(\mathcal{D})^N$ into X. Let f be our generic notation for a rule.

We say that a rule f is *approval-invariant* if for each $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{P}^N$ such that $R_i = R_i'$ for each $i \in N$, we have $f(\boldsymbol{p}) = f(\boldsymbol{p}')$. So, an approval-invariant rule depends only on the linear orderings part of preference-approvals and ignore the positions of approval thresholds.

We call an agent i as *decisive for* $x \in X$ if for each $\boldsymbol{p} \in \mathcal{P}^N$ such that $A_i = \{x\}$, $f(\boldsymbol{p}) = x$. Agent i is *decisive* if he is decisive for each alternative. So an agent who is not approving any alternative cannot be decisive for the outcome of the rule, which will be compatible with the notion of non-manipulability we work in this chapter.

Furthermore, we define the following standard axioms for our framework.

- **Efficiency.** For each distinct pair $x, y \in X$ and each $\boldsymbol{p} \in \mathcal{P}^N$ such that $x \, R_i \, y$ for each $i \in N$, we have $f(\boldsymbol{p}) \neq y$.

  So, *efficiency* simply means that when an alternative is dominated by another alternative for every agents, then the dominated one cannot be the social outcome.

- **Unanimity.** For each $x \in X$ and each $\boldsymbol{p} \in \mathcal{P}^N$ such that $r_1(R_i) = x$ for each $i \in N$, we have $f(\boldsymbol{p}) = x$.

  Unanimity, as a weaker condition than *efficiency*, means that when the top ranked alternatives are the same for every agent, the rule respects this agreement.

- **Anonymity.** For each $\boldsymbol{p} \in \mathcal{P}$ and each permutation π of N, we have $f(\boldsymbol{p}) = f(\boldsymbol{p}')$, where $p_i' = p_{\pi^{-1}(i)}$ for each $i \in N$.

  Anonymity simply means that the agents are treated symmetrically and name-tags of them should not matter.

Now, we introduce our notion of manipulability in the preference-approval framework.

Definition 4.2.2

*A rule is **manipulable** if there are $\boldsymbol{p} \in \mathcal{P}^N$, $i \in N$, and $p_i' \in \mathcal{P}$, such that*

$$f(\boldsymbol{p}) \notin A_i \ \& \ f(p_i', \boldsymbol{p}_{-i}) \in A_i.$$

*We say thay a rule is **non-manipulable** if it is not manipulable.*

According to the above formulation, each agent i manipulates the social outcome if and only if he can change the social outcome from an unacceptable one for himself to an acceptable one.

In this sense, one can note that the above definition of *nonmanipulability* is quite different than the standard notion of *strategy-proofness* which would be defined in this framework as for each $\boldsymbol{p} \in \mathcal{P}^N$, each $i \in N$, and each $p_i' \in \mathcal{P}$, $f(\boldsymbol{p}) \; R_i \; f(p_i', \boldsymbol{p}_{-i})$. On the other hand, we will show the relation between these two notions in Section 3, where we provide our results.

4.2.3   Circular domains

In this section, we introduce a domain condition, which is first proposed by Sato [70].

A set of preferences is called a *circular domain* if the alternatives can be arranged on a circle so that for every alternative on the circle, we have two preferences in the domain in which this alternative is top ranked, and additionally, the second ranked alternative in one of these preference is the bottom ranked in the other one and the bottom ranked alternative in the considered preference is the second ranked in the other one.

Formally, we say the following:

Definition 4.2.3

$\mathcal{D} \subseteq \mathcal{L}$ is circular *if the alternatives can be indexed* $x_1, x_2, \ldots, x_m$ *so that for each* $k \in \{1, \ldots, m\}$, *there exist two preferences* R *and* R' *in* $\mathcal{D}$ *such that*

1. $r_1(R) = x_k$, $r_2(R) = x_{k+1}$, $r_m(R) = x_{k-1}$,

2. $r_1(R') = x_k$, $r_2(R') = x_{k-1}$, *and* $r_m(R') = x_{k+1}$.

