



**HAL**  
open science

# Hintikka's defence of realism and the constructivist challenge

Radmila Jovanovic

► **To cite this version:**

Radmila Jovanovic. Hintikka's defence of realism and the constructivist challenge. Philosophy. Université Charles de Gaulle - Lille III; Univerzitet u Beogradu, 2015. English. NNT : 2015LIL30002 . tel-01216147

**HAL Id: tel-01216147**

**<https://theses.hal.science/tel-01216147>**

Submitted on 15 Oct 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

University Charles de Gaulle - Lille 3

# **Hintikka's defence of realism and the constructivist challenge**

Radmila Jovanović

A dissertation submitted in partial fulfillment of the requirements for the degree of  
Doctor of Philosophy

with direction of

Prof. dr. Shahid Rahman  
Prof. dr. Miloš Arsenijević

February, 9<sup>th</sup>, 2015.



## Jury members:

Prof. Dr Shahid Rahman (mentor), University Charles de Gaulle - Lille 3

Prof. Dr Miloš Arsenijević (mentor), University of Belgrade

Prof. Dr Gerhard Heinzmann, University of Lorraine

Dr Mark Van Atten, CNRS/Paris IV

Dr Michel Bourdeau, CNRS-Paris1-ENS

Dr Andrej Jandrić, University of Belgrade

## **Acknowledgements:**

It is my great privilege to express my deepest gratitude to Prof. dr Shahid Rahman for giving me the lifetime opportunity to work under his guidance. I am deeply grateful for the constructive criticism, constant encouragement and his dedication without limits.

I would also like to thank to Prof. dr Miloš Arsenijević for the support, intellectual guidance and encouragement, which is deeply appreciated.

*Dedicated to my beloved parents, for their love and support without limits.*

# *Contents*

## *Introduction*

## *Thesis overview*

## *I. Independent Friendly Logic*

### *I.1.1 Classical first - order logic*

### *I.1.2. Models or structures*

## *I.2. Independent friendly logic - syntax*

## *I.3. Game theoretical semantics (GTS)*

## *I.4. Game – theory framework*

### *I.4.1. Strategic and extensive games*

### *I.4.2. Extensive games with perfect information*

### *I.4.3. Strategies*

### *I.4.4. Extensive games with imperfect information*

### *I.4.5. Semantic games for IF logic*

## *I.5. Properties of IF logic*

### *I.5.1. Notion of strategy*

### *I.5.2. Axiom of choice*

### *I.5.3. Skolemisation procedure*

### *I.5.4. Kreisel counterexamples*

### *I.5.5. The law of the excluded middle*

### *I.5.6. Expressive power of IF logic*

### *I.5.7. Double negation law*

### *I.5.8. De Morgan's laws*

*I.5.8. A remark on the equivalence*

*I.5.9. Compactness*

*I.5.10. Löwenheim-Skolem theorem*

*I.5.11. Separation theorem*

*I.5.12. Failure of axiomatisability*

*I.5.13. Question of compositionality*

*I.5.14. Truth predicate*

*I.5.15. Extended IF logic*

*I.6. Some problems in IF logic*

*I.6.1. Signalling*

*I.6.2. Is IF logic really a first-order logic?*

*I.7. Logicism revitalised*

## *II Constructive type theory in a dialogical framework*

*II 1. Constructive Type Theory (CTT)*

*II 2. Dialogical approach to logic*

*II.3. Dialogical approach to CTT*

*II. 3.1. Rules of formation of propositions*

*II. 3. 2. Rules of substitution*

*II. 3. 3. Particle rules*

*II. 3. 4. Structural rules*

## *III Hintikka's take on the axiom of choice and the constructivist challenge*

*III. 1. Recapitulation: Hintikka's take on the axiom of choice*

*III.2. Martin-Löf on the axiom of choice*

*III.3. Axiom of choice in a dialogical framework*

*III 4. Conclusion: The antirealist rejoinder of Hintikka's takes on the axiom of choice*

#### *IV. Anaphora*

*IV. 1. GTS approach to anaphora*

*IV.2 Dialogical Approach to Anaphora*

*VI. 3. Branching quantifiers*

*IV. 4. Final remarks*

*Conclusions*

*Appendix*

*Bibliography*

## ***Introduction***

Mathematical game theory was first introduced by John von Neumann and Oskar Morgenstern in 1944<sup>1</sup>. During the second half of the twentieth century the link was made between games and logic and since then the game – theory has proven to be the most fruitful framework for different areas of logic and mathematics, as well as for the theory of argumentation. A specific kind of games, two – persons, win or lose, found their application in providing a new kind of rule – based semantics and since then, they have served for different purposes in logic, among others, for providing the definition of truth or for providing the account of formal proofs.

The game theoretically oriented semantics, which we are interested in, provide an alternative to the traditional Tarski – style semantics, implementing Wittgenstein’s idea of *the meaning as use*. The basic idea is that meaning is obtained in a game between two players, one trying to defend and the other trying to falsify the expression at stake. The notion of truth, or that of validity, is based on the existence of a *winning strategy* of the initial verifier in a game. The direction is opposite to the one in the Tarski – style semantics: the game starts with the entire expression until its component parts are reached. In this thesis we will be interested in two different game theoretical traditions: Game Theoretical Semantics developed by Hintikka and Sandu and Dialogical logic, first introduced by Lorenz and Lorenzen and developed by Rahman and his associates.

---

<sup>1</sup> (von Neumann, Morgenstern, 1944)

By 1960 *Dialogical logic* was developed by Paul Lorenzen and Kuno Lorenz, as a solution to some of the problems that arose in Lorenzen's *Operative Logik* (1955).<sup>2</sup> Herewith, the epistemic turn initiated by proof theoretic approaches was linked with the notion of games that provided the dynamic features of traditional dialectical reasoning. Inspired by Wittgenstein's *meaning as use*, the basic idea of the dialogical approach to logic is that the meaning of the logical constants is given by the norms or rules for their use. The approach provides an alternative to both model - theoretic and proof-theoretic semantics. Originally, dialogues were developed in the context of constructive mathematics and logic but since then they have served for comparison and combination of different logical systems.

In 1968 Jaakko Hintikka combined the model - theoretical and the game - based tradition to create Game Theoretical Semantics<sup>3</sup> (GTS) that, like in the dialogical framework, grounds the concepts of truth or validity on the game - theoretic concepts, such as the existence of a winning strategy for a player, though differently to the dialogical framework it is build up on the notion of model. Hintikka claims that his semantic games are exact codifications of language - games in Wittgenstein's sense, at least if one accepts that the activities associated with quantifiers are *looking for* and *finding*. The inspiration came partially from Peirce, who had already suggested in his second Cambridge Conference lecture of 1898<sup>4</sup> that the difference between *every* and *some* can be explained in terms of who chooses an object. In the late fifties Henkin proposed that we could understand in game terms some sentences

---

<sup>2</sup> The main original papers are collected in Lorenzen/Lorenz (1978). Other papers have been collected more recently in Lorenz (2010a,b).

<sup>3</sup> (Hintikka 1962, 1973, 1996a), (Hintikka, Sandu, 1997). See also (Hintikka, 1999) and in particular (Hintikka et al., 1999). Rahman and Tulenheimo (2006) studied the relation between dialogical logic and GTS.

<sup>4</sup> In (Peirce, 1898).

which did not have a satisfactory treatment in Tarski's semantics, such as those containing an infinite string of quantifiers<sup>5</sup>. Henkin came up with the idea of a game for such a string of quantifiers between two players who are choosing the objects to replace the bound variables and he suggested that the truth conditions of such a sentence could then be expressed by its Skolem version. Hintikka generalized this idea to come up with a game theoretical semantics for classical first - order logic, which is proven to be equivalent to Tarski's semantics, given that the axiom of choice holds. Namely, linking a first – order sentence with its truth conditions, that is, its Skolem form, yields a version of the axiom of choice, which proves, according to Hintikka, that the axiom of choice is an unproblematic first – order principle. Games for classical first - order logic are deterministic games of finite length<sup>6</sup>, and they are games of *perfect information*, where the players are aware of all the previous moves in a game. In 1989 Hintikka and Sandu came up with games with *imperfect information*, where the restriction can be made on the information available to a player – so the player has to play sometimes not knowing (or forgetting) the previous moves made by the opponent. Those games yielded *Independence friendly first - order logic (IF logic)*, exceeding in the expressive power classical first - order logic. The great expressive power is due to the possibility to express in IF language more dependency patterns between quantifiers and other logical constants than standard logic. It is expressive enough to enable formulating linearly and at first - order level, sentences containing branching quantification. Because of this characteristic, Hintikka claims that IF logic is most suitable for at least two main purposes:

- 1) To be the logic of the first - order fragment of natural language; and
- 2) To be the medium for the foundation of mathematics.

---

<sup>5</sup> In (Henkin, 1961). See (Hodges, 2013).

<sup>6</sup> Which is proven by the Gale-Stewart theorem.



This thesis aims to explore the above uses of IF logic.

As for the first point, Hintikka's argument is that, being more expressive than standard classical logic, in IF language some natural language sentences (that are perfectly first - order) can be expressed which otherwise couldn't be expressed in classical first - order logic. Though GTS provides a satisfactory account of some language phenomena, such as anaphora, to which it is difficult to give a suitable interpretation otherwise, Hintikka claims that at least some sentences containing anaphoric expressions can only be treated in a satisfactory manner with its IF logic formalization.

In relation to the second use of IF logic, that of being a suitable logic for mathematical reasoning, Hintikka developed it in detail in his work *The Principles of Mathematics Revisited* and in many papers following this major work. Hintikka argues there that IF logic is a superior alternative to set theory (usually formalized with classical first - order logic) and that, linked with game – theoretical semantics, it provides a realist foundation of mathematics, embracing the great part of classical mathematics. It is, in his opinion, *a dream of logicians come true*, a framework allowing conceptual reduction of mathematic to logic, as the very title of the main work suggests, being a clear allusion to Russell and Whitehead's work that set a landmark in the field. Moreover, Hintikka insists that in GTS, as mentioned above, Zermelo's axiom of choice is justified at first - order level in a way that should be perfectly acceptable for constructivists and that the IF component displays the grounds for the constructivist rejection of this axiom.

However, IF logic has some serious limitations and we will discuss them in this thesis. One of the biggest is the impossibility of the full axiomatisation. The lack of a complete proof system makes IF logic inevitably linked – if not reducible - to model theory. Although Hintikka doesn't recognise it as a problem, it seems to clash with his own claim that *IF-logic makes the logicist dream true*.

Moreover, game theoretical semantics allows a constructivist reading as well, as pointed out by Ranta and Sundholm, but Hintikka is not in favour of a constructivist approach. One of the principle reasons for his rejection of constructivist, proof – conditional meaning is Hintikka's claim that in this framework one cannot give a suitable account of branching quantifiers. We will prove him wrong in regard to this point.

In our dissertation we will present, analyse and criticize the above mentioned uses that Hintikka makes of IF logic. Before we elaborate on our main thesis, let us briefly cover the dialogical tradition we have previously mentioned and its recent developments that will be our main tool for refuting Hintikka's theory. The formal tool we will use is a dialogical approach to *Constructive Type Theory* by Per Martin – Löf.

In opposition to the model – theoretic tradition based of Frege's and Tarski's work, where set theory was linked with classical first – order logic, in the second half of the twentieth century a constructivist approach to mathematics and logic was born in the context of Brouwer's intuitionistic foundations of mathematics. In this stream an epistemic approach to meaning was put to the fore. The emphasis was put rather on

the *knowledge* and on the conditions for a proposition to be *asserted* than on the *truth* of a proposition, like in the old tradition. This way of thinking demanded a *proof – theory* in opposition to the model – theoretic semantics. In this tradition Martin – Löf developed Constructive Type Theory (CTT) where the principle “*propositions – as – types*” has a central place. According to this principle, propositions are identified with sets, types or sets of their proof – objects. CTT has intuitionistic logic at its core and it is fully predicative. Some of the important features of CTT are:

- In CTT a clear distinction is made between judgments (or assertions) and propositions expressed by those judgments.
- Classical logic can be developed in the CTT framework (see Ranta, 1994: chapter 2).
- As opposed to the model – theoretic approach to meaning, where the elements of language are linked to the world (or interpreted) via some meta – level means, in CTT judgments and proof objects are embedded at the object – level, and they account, by means of inferential rules, of the meaning of every expression, so a fully interpreted language results. This carries out Frege’s idea of a *Begriffsschrift* – that should provide a full and explicit formalization of mathematics.

Recently, a link has been established between CTT and dialogical logic in (Rahman, Clerbout, 2013, 2014), thus putting CTT in a game theoretic framework. It shows to be a powerful formal tool, although the work on it has just started. As mentioned above, the idea behind CTT is to render meta - logical features explicit at the object – language level. This is actually very close to Wittgenstein’s arguments against formal semantics in the style of Tarski,<sup>7</sup> and that is one of Wittgenstein’s

---

<sup>7</sup> For a discussion on this see (Rahman, Clerbout, Jovanović, forthcoming).

tenets that Hintikka (and other model-theoreticians) rejects. Moreover, the aim of rendering explicit the rules of meaning-explanation is inherent to the dialogical approach, where the meaning is constituted within certain interactions between players. In such a context it is natural to request for those interactions to be rendered explicit in the object – language. It should also be pointed out that though dialogical logic shares some basic elements with GTS there are two main distinctions, namely, (1) it does not assume an underlying model - as GTS does - and this links dialogical logic to the constructivist approaches in general, (2) it distinguishes between *local meaning* obtained at the play – level, and *global meaning*, obtained at the level of strategies. As it happens, the combination of CTT with dialogical logic proves Hintikka wrong in his claim that constructivist approaches can provide neither an account of branching quantification at the first - order level nor outside-inside reading of expressions containing such kind of quantification.

Finally, let me list the main theses to be developed in the present dissertation:

- 1) We will present and analyse two main uses Hintikka makes of IF logic and GTS: one as the medium for mathematical reasoning and the other as the means of formalisation and analysis of natural language. The properties of IF logic will be discussed, as well as the advantages of this approach such as the possibility to take account of (in)dependency relations among variables; GTS account of two different notions of scope of quantifiers; the “outside – in” direction in approaching the meaning, which turns out to be advantageous over the traditional “inside - out”; the usefulness of game – theoretic reasoning in mathematics; the expressiveness of IF language which allows formulating

branching quantifiers in the first - order level, as well as defining the truth predicate in the language itself. We will defend Hintikka's standing at the first - order character of IF logic against some critics on this point.

2) However, the weak points will also be discussed: first and foremost, the lack of a full axiomatisation for IF logic and second, the problem of signalling, the problematic phenomenon related to the introduction of the possibility of imperfect information in a game.

3) We will turn to another game – theoretically oriented semantics, that of Dialogical Logic linked with the Constructive Type Theory, in which dependency relations can be accounted for, but without using more means than constructive logic and dialogical approach to meaning have to offer. We will use this framework first to analyse Hintikka's take on the axiom of choice and second, to analyse the GTS account of anaphora.

4) We will compare and contrast Per Martin- Löf's analysis of the axiom of choice with Hintikka's standing on this axiom. Hintikka claims that his game theoretical semantics (GTS) for Independence Friendly Logic justifies Zermelo's axiom of choice in a first-order way perfectly acceptable for the constructivists. Martin- Löf's results, however, show that Hintikka's preferred version of the axiom of choice is indeed acceptable for constructivists and its meaning does not involve higher order logic only if it is based on an intensional take on functions. Extensionality is the heart of the classical understanding of Zermelo's axiom and this is the real reason behind the constructivist rejection of it. Furthermore, we will show that dependence and

independence features that motivate IF-Logic, can be formulated within the frame of constructive type theory (CTT) without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference.

5) We will analyse the GTS account of anaphora. We will argue that, even though the GTS approach has considerable advantages over other existing theories in dealing with anaphora, the extension of the dialogical framework contains both the contentual (first-order) features of CTT and the interactive aspects of GTS. In the dialogical approach, choice dependences are expressed at the object language level and they can be expressed in such a way that: 1. no more means is needed than what constructive (or even classical logic) has to offer, 2. without the possibility of some non – desirable phenomenon, such as *signalling*.

Consequently, the aim of this thesis will be to show how a satisfactory treatment can be made in CTT-dialogues for both anaphora and branching quantifiers at the first - order level, and how the relation of dependencies between quantifiers can be accounted for, but all that without sacrificing inferential role. It will be equally clear that neither branching quantifiers nor the anaphora push us toward a model theory.

In the the following chapter we provide a detailed overview of the thesis.

## ***Thesis overview***

In part ***I*** of this thesis we present *Independent Friendly logic (IF)* and two of the most cherished uses Hintikka makes of it: the one in foundation of mathematics and the other in formalization of a natural language in general and of anaphoric expressions in particular.

Firstly, we introduce IF language, starting with classical first - order language and enlarging it with a new syntactic device which makes a crucial change: a slash “/” indicating a relation of independence of a quantifier or a connective in relation to some quantifier in whose scope it lies.

The semantics of IF is given in section ***I.3.***, where we present *Game theoretical semantics (GTS)*. The traditional Tarski – style, compositional semantic is inapplicable in IF logic where formal barriers, involving the slash sign, prevent it from conforming to such a compositional framework. A satisfactory, non – compositional, model - theoretic semantics is given in a game – theoretical framework: the satisfaction of an IF sentence is checked in a model through a game between two players, one trying to verify and other to falsify the sentence. The game rules are thus presented and explained in this section. In general we follow a presentation Hintikka gave himself but we introduce smaller changes in some places, following other works dealing with IF logic, with appropriate theoretic justifications.

In section ***I.4.*** we discuss in more details games in question, presenting them in a general game - theoretic framework. We will be using for this two sources: the book of (Mann, Sandu, Sevenster, 2011) and the thesis of (Dechesne, 2005). The games related to IF logic are classified as *extensive games with imperfect information*.

We explain in this section the crucial notions related to IF logic and to GTS: that of restriction in information available to a player and that of a strategy.

In section *I.5.* we discuss some of the properties of IF logic. This section is thus divided into fifteen sub – sections, in each of which we are shedding some light on different moments in IF logic and GTS that will be of relevance for discussions that take place in other chapters of the thesis (the analysis of IF logic could certainly be enlarged upon, but our discussion is limited to what we considered to be enough for our purpose). We start with the notion of a *strategy* in a game, which leads us to the analysis of the *axiom of choice* included in the very definition of truth in GTS. Namely, the truth of a given sentence is defined as the existence of the winning strategy of the initial verifier in a game for that sentence and strategies are expressed by Skolem functions. Linking the sentence with its truth conditions, that is, with the assertion that such a choice function exists, yields a variant of the axiom of choice. One of Hintikka’s favorite examples of the supremacy of GTS is what he considers to be a justification of Zermelo’s axiom of choice in a way acceptable for the constructivists. Hintikka’s take on the axiom of choice will be one of our main targets in this thesis.

We’ll move then on to the Skolemization procedure, related to the truth conditions of a sentence and the procedure of Kreiselization, related to its falsity conditions. Then we follow up with a discussion of the failure of the law of the excluded middle in IF logic. We explain the reasons for the failure of this classical law and we compare it to the case of intuitionistic logic where this law happens not to be valid for entirely different reasons.

Then we make some remarks on the expressive power of IF logic, which significantly exceeds that of classical first - order logic, even though, according to



Hintikka, no commitment to higher order entities is involved in it. That is the reason for Hintikka to claim that IF logic is more convenient for formalization of first - order fragment of natural language. IF logic is expressive enough to embrace branching quantification at the first - order level, thus making it the best candidate both for foundation of mathematics and for formalization of a natural language: all important mathematical concepts can be expressed in it, as well as some natural language sentences involving *anaphoric expressions* that couldn't be expressed otherwise without involving a quantification over higher order entities. Both proposed uses of IF logic are discussed in this thesis.

Furthermore, some other properties are proved to apply to IF logic, as the law of double negation, de Morgan's laws, compactness, the Löwenheim-Skolem theorem and separation theorem. We make a short remark of the notion of equivalence in IF logic which is specific, given that the law of the excluded middle doesn't hold.

We continue by examining the failure of axiomatizability and the impossibility to give a compositional interpretation of IF logic. Both of them Hintikka doesn't consider to be a handicap (the lack of compositionality is rather taken as an advantage). His justification for the first is made from a model – theoretic perspective while the lack of compositionality is justified by the desirability of dependence of the meaning on the context in which the language is used. We will make some remarks about it.

In the next sub – section we discuss the possibility of formulating the truth predicate for IF language in the language itself – one of the biggest advantages of IF language, according to Hintikka. Namely, as per famous Tarski's result from 1936, the truth predicate cannot be defined for a first - order language within that same language because adding the truth predicate in some first - order language would

produce paradoxes. The possibility of formulating the truth predicate for IF first - order language is another important argument Hintikka is using in favour of application of this logic in foundation of mathematics.

At the end of section **1.5**, we consider an extension of IF logic where another kind of negation is introduced – which ought to enlarge even more the expressivity of IF logic and make it thus completely suitable for mathematical purposes.

Some problems of IF logic are discussed in section **1.6**. The first problem considered is a phenomenon appearing in some games for IF sentences called *signalling*. In many cases this phenomenon is undesirable and Hintikka sought to prevent it using certain formal means. However, it seems that the problem is not so easily ruled out because, as it happens, signalling is actually needed in some cases - for without it, the truth value of some classical first - order sentences in GTS would differ to the one under Tarskian interpretation, which is off course unacceptable.

Another problem concerns the order of IF language. Hintikka's claim that we are dealing here with first - order logic has been disputed by some authors. Feferman (2006) and Väänänen (2001) raised the question whether IF logic is really the first - order logic? Tulenheimo (2009) provides some elements to defend Hintikka's view. Curiously, Sundholm (2013) shows that those dependences and independences that motivate Hintikka's introduction of IF can be formulated in CTT-first - order logic, thus defending Hintikka's ground. With regard to this issue, we introduce a few earlier arguments concerning the order of branching quantifiers: that of Patton, Quine and Hand.

The last part of the first section serves to connect the dots we gave before in order to make a clearer picture about Hintikka's take on the role of IF logic in general and in the project of foundation of mathematics in particular. Hintikka's position in

the foundation of mathematics is analysed: specific realist position and even logicist, although in a modified and less ambitious version: it is a conceptual reduction (not a translation) of mathematics to IF logic. We discuss as well Hintikka's standing on the constructivism and his view on the ontological status of mathematical objects. With this section the first chapter is completed. In what follows some of Hintikka's crucial points will be contested.

In the second chapter of this thesis we present Per Martin- L f's Constructive Type Theory in a dialogical framework. The aim of this chapter is to give us some convenient formal tools, which we can use in the following chapters for questioning some of Hintikka's points.

In the first part some characteristics of the Constructive Type Theory (CTT) are briefly commented on. In the second we give an outline of dialogical approach to logic and we make a few remarks on the relation between GTS and dialogues. Being both game – theoretical semantics, they have apparently a lot in common, but there are some important differences to be stressed as well. Dialogical logic for classical first - order logic is presented in the Appendix.

In section *II.3.* we present recent developments in dialogical logic where the link is made between dialogical logic and CTT from (Rahman, Clerbout, 2013, 2014) and (Rahman, Clebourn, Jovanovi , forthcoming). Accordingly, in this section of the dissertation we present the rules for dialogical CTT- games: they include rules for the formation of propositions (which are necessary to satisfy the request of a fully interpreted language), particle rules (that yield, together with formation rules, the local meaning) and structural rules (which determine a general course of dialogical game - where global meaning is achieved).

In chapter *III* of this thesis we compare and contrast Per Martin-Löf's analysis of the axiom of choice with Jaakko Hintikka's standing on this axiom. Hintikka claims that his game theoretical semantics for IF logic justifies Zermelo's axiom of choice in a first-order way perfectly acceptable for the constructivists. In fact, Martin-Löf's results lead to the following considerations:

- 1) Hintikka's preferred version of the axiom of choice is indeed acceptable for the constructivists and its meaning does not involve higher order logic.
- 2) However, the version acceptable for constructivists is based on an intensional take on functions. Extensionality is the heart of the classical understanding of Zermelo's axiom and this is the real reason behind the constructivist rejection of it.
- 3) More generally, dependence and independence features that motivate IF-Logic, can be formulated within the frame of constructive type theory (CTT) without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference – logic is about inference after all.

Thus, we first make a recapitulation of Hintikka's take on the axiom of choice and then we contrast it with Martin-Löf's point from (Martin-Löf, 2006): that there are indeed some versions of the axiom of choice that are perfectly acceptable for a constructivist, namely one where the choice function is defined *intensionally*. In order to see this, the axiom must be formulated within the frame of a CTT-setting. Indeed such a setting allows comparing the extensional and the intensional formulation of the axiom. It is in fact the extensional version that implies the law of the excluded middle, whereas the intensional one is constructive.

In section *III.3.*, we present Rahman’s and Clerbout’s dialogical proof of the constructive formulation of the axiom of choice from (Rahman, Clerbout, 2014). They proved thus Martin Löf’s point in a game – theoretical framework. This result leads us to conclude that the CTT approach to meaning in general, and to the axiom of choice in particular, is very natural to game theoretical approaches where (standard) meta - logical features are explicitly displayed at the object - language level. Thus, in some way, this vindicates, although in a different manner, Hintikka’s plea for the fruitfulness of game-theoretical semantics in the context of the foundations of mathematics. We conclude that the proof of the axiom of choice is constructive but its game theoretical interpretation is antirealist after all.

In chapter *IV* we examine Hintikka’s and associates’ work on anaphora, based on GTS. In this section we compare the GTS approach to the problem of anaphora with a solution for anaphora in the dialogical approach to CTT. We believe that the GTS approach has considerable advantages over other existing theories in dealing with anaphora. However we show that the extension of the dialogical framework discussed in the preceding sections contains both the contentual (first-order) features of CTT and the interactive aspects of GTS. The dialogical approach provides a first - order solution that does not require other devices but the ones of constructive or even classical first - order logic, when formulated within a suitably adapted CTT-frame. We finish the chapter by showing how in the dialogical framework for CTT the more difficult examples involving branching quantifiers can also be handled in a satisfactory manner.

At the end, we sum up the conclusions of this thesis and we add some remarks about further research.