(Let $x_{m+1} = x_1$ and $x_0 = x_m$.) $\mathcal{P}(\mathcal{D})$ is circular *if $\mathcal{D}$ is circular.*

It is important to note that this condition is a restriction for only the linear order part of preference-approvals.

Example: For a set of three alternatives, $\{x_1, x_2, x_3\}$, the *minimal circular domain* would be the following set of preferences where the most preferred alternative is written as the leftmost one:

$$\{x_1 x_2 x_3, \ x_1 x_3 x_2, \ x_2 x_1 x_3, \ x_2 x_3 x_1, \ x_3 x_1 x_2, \ x_3 x_2 x_1\}$$

By Sato [70], we know that on any circular domain, any strategy-proof and unanimous social choice function should be dictatorial. So, a natural question is whether the above result extends to the preference-approval framework with the non-manipulability notion that we use in this chapter.

Before investigating this question in the next section, we note some properties of circular domains.

- The universal domain is a circular domain.

- The minimal circular domains consist of $2n$ preferences since each alternative should be top ranked in at least two distinct preferences.

- One of the necessary conditions for a domain $\mathcal{D}$ to be circular is that for every $x \in X$, there exists $y \in X$ such that $r_1(R) = x, r_2(R) = y, r_1(R') = x$, and $r_m(R') = y$ for some $R, R' \in \mathcal{D}$. If we cannot find such y for some x, then the domain cannot be circular.

4.3   Results

We present possibility and impossibility results on constructing *nonmanipulable* rules.

First, we show that for each approval-invariant rule on each domain, our *nonmanipulability* definition is logically equivalent to *strategy-proofness*.

Theorem 4.3.1

Let $\mathcal{D} \subseteq \mathcal{L}$. Let f be an approval-invariant rule on $\mathcal{P}(\mathcal{D})^N$. Then, f is nonmanipulable if and only if it is strategy-proof.

Proof. It is trivial to show that *strategy-proofness* implies *non-manipulability*. Thus, we will show the only if part,

non-manipulability implies *strategy-proofness*. We prove the contrapositive. So, assume that f violates *strategy-proofness*. Then, there exist $\boldsymbol{p} \in \mathcal{P}^N, i \in N$, and $p_i' \in \mathcal{P}$ such that $f(p_i', \boldsymbol{p}_{-i}) \; P_i \; f(\boldsymbol{p})$. Let $p_i^* = (R_i^*, A_i^*) \in \mathcal{P}$ be such that $R_i^* = R_i$ and $f(p_i', \boldsymbol{p}_{-i}) \in A_i^*$, and $f(\boldsymbol{p}) \in U_i^*$. For f is approval-invariant, we have $f(\boldsymbol{p}) = f(p_i^*, \boldsymbol{p}_{-i})$. Then, we get $f(p_i^*, \boldsymbol{p}_{-i}) \in U_i^*$ and $f(p_i', \boldsymbol{p}_{-i}) \in A_i^*$ implying that f is manipulable. ∎

By Theorem 4.3.1, since approval-invariant rules are as the standard rules of Arrovian

framework, the Gibbard–Satterthwaite theorem implies that only dictatorship is *nonma-nipulable* on $\mathcal{A}^N$. Hence, for a *nonmanipulable* and *nondictatorial* rule on $\mathcal{A}^N$, one has to investigate among rules that essentially depend on position of the cut-off lines between acceptable and unacceptable alternatives.

To state differently, for a positive result of non-manipulable rules, approval information should be taken into account for the rules under consideration.

Our next result shows an example of such a rule.

Theorem 4.3.2

Let $n \geq 3$ *and* $R, R', R'' \in \mathcal{D}$ *be such that the top ranked alternatives in these preferences are distinct from each other. On* $\mathcal{P}(\mathcal{D})$*, there exists an* efficient *and* nonmanipulable *rule under which no agent is decisive.*

Proof. Let $\boldsymbol{p} \in \mathcal{P}^N$. We consider the following steps for constructing the rule.