## ***I. Independent Friendly Logic***

*Independence friendly first - order logic* (a.k.a. *IF logic*) was introduced by Jaakko Hintikka and Gabriel Sandu in their article ‘Informational Independence as a Semantical Phenomenon’ (1989). Early works on it are (Hintikka, 1991) and (Sandu,1991). IF logic is an extension of first - order logic, where a new symbol is introduced, ‘/’ (slash, independence indicator), which serves to express at the object language level the relation of independence of a quantifier or a connective in relation to a quantifier in whose scopes it lies.

### ***I. 1.1. Classical first - order logic***

We start defining classical first - order language in a usual way with the following sets:

$S_1 = \{x, y, z, \dots\}$  – a countable set of variables.

$S_2 = \{\neg, \wedge, \vee, \exists, \forall\}$  – a set of logical symbols for negation, conjunction, disjunction, existential and universal quantifier, respectively.

$S_3 = \{R_n : n \text{ is an element of the set of natural numbers } N\}$  - a set of relational symbols. Every such relational symbol  $R$  has a natural number  $n$  that indicates its arity. Unary relational symbols are *predicates*.

$S_4 = \{f_m : m \text{ is an element of the set of natural numbers } N\}$  – a set of functional symbols. Every such a functional symbol  $f$  has a natural number  $m$  that indicates its arity. Nullary function symbols are *constant symbols*, which will be defined in a separate set.

$S_5 = \{c_1, c_2 \dots\}$  – a set of constant symbols.

$S_6 = \{(), =\}$  – a set of additional symbols.

We will introduce the set  $\sigma$  as a *signature* that consists of a union of  $S_3$ ,  $S_4$  and  $S_5$ , or the set of all non-logical symbols. Let it be the language  $L$ . Now we define  $L$ -terms.

-*Definition 1* : a) Every variable and every constant symbol belonging to  $\sigma$  is an  $L$ -term; b) If  $f$  is an  $n$ -ary functional symbol from  $\sigma$  and  $(t_1 \dots t_n)$  are  $L$ -terms then  $f(t_1 \dots t_n)$  is an  $L$ -term. We will simply write *terms* from now on.

By combining terms with relational symbols we get *atomic  $L$ -formulas*. By combining formulas with logical connectives and quantifiers we get *compound  $L$ -formulas*. For the sake of simplicity we will refer to them as *formulas*.

-*Definition 2*: If  $R$  is  $n$ -ary relational symbol from  $\sigma$  and  $(t_1 \dots t_n)$  are  $L$ -terms then  $R(t_1 \dots t_n)$  is an atomic formula.

Formulas are defined as follows:

-*Definition 3*: a) every atomic formula is a formula.

b) If  $t_1$  and  $t_2$  are terms then  $(t_1 = t_2)$  is an atomic formula.

c) If  $\varphi$  is a formula then  $\neg\varphi$  is a formula.

d) If  $\varphi$  and  $\psi$  are formulas then  $(\varphi \wedge \psi)$  and  $(\varphi \vee \psi)$  are formulas.

e) If  $\varphi$  is a formula then  $(\exists x\varphi)$  and  $(\forall x\varphi)$  are formulas.

f) Nothing else is a formula.

When it is not necessary we will leave the brackets out. We need a definition of a *free* and *bound* occurrence of a variable.

-*Definition 4*: The occurrence of a variable  $x$  is bound if it lies in the scope of the quantifiers  $\exists x$  or  $\forall x$ , otherwise it is free. Sets of all free and of all bound variables of a formula  $\varphi$  we call  $\text{Free}(\varphi)$  and  $\text{Bound}(\varphi)$ , respectively. In atomic formulas all variables occur free. For compound formulas:

- *Definition 5*:  $\text{Free}(\neg\varphi) = \text{Free}(\varphi)$ ;

$\text{Free}(\varphi \wedge \psi) = \text{Free}(\varphi) \cup \text{Free}(\psi)$ ;

$\text{Free}(\varphi \vee \psi) = \text{Free}(\varphi) \cup \text{Free}(\psi)$ ;

$\text{Free}(\exists x\varphi) = \text{Free}(\varphi) - \{x\}$ ;

$\text{Free}(\forall x\varphi) = \text{Free}(\varphi) - \{x\}$ .

A formula with no free occurrence of variables is a *sentence*.

### ***1.1.2. Models or structures***

A structure or a model of language  $L$  is a non-empty set of objects with relations and operations on them that interprets the symbols in the set  $\sigma$ .

-*Definition 6*: A structure or a model of vocabulary  $L$  is an object  $\mathbb{M} = (M; R^{\mathbb{M}} \dots, f^{\mathbb{M}} \dots, c^{\mathbb{M}} \dots)$  where  $M$  is a *universe* of  $\mathbb{M}$  which is non-empty;

$R^{\mathbb{M}}$  is an  $n$ -ary relation in the universe, which is an interpretation of an  $n$ -ary relational symbol  $R$ ;

$f^{\mathbb{M}}$  is an  $n$ -ary function on  $M$ ,

$f^{\mathbb{M}}: M^n \rightarrow M$ , that is an interpretation of functional symbol  $f$ ;

$c^{\mathbb{M}}$  is an element of the universe  $M$  that is an interpretation of a constant symbol  $c$ .

We still need to deal with variables.

-*Definition 7*: A function from set of individual variables to the universe  $M$  for a given structure  $\mathbb{M}$  is a *valuation*  $v$ . If  $a$  is an element of  $M$  and  $v$  is a valuation then

$v(x_i / a)$  is defined as:  $v(x_i / a)(x_j) = v(x_j)$  if  $i \neq j$ ; otherwise:  $v(x_i / a)(x_j) = a$  if  $i = j$ .

We will allow for a valuation to include also a mapping from terms to individuals of the universe, so we have:  $v(c) = c^{\mathbb{M}}$  and  $v(f(t_1 \dots t_n)) = f^{\mathbb{M}}(v(t_1) \dots v(t_n))$ .



-*Definition 8:* We say that a model is *suitable* for a formula  $\phi$  if it has interpretations of all relational and functional symbols in that formula.

Now we can say when one atomic formula is satisfied in a model.

-*Definition 9:*  $\mathbb{M}, v \models (t_1 = t_2)$  iff  $v(t_1) = v(t_2)$  ;

$\mathbb{M}, v \models R(t_1 \dots t_n)$  iff  $(v(t_1) \dots v(t_n)) \in R^{\mathbb{M}}$ .

## ***1.2. Independent friendly logic - syntax***

As previously mentioned, syntactically, IF logic has all the elements of classical first - order logic, with addition of one new symbol. The only new symbol that appears in the IF logic is the slash (/), which serves to express the independence of a quantifier or a connective in relation to some quantifier in whose scopes it lies.

For example:

$$1) \forall x (\exists y / \forall x) S(x, y); \exists x (\forall y / \exists x) S(x, y)$$

$$2) \forall x (Sx (v / \forall x) Rx); \exists x (Sx (\wedge / \forall x) Rx)$$

We will explain the meaning of the slash sign in detail in the section regarding semantics but let us just touch on it briefly. In classical logic, the only means of expressing relations between quantifiers are brackets and the nesting of their scopes. For Hintikka that is one important limitation, especially when it comes to mathematics. In some sense, the most interesting part of logic is precisely the understanding of relations of dependence and independence among quantifiers (and other logical operators) and the logic that allows expressing these relations has an undoubtable advantage. Expressing relations of dependency entails a completely

different understanding of the nature of quantifiers. According to Hintikka, a simplified view on quantifiers is a mistake rooted in Frege's work (Hintikka, 2008: 1). Frege interpreted the existential quantifier  $\exists x$  in the expression  $\exists xS(x)$  as a second order predicate that indicates if the first - order predicate  $S$  is empty or not. The expression is then satisfied if there is an individual  $a$  to which predicate  $S$  applies such that  $S(a)$  is true. However, this is not the only role quantifiers might have: the other one, equally important, is to express dependencies between variables. When a quantifier is applied to more than one-place predicate, as in the example  $\forall x\exists yS(x,y)$ , we face a dilemma: to find a value of the variable  $y$ , do we look for it knowing or not knowing the value of  $x$ ? According to (Hintikka, 2008: 3), it was Pierce who actually recognised this important semantic aspect manifested in the ordering of quantifiers, which led him to the concept of a semantic game for quantifiers played by two players. It was one of the inspirations for Hintikka to come up with *Game Theoretical Semantics*.<sup>8</sup> IF logic takes account of quantifiers' dependency relations and so allows freeing the expressivity of first-order language.

Now let's go back to the syntax of IF logic. We have to make some changes in definition 3 for IF formulas:

- Definition 3\**: a) Every atomic formula is a formula.
- b) If  $t_1$  and  $t_2$  are terms then  $(t_1 = t_2)$  is an atomic formula.
- c) If  $\varphi$  is a formula then  $\neg\varphi$  is an atomic formula.
- d) If  $\varphi$  is a formula of the form  $p\wedge q$  and connective  $\wedge$  occurs in the scope of a number of existential quantifiers of the form  $(\exists y_1\dots \exists y_n)$ , then  $(\exists y_1\dots \exists y_n) p(\wedge\exists y_1\dots \exists y_n)q$  is a formula.

---

<sup>8</sup> As mentioned in the introduction, further development of Pierce's idea was made in the late Fifties by Henkin.

- e) If  $\varphi$  is a formula of the form  $p \vee q$  and connective  $\vee$  occurs in the scope of a number of universal quantifiers of the form  $(\forall y_1 \dots \forall y_n)$ , then  $(\forall y_1 \dots \forall y_n) p(\vee / \forall y_1 \dots \forall y_n) q$  is a formula.
- f) If  $\varphi$  is a formula and  $(\exists x)$  occurs in the scope of a number of universal quantifiers of the form  $(\forall y_1 \dots \forall y_n)$ , then  $(\forall y_1 \dots \forall y_n) (\exists x / \forall y_1 \dots \forall y_n) \varphi$  is a formula.
- g) If  $\varphi$  is a formula and  $(\forall x)$  occurs in the scope of a number of existential quantifiers of the form  $(\exists y_1 \dots \exists y_n)$ , then  $(\exists y_1 \dots \exists y_n) (\forall x / \exists y_1 \dots \exists y_n) \varphi$  is a formula.
- h) Nothing else is an IF formula.

One remark is in order. Hintikka did not introduce clauses d) and g) – in his presentation, only the existential quantifier and disjunction can be independent from previous universal quantifiers in whose scopes they lie. We allow the independence of universal quantifier and conjunction following Dechesne in her thesis (Dechesne, 2005), in order to render the syntax symmetrical. That is not an essential change because every IF formula with a slashed universal quantifier or slashed conjunction can be transformed in an IF formula without it, as we shall see later. However, there are some reasons to prefer one syntax to the other and we will discuss it later on in this thesis.

Now, in clauses d), e), f) and g) the set of quantifiers behind the slash sign can be empty. In that case we get the usual first - order quantifiers  $\exists x$ ,  $\forall x$  or connectives  $\wedge$ ,  $\vee$ . Classical first - order expressions are special cases or the shorthand of IF expressions.

There are some changes to be made in *Definition 5* as well for free occurrence of variables for quantifiers:

- *Definition 5\**:  $\text{Free}(\neg \varphi) = \text{Free}(\varphi)$ ;

$\text{Free}(\varphi \wedge \psi) = \text{Free}(\varphi) \cup \text{Free}(\psi)$ ;  $\text{Free}(\varphi \vee \psi) = \text{Free}(\varphi) \cup \text{Free}(\psi)$ ;

$\text{Free}(\exists x/\forall y_1 \dots \forall y_n)\varphi = (\text{Free}(\varphi) - \{x\}) \cup \{y_1 \dots y_n\}$ ;

$\text{Free}(\forall x/\exists y_1 \dots \exists y_n)\varphi = (\text{Free}(\varphi) - \{x\}) \cup \{x_1 \dots x_n\}$ .

We need one more definition, that of a subformula:

*-Definition 10:* The set of subformulas  $\text{Sub}(\varphi)$  of an IF formula  $\varphi$  is the smallest set  $S$  such that:

$\varphi \in S$ ;

If  $\neg \psi \in S$  then  $\psi \in S$ ;

If  $\psi_1(\vee/\forall y_1 \dots \forall y_n)\psi_2 \in S$  then  $\psi_1, \psi_2 \in S$ ; If  $\psi_1(\wedge/\exists y_1 \dots \exists y_n)\psi_2 \in S$  then  $\psi_1, \psi_2 \in S$ ;

If  $(\exists x/\forall y_1 \dots \forall y_n)\psi \in S$  then  $\psi \in S$ ; If  $(\forall x/\exists y_1 \dots \exists y_n)\psi \in S$  then  $\psi \in S$ .

If there is more than one occurrence of the same element in a formula  $\varphi$ , (for example  $\varphi = \psi \vee \neg \psi$ ) we must treat every occurrence as a different subformula. The reason for that will be clear when we introduce the semantic games. Every such subformula produces a different subgame.

In Hintikka's presentation of IF logic (Hintikka, 1996a) it is assumed that all formulas are in *negation normal form*, meaning that all negation signs occur in front of atomic formulas. It is a restriction introduced for the sake of simplicity and it is not a big restriction, because for every IF formula there is an equivalent IF formula in negation normal form.<sup>9</sup> However, following (Dechesne, 2005) we think that it is better to leave out this assumption, even if it renders things technically little more complicated, because of the treatment of negation in game theory semantics. We will make a remark about it in the following chapter.

---

<sup>9</sup> De Morgan's laws are holding for IF logic, as we will prove later, so we are using them to push the negation signs inwards.

We will introduce one more definition of a *regular* IF formula:<sup>10</sup>

-*Definition 11: Regular* IF formulas are IF formulas meeting two conditions: 1) If a quantifier is slashed it has to be subordinate to the quantifier it is independent of. 2) There is no double quantification - two nested quantifiers cannot carry the same variable. Therefore, formulas  $\forall x (\exists y/\forall x) \varphi$  and  $(\forall y_1 \dots \forall y_n) \varphi$  ( $\forall/\forall y_1 \dots \forall y_n$ )  $\psi$  are examples of regular formulas, while  $(\exists y/\forall x) \varphi$  and  $\forall x (\varphi \wedge \exists x \psi)$  are not.

We have now all the necessary elements of syntax, so we can move on to *Game Theoretic Semantics*, the semantics that Hintikka proposed for IF logic as its most natural semantics. Some alternative semantics have also been proposed for IF logic, as one version of compositional semantics constructed by Hodges<sup>11</sup> and Skolem semantics. All those have been proven to be equivalent.<sup>12</sup>

### ***1.3. Game theoretical semantics (GTS)***

Recursive definition of truth à la Tarski, relies on the concept of valuation. It satisfies compositionality, going from the inside out (from atomic formulas to complex ones). Truth is relative to a model and a valuation function which attributes a certain type of entity to every constant and variable in a given language. This definition of truth, however, must be expressed in a higher order language, that is, in a more powerful language where we can quantify over valuations. As per Tarski's famous result from 1936, it is not possible to define the concept of truth for any formal arithmetic, using the expressive means of that arithmetic alone. Despite

---

<sup>10</sup> See (Caicedo, Dechesne, and Janssen, 2009). This definition is important for preventing some non desirable effects in IF logic such as *signalling*. We will discuss it in section **I.6.1** of this thesis.

<sup>11</sup> Although, not Tarskian – type semantics, see section **I.5.13**.

<sup>12</sup> See (Mann, Sandu, Sevenster, 2011).

Tarski's result, Hintikka thinks that attempts to seek a realist definition of truth for the language of mathematics in the language itself might not be in vain, but this cannot be done while respecting compositionality. The semantic theory which attributes the meaning "from inside out" is not the adequate one for IF logic. The reverse way is much more promising – where the meaning is obtained through a semantic game between two players, *Verifier* and *Falsifier*<sup>13</sup>, starting from the entire sentence and descending to the atomic expressions, whose authenticity is then checked in the model. According to Hintikka, compositionality and the “inside – out” approach amount to the fact that the sentence is considered independently of the context in which it is stated. The meaning of the sentence depends only on the meanings of its constituent parts and its structure. Different semantics, in which we start with the entire sentence until we reach its basic components, allows taking into account the dependence on context (Hintikka, Sandu, 1997).

Game theoretic approach appeared already in 1960 in *Dialogical logic* developed by Paul Lorenzen and Kuno Lorenz, as a solution to some of the problems that arose in Lorenzen's *Operative Logik* (1955).<sup>14</sup> This approach offered an epistemic turn that was initiated by the proof theoretic approaches and combined with the notion of games. In this way the dynamic features of the traditional reasoning were obtained. Inspired by Wittgenstein's *meaning as use* the basic idea of the dialogical approach to logic is that the meaning of the logical constants is given by the norms or rules for their use. This approach provides an alternative to both model - theoretic and proof - theoretic semantics.

In 1968 Jaakko Hintikka combined the model - theoretical and the game - based traditions by means of the development of what is now known as Game

---

<sup>13</sup> In his earlier papers, Hintikka named the players *Myself* and *Nature*.

<sup>14</sup> The main original papers are collected in (Lorenzen, Lorenz, 1978). Other papers have been collected more recently in (Lorenz, 2010a, b).

Theoretical Semantics (GTS) that, like in the dialogical framework, grounds the concepts of truth or validity on game theoretic concepts, such as the existence of a winning strategy for a player, though differently to the dialogical framework it is build up on the notion of model.<sup>15</sup> Hintikka claims that his semantic games are exact codifications of language games in Wittgenstein's sense, at least if one accepts that the activities associated with quantifiers are 'looking for' and 'finding'.

The central idea is that the language gets its meaning only within certain actions of participants in a language practice. Particularly interesting is the meaning of quantifiers.<sup>16</sup> Activities in which their meaning is obtained can be understood as search for (and discovery of) objects (or "individual - witnesses") the quantifiers range over. This game of "seeking and finding" is formalized as a semantic game for quantifiers. In general it is a game between *Verifier*, who proposes the sentence and tries to defend it, and *Falsifier*, who challenges it. Hintikka considers GTS to be the most natural semantics for IF logic.

The game is defined as follows:

*-Definition 12:* Let Eloise and Abelard be the players in a game. Eloise is the initial Verifier, trying to defend the sentence at stake and Abelard is the initial Falsifier, trying to deny it. A semantic game  $G(\varphi)$  for the sentence  $\varphi$  begins with  $\varphi$ . The game is played in respect to the model  $M$  with a given language  $L$ . Through various stages of game, players will consider either the sentence  $\varphi$  or other the sentence  $\varphi_1$  obtained from it through the game. The game is played with well-defined rules.

$R \vee$  - disjunction rule:  $G(\varphi_1 \vee \varphi_2)$  starts by the choice of a player in the role of verifier for  $\varphi_i$  ( $i = 1$  or  $2$ ). The game continues as  $G(\varphi_i)$ .

---

<sup>15</sup> (Hintikka, 1962, 1973, 1996a), (Hintikka, Sandu, 1997). See also (Hintikka, 1999) and in particular (Hintikka et al., 1999). See (Rahman, Tulenheimo, 2006) for the relation between dialogical logic and GTS.

<sup>16</sup> See for example (Hintikka, 1968).

R  $\wedge$  - conjunction rule:  $G(\varphi_1 \wedge \varphi_2)$  starts by the choice of a player in the role of falsifier for  $\varphi_i$  ( $i = 1$  or  $2$ ). The game continues as  $G(\varphi_i)$ .

R  $\exists$  - existential quantifier rule:  $G(\exists x Sx)$  starts by the choice of a player in the role of verifier of one member from the domain for  $x$ . If the individual is  $a$ , the game is played as  $G(Sa)$ .

R  $\forall$  - universal quantifier rule:  $G(\forall x Sx)$  starts by the choice of a player in the role of falsifier of one member from the domain for  $x$ . If the individual is  $a$ , the game is played as a  $G(Sa)$ .

R  $\sim$  - negation rule:  $G(\sim\varphi)$  is played the same as  $G(\varphi)$  except that the players change their roles.

R - atomic sentences rule: if  $A$  is an atomic sentence that is true, the verifier wins. If the sentence is false the falsifier wins.

Each application of the rules eliminates one logical constant, so in a finite number of steps we come to the rule for atomic sentences. Truth of an atomic sentence is determined in the model  $M$  with respect to which  $G(\varphi)$  is played. That is allowed by interpretation of all non - logical constant terms in the model. The interpretation is an integral part of the model  $M$  and it provides the meaning to the primitive symbols of a given interpreted first - order language.

In Hintikka's presentation (Hintikka, 1996a) he is speaking directly of Verifier and Falsifier. Once again, following (Dechesne, 2005) we want to give a more general presentation, so we introduced Eloise and Abelard as players in a game and we assigned to them the roles of Verifier and Falsifier. As Dechesne points out, this presentation puts to the fore the importance of the negation rule as a switch of roles of players in a semantic game. Namely, Hintikka initially takes sentences to be in a



negation normal form, so the change of roles of the players does not really happen. As the negation sign appears only in front of atomic expressions, for negated expressions there is no play - their status is determined directly in the model<sup>17</sup>. In a more general presentation, the rule of negation is shown in all its importance.

Now, we can state the truth and falsity conditions for an IF sentence.

*-Definition 13:* a) Sentence  $\varphi$  is true in model  $\mathbb{M}$  ( $\mathbb{M} \models \varphi$ ) if and only if there is a winning strategy for Eloise in the game  $G(\varphi)$  played in  $\mathbb{M}$ .

b) Sentence  $\varphi$  is false in model  $\mathbb{M}$  ( $\mathbb{M} \not\models \varphi$ ) if and only if there is a winning strategy for Abelard in the game  $G(\varphi)$  played in  $\mathbb{M}$ .

One more thing should be noted. The players in a game are idealised agents who are always capable of finding a winning strategy if there is one, so the game does not depend of the intellectual capacity of players. What counts is the mere existence of certain function in the model yielding a victory for a player.

#### ***1.4. Game – theory framework***

We have presented game theoretic semantics in the way Hintikka did it himself<sup>18</sup>. More precise formalisation is given in (Mann, Sandu, Sevenster, 2011) and (Dechesne, 2005). We will rely on those two works in the following section. The semantic games at stake are classified in the literature as *extensive, non-cooperative, zero-sum games with imperfect information*.

---

<sup>17</sup> Also recall the *definition 3\** c: if  $\varphi$  is an atomic formula the  $\neg\varphi$  is also considered to be an atomic formula.

<sup>18</sup> We have introduced a few changes compared to Hintikka's presentation with remarks about them.

### *1.4.1. Strategic and extensive games*

The semantic games are classified as *extensive* in opposition with *strategic* games. (Mann, Sandu, Sevenster, 2011: 9) In a strategic game players decide on their movements at the beginning of the game and then the outcomes are revealed. To define the game one needs to specify: a list of players, a set of possible actions every player can adopt and players' preferences over possible outcomes. The players are choosing their strategies simultaneously, without knowing the strategies chosen by their opponents, therefore, those games essentially are the games with *imperfect information*. The famous example is the Prisoner dilemma.<sup>19</sup>

Extensive games, on the other hand, offer a dynamic approach. In those games players move one after another so they choose their actions at every step of game. The example of extensive games would be the game of chess.

The game is *non-cooperative* or *strictly competitive* in the sense that players don't have an interest to cooperate with each other. It is a zero - sum game because the sum of payoffs for players is constant and it is zero, so if one player wins the other one must lose.

This brings us to the important notion of information which is central for the understanding of IF logic. Semantic games for classical first - order logic are extensive games with perfect information. Semantic games for IF logic are extensive

---

<sup>19</sup> A prisoner dilemma is a story of two criminals arrested and investigated separately in two different cells. The police have enough evidence to convict them of a minor offence but not enough to convict them of a serious offence, unless some of them talks. Each of them has two options: to cooperate with a police or to plead innocent. If they both plead innocent and deny the crime, they both get one year in prison for a minor offence. If one cooperates with the police and the other doesn't, the one who cooperates goes free as a witness against the other and the one who doesn't cooperate gets four years in prison. If they both cooperate they both get three years. The optimal choice in this game turns out to be that they both cooperate.

games with imperfect information. Let us start off with games with perfect information.

#### ***1.4.2. Extensive games with perfect information***

*-Definition 14:* An extensive game with perfect information is defined by specification of the following elements:

-  $P$ , the set of players.

-  $H$ , the set of *plays* or *histories* that are finite sequences. Let  $(h_1 \dots h_i)$ , the elements of  $H$ , be the *initial sequence*. Then for a sequence  $(h_1 \dots h_n) \in H$ , and for a sequence  $(h_1 \dots h_m) \in H$ , such that  $i < m < n$ , we say that  $(h_1 \dots h_n)$  is an *extension* of  $(h_1 \dots h_m)$ .

*Terminal history* is the sequence that has no extension in  $H$ . The set of terminal histories we denote  $T$ . The set of non-terminal histories we denote  $N$ .

- *The player function*  $f_p$  that assigns a player from the set  $P$  to every non-terminal history in  $N$ ,  $f_p: N \rightarrow P$ . This function shows which player is making a move.

- The set of *actions*  $A = \{a_1 \dots a_n\}$ . The *action* is a step from the history  $h = (h_1 \dots h_n)$  to the successor history  $h * h = (h_1 \dots h_n, h)$ .

- *The utility function*,  $f_u$  that assigns a payoff to every player  $p$  in the game,  $f_u: T \rightarrow R^p$ , where  $R$  is a set of possible payoffs for the player  $p$  in the game.

Formally, the game is strictly competitive if for all  $h, h'$  from  $T$  the following holds:

$$f_{u1}(h) \geq f_{u1}(h') \text{ iff } f_{u2}(h') \geq f_{u2}(h).$$

The game is zero-sum if the sum of players payoffs is zero so there is  $c \in R$  such that

$$f_{u1}(h) + f_{u2}(h) = c \text{ and } c = 0.$$

### 1.4.3. Strategies

Let us now have a look at the definition of *strategy*. A strategy for a player is a description of that player's move in any possible state of a game. It is a *choice function* that suggests to the player what move he should choose whenever it is that player's turn to play.

-*Definition 15*: For the player  $p$  let the set  $H_p = f_p^{-1}(p)$  be the set of histories where it is  $p$ 's turn to play. A strategy  $\sigma$  is defined as:  $\sigma \in \prod_{(h \in H_p)} A(h)$ .