STEP 1: If $\bigcap_{i=1}^{n} A_i \neq \emptyset$, let $f(\boldsymbol{p})$ be any *efficient* alternative[6] in $\bigcap_{i=1}^{n} A_i \neq \emptyset$. If $\bigcap_{i=1}^{n} A_i = \emptyset$, proceed to the next step.

STEP k ($1 < k < n-1$): If $\bigcap_{i=k}^{n} A_i \neq \emptyset$, let $f(\boldsymbol{p})$ be any *efficient* alternative in $\bigcap_{i=k}^{n} A_i \neq \emptyset$. If $\bigcap_{i=k}^{n} A_i = \emptyset$, proceed to the next step.

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[6]Given a profile $\boldsymbol{p}$, an alternative x is *efficient in* $Y \subset X$ if there is no $y \in Y$ such that $y \, R_i \, x$ for each $i \in N$.

STEP $n-1$: If $\bigcap_{i=n-1}^{n} A_i \neq \emptyset$, let $f(\boldsymbol{p})$ be any *efficient* alternative in $\bigcap_{i=n-1}^{n} A_i \neq \emptyset$.

If $f(\boldsymbol{p})$ is not determined after Step $n-1$, $f(\boldsymbol{p})$ is decided according to the following.

If $A_i \neq \emptyset$ for some $i \in N$, let i^* be the agent with the least index such that $A_{i^*} \neq \emptyset$, and let $f(\boldsymbol{p}) = r_1(R_{i^*})$. If $A_i = \emptyset$ for each $i \in N$, let $f(\boldsymbol{p})$ be any *efficient* alternative.

CLAIM 1: f is *efficient*.

Proof of Claim 1. By construction, f always chooses *efficient* alternatives. Hence, f is *efficient*.

CLAIM 2: f is *nonmanipulable*.

Proof of Claim 2. Let $\boldsymbol{p} \in \mathcal{P}^N$. Assume that the social choice is determined at Step $k \in \{1, \ldots, n-1\}$. Then, for each $i \in \{k, \ldots, n\}$, $f(\boldsymbol{p})$ is acceptable for agent i. Thus, agent i does not have an incentive to lie. Let $i \in \{1, \ldots, k-1\}$. Let $p_i' = (R_i', A_i') \in \mathcal{P}$. The social choice changes only if A_i' is such that $A_i' \cap \bigcap_{j=i+1}^{n} A_i \neq \emptyset$.

Let $B = A_i' \cap \bigcap_{j=i+1}^{n} A_i$. Since $A_i \cap \bigcap_{j=i+1}^{n} A_i = \emptyset$, $B \subset X \setminus A_i$. Thus, $f(p_i', \boldsymbol{p}_{-i}) \in X \setminus A_i$.

In the remaining case where $A_{n-1} \cap A_n = \emptyset$, we can see that each agent does not have an incentive to lie.

CLAIM 3: There is no decisive agent under f.

Proof of Claim 3. Let $x, y, z \in X$ denote distinct alternatives which are top ranked at

some preference relation in $\mathcal{D}$. (Such x, y, z exist by the assumption.)

Let $\boldsymbol{p} \in \mathcal{P}$ be such that $A_i = \{x\}$ for each $i \in \{1, \ldots, n-2\}$, and $A_{n-1} = A_n = \{y\}$. Then, $f(\boldsymbol{p}) = y$. Thus, each $i \in \{1, \ldots, n-2\}$ is not decisive.

Let $\boldsymbol{p}' \in \mathcal{P}$ be such that $A_i' = \{x\}$ for each $i \in \{1, \ldots, n-2\}$, $A_{n-1}' = \{y\}$, and $A_n' = \{z\}$. Then, $f(\boldsymbol{p}') = x$. Thus, neither agent $n-1$ nor agent n is decisive. ■

Under the rule f constructed in the proof of Theorem 4.3.2, an agent with a larger index is treated better than those with smaller indices.