We will yet use the following notation:

-*Definition 16*: The set  $H_\sigma$  denotes the set of histories where the particular strategy  $\sigma$  is followed.

The set  $T_\sigma = H_\sigma \cap T$  denotes the set of terminal histories where the particular strategy  $\sigma$  is followed.

The set  $T_p = f_u^{-1}(p)$  denotes the set of terminal histories won by the player  $p$ .

The strategy  $\sigma$  for the player I is the *winning strategy* if, following  $\sigma$ , the player I wins in every terminal history, regardless of the moves player II makes.

The game  $G$  is *determined* if one of the players has the winning strategy in  $G$ . From the *Gale-Stewart theorem* it follows that every two - players, win - lose game with perfect information and finite depth is determined (Gale, Stewart, 1953).

Semantic games for classical first - order formulas are extensive, two-players, win - lose games with perfect information. They are determined in the sense that one of the players always has a winning strategy if the game is of finite length, which of course must be the case because the law of the excluded middle holds for classical first-order logic. Semantically, for classical first - order formulas the game for two players is transparent. They have perfect information about the course of the game

and they can distinguish between every action they can undertake for a given history. The slash sign in IF logic brings a limitation in information available to a player in the game, so the games for IF logic are *extensive games with imperfect information* to which we turn now.

#### ***1.4.4. Extensive games with imperfect information***

-*Definition 17:* An extensive game with imperfect information is defined in the same way as an extensive game with perfect information (*definition 13*) with one additional element:

- The equivalence relation  $\approx_p$  on the set of histories  $H$ . When there is such a relation between two histories  $h$  and  $h'$  the actions for those two histories are indistinguishable for player  $p$ . We denote it  $h \approx_p h'$ . In that case, the player cannot distinguish between two actions available to him, so  $A(h) = A(h')$ .

The player I is actually making a move without knowing the move player II made, so player I is playing regardless of the move of his opponent.

The concept of strategy in extensive games with imperfect information is the same as the one previously defined, except when there is a limitation on information available for player  $p$ : then two histories are indistinguishable to him and so are the strategies concerning those two histories,  $\sigma(h) = \sigma(h')$ .

#### ***1.4.5. Semantic games for IF logic***

Let us specify a semantic game for IF logic in a given framework.

-*Definition 18:* If  $\varphi$  is an IF formula,  $\mathbb{M}$  a suitable model and  $v$  a valuation on a domain that contains  $\text{Free}(\varphi)$ , the game is defined with a specification of following elements:

- $P = \{\forall, \exists\}$ , the set of players, consisting of two players, Abelard and Eloise.

- $H$ , the set of *plays* or *histories* that are finite sequences which show possible moves in a game. We allow the game to have only one *initial history*. Starting from the initial history, the players can either be in a position where the game - rules suggest to one of them to make a move (*a position of decision*) either they can be in a position where no move is available to any player so the game comes to its end and the payoffs are revealed (*terminal history*, the sequence that has no extension in  $H$ ).  $T$  denotes the set of terminal histories whereas  $N$  denotes the set of non-terminal histories.

In a position of decision there are two kinds of moves available to a player. If he arrives to a quantifier he chooses an element of a domain. If he arrives to a connective he chooses between subformulas in question. The order and the kind of a move at stake are determined by the syntax, starting with the entire sentence and reaching its component parts. We will define the set of histories  $H$  recursively, by defining a set of finite histories  $H_\psi$  for every  $\psi \in \text{Sub}(\varphi)$ :

$$H_\varphi = \{v, \varphi\};$$

If  $\psi$  is atomic the induction stops;

$$\text{If } \psi = \psi_1 \vee \psi_2 \text{ then } H_{\psi_i} = \{h^* \psi_i : h \in H_{\psi_1 \vee \psi_2}\};$$

$$\text{If } \psi = \psi_1 \wedge \psi_2 \text{ then } H_{\psi_i} = \{h^* \psi_i : h \in H_{\psi_1 \wedge \psi_2}\};$$

$$\text{If } \psi = (\exists x / \forall y_1 \dots \forall y_n) \psi_1 \text{ then } H_{\psi_1} = \{h^*(x, a) : a \in M, h \in H_{(\exists x / \forall y_1 \dots \forall y_n)}\};$$

$$\text{If } \psi = (\forall x / \exists y_1 \dots \exists y_n) \psi_1 \text{ then } H_{\psi_1} = \{h^*(x, a) : a \in M, h \in H_{(\forall x / \exists y_1 \dots \exists y_n)}\};$$

$$\text{then } H = \bigcup_{\psi \in \text{Sub}(\varphi)} H_\psi.$$

Every  $h \in H$  defines a valuation  $v_h$  for the expression  $\psi_h$ .

$v_{h'} = v$  for  $h' = (v, \varphi)$ ;

$v_{h'} = v_h$  for  $h' = h^* \psi$ ;

$v_{h'} = v_h(x/a)$  for  $h' = h^*(x, a)$ .

- *The player function*  $f_p$  assigns a player from the set  $P$  to every non-terminal history in  $N$ ,  $f_p: N \rightarrow P$ . The function shows which player is making a move. If a history at a position of decision is a subformula with a disjunction or existential quantifier then the player function assigns Eloise to make a move. If a subformula is a conjunction or universal quantifier then it is Abelard who is playing.

$f_p(h) = \exists$  if  $h = \psi_1 \vee \psi_2$  or  $h = (\exists x / \forall y_1 \dots \forall y_n) \psi_1$ ;

$f_p(h) = \forall$  if  $h = \psi_1 \wedge \psi_2$  or  $h = (\forall x / \exists y_1 \dots \exists y_n) \psi_1$ .

- *The set of actions*  $A = \{a_1 \dots a_n\}$ . The action  $a$  is a step from the history  $h = (h_1 \dots h_n)$  to the successor history  $h^*h = (h_1 \dots h_n, h)$ .

- *The utility function*,  $f_u$  which assigns a payoff to a player  $p$  in the game,  $f_u: T \rightarrow R^p$ , where  $R$  is a set of possible payoffs for the player  $p$  in the game. When atomic formulas are reached the game ends and the outcomes are revealed. If the atomic expression  $\psi_h$  reached at the end is satisfied in the model under given valuation then Eloise wins, otherwise Abelard is the winner.

$f_u(h) = \exists$  if  $\psi_h$  is satisfied in  $\mathbb{M}$  under  $v_h$ ;

$f_u(h) = \forall$  if  $\psi_h$  is not satisfied in  $\mathbb{M}$  under  $v_h$ .

- *The equivalence relation*  $\approx_p$  on the set of histories  $H$ , which provides the meaning to the slash sign. Such a relation means that actions for those two histories are indistinguishable for the player  $p$ , or  $h \approx_p h'$ , so  $A(h) = A(h')$ .

Informally, when a player is to make a move for a subformula with the slash sign he must do it without knowing the values of the variables under the slash sign that has already been chosen.

If  $\psi$  is not atomic and  $\psi \in \text{Sub}(\varphi)$  then let  $Y$  denote a set of variables occurring under the slash sign in  $\psi$  (that set can be empty). Let  $V$  be the set of valuations for  $\psi$ . Then two valuations  $v$  and  $v'$  in  $V$  are equivalent if they assigns the same values for the variables in  $\psi$  except for the variables in  $Y$ , or

$$v \approx v' \text{ iff } v(x) = v'(x) \text{ for all } x \in \text{Free}(\varphi) - Y$$

The equivalence relation between two valuations produces the equivalence relation on the set of histories in  $\psi$ ,  $H_\psi$ .

$$\text{For } h, h' \in H_\psi: h \approx_p h' \text{ iff } v_h \approx v_{h'}.$$

It should be noted that the restriction in information available to a player in a semantic game doesn't forbid the player to perform any action. It rather concerns possible strategies available to him. Therefore, there is no specific game rule for the slash sign.

*Definition 119:* For the player  $p$  let the set  $H_p = f_p^{-1}(p)$  be the set of histories where it is  $p$ 's turn to play. The strategy  $\sigma$  is defined as  $\sigma \in \prod_{(h \in H_p)} A(h)$ .

*-Definition 20:* The set  $H_\sigma$  denotes the set of histories where the particular strategy  $\sigma$  is followed.

The set  $T_\sigma = H_\sigma \cap T$  denotes the set of terminal histories where the particular strategy  $\sigma$  is followed.

The set  $T_p = f_u^{-1}(p)$  denotes the set of terminal histories won by  $p$ .

We have already defined the satisfaction of IF formulas in a model. To complete our presentation we will write it again.

*-Definition 13:* a) An IF formula  $\varphi$  is true in the model  $\mathbb{M}$  ( $\mathbb{M} \models_t \varphi$ ) if and only if there is a winning strategy for Eloise in the game  $G(\varphi)$  played in the model  $\mathbb{M}$ .

b) Formula  $\varphi$  is false in the model  $\mathbb{M}$  ( $\mathbb{M} \models_f \varphi$ ) if and only if there is a winning strategy for Abelard in the game  $G(\varphi)$  played in the model  $\mathbb{M}$ .



We will present some examples of games with perfect and with imperfect information. First let us observe an expression without slash.

*-Example 1:* Let  $\varphi$  be  $\forall x \exists y (x=y)$  and the model  $\mathbb{M} = \{0,1\}$ .

The initial history is  $h = (\emptyset, \varphi)$ . As the formula starts with the universal quantifier the player function attribute to Abelard to make the first move and to choose the value for  $x$ . He has two possibilities,  $h_1 = (\emptyset, \varphi, (x,0))$  and  $h_1' = (\emptyset, \varphi, (x,1))$ . In the continue, there is the existential quantifier so it is Eloise who plays and chooses the value for  $y$ . As there is no slash sign, Eloise has the information about the value Abelard has chosen for  $x$  and all the histories are distinguishable for her. Her strategy is simple - to choose the same value for  $y$  as Abelard has chosen for  $x$ .

$\sigma(h_1) = (y, 0)$ ;  $\sigma(h_1') = (y, 1)$ .

Therefore, we have two terminal histories:  $h_2 = (\emptyset, \varphi, (x,0), (y, 0))$  and  $h_2' = (\emptyset, \varphi, (x,1), (y, 1))$ . In both of them Eloise has a wining strategy, so we conclude that  $\varphi$  is satisfied in  $\mathbb{M}$ .

*-Example 2:* Let  $\varphi$  be  $\forall x (\exists y / \forall x) (x=y)$  and the model  $\mathbb{M} = \{0,1\}$ .

The initial history is the same as before:  $h = (\emptyset, \varphi)$ . We start with the universal quantifier, so first it is Abelard's turn to play. Again, he has two possibilities:  $h_1 = (\emptyset, \varphi, (x,0))$  and  $h_1' = (\emptyset, \varphi, (x,1))$ . Then it is Eloise who is in the position of decision but this time the existential quantifier is independent of the universal quantifier and she has to make her choice for  $y$  without knowing the value Abelard has chosen for  $x$ . Two histories  $h_1$  and  $h_1'$  are indistinguishable to her:  $h_1 \approx_{\exists} h_1'$ . Her strategy  $\sigma$  cannot take as argument the value for  $x$ , so it must be a constant function.  $\sigma: (h_1) = \sigma(h_1') = (y, m)$ ;  $m = \{0,1\}$

Terminal histories are either

$h_2 = (\emptyset, \varphi, (x,0), (y, 0))$  and  $h_2' = (\emptyset, \varphi, (x,1), (y, 0))$  or

$h_2^* = (\emptyset, \varphi, (x,0), (y,1))$  and  $h_2'^* = (\emptyset, \varphi, (x,1), (y, 1))$ .

In either case Eloise does not win both terminal histories and consequently she does not have a winning strategy. Therefore,  $\varphi$  is not satisfied in the model. But it is clear immediately that Abelard does not have a winning strategy either - he cannot win both terminal histories. The law of the excluded middle doesn't work in IF logic. We will make some more remarks about it later. The Gale - Stewart theorem doesn't hold for the extensive games with imperfect information.

## ***1.5. Properties of IF logic***

We proceed with discussing some important properties of IF logic.

### ***1.5.1. Notion of strategy***

As presented previously, the GTS truth conditions for a sentence are defined in terms of strategies for two players who are trying to verify or falsify the sentence. The truth is, in Wittgenstein's manner, defined through the practice of verification and falsification within the given rules.

The strategie is expressed by the finite set of choice - functions or *Skolem functions* whose values say what individuals Verifier has to choose in his actions related to an existential quantifier and disjunction in order to win the game. The same holds for the Falsifier's strategy related to a universal quantifier and conjunction. Those functions belong to the existential part of second - order language that we

denote  $\Sigma_1^1$ .<sup>20</sup> We will show later that the expressive power of IF logic is exactly that of the  $\Sigma_1^1$  second - order fragment. The truth conditions for a sentence  $\varphi$  can be expressed in the  $\Sigma_1^1$  fragment. Such a translation is always feasible. Here is an example:

$$1) \forall x \exists y S(x,y)$$

The sentence 1) is true if there is a winning strategy for Verifier. Verifier's strategy shows him how to select  $y$  in function of the value of  $x$  ( $f(x) = y$ ) in order to win the game. To express the existence of such a strategy, we write the following:

$$2) \exists f \forall x S(x, f(x))$$

### ***1.5.2. Axiom of choice***

If we link 1) and 2) with a conditional we obtain the following formulation of the *axiom of choice*.

$$3) \forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x, f(x))$$

From the GTS point of view, the truth of 3) - which expresses one version of the axiom of choice - is in fact derived from the very definition of truth. In fact, as we will discuss further on, it is related to the truth of the universal.

---

<sup>20</sup> The existential second - order logic is the fragment of second order logic that consists of a formula in the form  $\exists x_1 \dots \exists x_n \Psi$ , where  $\exists x_1 \dots \exists x_n$  are second - order quantifiers and  $\Psi$  is a first-order formula.

Let us point out that the axiom of choice is essential for GTS, since without it the Tarski – style semantic and GTS for classical first - order logic would not be equivalent. The reason is that strategies in GTS are understood as *deterministic* strategies that impose to Verifier and Falsifier their choices, leaving no real options. Wilfried Hodges sees it as a weakness in Hintikka's approach and states that a more natural way to conceptualise GTS would be to use *non-deterministic* strategies so that no a priori presumption of the axiom of choice is needed (Hodges, 2013). However, Hintikka insists that there is nothing troubling with the axiom of choice and that it actually constitutes our conception of truth.

*This paradigm problem concerns the status of the axiom of choice. This axiom was firmly rejected by Brouwer and it was mooted in the controversies between the French intuitionists and their opponents....The axiom of choice is true. The idea of “choosing” or “finding” suitable individuals is systematised in what is known as game-theoretical semantics. For mathematicians, this semantics is no novelty, however, but little more than a regimentation and generalisation of the way of thinking that underlies mathematicians’ classical (or perhaps I should say Weierstrassian) epsilon-delta analyses of the basic concepts of calculus, such as continuity and differentiation...*

*...To return to the usual axiom of choice, it is thus seen to be unproblematically true. How can any intuitionist deny the axiom of choice...? What can possibly go wrong here? Moreover, evoking the concept of knowledge, either in the form of epistemic logic or informally, does not seem to help an intuitionistic critic of the axiom of choice at all, either. The discussion of the axiom of choice between intuitionists and classicists has*

*conducted at cross-purposes. It can only be dissolved by making distinction between knowing that and knowing what that neither party has made explicit.* (Hintikka, 2001)

*Zermelo did not begin to axiomatize set theory unselfishly from the goodness of his theoretical heart. His main purpose was to justify his well-ordering theorem. In practice, this largely meant to justify the axiom of choice. [ ... ]. But that is not the full story. Worse still: Zermelo's specific enterprise was unnecessary, in that the so-called axiom of choice turns out to be in the bottom a plain first-order logical principle.* (Hintikka, 2011)

Hintikka is arguing that GTS justifies the axiom of choice in a way perfectly acceptable for constructivists. Moreover, as per the last sentence in the previous quotation, the axiom of choice should be considered a first - order principle. In fact, as we will show in chapter **III**, the latter is true: the axiom of choice involves first - order logic. As for the former, we will argue later in this thesis that the axiom is indeed acceptable for constructivists only if we presuppose intensionality of the choice function. However, Hintikka would like to have a logic that renders classical mathematics. This is only possible if we assume extensionality. But to have both extensionality and the axiom of choice – without assuming unicity of the function – seems not to be possible.

We will address this issue and discuss it in more details later in this thesis. We will compare Per Martin- Löf's analysis of the axiom of choice with Jaakko Hintikka's standing on this axiom. We will show that dependence and independence features that motivate IF-Logic can be formulated within the frame of Constructive Type Theory

(CTT) without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference.

This point can be seen in the most subtle manner in the framework of dialogical logic. Moreover, we will point out that recent developments in dialogical logic show that the CTT approach to meaning in general and to the axiom of choice in particular is very natural to the game theoretical approaches where (standard) metalogical features are explicitly displayed at the object language level. In some way, this vindicates, although in a different manner, Hintikka's plea for the fruitfulness of game - theoretical semantics in the context of the foundation of mathematics.

### ***1.5.3. Skolemisation procedure***

We present in this section the skolemisation procedure for IF logic. We start off with a classical first - order formula  $\varphi$  in the negation normal form. The Skolem form of  $\varphi$  is obtained by the following transformations: If  $\exists y(\psi) \in \text{Sub}(\varphi)$  and it lies within the scope of a string of universal quantifiers  $\forall x_1 \dots \forall x_n$ , we take a n-ary function symbol  $f$ , we erase the existential quantifier  $\exists y$  and we replace all the occurrences of  $y$  in  $\psi$  with  $f(x_1 \dots x_n)$ . We do the same with all the existential quantifiers in  $\varphi$  and we obtain the first - order formula  $\varphi'$  in which all the functional symbols  $f_1 \dots f_n$  occur with the right arguments and there is no existential quantifier. In front of  $\varphi'$  we add the string  $\exists f_1 \dots \exists f_n$ . In this manner we obtain a  $\Sigma^1_1$  formula in the form  $\exists f_1 \dots \exists f_n(\varphi')$ . Below is the example for the first - order formula we had before:

*-Example:*  $\forall x \exists y S(x, y) \rightarrow \exists f \forall x S(x, f(x))$

The Skolem form of an IF sentence we obtain in the same manner with some changes when there is a slashed existential quantifier of the form  $(\exists y/\forall x_1 \dots \forall x_n)$ . Then we delete the existential quantifier, but this time the variables  $(x_1 \dots x_n)$  are not the arguments of a new introduced function because the existential quantifier is independent of the universal quantifiers under the slash sign.

*-Example:*  $\forall x_1 \forall x_2 (\exists y/\forall x_1) S(x, y) \rightarrow \exists f \forall x_1 \forall x_2 S(x, f(x_2))$

In addition, we have to provide the Skolem form of disjunction, because a disjunction is also a decision position for Eloise. Let  $\varphi$  be a classical first - order formula in the negation normal form, where disjunction  $\psi_1 \vee \psi_2 \in \text{Sub}(\varphi)$  occurs in the range of a string of universal quantifiers of the form  $\forall x_1 \dots \forall x_n$ . Then we choose a new n-ary functional symbol  $f$  with  $(x_1 \dots x_n)$  as arguments and with  $\{0, 1\}$  as its range. That implies that there are at least two elements in a model which are the interpretations of the constants 0 and 1. Then we replace  $\psi_1 \vee \psi_2$  with

$$(f(x_1 \dots x_n) = 0 \wedge \psi_1) \vee (f(x_1 \dots x_n) = 1 \wedge \psi_2).$$

The first disjunct corresponds to Eloise's choice of the left disjunct and the second to her choice of the right disjunct in the original formula. We do the same with every disjunction in the formula, thereby obtaining a new formula  $\varphi'$ . As previously explained, in front of  $\varphi'$  we add the string  $\exists f_1 \dots \exists f_n$  to obtain a  $\Sigma^1_1$  formula in the form  $\exists f_1 \dots \exists f_n (\varphi')$ .

*-Example:*  $\forall x_1 \forall x_2 (P(x_1, x_2) \vee R(x_1, x_2)) \rightarrow \exists f \forall x_1 \forall x_2 (f(x_1, x_2) = 0 \wedge P(x_1, x_2)) \vee (f(x_1, x_2) = 1 \wedge R(x_1, x_2))$ .

If  $\varphi$  is an IF formula where a disjunction is independent of a string of universal quantifiers in the form  $\forall x_1 \dots \forall x_n$ , the procedure is the same only the function symbols don't take as arguments variables  $x_1 \dots x_n$ , occurring under the slash sign. Here is the example:

*-Example:*  $\forall x_1 \forall x_2 (P(x_1, x_2) \vee \forall x_1 R(x_1, x_2)) \rightarrow \exists f \forall x_1 \forall x_2 (f(x_2) = 0 \wedge P(x_1, x_2)) \vee (f(x_2) = 1 \wedge R(x_1, x_2))$ .

Let us make a few remarks. We can see now how the skolemisation procedure yields the truth conditions for an IF sentence and why the axiom of choice is important in GTS. By GTS rules, existential quantifiers and disjunctions are decision positions for Eloise (i.e. the initial verifier) in a game for a given sentence. If she has a winning strategy in the game the sentence is true. When we transform the IF sentence in its Skolem form, the existential quantifier and disjunction disappear and instead, we explicitly see the choice functions that are guiding every Eloise's move. The arguments of those functions show previous Abelard's moves that Eloise's choices depend on. Here is the formal definition of Skolem form of an IF sentence.

*-Definition 14:* If  $\varphi$  is an IF sentence of vocabulary  $L$ , and  $\mathbb{M}$  is a model with at least two elements which provide interpretations of constant symbols 0 and 1 that are not in  $L$ , the Skolem form of  $\varphi$ ,  $Sk(\varphi)$  is obtained with the following rules:

$$sk(\varphi) [R(t_1, \dots, t_n)] = R(t_1, \dots, t_n).$$

$$sk(\varphi) [\sim R(t_1, \dots, t_n)] = \sim R(t_1, \dots, t_n).$$

$$sk(\varphi) [\psi_1 (\vee / \forall x_1 \dots \forall x_n) \psi_2] = (Sk(\varphi) [\psi_1] \wedge f(y_1 \dots y_n) = 0) \vee (Sk(\varphi) [\psi_2] \wedge$$



$f(y_1 \dots y_n) = 1$ ); where  $f$  is a new function symbol and the variables  $y_1 \dots y_n$  are bound by a string of universal quantifiers  $\forall y_1 \dots \forall x_n$  preceding  $\psi_1$  ( $\vee / \forall x_1 \dots \forall x_n$ )  $\psi_2$  in  $\varphi$ , but not among  $x_1 \dots x_n$ .

$$\text{sk}(\varphi) [\psi_1 (\wedge \exists x_1 \dots \exists x_n) \psi_2] = \text{sk}(\varphi) [\psi_1] \wedge \text{sk}(\varphi) [\psi_2].$$

$\text{sk}(\varphi) [(\exists y / \forall x_1 \dots \forall x_n) \psi] = \text{sk}(\varphi) [\psi]$  [ $y$  replaced with  $f(x_1' \dots x_n')$ ] where  $f$  is a new function symbol and the variables  $x_1' \dots x_n'$  are bound by a string of universal quantifiers  $\forall x_1' \dots \forall x_n'$  preceding  $(\exists y / \forall x_1 \dots \forall x_n) \psi$  in  $\varphi$ , but not among  $x_1 \dots x_n$ .

$$\text{sk}(\varphi) [(\forall y / \exists x_1 \dots \exists x_n) \psi] = \forall y \text{sk}(\varphi) [\psi].$$

As previously mentioned, in Hintikka's presentation of syntax for IF logic only existential quantifiers and disjunctions can appear slashed. Hintikka is putting emphasis on truth conditions that concern Eloise's moves in a game. Indeed, every IF sentence with slashed universal quantifiers and conjunctions can be translated into the sentence without it. However, we will follow authors<sup>21</sup> who think that falsity conditions for IF sentences are equally important. Therefore, we have presented a larger syntax where slashed universal quantifiers and slashed conjunctions are also allowed. It is justified by the fact that the law of the excluded middle doesn't hold for IF logic.

#### ***1.5.4. Kreisel counterexamples***

Winning strategies for Eloise are captured by the existence of Skolem functions. In the same manner, winning strategies for Abelard are captured by the existence of Kreisel counterexamples. A sentence in the Kreisel form provides the falsity conditions for the corresponding IF sentence (see Mann, Sandu,

---

<sup>21</sup> (Mann, Sandu, Sevenster, 2011) and (Dechesne, 2005)

Sevenster,2011: 75). The procedure of transforming an IF sentence  $\varphi$  into the Kreisel form is called *kreiselization* and it is actually the skolemisation of a dual sentence  $\sim \varphi$ .

-*Theorem*:  $\mathbb{M} \models_{\Gamma} \varphi$  iff  $\mathbb{M} \models_{\Gamma} \sim \varphi$

The proof follows immediately from the rule of negation as an exchange of roles for players in GTS.

The Kreisel form of IF sentences is defined in the following manner:

-*Definition 15*: If  $\varphi$  is an IF sentence of vocabulary  $L$ , and  $\mathbb{M}$  is a model with at least two elements that serve as interpretations of constant symbols 0 and 1 that are not in  $L$ , the Kreisel form of  $\varphi$  ( $kr(\varphi)$ ) is obtained with the following rules:

$$kr(\varphi) [R(t_1, \dots, t_n)] = \sim R(t_1, \dots, t_n).$$

$$kr(\varphi) [\psi_1 \vee \psi_2] = kr(\varphi) [\psi_1] \wedge kr(\varphi) [\psi_2]$$

$$kr(\varphi) [\psi_1 \wedge \psi_2] = kr(\varphi) [\psi_1] \vee kr(\varphi) [\psi_2]$$

$$kr(\varphi) [(\exists y/\forall x_1 \dots \forall x_n)\psi] = \forall y kr(\varphi) [\psi]$$

$$kr(\varphi) [(\forall y/\exists x_1 \dots \exists x_n)\psi] = kr(\varphi) [\psi] [y \text{ replaced with } f(x_1' \dots x_n')] \text{ where } f \text{ is a new function symbol and variables } x_1' \dots x_n' \text{ are bound by a string of existential quantifiers } \exists x_1' \dots \exists x_n' \text{ preceding } (\forall y/\exists x_1 \dots \exists x_n)\psi \text{ in } \varphi, \text{ but not among } x_1 \dots x_n.$$

### ***1.5.5. The law of the excluded middle***

The failure of law of the excluded middle in IF logic is due to possibility of imperfect information in a game. We remarked before that the Gale - Stewart theorem does not hold for the extensive games with imperfect information, so those games are not always determined. We have already presented an example of IF formula which is

neither true nor false ( $\varphi = \forall x(\exists y/\forall x) (x=y)$  and the model  $\mathbb{M} = \{0,1\}$ ). It is interesting to reflect on the nature of non - determinacy. Are we talking about third truth value?