For example, let $n = 10$, and $p_1 = p_2 = \cdots = p_8$ be such that $A_1 = A_2 = \cdots = A_8 = \{y\}$, and $p_9 = p_{10}$ be such that $A_9 = A_{10} = \{x\}$. Then, $f(\boldsymbol{p}) = x$. In this sense, f doesn't satisfy an equal treatment of the agents.

The next result shows that the agents cannot be treated equally under each *efficient* and *nonmanipulable* rule when n is even and $\mathcal{P}(\mathcal{D})$ is circular.

Theorem 4.3.3

Assume that n is even and $\mathcal{P}$ is circular. Then, there is no anonymous, efficient, *and* nonmanipulable *rule on $\mathcal{P}^N$.*

Proof. Let f be an *anonymous*, *efficient*, and *nonmanipulable* rule on $\mathcal{P}^N$. Let $\{N_1, N_2\}$ be a partition of N such that $|N_1| = |N_2|$. Assign a number from 1 to m to each alternative so that it makes $\mathcal{P}$ circular.

CLAIM 1: It is impossible that both N_1 and N_2 are decisive.

Proof of Claim 1. It is easy to derive a contradiction when N_1 and N_2 are both decisive.

Table 4.1: Profiles of preference-approvals

	$\boldsymbol{p}$		$\boldsymbol{p'}$		$\boldsymbol{p''}$	
	N_1	N_2	N_1	N_2	N_1	N_2
Best	x_k	$[x_{k+1}]$	x_k	$[x_{k+1}]$	x_k	$[x_{k+1}]$
2nd	$[x_{k+1}]$	x_{k+2}	$[x_{k+1}]$	x_k	x_{k-1}	x_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Worst	x_{k-1}	x_k	x_{k-1}	x_{k+2}	$[x_{k+1}]$	x_{k+2}

$\square$

CLAIM 2: Neither N_1 nor N_2 is decisive.

Proof of Claim 2. Suppose that one of N_1 and N_2 is decisive. Without loss of generality,

assume that N_1 is decisive. We claim that N_2 is also decisive. Let $x \in X$ and $\boldsymbol{p} \in \mathcal{P}$

be such that $A_i = \{x\}$ for each $i \in N_2$. Let π be a permutation of N such that for each

$i \in N_1$, $\pi(i) \in N_2$. By *anonymity*, $f(\boldsymbol{p}) = f(\pi(\boldsymbol{p}))$. Since N_1 is decisive, $f(\pi(\boldsymbol{p})) = x$.

Thus, $f(\boldsymbol{p}) = x$. This implies that N_2 is decisive for x. Since x was arbitrary, N_2 is

decisive. Therefore, both N_1 and N_2 are decisive, which is a contradiction to Claim 1. $\square$

CLAIM 3: For each $k \in \{1, \ldots, m\}$, either N_1 is decisive for x_k or N_2 is decisive for

x_{k+1}.

Proof of Claim 3. The following arguments are modification of those by Sato (2010).

Let $x_k \in X$. Assume that agent N_1 is not decisive for x_k. Then, at $\boldsymbol{p}$ in Table 4.1,[7]

$f(\boldsymbol{p}) \neq x_k$. By *efficiency*, $f(\boldsymbol{p}) = x_{k+1}$. Next, consider $\boldsymbol{p}'$ in Table 4.1. (For each $i \in N_1$, $p_i = p_i'$.) By *nonmanipulability*, $f(\boldsymbol{p}') = x_{k+1}$. Finally, consider $\boldsymbol{p}''$. (For each $i \in N_2$, $p_i' = p_i''$.) By *efficiency*, $f(\boldsymbol{p}'') \in \{x_k, x_{k+1}\}$. Since $f(\boldsymbol{p}'') = x_k$ is a contradiction to *nonmanipulability*, we have $f(\boldsymbol{p}'') = x_{k+1}$. By *nonmanipulability*, N_2 is decisive for x_{k+1}. $\qquad\square$

CLAIM 4: Either N_1 is decisive or N_2 is decisive.