Hintikka does not think that there is third truth value involved. According to him, the non - dterminacy indicates that not all types of choice functions exist in a given model; it is thus a structural characteristic of model. The argument is provided in (Tulenheimo, 2009, note 27.):

*No third truth-value is stipulated in the semantics of IFL. Truth-value gaps may arise when neither truth nor falsity can be meaningfully ascribed to a sentence (because a presupposition of the sentence is not satisfied). This is not what non-determinacy means. Non-determinacy is a model-relative complex negative property, which can be ascribed to a sentence. A sentence has this property iff neither of the simple positive properties of truth and falsity can be correctly ascribed to the sentence. An ascription of non-determinacy is correct or incorrect depending exclusively on the model relative to which the ascription is effected.*

However, some other authors speak about third truth - value (Pantasar, 2009, Mann, Sandu, Sevenster, 2011).

We can compare the failure of law of the excluded middle in IF logic with the case of intuitionistic logic. In intuitionism the reason for the failure of law of the excluded middle is rather epistemological. It concerns the ability to construct a proof for a given formula. The case is different in IF logic. Following Hintikka, the non – existence of a certain function is a structural property of the model in question, so we are rather on ontological level. In Hintikka’s own words from (Hintikka, 2002a: 589):

*Secondly, the failure of the excluded middle is purely structural, combinatorial matter. Whether or not  $(S \vee \sim S)$  is true depends on whether there exists a winning strategy for either player in  $G(S)$ . And this depends only on the structure of the model (“possible world”) on which  $G(S)$  is played, i.e. has nothing to do with the limitations of human knowledge or the knowledge of any particular human being.*

This might be a reason to speak about third truth value in IF logic after all.

Considering the failure of law of the excluded middle, Hodges makes a remark in (Hodges, 2013) that the rule for dual negation, as a switch of roles of players in a game, is no longer justified in IF logic. According to this view, the rule for dual negation makes sense only in classical first - order logic where the law holds. Hintikka’s brief direct response in (Hintikka, 2006a) was that no justification is needed because the rule for dual negation captures the very intuitive meaning of negation in a game theory. The justification of dual negation can be found in (Hintikka, 2002a): that IF logic is an extension of classical first - order logic with a number of classical properties – De Morgan’s laws, interdefinability of quantifiers, double negation law, distribution laws and the metatheorems such as the compactness, the Löveinheim- Skolem theorem, the separation theorem and so on. A classical first - order sentence is understood as a special case of an IF sentence – the one where the set of variables under the slash sign is empty. In Hintikka’s own words:

*IF logic is thus reached not by changing anything in the classical (read: old-fashioned) first - order logic, but by liberalising it in the spirit of the original enterprise. (Hintikka, 2002a: 589)*

From this point of view, all the game rules for classical first – order logic must remain the same in IF logic. According to Hintikka, IF logic shouldn't be consider non - classical but rather *hyperclassical* logic (Hintikka, 2002a).

It should be noted that dual negation in front of atomic formulas behaves in the same way as the classical negation – atomic expressions are either true or false. There is a certain gap in the treatment of atomic and that of complex formulas on this point, due to the introduction of imperfect information in a game. Truth and falsity conditions for IF sentences can be formulated in the  $\Sigma^1_1$  second-order fragment and this fragment is not closed under classical negation.

On this point we can underline another difference between IF and the intuitionistic logic when it comes to the failure of law of the excluded middle. In intuitionistic logic atomic formulas might also be undetermined. In IF logic atomic sentences are always determined in the model, in the classical way. The reason is that in front of atomic sentences dual negation behaves as classical, contradictory negation.

One more difference is that, unlike in the intuitionistic logic, in IF logic the law of double negation holds ( $\sim\sim A \rightarrow A$ ), as Dechesne points out in (Dechesne, 2005). The double negation means that the players are changing their roles in a game twice in a row, so the game for  $\sim\sim A$  is identical with the game for  $A$ .

### 1.5.6. Expressive power of IF logic

In some cases, the slash sign does not contribute to anything that could not be expressed without it, but sometimes it allows expressing the structures that would not be expressible in the first - order logic. The classic example is the expression with Henkin's branching quantifiers. It can be represented as:

$$\left. \begin{array}{l} \forall x \exists y \\ \\ \forall z \exists u \end{array} \right\} S(x, y, z, u)$$

But there is no linear way to express this formula in classical first-order logic. The only mean of expressing dependences between quantifiers in classical first – order logic is the nesting of quantifiers' scopes - and those scopes cannot be arranged in such a way that they overlap only partially. The Skolem form of the previous sentence is:

$$\exists f \exists g \forall x \forall z S(x, f(x), z, g(z))$$

and it has a translation in IF logic in the following manner:

$$\forall x \forall z (\exists y / \forall z) (\exists u / \forall x) S(x, y, z, u).$$

In his study of branching quantifiers, Walkoe showed that the expressive power of formulas with branching quantifiers is precisely that of the existential

second - order logic (Walkoe, 1970). Independently, Walkoe and Enderton also showed that every existential second - order sentence  $\Sigma^1_1$  is equivalent to the second - order truth or falsity condition of an IF sentence (Walkoe, 1970, Enderton, 1970). Therefore, IF logic captures exactly the expressive power of Henkin's branching quantifiers.<sup>22</sup>

We have already shown that every IF sentence can be translated as an  $\Sigma^1_1$  sentence through the procedure of skolemisation. Now we should describe the reverse process of translation of  $\Sigma^1_1$  sentences into IF sentences.<sup>23</sup>

Let  $\Phi$  be an  $\Sigma^1_1$  formula. We rewrite  $\Phi$  in the form  $\exists f_1 \dots \exists f_k \forall x_1 \dots \forall x_n (\varphi)$ , where  $\varphi$  is a quantifier - free classical first - order sentence and the function symbols  $f_1 \dots f_k$  and variables  $x_1 \dots x_n$  occur in such a manner that functional symbols are not nested and that every function symbol has only one string of arguments. Let us denote the new formula  $\Phi'$ .  $\Phi'$  is translated in IF sentence with the following steps: we remove the string of quantified second - order function symbols  $\exists f_1 \dots \exists f_k$  and every  $\exists f_i$  we replace with a slashed first - order quantifier  $(\exists y_i / \forall x_1 \dots \forall x_m)$  in front of  $\varphi$ , where  $x_1 \dots x_m$  are the variables from the set  $\{x_1 \dots x_n\}$  that don't occur as arguments of function symbol  $f_i$ . Then in  $\varphi$  we replace every functional symbol  $f_i$  and its arguments with  $y_i$ . In that manner we obtain an IF sentence.

We have yet to show how the sentence  $\Phi'$  is obtained from the sentence  $\Phi$ . Let  $L$  be the language that contains the equality sign  $=$ , and let  $\mathbb{M}$  be the model that has at least two elements which are the interpretations of the constant symbols 0 and 1. First we need a sentence in the form  $\exists f_1 \dots \exists f_k \forall x_1 \dots \forall x_n (\varphi)$ . So, if there is a predicate bound by an existential quantifier in  $\Phi$ , we present the predicate as its characteristic function.

<sup>22</sup> More about branching quantifiers can be found in (Blass, Gurevich, 1986).

<sup>23</sup> Hintikka explains the translation procedure in (Hintikka, 1996a: 62-63).

For example, if  $\Phi$  is  $\exists R \forall x (Rx)$ ,  $\Phi'$  will be  $\exists f \forall x (f(x) = 1)$ . Second, if  $\Phi$  is  $\exists f_1 \exists f_2 \forall x (\varphi(f_2(f_1(x))))$ , that is, if in  $\Phi$  two function symbols are nested, we introduce new variable to get  $\Phi'$  in the form:  $\exists f_1 \exists f_2 \forall x_1 \forall x_2 (x_2 = f_1(x_1) \rightarrow \varphi(f_2(x_2)))$ . And finally, if  $\Phi$  is in the form  $\exists f_1 \forall x_1 \forall x_2 (\varphi(f_1(x_1), f_1(x_2)))$ , that is, the same function symbol has two different arguments, we transform it to  $\Phi'$  by introducing a new function symbol:

$$\exists f_1 \exists f_2 \forall x_1 \forall x_2 (x_1 = x_2 \rightarrow f_1(x_1) = f_2(x_2) \wedge \varphi(f_1(x_1), f_1(x_2))).$$

Consequently, IF logic has a bigger expressive power than classical first - order logic. An example of a property of a domain  $\mathbb{M}$  that can be expressed in IF logic, but not in classical first - order, is the Dedekind - infinity.

*-Example:* The domain is Dedekind - infinite if there is an injective function from the domain to its proper subset. In the second - order logic we can express it with the following sentence:

$$\exists f_1 \exists f_2 \exists m \forall x_1 \forall x_2 ((f_1(x_1) = f_2(x_2) \leftrightarrow x_1 = x_2) \wedge f_1(x_1) \neq m).$$

This sentence expresses truth conditions of IF first - order formula:

$$\exists y \forall x_1 \forall x_2 (\exists y_1 / \forall x_2) (\exists y_2 / \forall x_1) (x_1 = x_2 \leftrightarrow y_1 = y_2 \wedge y \neq y_1).$$

Other properties that can be expressed in IF logic but not in classical first - order logic are: accountability of a structure, equicardinality of predicates of two first - order formulas, ill- foundedness of a linear ordering, topological notion of open set and so on (see Hintikka, 1996a). Hence, in Hintikka's opinion it is one of the biggest advantages of IF logic: we can express in it some important mathematical properties



but to stay at the first - order level. That makes IF logic the best candidate for the foundation of mathematics. Better still, IF logic has some important and desirable properties.

Beside the application in mathematics, Hintikka and his associates think that IF logic and GTS are more suitable for formalisation of natural languages. One of the Hintikka's favourite examples of language phenomena that can be treated in a satisfactory manner with GTS (and sometimes exclusively with IF logic) is anaphora. IF logic allows taking into account different patterns of dependency among logical expressions and therefore, it is more appropriate for the formalisation of natural languages. Moreover, all that can be done at the first - order level.

The following is the example of natural language sentence that involves branching quantifiers from (Hintikka, 1973: 344):

*Some relative of each villager and some relative of each townsman hate each other.*

The sentence is expressed by the existential second - order sentence:

$$\exists f \exists g \forall x \forall z ((\text{Villager}(x) \wedge \text{Townsmen}(z)) \rightarrow \text{Relative}(x, f(x)) \wedge \text{Relative}(z, g(z)) \wedge \text{Hate}(f(x), g(z))),$$

which has a translation in the first - order IF language, but otherwise cannot be expressed at the first - order level<sup>24</sup>. The GTS approach together with IF logic provide thus an interpretation of those complicated examples of anaphoric expressions.

---

<sup>24</sup> This claim was objected in (Gierasimeczuk, Szymanik, 2009). They tried to offer an alternative reading to Hintikka's examples involving branching quantifiers which makes them expressible at the first - order level in a linear representation, without using more syntactic tools than those from the

Later in this work we will compare the GTS approach to anaphora with the dialogical approach that makes use of CTT. We will present in more detail the GTS approach to anaphora and we will argue that this approach has considerable advantages over other existing approaches. However, we will show that in the dialogical approach we can give a natural interpretation of anaphoric expressions without paying a price of using IF logic that is neither axiomatizable nor has an underlying theory of inference; logic is about inference after all. This important example will also help to shed some light on the way the meaning is obtained in general.

We will proceed now with properties of IF logic.

### ***1.5.7. Double negation law***

*-Theorem:* If  $\varphi$  is an IF sentence then the following equivalence holds:

$$\varphi \equiv \sim\sim\varphi$$

*-Proof:* It follows directly from the GTS rule of negation. Double negation means that players are changing their roles twice in a row in a game, so the strategy for winning  $G(\sim\sim\varphi)$  is exactly the same as the strategy for winning  $G(\varphi)$ .

---

classical first – order logic. However interesting their analysis might be, it is not relevant for our further discussion of branching quantifiers.

### 1.5.8. De Morgan's laws

- *Theorem:* Let  $\psi$  be a subformula of an IF sentence  $\phi$ . Then in IF logic the following equivalences hold:

$$\text{a) } \sim(\psi_1 (\vee / \forall x_1 \dots \forall x_n) \psi_2) \equiv \sim\psi_1 (\wedge / \exists x_1 \dots \exists x_n) \sim\psi_2;$$

$$\text{b) } \sim(\psi_1 (\wedge / \exists x_1 \dots \exists x_n) \psi_2) \equiv \sim\psi_1 (\vee / \forall x_1 \dots \forall x_n) \sim\psi_2;$$

$$\text{c) } \sim(\exists x / \forall y_1 \dots \forall y_n) \psi \equiv (\forall x / \exists y_1 \dots \exists y_n) \sim\psi;$$

$$\text{d) } \sim(\forall x / \exists y_1 \dots \exists y_n) \psi \equiv (\exists x / \forall y_1 \dots \forall y_n) \sim\psi.$$

-*Proof:* It follows from the GTS rules of connectives, quantifiers and negation. In a) in the first part of the equivalence Abelard choose which disjunct he will challenge (as the subformula is the negation). In the equivalent subformula we have the same thing: by the rule of conjunction, it is Abelard's turn to choose between  $\sim\psi_1$  and  $\sim\psi_2$ .

Proofs for b), c) and d) are similar.

Moreover, sentences a), b), c) and d) are true in the sense that the strategies for Abelard and Eloise are exactly the same in the games for both parts of each equivalence.

Given the double negation and De Morgan's laws, we can easily note how every IF sentence can be brought up to the *negation normal form*. Those laws allow to push the negation sign until reaching its atomic expressions.

### 1.5.8. A remark on the equivalence

The preceding formulas are equivalent in the strongest possible way, meaning that two formulas are true in the same models and false in the same models. But the failure of law of the excluded middle yields another two weaker concepts of equivalence: when two formulas are true but not false in the same models and when two formulas are false but not true in the same models. The former case we denote  $\equiv_t$  and the later we denote  $\equiv_f$ . Hintikka actually understands equivalence as  $\equiv_t$  equivalence:

*Are two sentences logically equivalent if and only if they are true in the same models (on the same interpretation), or are they to be so called if and only if they are true and false in the same models? In this work, I will consistently opt for the former alternative and for its analogies with other concepts of the metatheory of logic (Hintikka, 1996a: 65).*

That is not a big surprise if we keep in mind Hintikka's strong focus on truth – in his presentation of the IF syntax only existential quantifiers and disjunctions can appear slashed which are the decision positions for Eloise.<sup>25</sup> From Hintikka's model - theoretic perspective, material truth is the most important in mathematics and not the *logical* truth (Hintikka, 1996a: 66).

It should be noted that the equivalence between IF logic and the  $\Sigma^1_1$  second - order logic is the  $\equiv_t$  equivalence. IF sentences and existential second - order sentences are always true in the same models, but not necessarily false in the same models. It is

---

<sup>25</sup> Noted by (Dechesne, 2005: 32)

clear already from the fact that the law of the excluded middle holds for the second - order logic but fails in IF.

### ***1.5.9. Compactness***

*-Theorem:* A set of IF sentences has a model if all its finite subsets have a model.

*-Proof<sup>26</sup>:* Let us denote a set of IF sentences  $\alpha$ . We can translate those sentences into existential second - order sentences, using the skolemisation procedure described earlier. We obtain sentences in the form  $\exists f_1 \dots \exists f_k \forall x_1 \dots \forall x_n (\varphi)$ , where  $\varphi$  is a classical first - order formula. Let us denote this set  $\beta$ . Now in all sentences in  $\beta$  we omit the quantification over function symbols to get a new set  $\gamma$  of classical first - order formulas. If every finite subset of  $\gamma$  has a model then every finite subset of  $\beta$  and every finite subset of  $\alpha$  has a model as well. The compactness theorem holds for classical first - order logic so  $\gamma$  has a model if every finite subset of  $\gamma$  has a model. If  $\gamma$  has a model then  $\alpha$  and  $\beta$  also have a model. Thus, if every finite subset of  $\alpha$  has a model, then  $\alpha$  has a model, which we had to prove.

### ***1.5.10. Löwenheim-Skolem theorem***

*-Theorem:* If a set of IF sentences has an infinite model then it has a model of any infinite cardinality.

*-Proof:* The proof is similar as the previous one. We use sets  $\alpha$ ,  $\beta$  and  $\gamma$ , as per before. Sets of sentences  $\alpha$ ,  $\beta$  and  $\gamma$  are satisfiable in the same models. As the Löwenheim-

---

<sup>26</sup> from (Hintikka, 1996a: ch 3)

Skolem theorem holds for classical first - order logic, if  $\gamma$  has an infinite model it has a model of any infinite cardinality. Then, if  $\alpha$  has an infinite model it has a model of any infinite cardinality, which we had to prove.

### ***1.5.11. Separation theorem***

*-Theorem:* Let  $\varphi$  and  $\psi$  be two contrary IF sentences<sup>27</sup> of vocabularies  $\sigma$  and  $\sigma'$ , respectively. Then there is a classical first - order formula  $\tau$  of vocabulary  $\sigma \cap \sigma'$  such that  $\varphi \equiv_{\tau} \tau$  and the sentences  $\psi$  and  $\tau$  are the contrary sentences.

*-Proof:* The method is the same as in the previous proof. With skolemisation procedure we get the existential second - order sentences  $\varphi'$  and  $\psi'$ . By omitting the quantification over function symbols in  $\varphi'$  and  $\psi'$  we get classical first - order sentences  $\varphi''$  and  $\psi''$ . The separation theorem holds for classical first - order logic, so there is a first - order sentence  $\tau$  such that  $\varphi'' \equiv_{\tau} \tau$  and the sentences  $\psi''$  and  $\tau$  are the contrary sentences. Sentences  $\varphi$ ,  $\varphi'$ ,  $\varphi''$  are true in the same models as well as  $\psi$ ,  $\psi'$  and  $\psi''$ . So  $\tau$  is a first - order sentence such that  $\varphi \equiv_{\tau} \tau$  and  $\psi$  and  $\tau$  have no model in common.

We have seen so far that there are number of "pleasant" logical properties that IF logic shares with classical first - order logic. Now we will turn our attention to differences between them. The first is the failure of law of the excluded middle. We will address other differences in the following paragraphs.

---

<sup>27</sup> Two sentences are contrary if they have no model in common.

### ***1.5.12. Failure of axiomatisability***

Kurt Gödel proved in 1930 that classical first - order logic has a complete axiomatisation, that is, it has a sound and complete proof procedure. The same property doesn't apply to IF logic. The failure of axiomatisability is actually due to its great expressive power, so Hintikka doesn't consider it to be a big disadvantage. In (Hintikka, 1996a: 66- 67) he states:

*Some philosophers have seen in the absence of a complete proof procedure the basis of a serious objection to IF first-order logic. Such a way of thinking is based on a number of serious mistakes. Apparently the skeptics think that they cannot understand a language unless they understand its logic; that they do not understand its logic unless they understand what its logical truths are; and that they do not understand the logical truths of a part of logic unless they have a complete axiomatization of such truths (valid formulas). The last two steps of this line of thought are both fallacious...The subtler mistake seems to be that in order to understand logic one has to understand its logical truths. This is at best a half-truth. One possible mistake here is to think that logical truths are merely a subclass of truths. Hence, according to this line of thought, a satisfactory account of truth for a language ought to yield as a special case an account of logical truth for this language. But logical truth simply is not a species of (plain) truth, notwithstanding the views of Frege and Russell. Logical truths are not truths about this world of ours. They are truths about all possible worlds, truths on any interpretation of nonlogical constants. This notion is an animal of an entirely different color from ordinary truth in some*

*one world. In this sense, to call valid sentences "logically true" is a misnomer. Several philosophers who swear by the ordinary notion of truth are nonetheless more than a little skeptical of the ultimate viability of the very notion of logical truth.*

In the lack of a complete proof system, IF logic is inevitably linked to a model theory. Hintikka's focus is on the material truth in a model. Validity is conceived as the truth in all models. We will show later that the idea behind the expressivity of IF logic is to be able to reduce the problem of deciding whether a second - order mathematical sentence is valid to the problem of validity of its equivalent IF sentence. However, it seems strange to propose a reduction of mathematics to a logic lacking proof system<sup>28</sup>. We will still touch on this point later.

In order to justify his model - theoretic approach, Hintikka investigates in (Hintikka, 1996a: ch 1) the role of logic in mathematics. In examining this question, Hintikka points out that the role of logic in mathematics is in fact multiple and that some of its aspects are neglected. What we usually mean when we speak of logic in mathematics is its *deductive* role or what Hintikka calls *a theory of demonstration*. Regardless of whether we are dealing with an interpreted or non-interpreted mathematical system, inferences of theorems are completely logical. An axiomatized logic serves thus to secure inferences in mathematics. Then, we can ask whether these inferences can be fully covered by computable, recursive formal rules.

Hintikka points out to some other roles of logic. One of them is often disputed but for him the most important - he calls it *descriptive* role - and it is performed by model theory. Again, regardless of whether we are dealing with an interpreted or non-

---

<sup>28</sup> We will explain later in this thesis the specific logicist programme Hintikka advocates.



interpreted mathematical system, we use logical terms to analyse the meaning of mathematical axioms, with the idea that the meaning is determined by a class of structures or models that can be assigned to them. Thus, for a sentence  $\varphi$  we specify a class of models  $M(\varphi)$ , where, simply put,  $M$  is a model for  $\varphi$  if and only if  $\varphi$  is true in  $M$ . If we approach mathematics through the model theory, the problem of definition of truth for mathematical theory comes to the fore, and we will address that question later.

The third function of logic in mathematics is its role as a medium for the axiomatic set theory.

In parallel with different roles of logic in mathematics, we can talk about different types of completeness of a deductive system. Hintikka distinguishes in (Hintikka, 1996a: ch 5) several of them, among others (in his, not quite usual, terminology): *semantic, deductive and descriptive completeness*. The point is that they apply to different things.

1. *Semantic completeness* can be a property of an axiomatized system of logic or some of its fragments if the set of valid expressions of the system is recursively enumerable. This property can also be expressed by saying that the system has a full axiomatisation. As previously mentioned, classical first-order logic has this property while IF logic doesn't.

2. *Deductive completeness* is related to a non-logical deductive system with axiomatized system of logic. This property guarantees that, starting from a given non-logical system and applying given logical means, we can prove the truth or falsity of any formula of the system. Gödel's proof of the incompleteness of arithmetic is about deductive completeness.

3. But there is another type of completeness that corresponds to the descriptive (and, according to Hintikka, the most important) role of logic in mathematics - Hintikka refers to it as *descriptive completeness*, which is actually *categoricity* in Dedekind's sense<sup>29</sup>. It applies to an axiomatized non - logical system and it is related to the model - theoretic approach. The descriptive completeness of the system exists if there is only one model, or - if there are more models - if we can prove that there is an isomorphism between them.

The impact of Gödel's result of incompleteness of arithmetic has been widely discussed and Hintikka believes that it actually has a limited importance. In Hintikka's view, for the foundation of mathematics it is categoricity of a system that we should care the most about, because it brings the insight into the relationship between the theory and its instantiation i.e. structures that satisfy it. Classical first-order logic, although semantically complete in the sense described above, cannot provide the categoricity of arithmetic. If we remain at the first-order level, there is no immunity for the emergence of non-standard models of theory, as follows from the Löwenheim–Skolem theorem. However, if we sacrifice the semantic completeness of logic, we may be able to provide the desired isomorphism between models of arithmetic. This we can do by passing to the second - order language or to the set theory, as did Dedekind, Peano and Frege, but although in that way the categorical system of arithmetic is obtained, we meddle ourselves in other problems. According to Hintikka, a commitment to higher order entities is not desirable. He is a defender of nominalism in mathematics. On the other hand, the set theory formalized with classical first - order logic in the usual way is a poor medium for the model theory because the truth cannot be defined in it (Hintikka, 1996a, 2006 etc.). Hintikka would

---

<sup>29</sup> see (Dedekind, 1996).

like to find the half - way between the two: on the one side, to have a logic with expressiveness of second - order logic and that, like in it, we can guarantee isomorphism between models of arithmetic, but on the other hand, to stay at the first-order level. According to Hintikka, IF logic is the one we are looking for, so the failure of axiomatisability is not the issue. We should mention however that IF logic has the complete *disprove procedure*; the invalid IF sentences can be recursively enumerated.

Later in this thesis we will compare Hintikka's approach with the dialogic approach making use of Constructive Type Theory. Both approaches are game-theoretic. However, as opposed to the model- theoretic approach, we will argue for the approach where games are oriented toward formal proofs. Hintikka was aware of a close link between GTS and dialogues but he saw the formal- proof orientation of the dialogical approach as its weakness. He argues the following in (Hintikka,1973: 81):

*In contrast to our games of seeking and finding, the games of Lorenzen and Stegmüller are 'dialogical games' which are played 'indoors' by means of verbal 'challenges' and 'responses'. [...].*

*[...]. If one is merely interested in suitable technical problems in logic, there may not be much to choose between the two types of games. However, from a philosophical point of view, the difference seems to be absolutely crucial. Only considerations which pertain to 'games of exploring the world' can be hoped to throw any light on the role of our logical concepts in the meaningful use of language.*

However, the lack of complete system of inference for IF logic conflicts Hintikka's claim that IF logic realise the logicist programme. We will come to that point a little later in this thesis.

It is worth noting a few remarks made by Tennant in (Tennant, 1998) concerning Hintikka's easy dismissing of deductive aspect of logic. In his review of *Principles of Mathematics Revisited*, Tennant justly observed that the only axiomatisation Hintikka mentioned there is the Hilbert – style axiomatisation, while *no mention is made of the much more natural and entirely rule-theoretic presentations given by Gentzen* (Tennant, 1998: 97). According to the author, it seems as an omission, given that Hintikka holds on the *naturalness* of GTS.

Another Tennant's critic concerns the lack of a clear link between truth and falsity conditions in GTS with a usual requirement for the inference - patterns in logic. Even more so when the negation is taken into consideration, because the dual negation leaves different possibilities for defining validity: it would be one concept when Eloise has a winning strategy in every model and another when Abelard has no winning strategy in any model. It is a serious ambiguity according to (Tennant, 1998: 98). Moreover, previously suggested complete disproof procedure suffers from the same problem.

Hintikka's sharp (and one might say, quite rude) response came out in 2000, where he accused Tennant of a complete misunderstanding of the game theoretical semantics and of his using *in his recent animadversions on IF logic* (Tennant 1998) *a disproportional amount of space and printer's ink to discuss rules for formal proofs of validity, which in the best of circumstances yield only indirect indications of the truth conditions that are the real life blood of logic* (Hintikka, 2000: 135). No concrete response to Tennant's objections was offered to our knowledge.

We would like to make a final remark. A discussion about advantages of a proof - theoretic approach over a model - theoretic one (or the other way around) is a big issue and we cannot address it in this thesis. The questions concerning it are complex and they would demand an independent discussion. In this work we will undertake a more modest task: we will argue that neither the treatment of branching quantifiers nor that of anaphora really demands a model- theory.

### ***1.5.13. Question of compositionality***

Another issue to be open is the failure of *compositionality* in IF logic. The principle of compositionality is a demand that the meaning of a complex expression is determined by the meaning of its constituent parts and by its structure. In the Tarski – type semantics the meaning is built in such a way: we start off with atomic expressions until the whole sentence is determined, which is the so called “inside – out” approach. In IF logic, on the other hand, this demand is difficult to meet: in an IF sentence an existential quantifier can be independent of some of the universal quantifiers in whose scope it lies and dependent of some others. The interpretation of the sentence will then violate the compositionality principle. Consequently, game theoretical semantics is much more natural for IF logic: it offers a different approach to meaning where we start off with the entire sentence until its constituent parts are reached.

Hintikka insists on the fact that the lack of compositionality is rather an advantage than a defect (Hintikka, 2002). Namely, he is relating compositionality to *context-independence* of a sentence while we should prefer semantics that provides

the *context-dependent* meaning. Therefore, in Hintikka's opinion, IF logic *shouldn't be* compositional.

Hodges proved in (Hodges, 1997a, b) that it is possible to create one variant of compositional semantics for IF logic<sup>30</sup>. However, Hodges' semantics is not a Tarski – style semantics because we deal in it with *sets of assignments* and not with a single assignment, which is rather considered as price to pay. It is a set of assignments of elements to free variables that makes a sentence true or false in a given model and in this way dynamic features offered by GTS are lost. Consequently, it is no wonder that Hintikka does not consider Hodges' result to be a success. Indeed, in (Cameron, Hodges, 2001) it is proven that no strict Tarski - type semantics can be made for IF logic. In (Hintikka, 2002: 411) he states the following:

*This parallelism between syntactical and semantical rules is irretrievably lost in IF logic. The failure of compositionality which has thus been diagnosed, is closely related to the inadequacies of the traditional notion of scope. I have discussed the vagaries of this notion (with an eye also on its uses in linguistics) in an earlier paper (Hintikka [1997]) entitled 'No scope for scope?' Its punch line is: 'In linguistics, once a day with scope does not do it.' Wilfrid Hodges and his associates have tried to construct a compositional semantics for IF languages. In spite of their great ingenuity, which has led to important new insights into IF logics, the upshot is essentially an impossibility result, as far as my IF logic is concerned.*

---

<sup>30</sup> A presentation of compositional semantics for IF logic can also be found in (Mann, Sandu, Sevenster, 2011).

We will still come back later to the point Hintikka makes about *inadequacies of the traditional notion of scope*. In (Hintikka,1997a), he is arguing that the distinction should be made between the *priority* and *binding scope* of quantifiers. The priority scope concerns the relations of dependences between quantifiers and other logical operators while the binding scope indicates that variable is bound by a certain quantifier. In traditional approaches there is a confusion about them because they are both expressed by the same syntactic device. According to Hintikka, without a proper understanding of the two notions of scope we can never explain in a satisfactory manner some natural language phenomena such as anaphora. In (Hintikka, Kulas, 1983) it has been already argued that in order to provide a satisfactory semantics of natural language, compositionality has to be abandoned. In (Hintikka, 1996a: 108 – 109) there are some natural language counterexamples for compositionality – compositionality is labeled there *a good established dogma*. Hintikka concludes with a point:

*Conversely, to look at the bright side of things, the very existence of IF first-order logic is an eloquent proof that a rejection of compositionality is no obstacle to the formulation of a simple and powerful logic. Indeed, the best argument against compositionality as a general linguistic principle is the success of independence friendly logic in its different variants in the logical analysis of various important concepts and of their manifestations in natural languages. This is not the occasion to tell that success story, which is in fact still continuing. A cumulative evidence of different applications is in any case sufficiently impressive for me to rest my case against compositionality for the purposes of this book. (Hintikka, 1996a: 110).*

We will argue in the following chapters of this thesis that the “outside – in” approach in game - theoretically oriented semantics actually has significant advantages over the traditional “inside – out” perspective. It will be illustrated later when dealing with anaphora. In addition, the distinction Hintikka makes between the *priority* and *binding scope* is indeed essential. However, we will compare and contrast GTS with the dialogical game theoretic approach to show that the *parallelism between syntactical and semantical rules* doesn’t have to be lost along the way. The dialogical semantics is a particular alternative both to the model - theoretic and the proof theoretic approach to meaning. Related to Constructive Type Theory, it takes seriously the meaning - dependence on the context in which the sentence is asserted and it has tools to distinguish between the two notions of scope. But this can be done without neglecting the deductive role of logic. In the same time, a satisfactory interpretation can be provided for all of the Hintikka’s examples of problematic anaphoric expressions.

#### ***1.5.14. Truth predicate***

As per Tarski’s well - known impossibility result from 1936, the truth predicate cannot be defined for a classical first - order language within the same language. Adding the truth predicate to a first - order language would produce paradoxes such as the famous *liar’s paradox*.<sup>31</sup> In other words, first - order languages are not expressive enough to express their own truth. To be able to formulate it, a

---

<sup>31</sup> See (Beall, Glanzberg, 2014).



richer language is needed, a stronger meta - language. Hintikka labels the above result "Tarski's curse" (Hintikka, 1996a: 116).

The expressive power of IF logic exceeds that of classical first - order logic, enough to be able to define the truth predicate within the language itself. Still, with IF logic we remain at the first - order level. In Hintikka's words, IF language allows to "exorcise Tarski's curse" (Hintikka, 1996a: 117); it is rich enough to be able to speak of itself. In (Hintikka, 1996a: 113) an adequate truth predicate is formulated for IF language using Gödel numbering.

When we add the truth predicate to the IF language the liar's paradox does not occur because the liar sentence will be neither true nor false - its truth value will be undetermined. The possibility of formulating the truth predicate in IF language is actually due to the failure of law of the excluded middle and to the fact that the negation used is not classical but stronger, dual negation. However, if we take that Tarki's impossibility result is restricted to first - order languages with classical negation, then the case of IF logic is not inconsistent with it.

The fact that the truth can be defined in IF language is very important for the realist position Hintikka defends, certainly if we bear in mind his model – theoretic perspective<sup>32</sup>. It is closely related to game theoretical semantics where the existence of a winning strategy for Eloise, which amounts to the truth in a model, includes the axiom of choice. When a sentence is true, Skolem functions exist for it, which renders it possible to express the truth in the language itself. But in the set theory formalised with classical first - order logic it is not possible, which makes the set theory, according to the author, inadequate for a model theory in general. In (Hintikka, 2008: 25) he says:

---

<sup>32</sup> See the discussion in (Hintikka, 2008: 24)

*This is a striking result in that it contradicts the widespread idea of axiomatic set theory as the natural medium of all model theory. This idea is simply wrong. For any halfway adequate model theory you need the notion of a truth, which just is not available in a set theory using traditional first-order logic. First-order axiomatic set theories are poor frameworks even for their own model theory. A fortiori, there are likely to be poor frameworks for any theory formulated in their terms.*

Hintikka's claim of definability of truth at the first - order level in IF logic has been disputed. Marcus Pantsar presents arguments from (De Rouilhan, Bozon, 2006) that even if the truth predicate for IF language is the adequate one, we can only show it to be the case in one richer language. Therefore, the hierarchy of languages cannot be avoided. In (Pantasar, 2009: 164) he argues:

*Finally, needless to say, first-order IF languages do not contain enough expressive power to carry out the sort of theorizing that Hintikka does in natural language. In this respect, De Rouilhan's and Bozon's criticism resembles my criticism of Field. It is all well to use axioms to define concepts in formal languages, but we should not forget that all this is initially done in our pre-formal languages. When we switch from pre formal to formal concepts, we cannot fool ourselves into thinking that the former never existed. Moreover, we cannot forget that they are the reason why the formal concepts are possible in the first place. Similarly, Hintikka's game-theoretic definition of truth for IF languages is adequate, but we cannot look at this result*

*independently from all the background needed to establish it. De Rouilhan and Bozon argue that a first-order IF language itself is not able to express this, or any of the desired results around it. That is why Tarski's hierarchy exists also in IF. For us to be able to say that the definition of truth for IF is adequate, we need to have a richer language. The main problem for Hintikka (as well as for others that want to escape the Tarskian hierarchy) is that everything needs to be done in a single language. De Rouilhan and Bozon call this the monolingual speaker problem, and it follows us wherever we go with projects like Hintikka's.*

Another argument of De Rouilhan and Bozon in (Pantasar, 2009: 165) is that

*the model-theoretic concepts of logical truth, logical implication and logical equivalence are all definable in IF languages only in a very weak sense. Indeed, as they point out, here IF loses something from classical first-order languages where those concepts are fully definable.*

However, as Pantasar notices in continue, if the hierarchy of languages is still needed it is for different reasons than in the case of classical first - order logic. The possibility to express the truth in the language itself is still an important characteristic of IF logic. We will still discuss this point in relation with the axiom of choice in chapter **III**. However, contrary to what Hintikka says in (Hintikka, 2008: 6), to make account of relations of dependences and independences between quantifiers at the first - order level and to be able to formulate the truth predicate in the language IF logic is not *indispensable*. In chapter **II** of this thesis it will be shown that the same can be

carried out at the first - order level in the dialogical framework. A model theory is not indispensable either.

### ***1.5.15. Extended IF logic***

The most interesting element in IF logic is perhaps the rule of so - called *dual* negation. As presented previously, this rule requires a change of roles for players in a given game. However, a problem is that IF logic with dual negation still proves to be insufficient to express all the important mathematical properties (among others, mathematical induction). Therefore, Hintikka proposes an extended version of IF logic, corresponding to a larger fragment of second-order logic, whose expressivity is fully satisfactory when it comes to mathematics. As Hintikka puts it in a witty way:

*At this point of my mystery story of the missing negation, I fully expect Lieutenant Colombo to scratch his head and say to me: "Yes, but there is still one question that bothers me. Why cannot we simply introduce a contradictory negation by a fiat? All we need to give a semantical meaning to such a negation, say  $\neg S$ , is to stipulate that  $\neg S$  is true just in case  $S$  is not true. What's wrong with such a procedure?" (Hintikka, 1996a: 147)*

The extended version of IF logic is created by distinguishing two types of negation: *dual* or *strong* negation, which we denote ( $\sim$ ) and the other, *contradictory* or *weak* negation ( $\neg$ ), that can occur only as a negation of an entire IF sentence. Semantically, this negation does not behave like other connectives, because there is no rule for it in terms of game theory. Semantic explanation is that  $\neg\phi$  is true if and

only if  $\varphi$  is not true, in other words, if it is false or undetermined. The law of the excluded middle holds for a contradictory negation. Contradictory negation in a sentence  $\neg\varphi$  indicates actually that there is no winning strategy for the initial Verifier in a game played for  $\varphi$ .

The extended version of IF logic is even more expressive than the regular IF logic. It corresponds to a broader fragment of second-order logic, the fragment that is usually denoted  $\Pi^1_1$ . Contradictory negation fits a second-order sentence in the form  $\forall x_1 \dots \forall x_n \Psi$ , where  $\Psi$  is a first-order formula. Together IF and the extended IF logic are equivalent to the  $\Sigma^1_1 \cup \Pi^1_1$  part of second - order logic. This language is sufficient to express the most of important mathematical properties. We can express mathematical induction, finite domains, well-ordering and so on. The elementary number theory can be formulated with IF and the extended IF logic and moreover, its categoricity is guaranteed.

All Peano's axioms can be formulated, including the axiom-schema of mathematical induction. In its second - order version that axiom has the following form:

$$\forall X ((X(0) \wedge \forall y (X(y) \rightarrow X(y+1))) \rightarrow \forall y X(y)),$$

which is an  $\Pi^1_1$  sentence.

However, introducing the contradictory negation changes a lot when it comes to the properties of IF logic. Pleasant properties like compactness, separation theorem, Löwenheim-Skolem theorem and the existence of a complete disproof procedure are lost.

## ***1.6. Some problems in IF logic***

We will move on to examine some of the problems in IF logic. The first is the so - called *signalling problem* that was first observed in (Hogdes, 1997b).

### ***1.6.1. Signalling***

The big novelty in IF logic is the slash sign that introduces the imperfect information in a game played for a given IF sentence. The slash sign indicates that some information is hidden from one of the players in the game. Through the phenomenon of signalling it is possible that the information that is supposed to be hidden from the player becomes available in the game via another variable.

*-Example:* Let the IF sentence  $\varphi$  be  $\forall x(\exists y/\forall x) (x=y)$  and the model  $\mathbb{M} = \{0, 1\}$ .

Neither Eloise nor Abelard has the winning strategy - the truth value of the expression is undetermined. But if we introduce a vacuous quantifier the situation will change.

Note the following expression:

$$\varphi': \forall x\exists z(\exists y/\forall x) (x=y).$$

In the game  $G(\varphi')$  it is Abelard who first makes his choice for variable  $x$ . Then Eloise makes her choice for  $z$ . In order to win the game, she will choose the value of  $z$  to be equal to the value Abelard has already chosen for  $x$ . Then again, it is Eloise's turn to move. She has to choose the value of  $y$ , not knowing the value of  $x$ . But this time it is not the problem because she knows the value she has already chosen for  $z$ , that equals  $x$ , so she can choose the same value for  $y$  and win the game. Eloise has her winning

strategy in the game for  $\varphi'$  due to the introduction of the vacuous quantifier  $\exists z$ . Formula  $\varphi'$  is now true in  $\mathbb{M}$ .<sup>33</sup>

The introduction of a vacuous quantifier cannot change the truth value of one classical first - order sentence. Games for classical first - order sentences are the games with perfect information, so no signalling phenomenon is possible.

To prevent the signalling problem Hintikka introduced a convention: existential quantifiers and disjunctions are always independent of previous existential quantifiers in a sentence. In other words, Eloise forgets all her previous moves. It is explained in (Hintikka, 1996a: 63):

*The small extra specification that is needed is that moves connected with existential quantifiers are always independent of earlier moves with existential quantifiers. This assumption was tacitly made in the second-order translation of IF first-order sentences explained earlier in this chapter. The reason for this provision is that otherwise "forbidden" dependencies of existential quantifiers on universal quantifiers could be created through the mediation of intervening existential quantifiers.*

This restriction works nicely in the previous example but unfortunately the problem does not end there.

Janssen showed many other examples of signalling in (Janssen, 2002) and raised new questions related to this non - desirable phenomenon. In (Janssen, Dechesne, 2006), they observed that the restriction made by Hintikka does not work

---

<sup>33</sup> The example is used first in (Hodges, 1997b).

well in some other cases where the signalling is actually *needed*. That can be illustrated on the following valid classical first - order formula:

$$\forall x \exists y (y = x \wedge (y = 1 \vee y \neq 1)).$$

It is easy to note that, with restrictions introduced by Hintikka, Eloise does not have a winning strategy. If she forgets her previous moves then she must make her choice of a disjunct not knowing her previous choice for  $y$ , so she cannot win the game. The signalling phenomenon seems to be an essential part of GTS. Jansen and Dechesne conclude their article with a pessimistic observation:

*Signalling is a tricky business. It disturbs several extrapolations from classical logic (change of bound variables, prenex normal form), and the interaction of signalling and implicit independence causes that Hintikka's IF is not a conservative extension of predicate logic.*

In (Dechesne, 2005: 104) it is argued that IF logic rests conservative extension of classical first - order logic with a restriction: a classical first - order formula must be in its *regular form* we defined in *definition 10*, which ensures that there is no nested quantification over the same variable in the formula and that the same variable must not have both free and bound appearances in the formula. The paper of (Janssen, Dechesne, 2006) suggests however that not all the examples of signalling can be prevented in such a way.

We will quote at the end of this section observation from (Mann, Sandu, Sevenster, 2011: 74):



*...one can think of the semantic game for any IF sentence as a generalized signalling problem in which each existential quantifier corresponds to an agent who can both send and receive signals, while the universal quantifiers correspond to states of affairs beyond the agents' control.*

### ***1.6.2. Is IF logic really a first-order logic?***

According to Hintikka, the biggest advantage of IF logic is the possibility to express important mathematical properties at the first - order level. That claim has been challenged by authors such as Feferman, Väänänen, De Rouilhan and Wolenski. The basic question is whether IF logic is really a first - order logic? Feferman claims that even if IF logic remains at the first - order level syntactically, it still does not mean that its semantics does not involve higher order entities (Feferman, 2006). Hintikka's response was simple: the only criterion whether we are dealing with a first or a higher - order logic is whether variables range over individuals only or not. Tulenheimo reformulates this criterion in the following manner:

*...a logic is of first order, if any play of a semantic game associated with a formula of this logic only involves (in addition to choices interpreting conjunctions and disjunctions) choices of individuals, as opposed to choices of higher-order entities. (Tulenheimo, 2009)*

It is a traditional view on this distinction, relying on Quine's motto *to be is to be a value of a bound variable*, but Feferman and Väänänen put that criterion in question.

According to Feferman, the descriptive role of logic that Hintikka favours so much in mathematics turns out to be insufficient. In mathematical reasoning, the deductive role of logic inevitably comes into play. When proving that a formula is a theorem of a system, we are dealing with the question of whether that formula follows deductively from the axioms of arithmetic. We can observe the axioms of arithmetic that can be translated into extended IF language. Let us call such a sentence  $\varphi^*$ . It has a form of a contradictory negation of a sentence  $\varphi$  of unextended IF language. We wonder whether the sentence  $\psi$  is a consequence of that axiom. That is formulated as the question whether the implication  $\varphi^* \rightarrow \psi$  is valid, which is equivalent to the question whether the disjunction  $\neg \varphi^* \vee \psi$  is valid (a formula of the unextended IF logic). According to Hintikka, this is the way to reduce practically all of mathematics, or its great part, to the question of validity in IF logic. But note that we don't speak any more about the *satisfaction* of the sentence  $\neg \varphi^* \vee \psi$  in a model, we speak now of its *validity* i.e. of the truth in all models. Väänänen argues that when it comes to validity of those formulas, we are committed to the entire second-order logic (Väänänen, 2001).

To determine validity of an IF logic sentence is to show that there is a winning strategy for Eloise in the game played for that sentence *in any model*. Tero Tulenheimo gives an example in (Tulenheimo, 2009) of the sentence  $\neg \forall x \exists y \forall z (\exists w / \forall x) R(x, y, z, w)$  and ask if we can avoid commitment to higher-order entities when we talk about its validity? If such a sentence is valid, it means that in each model there are functions  $f: M \rightarrow M$  and  $g: M \rightarrow M$  such that the condition  $R(a, f(a), b, g(b))$  is satisfied for the individuals  $a$  and  $b$ . To prove validity of this sentence we must assume that we have some understanding of *all functions* and *all subsets*, which is certainly a higher-order idea.

IF logic is necessarily oriented toward a model theory because it doesn't have a complete axiom system. As a consequence, we cannot really *do* mathematics in IF logic and Hintikka is aware of this fact. Our mathematical practice has to remain unchanged and Hintikka's aim is purely foundational ( Hintikka, 1996a: 206). The idea is that, by translating the most of important mathematical concepts in IF language, we get something important from the foundational point of view – we avoid the commitment to higher - order entities. We will discuss in more details Hintikka's foundational intentions in the following chapter. Hintikka states in (Hintikka, 1996a: 205 – 206):

*What I have argued is that most (and perhaps all) of mathematics can in principle be done by means of one and the same logic, IF first-order logic. Moreover, that logic is a genuine article and not a disguised version of set theory because it is first-order logic and therefore free of the philosophical problems that have beset set theory or type theory. It has the same - and a better - claim to the title of logic as ordinary first - order logic. Needless to say, I am advocating this purely logical nature of mathematical theorizing as a rational reconstruction calculated to solve philosophical and other theoretical problems. As to practicalities, presumably the most intuitive way of mathematical theorizing would be to carry it out on a second-order level, or maybe with a simple type theory as its logic; in other words, not unlike what is done in general topology. Only when theoretical problems that have to do with the status of higher-order entities that have to do arise will there be a reason to resort to the kind of reduction to IF first-order logic described in this chapter.*

In this thesis we will embrace Feferman's view of the importance of inferential aspects of logic in mathematics. However, we will not agree with his criticism concerning the order of IF logic. We will contrast it with Goran Sundholm's analysis to show that we have some good reasons to consider IF logic a first - order logic. In the following section of this thesis we will however show that we can obtain Hintikka's cherished goal and stay at the first-order level, without sacrificing the deductive role of logic in mathematics. Moreover, it can be done in a game- theory framework. But it will demand the dialogical approach to logic.

We will go back now to the question of order of IF logic. Similar questions had been raised earlier, in connection to ontological commitment related to branching quantifiers. Hand defended in (Hand, 1993) the standing that, if considered under the GTS interpretation, branching quantifiers demand no more ontological commitment than ordinary first - order quantifiers.<sup>34</sup> He argues against Quine's and Patton's arguments in favour of the second - order reading of branching quantifiers. Patton's arguments seem more interesting from our perspective and so it is worthy briefly mentioning some of them. The first one is that a strategy in GTS is *a function*, a second - order entity and consequently, semantics employing such a notion makes us committed to second - order entities. The argument is not so sharp and Hand does not have much trouble dismissing it in (Hand, 1993: 428): in that case, one would have to say that under GTS interpretation even classical first - order logic is second – order logic because the truth definition for a classical first – order sentence involves a choice function. That would be a strange consequence indeed.

---

<sup>34</sup> Hand is employing the GTS interpretation of branching quantifiers and he even speaks of games with imperfect information. Curiously, he does not mention Independent Friendly Logic in this paper.

Second Patton's argument for the second - order reading of branching quantifiers is more interesting. Patton says in (Patton, 1991: 216) that actually branching quantifiers should not be considered quantifiers at all because a necessary condition to be a quantifier is to have instances. Let us look once more at the typical example of branching quantifiers we presented previously:

$$\left. \begin{array}{l} \forall z \exists u \\ \forall x \exists y \end{array} \right\} S(x,y,z,u)$$

There is no instance of such a sentence because what ever one could offer as its instance would have to be a linear sentence. However, any such a linear sentence would be rather an instance of a linear universal sentence and not of the given branching sentence (see Hand, 1993: 429). As an answer to this challenge, Hand proposes a re - evaluation of such a concept of quantifier, made for classical first - order logic. According to him, we should extend the concept of quantifier so that it can include branching quantifiers. However, we think that Patton's argument can be dismissed without extending the ordinary concept of quantification. We will show in chapter *II* of this thesis that under dialogical interpretation of branching quantifiers, those are indeed first - order quantifiers and they indeed have instances. It is one more reason to favour the dialogical semantics over GTS when it comes to the interpretation of branching quantifiers.

Ekland and Kolak in (Ekland, Kolak, 2002) remark that Hand's defence of the first - order character of branching quantifiers is strictly related to its GTS interpretation. They tried to avoid this by defending IF logic on its own, even under

some other semantic interpretations. However, their defence rely on a certain conception of logic (however weak it might be) and on a certain claim about the natural language – the claim they share with Hintikka. Their point is that IF logic should be considered the first - order logic on the basis that this logic captures the fragment of the natural language that is usually considered to be a first - order fragment. Moreover, they follow Hintikka in his claim that IF logic can capture this first - order fragment while classical first - order logic cannot. They provided the examples of natural language sentences that are considered first - order but cannot have a satisfactory interpretation in classical first - order logic; namely, sentences containing anaphoric expressions. We will elaborate Hintikka's take on anaphora later in this thesis. However, Eklund and Kolak finally arrived to a sort of relativisation of the very distinction between a first and a second - order logic. Indeed, the question of ontological commitment is not the easy one and the answer is not self - evident.

Another sceptical argument about the first - order character of IF logic is made by Shapiro in his review of *The Principles of Mathematics Revisited* (Cook, Shapiro, 1998: 314, 315). He discusses Hintikka's claim that IF logic has the strongest case of being a first - order logic in the simple fact that we avoid in it a quantification over functions and other second order entities. Shapiro points out that, even if we accept that, there is still a problem with extended IF sentences. In the interpretation of those sentences there is rather an explicit quantification over functions. The contradictory negation does not have the same treatment in GTS as other connectives do - there is no game rule for it. A sentence that begins with the contradictory negation actually explicitly states the lack of Verifier's winning strategy, i.e. it states the non-existence of a function. Therefore, there is not implicit but rather explicit referring to a higher order entity.

In our opinion, there is a very effective defence of the first – order character of IF logic against all previously mentioned arguments. It was formulated by Sundholm in (Sundholm, 2013). Sundholm questions the very notion of a function in mathematics, stating that we should be careful when interpreting functions, sets and other second - order notions. Sundholm makes use of Per Martin-Löf’s Constructive Type Theory<sup>35</sup> to show that functions we quantify over in the Skolem form of IF sentences actually belong to the lowest level in the type - hierarchy. In CTT we can see that in the Skolem- form of IF sentences there are quantifications over the *function sets* and not over the *function types*. But *function sets* belong to the lowest level in the hierarchy because they are predicative and generated from below – as opposed to *function types* where it is not the case. A set in Constructive Type Theory is interpreted in terms of its canonical elements and of the way the set is composed out of those elements. As the author points out, the constructive conception of set remains the same whether the law of the excluded middle is added or not. In such a framework, a set is not interchangeable with its characteristic function. Therefore, if we consider IF logic in the Constructive Type Theory framework we can note that no quantification over function types is needed and thus, there is no reference to higher order entities. Hintikka can keep his ground.