Proof of Claim 4. For each $x_k \in X$, either N_1 is decisive for x_k or N_2 is decisive for x_k. (If not, then by Claim 3, N_1 is decisive for x_{k-1} and N_2 is decisive for x_{k+1}. However, this cannot be the case.) Let $x \in X$. Then, either N_1 is decisive for x or N_2 is decisive for x. Consider the former case. Let $y \in X \setminus \{x\}$. Then, either N_1 or N_2 is decisive for y. Since N_2 cannot be decisive for y, N_1 is decisive for y. This implies that N_1 is decisive. By similar arguments, when N_2 is decisive for x, N_2 is decisive. $\qquad\square$

Clearly, Claim 4 is a contradiction to Claim 2. $\qquad\blacksquare$

When $n = 2$, the impossibility in Theorem 4.3.3 disappears if *efficiency* is replaced by *unanimity*.

Proposition 4.3.4

Assume $N = \{1, 2\}$. There is an anonymous, unanimous, *and* nonmanipulable *rule.*

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[7] In Table 4.1, the horizontal lines between alternatives represent a boundary between the acceptable and the unacceptable range. The alternative between the brackets is a social outcome at each profile.

Proof. Let $x^* \in X$ be fixed in the following. Let $\boldsymbol{p} \in \mathcal{P}^N$.

CASE 1: There is $x \in X$ such that $r_1(R_i) = x$ for each $i \in N$. Let $f(\boldsymbol{p}) = x$.

CASE 2: $A_1 \cap A_2 \neq \emptyset$. Let $f(\boldsymbol{p})$ be any efficient alternative in $A_1 \cap A_2$.

CASE 3: One of A_1 and A_2 is empty and the other is nonempty. Let $A_i \neq \emptyset$. Then, let

$$f(\boldsymbol{p}) = r_1(R_i).$$

CASE 4: Cases 1 through 3 do not apply. Let $f(\boldsymbol{p}) = x^*$.

For each $\boldsymbol{p} \in \mathcal{P}^N$, check from Case 1 to Case 4, and determine $f(\boldsymbol{p})$ according to the first case to which $\boldsymbol{p}$ can be applied. Then, the rule f is *anonymous*, *unanimous*, *nonmanipulable*.

Since *anonymity* and *unanimity* of f are clear, we prove *nonmanipulability*. Let $\boldsymbol{p} \in \mathcal{P}^N$. If one of Cases 1, 2, and 3 determines $f(\boldsymbol{p})$, then it is clear that each agent does not have an incentive to lie. Thus, assume that $f(\boldsymbol{p})$ is determined by Case 4. Since the Cases 1 through 3 are not applicable, either $A_1 = A_2 = \emptyset$ or [$A_1 \neq \emptyset$ and $A_2 \neq \emptyset$ and $A_1 \cap A_2 = \emptyset$]. In the former case, manipulation never occurs. Consider the latter case. Consider agent 1. If $x^* \in A_1$, then he has no incentive to lie. Assume $x^* \notin A_1$. To change the social choice, he has to report $p_1' \in \mathcal{P}$ such that one of Cases 1, 2, and 3 holds. However, in each case, $f(p_1', p_2) \in A_2$. Since $A_1 \cap A_2 = \emptyset$, such p_1' is not a profitable misrepresentation. By similar arguments, it can be seen that agent 2 does not have an incentive to lie. Thus, f is *nonmanipulable*. ∎

4.4 Concluding remarks

We analyze a type of manipulation in the preference-approval framework such that each agent i manipulates the social outcome if and only if he can change the social outcome from an unacceptable one to an acceptable one. We show that according to this definition, under some mild domain assumption, there exists an *efficient* and *nonmanipulable* rule under which no agent is decisive. However, when the number of the agents is even, we cannot have an *anonymous*, *efficient*, and *nonmanipulable* rule on each circular domain. For further research, it would be interesting to characterize the set of efficient and *nonmanipulable* rules in preference-approval framework.

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