### ***1.7. Logicism revitalised***

In this section we want to make a clearer picture about Hintikka’s take on the role of IF logic in general and in the project of foundation of mathematics in

---

<sup>35</sup> With an important remark ...*that the considerations of this note are entirely neutral with respect to classical versus constructive logic because we can easily add the law of excluded middle to the type theory in question and the point will still stand.* (Sundholm 2013)

particular. In this section we will conclude the presentation of IF logic and we will move on to the following chapter where we will challenge some of Hintikka's points. At the beginning, let us follow Hintikka's own line of thoughts.

According to Hintikka, IF logic has many uses, among others, it is the most appropriate logic for the study of natural language. Hintikka considers it to be the basic logic, more basic than classical first - order logic, because it allows expressing the relations between quantifiers that are not otherwise expressible at the first - order level. In (Hintikka, 2002) he labels IF logic *hyperclassical logic*. His point is that the fragment of natural language we associate traditionally with classical first - order logic is actually captured with IF language. The argument for this claim is that there are some sentences of natural language that cannot be captured by the classical first - order formalisation but they have their satisfactory expressions in IF language. As previously mentioned, those are natural language sentences involving branching quantification and anaphoric expressions. We will discuss this claim later on in this thesis.

In mathematical theorizing IF logic can play another important role. The fundamental question is that of appropriate logic for mathematics. According to Hintikka, to answer the question we need first to reflect on the role of logic in mathematics. We already underlined three roles of logic that had been distinguished by Hintikka: deductive, descriptive and the role as a medium for the axiomatic set theory. At first glance, classical first - order logic seems like the best candidate for those functions of logic, with all desirable logical properties it has, such as completeness, compactness, theorem of separation, theorem of interpolation, Beth's theorem and so on. But despite these advantages, classical first - order logic applied to mathematics actually suffers from a range of problems. One of them is that not all of



mathematical concepts can be expressed in a first - order language. First - order languages are not expressive enough for concepts such as cardinality, well - ordering, infinity, mathematical induction and so on. Another important problem, according to Hintikka, is the "curse of Tarski" - the impossibility to define the concept of truth within the same language.

Consequently, we cross to higher-order languages or to set theory. But there we find ourselves trapped into a new range of problems. In addition to paradoxes of set theory, there is a problem of ontological commitment to sets and the issue of quantification over infinite domains. Hence, it would be very desirable for variables to range over individuals only. Furthermore, Hintikka argues that beside the issue of ontological commitment, set theory does not have a clear account of its intended models. Therefore, a search for stronger assumptions is not guided by questions of truth and falsity but rather by some vague intuitions (Hintikka, 1996b: 206). As discussed in section *I.5.14.*, according to Hintikka, set theory is not a good candidate for a medium of model theory, what so ever. When it is formalised with classical first - order logic, the problem of definition of truth becomes essential. The impossibility of defining truth of classical first - order language in the language itself is one of Hintikka's key arguments against set theory. Recall that categoricity of mathematics is the crucial aim from the model - theoretic perspective. However, Tarski's result applies only to compositional formal languages. Hintikka refers to compositionality as *a good established dogma*. A non - compositional semantics allows formulating the truth predicate in IF first - order language.

IF logic provides categoricity of arithmetic and it is sufficiently expressive. Therefore, all important concepts of classical mathematics can be expressed in it. IF logic ought to have advantages of both first and second - order logic, and to eliminate

the problems. In addition, Hintikka believes that it allows fulfilling a dream of realists when it comes to foundations of mathematics.

We will reflect more on Hintikka's realist position because it is somewhat specific. However, we will not be able to enter very deep into a complicated discussion of meaning of mathematical realism – it is a large topic and it would demand a separate consideration. Let us just say that, according to a very general and very simplified view on this matter, realism is a position holding that the truth of a sentence is independent of our knowledge or our ability to prove it. The meaning of a proposition is then grasped via its truth – conditions. As opposed, in the constructivist approach, the meaning is not relied to an objective concept of truth, independent of human mind. The meaning is rather obtained in providing an evidence for the assertability of proposition. The evidence is the actual construction of proof for it. As pointed out in the following quote from (Rahman, Clebourn, Jovanović, forthcoming):

*Intuitionists, as pointed out by D. Prawitz (2012, p. 47) “avoid the term truth and reject the idea that intuitionism could replace ‘p is true’ with ‘there exists a proof of p’ understood in a realistic vein.” Indeed, the existence of a proof, as pointed out by Prawitz in the same text, is to be “understood epistemically as the actual experience of the construction intended by the proposition, not as the existence of an ontological fact”. More generally, from the intuitionistic point of view proof-theory provides the means for the development of an epistemic approach to meaning rooted in assertions (rather than propositions).*

Recall that the truth of a sentence in GTS amounts to the existence of a winning strategy for Eloise in a game played for that sentence. The existence of such a strategy is a fact concerning the structure of a model in respect to which the game is played. In a realist manner, it is the fact independent of actual knowledge of a strategy function. As Ranta explains it in (Ranta, 1988: 383):

*The question is then whether we can introduce all games by stating their strategy prescriptions. For first - order games, the answer is affirmative. We can use a canonical notation for strategies, which enables us to read from a strategy term what game it is a strategy in. Conversely, given a first - order game, we can express in the canonical notation the form of a winning strategy of Myself in it. We can thus "bracket" the semantical games and say that a game is the set of winning strategies of Myself in it, as follows: given what canonical winning strategies of Myself in atomic games should be like, canonical winning strategies in complex games are defined in terms of them. A winning strategy in general, then, is an entity whose existence guarantees the existence of a canonical winning strategy. In constructive semantics this demand of guarantee is the strict demand that a non - canonical strategy be computable to a canonical one in order to count as a strategy.*

In chapter 10 in (Hintikka 1996a), he presented a constructivist version of IF logic<sup>36</sup> – in which choice functions are restricted to recursive functions or even to functions *known* to a player<sup>37</sup>. According to Hintikka, the former restriction yields the

---

<sup>36</sup> Hintikka mentioned the constructive version of GTS much earlier, see ( Hintikka,1979: 21, 73)

<sup>37</sup> However difficult it might be to define the later restriction.

constructivist approach and the intuitionist approach the later<sup>38</sup>. In constructivist IF logic it is required that all functions have to be *computable* or, in the game - terminology, strategies have to be *playable*.

The restriction yielding allegedly constructive IF logic is discussed in (Ranta, 1988). The author notes that *this restriction somehow yields intuitionistic logic* (Ranta, 1988: 387). In our opinion, there are some good arguments that it is not the case. First the “somehow” in the previous quote is not so innocent. It seems that there is something more needed to make the reading of GTS acceptable for a constructivist than the restriction to recursive functions. The arguments are listed below:

- First, as Ranta points out, a constructivist cannot accept GTS rule for negation as change of roles of the players in a game. This rule yields  $\neg\neg(A \vee \neg A)$  equivalent with  $(A \vee \neg A)$ , because the winning strategy for the former can be easily transformed into the winning strategy for the later. However, the first expression is constructively provable while the other is not!

Instead, the constructive reading of negation is needed,  $\neg A$  should be interpreted as  $A \rightarrow \perp$ .<sup>39</sup>

- Another important remark that Ranta makes is that in Hintikka’s GTS *...language games are entirely theoretical entities with no connection with language use* (Ranta, 1988: 387).

In the actual use of language, sentences are *assertions, questions, commands* and so on. In the constructivist approach we have to give account of a *force* they have in a conversation. Therefore, when we analyse a sentence we should

---

<sup>38</sup> Hintikka makes a distinction between *constructivism* and *intuitionism* in (Hintikka, 2001).

<sup>39</sup> In the dialogical framework we will see that negation is interpreted as  $A \rightarrow \perp$ , and with the particle rule for implication it amounts in a way to the change of roles of the players in a game.

separate two moments: a proposition and a force the sentence actually have in argumentation<sup>40</sup>.

- We will add the third, very strong point. In chapter *III* of this thesis we will examine Hintikka's take on the axiom of choice. We will argue that for a constructive reading of the axiom it is actually essential to have an *intensional* reading of the choice function. It is the way the function is defined that actually matters.

Let us consider the last argument against Hintikka's plea that the restriction to recursive functions yields constructive version of IF logic. It is supported with arguments from (Skolem, 1955), (Heyting, 1962) and finally, (Bishop, 1967). The arguments are very well presented in (Coquand, 2014) where Coquand shows that there is a conceptual gap between the notion of constructivity and the notion of recursive function. He concludes the article with the point that the notion of function in constructive mathematics cannot be defined in terms of recursivity. The main insight from the papers of Skolem, Heyting and Bishop is the following<sup>41</sup>. Let us consider the definition of recursive function: If  $g(u_1 \dots u_n, x)$  is a recursive function such that for all  $u_1 \dots u_n$  there exist  $x$  such that  $g(u_1 \dots u_n, x) = 0$ , then the function  $f$  which for  $u_1 \dots u_n$  provides the least  $x$  such that  $g(u_1 \dots u_n, x) = 0$  is a recursive function. The problem is the existential quantifier in the previous definition – it has to be understood constructively in order to have a constructive reading. But then the definition of recursivity already presupposes the calculability of function. On the other hand, if the existential quantifier is not understood constructively the link between

---

<sup>40</sup> This requirement is also met in the dialogical framework.

<sup>41</sup> We are following here the interpretation from (Coquand, 2014).

recursivity and computability of function is lost.<sup>42</sup> Bishop suggested in (Bishop, 1967) that functions in constructive mathematics should rather be understood in terms of “rules” and not in terms of recursivity. Bishop’s suggestion supports our previous argument: the reading of the axiom of choice in GTS is constructive if the choice function is defined *intensionally*. Recursivity of function is not relevant for the constructive reading of the axiom.

In chapter *II* of this thesis we will present the dialogical approach to logic linked with Constructive Type Theory and we will still clarify the points previously mentioned. In dialogical logic all the requirements are met to make a constructivist reading possible in a game theoretic approach. We will still reflect on that point later in this work.

However, Hintikka argues against constructivist reading of GTS. He is in favour of embracing the whole of classical mathematics – curiously, by making use of a non - classical logic. Here is one of his arguments against constructivism:

*Even though many constructivists emphasize the role of human thinking and human constructions in the "game" of mathematics, the main argument on their behalf virtually amounts to arguing that since robots have to be constructivists, we humans, too, have to be constructivists. For the only strategies that a digital automaton can be programmed to play in accordance with are the recursive ones. This is precisely what the proconstructivist argument sketched above has alleged to be the inevitable human predicament. The constructivists are in effect imputing to human beings those very limitations that characterize digital computers. It is thus the classical*

---

<sup>42</sup> The argument is from (Heyting, 1962).

*mathematicians who have more faith in human creativity than constructivists, contrary to the occasional claims of the latter.* (Hintikka, 1996a: 227)

As pointed out in (Ranta, 1988: 379) Hintikka contests the constructivist proof – conditional semantics but his main argument is in fact erroneous: *that proof conditions cannot be stated to branching quantifier sentences*. We will show later that it is not true in the light of Martin – Löf’s Constructive Type Theory.

Let us reflect more on the question of ontological commitment. Models of mathematical theories are structures made of certain elements. The structure of natural numbers is such a model for the elementary number theory. We should give an outline of the ontological status of those primitive elements in Hintikka’s theory. With regard to that issue, his realism goes hand in hand with nominalism, as he himself emphasis in (Hintikka, 1996a : ch 10 and Hintikka, 2006b). In Hilbert’s manner, individuals in a model are understood as individual extra - logical objects, directly accessible to human experience, as records on a paper or on a board. Curiously, according to Hintikka, Hilbert should be rather considered a realist - formalism often associated with him is erroneous.<sup>43</sup> Hintikka’s reading of Hilbert focus on his finitism and his *preference of combinatorial over set - theoretical reasoning in the foundations of mathematics* (Hintikka, 1997b: 16). Combinatorial reasoning is introduced via epsilon – terms interpreted as choice functions although, Hilbert’s weak point following this interpretation is that relations of dependences are neglected in these choices. However, Hilbert’s endeavor should be first and foremost understood as a justification of nominalism in mathematics – the primitive elements forming structures must be first – order individuals and not sets, because sets, having their own structures, would

---

<sup>43</sup> See (Hintikka, 1997b).

make a difference in a reformulation of axioms and in performing deductions from them (Hintikka, 2006b). It is disputable whether this is a good interpretation of Hilbert. It seems that Hilbert's purely proof – theoretical orientation in proving consistency of mathematical theories is easily characterized as a less important technical tool to prove the existence of models for those theories and to deal with them in a combinatorial manner.

In IF logic we reduce quantifications over higher - order entities to combinatorial problem of *seeking and finding* individuals in a model. The word *combinatorial* is understood in a specific sense to refer to combinations of objects, not formulas. The advantage is that in IF logic we have ontological commitment only to first - order entities. In a sense, this reduction for Hintikka is "a dream come true" of logicians, in a modified and less ambitious version: it is a conceptual reduction - not a translation of mathematics to logic. The idea is that classes of models of the most of mathematical theories can be grasped with IF logic and extended IF logic. Hintikka argues that this reduction is the real foundational reduction. All questions concerning the existence of higher - order entities can be thus reduced to issue of truth or validity of IF sentences. Mathematics is considered to be a research of structures or models of a theory where structures are made up of individuals and not of abstract entities. In (Hintikka, 1996a: 206), he states:

*My reduction of all mathematical problems to questions concerning the validity of sentences of an IF first-order language has both mathematical and philosophical significance. One kind of mathematical significance is that it shows that practically all mathematical problems are at bottom combinatorial rather than set-theoretical.*



*This implies that the notion of truth applies in mathematical theories. If you look at set theory, especially in its familiar axiomatic dress, you have a theory whose intended models are not clearly understood, so that the choice of stronger assumptions seems not to be guided by questions of truth and falsity but by some vague "intuitions", or else by considerations of mathematical taste and expediency. In contrast, combinatorial problems are clear-cut. Either there exists a structure of a certain kind or else there does not exist one. Either your jigsaw puzzle or tiling task can be completed or else it cannot be. We have in such cases a razor-sharp characterization of the structures whose existence we are speaking of. This provides no reason whatsoever for dispensing with the notion of truth. The search for stronger deductive premises will be guided by one's combinatorial experience.*

In our opinion, there is an issue with proposed logicist programme, as we mentioned previously: IF logic has no underlying theory of inference. We cannot actually *do* mathematics in IF logic.

In Hintikka's own words, GTS is a semantics that overcome the distinction between the verificationist and vericonditionalist approach. The meaning is obtained, in Wittgenstein's manner, through certain actions. Games for quantifiers amount to seeking and finding individuals in a model and the truth of a sentence amounts to a combinatorial problem of finding suitable *witness* – *individuals*. According to Hintikka, GTS, in its core, should already be acceptable for a constructivist. As shown previously, the very notion of truth in GTS yields a version

of the axiom of choice. So Hintikka thinks that GTS shows how a constructivist should embrace this classical principle.

In what follows we will make use of Martin- Löf's analysis of the axiom of choice to show that there is indeed a version of the axiom of choice that is perfectly acceptable for a constructivist but it is a version of the axiom where the choice function is understood *intensionally*. However, to embrace entire classical mathematics the *extensionality* is required. To have both at the same time seems to amount to having a cake and eating it too.

Moreover, we want to suggest more than that. In another passage against the constructivist approach Hintikka states:

*First, in this way constructivistic ideas can be seen to play a major legitimate role in a mathematician's work in any case. They are needed to understand and to master the deductive task of logic in mathematics. At the same time, this role is different from what constructivistic philosophers like to think. For one thing, the role of constructivistic notions has nothing to do within the meaning of mathematical statements. Meaning is a matter of the descriptive function of logic in mathematics. It is a matter of sentence-model relationships, ultimately a matter of truth definitions. It is not a matter of deductive relationships between propositions. (Hintikka, 1996a: 232)*

We will present in the following chapter of this thesis a constructivist approach where the dialogical logic and Martin- Löf's Constructive Type Theory are linked together. We will contest Hintikka's view from the last quotation: the meaning can be explained in a satisfactory manner without sacrificing a complete proof system. To that goal we

will examine two of Hintikka's favorite examples - the axiom of choice and anaphora.

But let us first present the dialogical approach to logic.

## ***II Constructive type theory in a dialogical framework***

In this chapter we will present Per Martin-Löf's Constructive Type Theory in a dialogical framework. The aim of this chapter is to provide us with some convenient tools which we will use in the following chapter of this thesis to question some of Hintikka's points. The presentation will not be exhaustive. We will not be able to discuss it in all details but we will explain as much as necessary for our purposes in this work.

### ***II 1. Constructive Type Theory (CTT)***

The father of type theory was Russell. He introduced it as a cure for paradoxes arising in set theory. The first version of type theory is called *simple type theory* in opposition with the later *ramified type theory* from *Principia Mathematica*. Church made the standard formalization of simple type theory in terms of  $\lambda$  – calculus. An important contribution to further development of type theory was made by Curry and Howard, who used it in the theorization of computer programming. The first study of dependent type theory was done in 1960 by de Bruijn and his collaborators in connection with the AUTOMATH project. Those works were inspiring for Per Martin-Löf to formulate CTT<sup>44</sup>.

The central idea of Martin – Löf's CTT is the “*propositions – as – types*” principle. According to this principle, propositions are identified with sets, types or

---

<sup>44</sup> (Per Martin – Löf, 1984). It was further developed in (Sundholm, 1986) and (Ranta 1994).

sets of their proof – objects. CTT has intuitionistic logic in its core and it is fully predicative. There are several features of CTT that we would like to bring to the fore:

- In CTT the clear distinction is made between judgments (or assertions) and propositions expressed by those judgments. In other words, propositions are contents of judgments. Judgments are certain linguistic acts that cannot be considered independently of a *context* in which they are made. This important notion of context makes the CTT approach to meaning fundamentally pragmatic, as pointed out in (Ranta, 1994: 2). This feature, among others, makes it very appealing to make a link between CTT and the dialogical approach to logic, as we will show later.

- Classical logic can be developed in the CTT framework (see Ranta, 1994: chapter 2).

- As opposed to a model – theoretic approach to meaning, where elements of a language are linked to the world (or interpreted) via some meta – level means, in CTT judgments and inferences are embedded at the object – level and they account, by means of inferential rules, for the meaning of every expression, so a fully interpreted language results. This carries out Frege’s idea of a *Begriffsschrift* – that should provide a full and explicit formalization of mathematics.

- We saw in the previous chapter that GTS, worked out by Hintikka, is model – theoretic oriented and used by Hintikka to defend a realist position. However, a game – theoretic approach doesn’t have to be either of those: in what follows, we will present the dialogical game – theoretic approach that is not linked to a model theory and has a constructivist origin. More of the common ground for dialogical logic and CTT will be explained in the following sections.

## *II 2. Dialogical approach to logic*

The dialogical approach to logic, in its contemporary form, was first proposed by Paul Lorenzen in middle 20<sup>th</sup> century and then it was developed by Kuno Lorenz.<sup>45</sup> Most of recent developments in this field are done by Shahid Rahman and his colleagues.<sup>46</sup> Dialogical semantics is shown to be the most fruitful for a study of validity of propositional and first-order formulas, both in their classical and intuitionistic version<sup>47</sup>, but also as a powerful tool for a study of some non-classical logics. In somewhat different, rhetorical tradition, dialogues are shown to be significant for a study of natural languages and for the theory of argumentation.<sup>48</sup>

The dialogical approach to logic is a game – theory semantic framework. A *dialogical game* is played by two players: Proponent, who posits *the thesis* and defend it and Opponent, who tries to falsify it. Actions or moves of two players can be understood as elements in the argumentation or certain speech- acts and they are of two kinds: declarative utterances ( posits) and interrogative utterances ( requests). The actions are guided by fixed rules and those rules provide the *meaning* of expressions

---

<sup>45</sup> Their most important papers about dialogical logic are published in Lorenzen, P. and Lorenz, K., 1978: *Dialogische Logik*. Darmstadt: Wissenschaftliche Buchgesellschaft. See also Lorenz, K., 2001: “Basic objectives of dialogue logic in historical perspective” in *Synthese* April 2001, Volume 127, Issue 1-2, pp 255-263

<sup>46</sup> A detailed account of recent developments since (Rahman, 1993), can be found in (Rahman, Keiff, 2005), (Keiff, 2009) and (Rahman, 2012). For the underlying metalogic see (Clerbout, 2013a,b). For textbook presentations: (Kamlah, Lorenzen, 1972, 1984), (Lorenzen, Schwemmer, 1975), (Redmond, Fontaine, 2011) and (Rückert, 2011a). For the key role of dialogic in regaining the link between dialectics and logic, see (Rahman, Keiff, 2010). (Keiff, 2004a,b, 2007) and (Rahman, 2009) study Modal Dialogical Logic. (Fiutek et al., 2010) study the dialogical approach to belief revision. (Clerbout, Gorisse, Rahman, 2011) studied Jain Logic in the dialogical framework. (Popek, 2012) develops a dialogical reconstruction of medieval *obligationes*. (Rahman, Tulenheimo, 2006) study the links between GTS and Dialogical Logic. For other books see (Redmond, 2010) – on fiction and dialogic – (Fontaine, 2013) – on intentionality, fiction and dialogues – and (Magnier, 2013) – on dynamic epistemic logic (van Ditmarsch et al. 2007) and legal reasoning in a dialogical framework. (this reference is taken from (Rahman, Clerbout, Jovanović, forthcoming)

<sup>47</sup> Dialogical logic for first - order logic is presented in the Appendix.

<sup>48</sup> This approach is developed by Perelman. See (Keiff, 2009).

in such a way that the meaning is not independent of the very act of uttering the expressions. This is the pragmatic feature that dialogical logic shares with CTT.

A dialogical game for a sentence  $\varphi$  starts off with  $\varphi$  posited as the thesis by Proponent. Then the players make moves one after the other, according to the rules, and such a sequence of their moves is called *a play*. Dialogical game is then the set of all possible plays for  $\varphi$ . It can be provided a tree representation: the root is the thesis and every path starting with the root is a linear representation of a play.

The dialogical approach to meaning and GTS, being both game – theoretically oriented, have apparently a lot in common<sup>49</sup>. Both dialogues and GTS approach the meaning in the so - called “outside – in” direction, as opposed to the Tarski - style “inside – out” semantics. The meaning of an expression is obtained in a game played for that expression starting with the whole expression and then reaching its component parts. However, dialogues have a more subtle apparatus to approach the meaning and we will elaborate on it later on in this thesis.

As in GTS, in the dialogical approach the distinction is made between the *play – level* (that is, winning of a local play) and the *strategy level* (the existence of a winning strategy). The notion of validity in formal dialogues and the notion of satisfaction in material dialogues amount to existence of a winning strategy for Proponent in a given game.

However, there are some important differences between the dialogical approach to logic and GTS:

---

<sup>49</sup> The link between the two approaches was first presented in the paper of Esa Saarinen in 1978. Not much work was done in this field until 2006 when Rahman and Tulenheimo studied the relation between GTS and dialogues for the propositional and first - order logic in a systematic way.

- First, unlike in GTS, in the dialogical framework there are two kinds of rules: *particle rules*, which are the rules for logical constants, and *structural rules* which determine the general course of a dialogical game. This division allows making a difference between the *local* and *global* meaning. The local meaning is player – independent.
- Dialogical framework is not the model - theoretic one. A sentence is said to be valid if there is a winning strategy in a dialogical game played for the sentence *independently of any model*. Recall that the notion of validity in GTS is understood as the existence of a winning strategy for Eloise in *every model*.
- In dialogues the distinction is made between *formal dialogues*, where the validity of a formula is determined and *material dialogues* where the satisfaction of a sentence is at stake. In the former, plays are played independently of knowing the meaning of elementary sentences in the main thesis. In the later the game is relative to a model. However, the idea is still to avoid specifying the model explicitly. Instead, we introduce a list of additional hypotheses as *initial concessions* of Opponent. Those hypotheses form then the root of the dialogue, together with the thesis defended by Proponent<sup>50</sup>. As explained in (Rahman, Tulenheimo, 2006 ):

*By contrast, the idea behind material dialogues is to avoid having an extra component to dialogues (such as a specification of a model); they are meant to do with the resources of dialogues designed for dealing with validity, and the idea is to ‘approximate’ a characterization of truth by adding a sufficient amount of additional hypotheses — taken to be initial concessions of*

---

<sup>50</sup> See (Keiff, 2009).



*Opponent* — which will serve to specify a model by using the resources of the object language only.

In (Rahman, Tulenheimo, 2006) the connection is made between GTS and the intuitionistic material dialogues for first - order logic in a constructive manner. It has been shown precisely how to construct a winning strategy for Proponent in the dialogue  $D(\varphi)$  for a first - order sentence  $\varphi$ , starting with the winning strategy for Eloise in the semantic game  $G(\varphi)$  and conversely, how the winning strategy for Proponent in the material dialog  $D(\varphi)$  gives rise to the winning strategy for Eloise in the semantic game  $G(\varphi)$ .

### ***II.3. Dialogical approach to CTT<sup>51</sup>***

We move on to present recent developments in the dialogical logic where the link is made between dialogical logic and CTT.

As mentioned previously, the idea behind CTT is to render meta - logical features explicit at the object – language level. This is actually very close to Wittgenstein’s arguments against formal Tarski – style semantics,<sup>52</sup> and that is one of Wittgenstein’s tenets that Hintikka (and other model - theoreticians) rejects. Moreover, the aim of rendering explicit the rules of meaning - explanation is inherent to the dialogical approach, where the meaning is constituted within certain interactions between players. In such a context it is natural to request for those interactions to be rendered explicit in the object – language. Indeed, one of the main insights of Lorenz’ interpretation of the relation between the so-called *first* and *second*

---

<sup>51</sup> This section is based on Rahman/Clerbout (2013, 2014). It will also appear in (Rahman, Clerbout, Jovanović, forthcoming).

<sup>52</sup> For a discussion on this see (Rahman, Clerbout, Jovanović, forthcoming).

Wittgenstein is based on a thorough criticism of the metalogical approach to meaning (Lorenz 1970, pp. 74-79). As pointed out by Kuno Lorenz, the heart of Wittgenstein's philosophy of language is the internal relation between language and world. The internal relation is what language games display while they constitute meaning: there is no way to ground a logical language outside the language (recall the case of Neurath's sailor in his raft):

*Also propositions of the metalanguage require the understanding of propositions, [...] and thus cannot in a sensible way have this same understanding as their proper object. The thesis that a property of a propositional sentence must always be internal, therefore amounts to articulating the insight that in propositions about a propositional sentence this same propositional sentence does not express anymore a meaningful proposition, since in this case it is not the propositional sentence that it is asserted but something about it.*<sup>53</sup>

*Thus, if the original assertion (i.e., the proposition of the ground-level) should not be abrogated, then this same proposition should not be the object of a metaproposition, [...].*<sup>54</sup> (Lorenz 1970, p.75).

---

<sup>53</sup> Similar criticism has been raised in (Sundholm, 1997, 2001) who points out that the standard model-theoretic approaches to meaning turn semantics into a meta-mathematical formal object where syntax is linked to semantics by the assignation of truth values to uninterpreted strings of signs (formulae). Language does not any more *express content* but it is rather conceived as a system of signs that speaks *about* the world - provided a suitable metalogical link between signs and world has been fixed

<sup>54</sup> The quote is a translation of the following original text:

*Auch Metaaussagen, so können wir zusammenfassen sind auf das Verständnis von Aussagen, [...] angewiesen, und können dieses Verständnis nicht sinnvoll zu ihrem Gegenstand machen. Die These, dass eine Eigenschaft eines Aussagesatzes stets intern sein muss, besagt daher nichts anderes, als die Artikulation der Einsicht, dass in Aussagen über einen Aussagesatz selbst nicht mehr der Ausdruck einer sinnvollen Aussage ist, nicht er wird behauptet, sondern etwas über ihn.*

*Wenn also die originale Behauptung, die Aussage der Grundstufe nicht ausser Kraft gesetzt werden soll, darf sie nicht zum Gegenstand einer Metaaussage gemacht werden, [...].* (Lorenz 1970, p.75).

*While originally the semantics developed by the picture theory of language aimed at determining unambiguously the rules of “logical syntax” (i.e. the logical form of linguistic expressions) and thus to justify them [...] – now language use itself, without the mediation of theoretic constructions, merely via “language games”, should be sufficient to introduce the talk about “meanings” in such a way that they supplement the syntactic rules for the use of ordinary language expressions (superficial grammar) with semantic rules that capture the understanding of these expressions (deep grammar).<sup>55</sup> (Lorenz 1970, p.109).*

In (Hintikka, 1996b), he made an extension of van Heijenoort’s distinction between the *language as the universal medium* and *language as a calculus* - the point is that the dialogical approach shares some tenets of both conceptions. Indeed, on the one hand, the dialogical approach shares with the universalists the view that we cannot place ourselves outside our language, on the other, it shares with the anti - universalists the view that we can develop a methodical reconstruction of a given complex linguistic practice out of the interaction of simple ones.

---

<sup>55</sup> The quote is a Rahman’s translation of the following original text:

*Diente ursprünglich die mit der Abbildtheorie entworfene Semantik dazu, die Regeln der ‘logischen Syntax’, also die logische Form sprachlicher Ausdrücke, eindeutig zu bestimmen und damit zu rechtfertigen [...]-, so soll jetzt der Sprachgebrauch selbst, ohne Vermittlung theoretischer Konstruktionen, allein auf dem Wege über die ‘Sprachspiele’, zur Einführung der Rede von ‘Bedeutungen’ hinreichen und die syntaktischen Regeln zur Verwendung gebrauchsprachlicher Ausdrücke (Oberflächengrammatik) mit semantischen, das Verständnis dieser Ausdrücke darstellenden Regeln (Tiefengrammatik), ergänzen. (Lorenz 1970, p.109).*

### *II. 3.1. Rules of formation of propositions*

We start now with presentation of dialogical rules for Constructive Type Theory. First, we need the rules for formation of propositions. Those rules are the local – semantic rules and together with the particle rules they provide the meaning to the logical constants in game – theoretical terms. The formation rules are indispensable to meet the request for a fully interpreted language. By those rules the players are allowed to ask explicitly during a game for the status of expressions. Otherwise, if an expression is well formed or not, would have to be examined on some meta – level, which we want to avoid in this approach.

The formation rules actually allow Opponent to question both syntactic and semantic features of the thesis. Those rules bring the main requests of CTT to the fore: request that propositions are identified with sets and that those sets are constructible. So, if Proponent posits the thesis  $\varphi$ , Opponent can ask him about the formation of  $\varphi$  - then Proponent can defend his position by stating that  $\varphi$  is a proposition (written  $\varphi: \text{prop}$ ) under condition that  $A$  is a set ( $A: \text{set}$ ). Opponent can then concede that  $A$  is the set if Proponent shows how  $A$  was constructed out of its elements.

When it comes to material dialogues we need to have a rule concerning elementary sentences. In principle, an elementary sentence can be attacked only by Opponent, through the formation rules. Here is the rule:

*O's elementary sentences cannot be challenged, however, O can challenge an elementary sentence (posited by P) iff she herself (the opponent) did not posit it before.*<sup>56</sup>

The formation rules are the following:

<b>Posit</b>	<b>Challenge</b> [when different challenges are possible, the challenger chooses]	<b>Defence</b>
$\mathbf{X} ! \Gamma : \text{set}$	$\mathbf{Y} ?_{\text{can}} \Gamma$ or $\mathbf{Y} ?_{\text{gen}} \Gamma$ or $\mathbf{Y} ?_{\text{eq}} \Gamma$	$\mathbf{X} ! a_1 : \Gamma, \mathbf{X} ! a_2 : \Gamma, \dots$ ( $\mathbf{X}$ gives the canonical elements of $\Gamma$ ) $\mathbf{X} ! a_i : \Gamma \Rightarrow a_j : \Gamma$ ( $\mathbf{X}$ provides a generation method) ( $\mathbf{X}$ gives the equality rule for $\Gamma$ )
$\mathbf{X} ! \varphi \vee \psi : \text{prop}$	$\mathbf{Y} ?_{\text{F}\vee 1}$ or $\mathbf{Y} ?_{\text{F}\vee 2}$	$\mathbf{X} ! \varphi : \text{prop}$  $\mathbf{X} ! \psi : \text{prop}$
$\mathbf{X} ! \varphi \wedge \psi : \text{prop}$	$\mathbf{Y} ?_{\text{F}\wedge 1}$ or $\mathbf{Y} ?_{\text{F}\wedge 2}$	$\mathbf{X} ! \varphi : \text{prop}$  $\mathbf{X} ! \psi : \text{prop}$
$\mathbf{X} ! \varphi \rightarrow \psi : \text{prop}$	$\mathbf{Y} ?_{\text{F}\rightarrow 1}$ or $\mathbf{Y} ?_{\text{F}\rightarrow 2}$	$\mathbf{X} ! \varphi : \text{prop}$  $\mathbf{X} ! \psi : \text{prop}$
$\mathbf{X} ! (\forall x : A) \varphi(x) : \text{prop}$	$\mathbf{Y} ?_{\text{F}\forall 1}$ or $\mathbf{Y} ?_{\text{F}\forall 2}$	$\mathbf{X} ! A : \text{set}$  $\mathbf{X} ! \varphi(x) : \text{prop } (x : A)$
$\mathbf{X} ! (\exists x : A) \varphi(x) : \text{prop}$	$\mathbf{Y} ?_{\text{F}\exists 1}$ or $\mathbf{Y} ?_{\text{F}\exists 2}$	$\mathbf{X} ! A : \text{set}$  $\mathbf{X} ! \varphi(x) : \text{prop } (x : A)$
$\mathbf{X} ! B(k) : \text{prop}$ (for atomic $B$ )	$\mathbf{Y} ?_{\text{F}}$	$\mathbf{X} \quad \text{sic} \quad (n)$ ( $\mathbf{X}$ indicates that $\mathbf{Y}$ posited it in move $n$ )

<sup>56</sup> Presentations of all rules are taken from (Rahman, Clerbout, Jovanović, forthcoming), based on (Rahman, Clerbout, 2014).

$X ! \perp : \text{prop}$	–	–
---------------------------	---	---

### II. 3. 2. Rules of substitution

Players are allowed to ask for a substitution in the context  $x_i : A_i$  either if a variable (or number of them) appears in a posit with a proviso (then the challenger posits an instantiation of the proviso) or in a formation – play. Here is the substitution – rule:

Posit	Challenge	Defence
$X ! \pi(x_1, \dots, x_n) (x_i : A_i)$	$Y ! \tau_1 : A_1, \dots, \tau_n : A_n$	$X ! \pi(\tau_1 \dots \tau_n)$

### II. 3. 3. Particle rules

Recall that in CTT propositions are identified with sets of their *proof – objects*. In the dialogical context, propositions are rather understood as sets of *play – objects*.<sup>57</sup> Notation  $p : \varphi$ , which is the form of players’ posits in a game, will be read as  $p$  is a *play-object for*  $\varphi$ . The meaning of those posits depends, in a game – theoretical manner, on the way how the play - objects are used in the game, that is, how those posits are challenged or defended. The form of play – objects depends on the structure

<sup>57</sup> Play – objects are actually different from proof – objects in CTT. It can be read from the treatment of the universal quantification and implication, which is different in the dialogical approach and in CTT. For discussion about it see (Rahman, Clerbout, 2014: 30) – as the authors point out there, the connection between dialogues and CTT is in the level of strategies rather than in the level of plays.

of an expression  $\varphi$  and they can be obtained of other play – objects during the game, according to the particle rules. The particle rules are the following:

Posit	Challenge	Defence
<b>X</b> $\phi$ (where no play-object has been specified for $\phi$ )	<b>Y</b> ? play-object	<b>X</b> $p : \phi$
<b>X</b> $p : \phi \vee \psi$	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $\phi \vee \psi : \text{prop}$
	<b>Y</b> ? $[\phi, \psi]$	<b>X</b> $L^\vee(p) : \phi$ <sup>58</sup> Or <b>X</b> $R^\vee(p) : \psi$ <b>[the defender has the choice]</b>
<b>X</b> $p : \phi \wedge \psi$	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $\phi \wedge \psi : \text{prop}$
	<b>Y</b> ? $[\phi]$ Or <b>Y</b> ? $[\psi]$ <b>[the challenger has the choice]</b>	<b>X</b> $L^\wedge(p) : \phi$ respectively <b>X</b> $R^\wedge(p) : \psi$
<b>X</b> $p : \phi \rightarrow \psi$	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $\phi \rightarrow \psi : \text{prop}$
	<b>Y</b> $L^\rightarrow(p) : \phi$	<b>X</b> $R^\rightarrow(p) : \psi$
<b>X</b> $p : \neg\phi$	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $\neg\phi : \text{prop}$
	<b>Y</b> $L^\perp(p) : \phi$	<b>X</b> $R^\perp(p) : \perp$
<b>X</b> $p : (\exists x : A)\phi$	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $(\exists x : A)\phi : \text{prop}$
	<b>Y</b> ? <sub>L</sub> Or <b>Y</b> ? <sub>R</sub> <b>[the challenger has the choice]</b>	<b>X</b> $L^\exists(p) : A$ Respectively <b>X</b> $R^\exists(p) : \phi(L(p))$
<b>X</b> $p : \{x : A \mid \phi\}$	<b>Y</b> ? <sub>L</sub> Or <b>Y</b> ? <sub>R</sub> <b>[the challenger has the choice]</b>	<b>X</b> $L^{\{\dots\}}(p) : A$ Respectively <b>X</b> $R^{\{\dots\}}(p) : \phi(L(p))$
<b>X</b> $p : (\forall x : A)\phi$	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $(\forall x : A)\phi : \text{prop}$
	<b>Y</b> $L^\forall(p) : A$	<b>X</b> $R^\forall(p) : \phi(L(p))$
<b>X</b> $p : B(k)$ (for atomic B)	<b>Y</b> ? <sub>prop</sub>	<b>X</b> $B(k) : \text{prop}$
	<b>Y</b> ?	<b>X</b> sic (n) (X indicates that Y posited it at move n)

<sup>58</sup> L and R are the abbreviations for the Left and Right part of the expression.



Note that in every rule there is first a challenge in the form “ $Y \text{ ?}_{\text{prop}}$ ”. With this rule  $Y$  is asking  $X$  to confirm that the expression he posited is a proposition in accord with the formation rules.

Expressions  $L(p)$  and  $R(p)$  are called *instructions*. They are used when the form of play – objects is not evident immediately. We read them as “take the left part of  $p$ ” and “take the right part of  $p$ ”.

The informal explanations of the particle rules are the following:

1. Disjunction: if  $X$  posits an expression in the form of disjunction  $\varphi \vee \psi$ , the play consists of two plays  $p_1$  and  $p_2$ , where  $X$  plays for  $\varphi$  in  $p_1$  and for  $\psi$  in  $p_2$ . It is up to  $X$  to switch from  $p_1$  to  $p_2$ , or the other way around, as he pleases.
2. Conjunction: if  $X$  posits an expression in the form of conjunction  $\varphi \wedge \psi$ , the play consists of two plays  $p_1$  and  $p_2$ , where  $X$  plays for  $\varphi$  in  $p_1$  and for  $\psi$  in  $p_2$ . It is  $Y$  who can switch from  $p_1$  to  $p_2$  or the other way around.
3. Implication: if  $X$  posits an expression in the form of implication  $\varphi \rightarrow \psi$ , then the play consists of two plays  $p_1$  and  $p_2$ , where in  $p_1$  it is  $Y$  who plays for  $\varphi$  and in  $p_2$  it is  $X$  who plays for  $\psi$ .
4. Negation: if  $X$  posits an expression in the form of negation  $\neg\varphi$  the play consists of two plays  $p_1$  and  $p_2$ , where in  $p_1$  it is  $Y$  who plays for  $\varphi$  and in  $p_2$  it is  $X$  who plays for  $\perp$ . It is up to  $X$  to switch from  $p_1$  to  $p_2$ , or the other way around, as he pleases. The negation is understood in intuitionistic manner, as the implication  $\varphi \rightarrow \perp$ . The negation rule amounts to switch of roles of players in a game.
5. The rules for quantifiers are somewhat more complicated. First, it should be noted that the quantifiers are written in the form  $(\exists x : A)\varphi$  and  $(\forall x : A)\varphi$ , like

in CTT. This form expresses the CTT idea that a quantification is always relative to a set of certain kind. Of course, we have to specify the kind of objects which form the set and that is done in a dialogue by means of the structural rule SR4.1. We will explain it later. This structural rule allows “resolving the instructions” and then the kind of objects in question is specified. Second, the rules for quantifiers underline two important ingredients of the meaning of quantifiers: the choice of a value of a bound variable and the instantiation of a formula when the bound variable is replaced with the chosen value.

6. There is one last rule left to explain, the one introduced between the rules for two quantifiers. It concerns a posit that X can make in the form  $\{x : A \mid \varphi\}$  which should be read as “those elements of A such that  $\varphi$ ”. Those expressions are used in CTT to deal with separate subsets. The idea behind such a posit is that there is at least one element  $L^{\{\dots\}}(p)$  of A that is a witness for  $\varphi(L^{\{\dots\}}(p))$ . This expression cannot be challenged by the question “Y ?<sub>prop</sub>” because there is no assumption that posit is a proposition: the expression stands here for a set.

Remark: Play objects for a conjunction and play - objects for an existentially quantified expression are of the same form. The same stands for play - objects for an implication and a universally quantified expression. There is a similarity between those logical constants in CTT, where the conjunction and existential quantifier are

two particular cases of the  $\Sigma$  operator, while the material implication and universal quantifier are two particular cases of the  $\Pi$  operator.<sup>59</sup>

There are some more rules to be introduced. First, there is the *definitional equality*.

$\forall$ -Equality rule : if player **X** posited the function  $f(x) : B(x)$  ( $x : A$ ) and if in a further move this function has been applied to the play - object  $k$ , then the challenger **Y** can force **X** to posit the equality  $f(k) = R^f(k)$ .

$\exists$ -Equality rule : If player **X** posited the pair of expressions  $k_1 : A$  and  $k_2 : B(k_1)$  (as response to a challenger on an  $\exists$  - move) , the challenger **Y** can ask **X** to posit:

$L^\exists = k_1 : A$  and/or  $R^\exists = k_2 : B(k_1)$ .<sup>60</sup>

The rules concerning the properties of sets are the following:

#### Reflexivity within set

Posit	Challenge	Defence
• <b>X</b> -- $A : \text{set}$	<b>Y</b> -? <sub>set</sub> - refl	<b>X</b> -- $A = A : \text{set}$

#### Symmetry within set

Posit	Challenge	Defence
• <b>X</b> -- $A = B : \text{set}$	<b>Y</b> -? <sub>B</sub> - symm	<b>X</b> -- $B = A : \text{set}$

#### Transitivity within set

Posit	Challenge	Defence
• <b>X</b> -- $A = B : \text{set}$	<b>Y</b> -? <sub>A</sub> - trans	<b>X</b> -- $A = C : \text{set}$

<sup>59</sup> For the full explanation of  $\Sigma$  and  $\Pi$  operators in CTT see (Ranta, 1994: chapter 2). For a discussion about the parallel with dialogical rules see (Rahman, Clerbout, 2014).

<sup>60</sup> See (Rahman, Clerbout, Jovanovic, forthcoming).

$X \dashv\vdash B = C : \text{set}$		
-------------------------------------	--	--

Reflexivity within A

Posit	Challenge	Defence
$\cdot$ $X \dashv\vdash a : A$	$Y \text{?-} a \text{ refl}$	$X \dashv\vdash a = a : A$

Symmetry within A

Posit	Challenge	Defence
$\cdot$ $X \dashv\vdash a = b : A$	$Y \text{?-} b \text{- symm}$	$X \dashv\vdash b = a : A$

Transitivity within A

Posit	Challenge	Defence
$\cdot$ $X \dashv\vdash a = b : A$ $X \dashv\vdash b = c : A$	$Y \text{?-} a \text{- trans}$	$X \dashv\vdash a = c : A$

Set-equality (or extensionality )

Posit	Challenge	Defence
$\cdot$ $X \dashv\vdash A = B : \text{set}$	$Y \text{?-}_{\text{ext}} a : A$	$X \dashv\vdash a : B$

Posit	Challenge	Defence
$\cdot$ $X \dashv\vdash A = B : \text{set}$	$Y \text{?-}_{\text{ext}} a = b : A$	$X \dashv\vdash a = b : B$

Set-substitution

Posit	Challenge	Defence
$\cdot$ $X \dashv\vdash B(x) : \text{set } (x : A)$	$Y \text{?-}_{\text{subst}} a = c : A$	$X \dashv\vdash B(a) = B(c) : \text{set}$

Posit	Challenge	Defence
• $\mathbf{X} \dashv\vdash C(x, y) : \text{set } (x : A, y : B(x))$	$\mathbf{Y} \text{-?}_{\text{subst}} a = c : A$	$\mathbf{X} \dashv\vdash C(a, y) = C(c, y) : \text{set}$

Posit	Challenge	Defence
• $\mathbf{X} \dashv\vdash C(x, y) : \text{set } (x : A, y : B(x))$	$\mathbf{Y} \text{-?}_{\text{subst}} b = d : B(x)$	$\mathbf{X} \dashv\vdash C(x, b) = C(x, d) : \text{set}$

Posit	Challenge	Defence
• $\mathbf{X} \dashv\vdash b(x) : B(x) (x : A)$	$\mathbf{Y} \text{-?}_{\text{subst}} - a = c : A$	$\mathbf{X} \dashv\vdash b(a) = b(c) : B(a)$

Before we present the structural rules there is one more rule that will be important for the treatment of branching quantifiers – the rule for the *function substitution*. This rule provides a mean to distinguish between a provisional clause where it is a defender who posits the expression within the clause and a provisional clause where it is a challenger who does it. If it is the defender who is committed to the proviso we use the notation  $\langle \dots \rangle$  and if it is the challenger, we use the notation  $(\dots)$ . It is a novelty comparing to standard CTT because we are formulating the rules in terms of a game theory and we have the players involved. Here is the rule:

Posit	Challenge	Defence
$\mathbf{X} ! p : \varphi [f(k_1)]$	$\mathbf{Y} f(k_1) ?$	$\mathbf{X} p : \varphi [k_2 / f(k_1)]$ $\langle \varphi [f(k_1)] = \varphi [k_2 / f(k_1)] : \text{set} \rangle$

Note that the defence requires of the defender to posit both the substitution

and the corresponding set equality. In other words, the defensive move is understood in the following manner:  $f(k_1)$  is substituted by  $k_2$ , provided that  $X$  posits  $\varphi [f(k_1)] = \varphi [k_2 / f(k_1)] : set$ . Thus, the rule could also be formulated as involving two challenges (and two defences). However, this might trigger two different plays (namely, when the challenger has chosen rank 1) and this might make the core quite complex (including variations in the order of the two challenges). The twofold formulation is the following:

Posit	Challenge	Defence
$X ! p : \varphi [f(k_1)]$	$Y L^{f(k_1)/?}$	$X p : \varphi [k_2 / f(k_1)]$
	$Y R^{f(k_1)/?}$	$X <\varphi [f(k_1)] = \varphi [k_2 / f(k_1)] : set >$

### II. 3. 4. Structural rules

As previously mentioned, the dialogical logic has a very subtle approach to meaning. The particle rules we presented earlier provide the *local meaning*. Now we will present the *structural rules* where the *global semantics* is achieved. Those rules determine a general course of a dialogical game.

**SR0 (Starting rule):** A dialogical game begins with the thesis posited by Proponent. Then both players choose their *repetition ranks*. A repetition rank determine for a player the number of challenges he can play related to the same move.

**SR1i (Intuitionistic Development rule):** After the thesis is posited and the repetition ranks chosen, the players continue playing one after the other. Each move is either a challenge or a defence in reaction to the previous move and it is guided by the particle rules. In the intuitionistic dialogues the players can answer only *the last non-answered*

challenge of the adversary.

[**SR1c (Classical Development rule)**: After the thesis is posited and the repetition ranks chosen, the players continue playing one after the other. Each move is either a challenge or a defence in reaction to the previous move and it is guided by the particle rules. In the classical dialogues the players can answer any challenge – even the one which had been already answered.]

**SR2 (Formation first)**: The first challenge Opponent lunches is the request '?<sub>prop</sub>'. Then the formation rules are applied and Proponent must defend that the thesis is indeed a proposition. After this challenge Opponent can play following other particle and structural rules.

**SR3 (Modified Formal rule)**: Proponent is not allowed to challenge Opponent's elementary sentences. Opponent is allowed to challenge Proponent's elementary sentences if Opponent didn't concede them himself earlier.

**SR4.1 (Resolution of instructions)**: If a player posits a move with the instructions  $I_1, \dots, I_n$ , his adversary can ask him to replace these instructions (or some of them) by the suitable play - objects. Those play – objects are chosen by the defender of the instructions, except in the following cases: if there is a challenge of a universally quantified expression or implication and the instruction (or the list of instructions) occurs at the right of the colon and the posit is the tail of that universally quantified expression or the tail of the implication. In those cases it is the challenger who can choose play-objects. The rule is illustrated bellow:

Posit	Challenge	Defence
		$X \pi(b_1, \dots, b_m)$ - if the instruction that occurs at

$X \pi(I_1, \dots, I_n)$	$Y I_1, \dots, I_m/? \quad (m \leq n)$	<p>the right of the colon is the tail of either a universal or an implication (such that <math>I_1, \dots, I_n</math> also occur at the left of the colon in the posit of the head), then <b><math>b_1, \dots, b_m</math> are chosen by the challenger</b></p> <p>- Otherwise <b>the defender chooses</b></p>
--------------------------	--	---

It can happen that instructions are embedded in the form  $I_1(\dots(I_k)\dots)$ . In that case the rule implies to make the substitutions in the opposite direction: from  $I_k$  to  $I_1$ .

**SR4.2 (Substitution of instructions):** If one of the players choose the play – object  $b$  for the instruction  $I$ , and player  $X$  posits  $\pi(I)$  at some point, then  $Y$  can ask  $X$  to substitute  $I$  with  $b$  in any posit  $X \pi(I)$ . The rule is illustrated below:

Posit	Challenge	Defence
<b>Player 1</b> $\pi_i(I)$ <b>Player 2</b> $I/?$ <b>Player 3</b> $\pi_i(b)$ ... <b>X--</b> $\pi_j(I)$	$Y ? b/I$	$X \pi_j(b)$

In the rules **SR4.1** and **SR4.2**, in which the play – objects are being substituted for the instructions, serve as a tool to interpret, in a satisfactory manner, some puzzling natural – language sentences, such as those where anaphoric expressions occur. We mentioned in the first section of this thesis that Hintikka conceived IF logic as the true logic of natural language (that is, of the natural – language fragment that we consider to be first – order). One of his arguments was that some natural – language sentences need the explicit treatment of dependences between quantifiers



and the specification of their scope in order to be well interpreted. According to Hintikka and his associates, GTS is the most satisfactory semantics of anaphoric expressions. We will argue later that with the explicit CTT language in the dialogical framework we have all the means needed to make the account of the quantifier dependencies and scopes and to give a satisfactory interpretation of anaphora. The last chapter of this thesis will be consacrated to this problem.

**SR5 (Winning rule for dialogues)** If one of the players posits “ $p : \perp$ ” in a dialogue, he looses the play. Otherwise the player who makes the last move in a dialogue wins.

At the end it remains to precise that the formula is valid if there is a winning strategy for Proponent in dialogical game played for that formula. In material dialogues we say that the sentence at stake is satisfied in a given model.

Strategy is defined in a similar manner as in GTS: it is function assigning to a player a move, in accordance with particle and structural rules, every time it is that player’s turn to move.

Now we have enough of tools at our disposal. We proceed to examine some of Hintikka’s favorite accomplishments of GTS: the justification of the axiom of choice and the interpretation of anaphoric expressions.

### *III Hintikka's take on the axiom of choice and the constructivist challenge*<sup>61</sup>

In this section we will compare and contrast Per Martin- L f's analysis of the axiom of choice with Jaakko Hintikka's standing on this axiom. Hintikka claims that GTS justifies Zermelo's axiom of choice in a first-order way, perfectly acceptable for the constructivists. In fact, Martin- L f's results lead to the following considerations:

- 1) Hintikka preferred version of the axiom of choice is indeed acceptable for the constructivists and its meaning does not involve higher - order logic.
- 2) However, the version acceptable for constructivists is based on an intensional take on functions. The property of extensionality is the heart of the classical understanding of Zermelo's axiom and this is the real reason behind the constructivists rejection of it.
- 3) More generally, dependence and independence features that motivate IF logic, can be formulated within the CTT frame, without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference – logic is about inference after all.

We will conclude pointing out that recent developments in dialogical logic show that the CTT approach to meaning in general and to the axiom of choice in particular is very natural to game theoretical approaches where (standard) meta - logical features are explicitly displayed at the object - language level. In some way, this vindicates Hintikka's plea for the fruitfulness of game-theoretical semantics in the

---

<sup>61</sup> This chapter is based on (Jovanovi , 2013).

foundations of mathematics. In fact, from the dialogical point of view, those actions that constitute the meaning of logical constants, such as choices, are the crucial element of its full-fledged local semantics.

### ***III. 1. Recapitulation: Hintikka's take on the axiom of choice***

As explained in the first chapter of this thesis, in GTS the truth and falsity conditions are defined via the notion of winning strategy. A strategy is defined by a finite set of choice functions or *Skolem functions*. Values of functions indicate which individuals Verifier has to choose in his actions when the game is played for an existential quantifier (or which disjunct when the game is played for a disjunction) in order to win the game and which individuals Falsifier has to choose when the game is played for a universal quantifier (or which conjunct when the game is played for a conjunction). Remember the example from section **I.5.2.**:

$$1) \forall x \exists y C(x, y)$$

This sentence is true if there is a winning strategy for Eloise. The winning strategy shows her how to select a value for  $y$  in function of the value of  $x$  ( $f(x) = y$ ). If we wish to express the existence of such a strategy, we state the following:

$$2) \exists f \forall x C(x, f(x))$$

If we link 1) and 2) with a conditional we obtain the following formulation of the *axiom of choice*:

$$3) \forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x,f(x))$$

From the GTS point of view, the truth of 3), expressing a version of the axiom of choice, is derived from the very definition of truth. In fact, as we will discuss further on, it is related to the truth of the universal.

Let us repeat that the axiom of choice is essential for GTS, since without it the Tarski- style semantic and GTS for the classical first - order logic would not be equivalent. The reason is that the strategies in GTS are understood as *deterministic* strategies which impose to Verifier and Falsifier their choices, leaving no real options. However, Hintikka insists that there is nothing troubling with the axiom of choice and that it actually constitutes our conception of truth. Recall quotation priviously given:

*This paradigm problem concerns the status of the axiom of choice. This axiom was firmly rejected by Brouwer and it was mooted in the controversies between the French intuitionists and their opponents....The axiom of choice is true. The idea of “choosing” or “finding” suitable individuals is systematised in what is known as game-theoretical semantics. For mathematicians, this semantics is no novelty, however, but little more than a regimentation and generalisation of the way of thinking that underlies mathematicians’ classical (or perhaps I should say Weierstrassian) epsilon-delta analyses of the basic concepts of calculus, such as continuity and differentiation...*

*...To return to the usual axiom of choice, it is thus seen to be unproblematically true. How can any intuitionist deny the axiom of choice...? What can possibly go wrong here? Moreover, evoking the concept of knowledge, either in the form of epistemic logic or informally, does not seem to help an intuitionistic critic of the axiom of choice at all, either. The*

*discussion of the axiom of choice between intuitionists and classicists has conducted at cross-purposes. It can only be dissolved by making distinction between knowing that and knowing what that neither party has made explicit. (Hintikka, 2001)*

*Zermelo did not begin to axiomatize set theory unselfishly from the goodness of his theoretical heart. His main purpose was to justify his well-ordering theorem. In practice, this largely meant to justify the axiom of choice. [ ... ]. But that is not the full story. Worse still: Zermelo's specific enterprise was unnecessary, in that the so-called axiom of choice turns out to be in the bottom a plain first-order logical principle. (Hintikka, 2011)*

In fact, as we will show below, the last sentence is true: the truth of the axiom of choice involves only a first-order logic. However, in contrast to what Hintikka argues against intuitionists, the axiom of choice is true in the context of intuitionistic first - order logic enriched with the axioms of ZF-set theory, formulated in the style of CTT.

### ***III.2. Martin-Löf on the axiom of choice***

Let us give briefly an outline of the axiom of choice. It is well known that it had been first introduced by Zermelo in 1904 in order to prove the well – ordering theorem<sup>62</sup>. Zermelo gave two formulations of this axiom, the first in 1904 and the

---

<sup>62</sup> The theorem states that every set can be well – ordered.

second in 1908. The second formulation is relevant for our discussion, since it is related to both Hintikka's and Martin-Löf's formalization:

*A set  $S$  that can be decomposed into a set of disjoint parts  $A, B, C, \dots$  each of them containing at least one element, possesses at least one subset  $S_1$  having exactly one element with each of the parts  $A, B, C, \dots$  considered. (Zermelo, 1908)*

The Axiom immediately attracted the attention and both of its formulations were criticized by the constructivists such as Baire, Borel, Lebesgue and Brouwer. The first objections were related to the non-predicative character of the axiom because a choice function was supposed to exist without showing constructively that it did. However, the axiom found its way into ZFC set theory and it was finally accepted by majority of mathematicians because of its usefulness in different branches of mathematics.

Martin-Löf produced a proof of the axiom in a constructivist setting bringing together two seemingly incompatible perspectives on this axiom, namely:

- 1) Bishop's surprising observation from 1967: *A choice function exists in constructive mathematics, because a choice is implied by the very meaning of existence.*
- 2) The proof by Diaconescu (in 1975) and by Goodman and Myhle (in 1978) that the Axiom of Choice implies the law of the excluded middle.

In the paper of (Martin-Löf, 2006), he shows that there are indeed some versions of the axiom of choice that are perfectly acceptable for a constructivist, namely the ones where the choice function is defined *intensionally*. In order to see

this, the axiom must be formulated within the frame of a CTT-setting. Indeed such a setting allows comparing the extensional and the intensional formulation of the axiom. It is in fact the extensional version that implies the law of the excluded middle, whereas the intensional version is compatible with Bishop's remark:

[...] *this is not visible within an extensional framework, like Zermelo-Fraenkel set theory, where all functions are by definition extensional.*" (Martin-Löf, 2006, p.349)

In CTT the truth of the axiom actually follows rather naturally from the meaning of quantifiers:

Take the proposition  $(\forall x: A) B(x)$  where  $B(x)$  is of the type proposition provided that  $x$  is an element of the set  $A$ . If the proposition is true; then there is a proof for it. Such a proof is in fact a function that for every element  $x$  of  $A$  renders a proof of  $B(x)$ . This is how Bishop's remark should be understood: the truth of a universal amounts to the existence of a proof, and this proof is a function. Thus, the truth of a universal amounts, in the constructivist account, to the existence of a function. The proof of the axiom of choice can be developed from it quite straightforwardly. If we recall that in the CTT-setting:

1) the existence of a function from  $A$  to  $B$  amounts to the existence of the proof-objects for the universal *every  $A$  is  $B$* , and that

2) the proof of the proposition  $Bx$ , existentially quantified over the set  $A$ , amounts to a pair such that the first element of the pair is an element of  $A$  and the second element of the pair is a proof of  $Bx$ .

The full-fledged formulation of the axiom of choice – where we make explicit the set over which the existential quantifiers are defined – is the following:

$$(\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\forall x : A) C(x, f(x))$$

The proof from (Martin-Löf, 1980, p. 50-51) is the following:

*The usual argument in intuitionistic mathematics, based on the intuitionistic interpretation of the logical constants, is roughly as follows: to prove  $(\forall x)(\exists y)C(x,y) \rightarrow (\exists f)(\forall x)C(x,f(x))$ , assume that we have a proof of the antecedent. This means we have a method which, applied to an arbitrary  $x$ , yields a proof of  $(\exists y)C(x,y)$ . Let  $f$  be the method which, to an arbitrarily given  $x$ , assigns the first component of this pair. Then  $C(x,f(x))$  holds for an arbitrary  $x$ , and hence, so does the consequent. The same idea can be put into symbols getting a formal proof in intuitionistic type theory. Let  $A : \text{set}$ ,  $B(x) : \text{set } (x : A)$ ,  $C(x,y) : \text{set } (x : A, y : B(x))$ , and assume  $z : (\Pi x : A)(\Sigma y : B(x))C(x,y)$ . If  $x$  is an arbitrary element of  $A$ , i.e.  $x : A$ , then by  $\Pi$ -elimination we obtain*

$$Ap(z,x) : (\Sigma y : B(x))C(x,y)$$

*We now apply left projection to obtain*

$$p(Ap(z,x)) : B(x)$$

*and right projection to obtain*

$$q(Ap(z,x)) : C(x,p(Ap(z,x))).$$

*By  $\lambda$ -abstraction on  $x$  (or  $\Pi$ -introduction), discharging  $x : A$ , we have*

$$(\lambda x) p(Ap(z,x)) : (\Pi x : A)B(x)$$

*and by  $\Pi$ -equality*

$$Ap((\lambda x) p(Ap(z,x), x) = p(Ap(z,x))) : Bx.$$

*By substitution [making use of  $C(x,y) : \text{set } (x : A, y : B(x))$ ,] we get*

$$C(x, Ap((\lambda x) p(Ap(z,x), x) = p(Ap(z,x)))) = C(x, p(Ap(z,x)))$$

*[that is,  $C(x, Ap((\lambda x) p(Ap(z,x), x) = p(Ap(z,x)))) : \text{set}$ ]*



and hence by equality of sets

$$q(Ap(z,x)): C(x, Ap((\lambda x) p (Ap(z,x), x))$$

where  $((\lambda x) p (Ap(z,x)))$  is independent of  $x$ . By abstraction on  $x$

$$((\lambda x) p (Ap(z,x))): (\Pi x: A)C(x, Ap((\lambda x) p (Ap(z,x), x))$$

We now use the rule of pairing (that is  $\Sigma$ -introduction) to get

$$(\lambda x) p(Ap(z,x)), (\lambda x) q(Ap(z,x)): (\Sigma f: (\Pi x: A)B(x))(\Pi x: A)C(x, Ap(f,x))$$

(note that in the last step, the new variable  $f$  is introduced and substituted for

$((\lambda x) p (Ap(z,x)))$  in the right member). Finally by abstraction on  $z$ , we obtain

$$(\lambda z)((\lambda x) p (Ap(z,x)), (\lambda x) q (Ap(z,x))): (\Pi x: A)(\Sigma y: B(x))C(x,y) \rightarrow$$

$$(\Sigma f: (\Pi x: A)B(x))(\Pi x: A)C(x, Ap(f,x)).$$

Curiously, this seems to be close to Hintikka's own formulation and even to his analysis that a winning strategy for a universal amounts to the existence of a (Skolem) function. It is curious since Martin-Löf's proof is developed within a constructivist setting. Moreover, what is shown in (Martin-Löf, 2006) is that what is wrong with the axiom, from the constructivist point of view, is its extensional formulation, that is:

$$(\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\text{Ext}(f) \ \& \ (\forall x : A) C(x, f(x)))$$

$$\text{Where } (\text{Ext}(f) = ((\forall i, j : A) (i =_A j \rightarrow f(i) = f(j)))$$

Therefore, from the constructivist point of view, what is really wrong with the classical formulation of the axiom of choice is the assumption that from the truth that *all of the A are B* we can obtain a function that satisfies the property of extensionality. In fact, as shown in (Martin-Löf, 2006), the classical version holds, even

constructively, if we assume that there is only one such choice function in the set at stake:

$$(\forall x : A) (\exists ! y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\text{Ext}(f) \ \& \ (\forall x : A) C(x, f(x)))$$

Let us retain that:

1. If we take  $(\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\forall x : A) C(x, f(x))$  to be the formalization of the axiom of choice, then that axiom is not only unproblematic for a constructivist but it is also a theorem. But this formalization is a full-fledged formulation of the version Hintikka's adopts.<sup>63</sup> Certainly, the point is that the CTT - formulation stresses explicitly that the choice function at stake has been defined by means of intensional equality but Hintikka seems to assume the extensionality. In fact it is the CTT-explicit language that allows a fine - grained distinction between, on the surface, the equivalent formulations. This is due to the expressive power of CTT that allows expressing at the object language level properties that in other settings are left implicit at the metalanguage. This leads us to the second point;
2. According to the constructivist approach functions are identified with proof-objects for propositions and are given in the *object-language*, as the objects of a certain type. Understood in that way, functions belong to the lowest - level of entities and there is no jumping to higher order. Once more, the truth of a first - order universal sentence amounts to the existence of a function that is defined by means of the elements of the set over which the universal

---

<sup>63</sup> Indeed, Martin-Löf's formalization follows from making explicit in Hintikka's formulation  $\forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x,f(x))$  the range of its quantifiers, that is:  $\forall x$  quantifies over, say the set A,  $\exists y$  quantifies over, say the set Bx, and  $\exists f$ , over the set  $(\forall x : A) Bx$ .

quantifies and the first-order expression  $Bx$ . The existence of such a function is the CTT - way to express at the object language level, that a given universal sentence is true.

Thus, Hintikka is right in defending that we need only a first - order language, but this does not really support his attachment to the classical understanding of it.

### III.3. Axiom of choice in dialogical framework

Let us use now the framework we have presented above where the dialogical approach to logic is linked with CTT. Shahid Rahman and Nicolas Clerbout worked out a dialogical proof of the constructive formulation of the axiom of choice in (Rahman, Clerbout, 2014). They proved thus Martin L of's point in a game – theory framework. Here is the proof<sup>64</sup>:

*First play: Opponent's 9<sup>th</sup> move asks for the left play object for the existential quantification on  $f$*

O		P	
	H1: $C(x, y) : set(x : A, y : B(x))$ H2: $B(x) : set(x : A)$		$p : (\forall x : A) (\exists y : B(x)) C(x, y) \rightarrow (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x, f(x))$ 0
1	m:= 1		n:= 2 2
3	$L^{\rightarrow}(p) : (\forall x : A) (\exists y : B(x)) C(x, y)$ 0		$R^{\rightarrow}(p) : (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x, f(x))$ 6
5	$v : (\forall x : A) (\exists y : B(x)) C(x, y)$	3	$L^{\rightarrow}(p) /?$ 4
7	$R^{\rightarrow}(p) /?$ 6		$(v, r) : (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x, f(x))$ 8
9	$L?$ 8		$L^{\exists}(v, r) : (\forall x : A) B(x)$ 10
11	$L^{\exists}(v, r) /?$ 10		$v : (\forall x : A) B(x)$ 12
13	$L^{\forall}(v) : A$ 12		$R^{\forall}(v) : B(w)$ 26

<sup>64</sup> From (Rahman, Clerbout, 2014). It will also appear in (Rahman, Clerbout, Jovanović, forthcoming). In the present proof two plays are extracted from the extensive tree of all the plays – those which constitute the so -called core of the strategy (that is, of the dialogical proof). The proof that these plays render the corresponding CTT demonstration can be found in (Rahman, Clerbout, 2014).

15	$w : A$		13	$L^{\forall}(v) : /?$	14
19	$R^{\forall}(v) : (\exists y : B(w)) C(w, y)$		5	$L^{\forall}(v) : A$	16
17	$L^{\forall}(v) /?$	16		$w : A$	18
21	$(t_1, t_2) : (\exists y : B(w)) C(w, y)$		19	$R^{\forall}(v) /?$	20
23	$L^{\exists}((t_1, t_2) : B(w))$		21	$L?$	22
25	$t_1 : B(w)$		23	$L^{\exists}(t_1, t_2) /?$	24
27	$R^{\forall}(v) /?$	26		$t_1 : B(w)$	28

**Description:**

**Move 3:** After setting the thesis and establishing the repetition ranks **O** launches an attack on material implication.

**Move 4:** **P** launches a counterattack and asks for the play object that corresponds to  $L^{\rightarrow}(p)$ .

**Moves 5, 6:** **O** responds to the challenge of 4 . **P** posits the right component of the material implication.

**Moves 7, 8:** **O** ' asks for the play object that corresponds to  $R^{\rightarrow}(p)$ . **P** responds to the challenge by choosing the pair  $(v, r)$  where  $v$  is the play object chosen to substitute the variable  $f$  and  $r$  the play object for the right component of the existential.

**Move 9:** **O** has here the choice to ask for the left or the right component of the existential. The present play describes the development of the play triggered by the left choice.

**Moves 10-26:** follow from a straightforward application of the dialogical rules. Move 26 is an answer to move 13, since **P** decided to have enough information to apply the characteristic – copy-cat method imposed by the formal rule.

**Move 27-28:** **O** asks for the play object that corresponds to the instruction posited by **P** at move 26 and **P** answers and **wins** by applying copy-cat to **O**'s move 25. Notice that 28 this is not a case of function substitution: it is simply the resolution of an instruction.

**Second play:** Opponent's 9<sup>th</sup> move asks for the right play object for the existential quantification on  $f$

<b>O</b>		<b>P</b>		
	H1: $C(x, y) : set(x : A, y : B(x))$ H2: $B(x) : set(x : A)$		$p : (\forall x : A) (\exists y : B(x)) C(x, y) \rightarrow (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x, f(x))$	0
1	m:= 1		n:= 2	2
3	$L^{\rightarrow}(p) : (\forall x : A)$	0	$R^{\rightarrow}(p) : (\exists f : (\forall x : A) B(x)) (\forall x : A)$	6

	$(\exists y : B(x)) C(x, y)$			$C(x, f(x))$	
5	$v : (\forall x : A) (\exists y : B(x)) C(x, y)$		3	$L^{\rightarrow}(p) / ?$	4
7	$R^{\rightarrow}(p) / ?$	6		$(v, r) : (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x, f(x))$	8
9	$R_?$	8		$R^{\exists}(v, r) : (\forall x : A) C(x, L^{\exists}(v, r)(x))$	10
11	$L^{\exists}(v, r) / ?$	10		$R^{\exists}(v, r) : (\forall x : A) C(x, v(x))$	12
13	$R^{\exists}(v, r) / ?$	12		$r : (\forall x : A) C(x, v(x))$	14
15	$L^{\forall}(r) : A$	14		$R^{\forall}(r) : C(x, v(w))$	32
17	$w : A$		15	$L^{\forall}(r) : / ?$	16
21	$R^{\forall}(v) : (\exists y : B(w)) C(w, y)$		5	$L^{\forall}(v) : A$	18
19	$L^{\forall}(v) / ?$	18		$w : A$	20
23	$(t_1, t_2) : (\exists y : B(x)) C(x, y)$		21	$R^{\forall}(v) / ?$	22
25	$L^{\exists}((t_1, t_2) : B(w))$		23	$L_?$	24
27	$t_1 : B(w)$		25	$L^{\exists}(t_1, t_2) / ?$	26
29	$R^{\exists}(t_1, t_2) : C(w, t_1)$		23	$R_?$	28
31	$t_2 : C(w, t_1)$		29	$R^{\exists}(t_1, t_2) / ?$	30
33	$R^{\forall}(r) / ?$	32		$t_2 : C(w, v(w))$	34
35	$v(w) / ?$	34		$t_2 : C(w, t_1)$ $< C(w, t_1) = C(w, t_1 / v(w)) : set >$	42
41	$C(w, t_1) = C(w, t_1 / v(w)) : set$		H1?subst	$v(w) = t_1 : B(w)$	36
37	$v(w) = t_1 : B(w) ?$	36		sic (39)	40
39	$v(w) = t_1 : B(w)$		5, 18, 21, 25	? $\forall$ -eq.	38

**Description:**

**Move 9:** Until move 9 this play is the same as the previous. In the present play, in move 9 the Opponent chooses to ask for the right-hand side of the existential posited by **P** at 8.

**Moves 10-34:** the Proponent substitutes the variable  $f$  by the instruction correspondent to the left-hand component of the existential, i.e.,  $L^{\exists}(v, r)$ . By this **P** accounts for the dependence of the right-hand part on the left-hand component. The point is that the local meaning of the existential requires this dependence of the right component to the left component even if in this play the Opponent, due to the restriction on rank 1, she can ask only for the right-hand part

The conceptually interesting moves start with 35, where the opponent asks to **P** to substitute the function. As already pointed out, in order to respond to 35 the opponent's move 31 is not enough. Indeed the proponent needs also to posit  $C(w, t_1) = C(w, t_1 / v(w)) : set$ . **P** forces **O** to

concede this equality (41), on the basis of the substitutions  $w / x$  and  $t_1 / y$  on  $H1$  (we implemented the substitution directly in the answer of  $\mathbf{O}$ ) given the  $\forall$ -equality  $v(w) = t_1$  in  $B(w)$  (36), and given that this  $\forall$ -equality yields the required set equality. Moreover,  $\mathbf{P}$ 's posit of the  $\forall$ -equality (36) is established and defended by moves 38-40.

Thus, in the dialogical framework the point is shown straightforwardly. The axiom of choice is valid in a constructive setting only if the choice function is defined intensionally.

A winning strategy for Proponent in a dialogical game for the implication at stake follows immediately from the very meaning of the antecedent. Proponent wins the play because he copies the choice that Opponent made for  $y$  in the antecedent and uses it for  $f(x)$  in the consequence. It is possible because both  $y$  and  $f(x)$  are objects of the same type:  $B(x)$ , given that  $x : A$ . It is important to notice that  $C(x, y)$  in the antecedent and  $C(x, f(x))$  in the consequence are two *intensionally* equal sets for every  $x : A$ . It is thus the intensionality that is sine qua non of the proof of the axiom of choice in a constructive setting, unless the uniqueness of the function is assumed.

Let us entertain for the moment the idea that  $C(x, y)$  were defined *extensionally*. In that case, for any extensionally equal  $a$  and  $b$  in  $A$ , we can substitute  $a : A$  occurring at the place of  $x$  in  $C(x, y)$  with  $b : A$ ; and the similar goes for every  $c : Bx$  ( $x : A$ ), as pointed out in (Rahman, Clerbout, 2014, appendix). Rahman and Clerbout (Ibid.) worked out the dialogical rule for extensionality of function in the following manner:

<b>Posit</b>	<b>Challenge</b>	<b>Defence</b>
$\mathbf{X} p : Ext(f)$ (where $f : A \rightarrow B$ )	$\mathbf{Y} - L^{Ext(f)}(p) : k_i =_A k_j$ ( where $k_i$ and $k_j$ are elements of $A$ chosen by the challenger)	$\mathbf{X} R^{Ext(f)}(p) : f(k_i) =_B f(k_j)$

The idea underlying the rule should be clear: The play object  $p$  is constituted by a pair, the left and the right side of  $p$ . If  $X$  posited that  $f$  ( $f : A \rightarrow B$ ) is extensional, then the left part of  $p$  is a play object for the extensional equality (in  $A$ ) - posited by  $Y$  - of two elements of  $A$  chosen by the challenger. The right part of  $p$  is a play object for the extensional equality (in  $B$ ) – posited by  $X$  - of the two functions that take as arguments the elements of  $A$  chosen by the challenger.

We will also need also a rule for substitution in extensional relations:

Posit	Challenge	Defence
• <b>X</b> - $p : Ext(C(k_i, k_z)) (k_i : A, k_z : B)$	<b>Y</b> - $L_1^{Ext(Cx,y)}(p) : k_j =_A k_i$	<b>X</b> - $R_1^{Ext(Cx,y)}(p) C(k_j, k_z)$
<b>X</b> - $p : Ext(C(k_i, k_z)) (k_i : A, k_z : B)$	<b>Y</b> - $L_2^{Ext(Cx,y)}(p) : k_w =_B k_z$	<b>Y</b> - $R_2^{Ext(Cx,y)}(p) : C(k_i, k_w)$

Now, the point is that, unless the unicity of the function is assumed, in a dialogical game for the extensional formulation of the axiom of choice Proponent cannot have a winning strategy because, even when Opponent concede that there is a function at stake (as he does in the intensional formulation of the axiom), he cannot be forced to concede the extensionality of that function. The extensional equality of two functions  $f(a)$  and  $f(b)$ , for some  $a$  and  $b$  chosen by Opponent (extensionally equal in  $A$ ), can be obtained from the earlier concessions of Opponent in the game only under the assumption that there is only one such function i.e. if Opponent conceded that there is only one element of the set  $Bx$ . In the last case, when the unicity of the

function is assumed, there is a dialogical proof for the validity of the extensional version of the axiom of choice (Rahman, Clerbout, *ibid*):

O		P		
	$H_1: t : (\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\forall x : A) C(x, f(x))$  $H_2: L^{\exists}(R^{\rightarrow}(t)) = L^{\exists}(R^{\rightarrow}(p)) = g$  $: (\forall x : A) Bx$  $H_3: Ext(C)$ (where $C$ occurs in $g$ )		$p : (\forall x : A) (\exists !y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) Ext(f) \wedge (\forall x : A) C(x, f(x))$	0
1	m:= 1		n:= 2	2
3	$L^{\rightarrow}(p) : (\forall x : A) (\exists !y : Bx) C(x, y)$	0	$R^{\rightarrow}(p) : (\exists f : (\forall x : A) Bx) Ext(f) \wedge (\forall x : A) C(x, f(x))$	6
5	$g : (\forall x : A) (\exists !y : Bx) C(x, y)$	3	$L^{\rightarrow}(p) / ?$	4
7	$R^{\rightarrow}(p) / ?$	6	$(g, r) : (\exists f : (\forall x : A) Bx) Ext(f) \wedge (\forall x : A) C(x, f(x))$	8
9	$R_?$	8	$R^{\exists}(g, r) : Ext(L^{\exists}(g, r) \wedge (\forall x : A) C(x, L^{\exists}(g, r)(x)))$	10
11	$L^{\exists}(g, r) / ?$	1 0	$R^{\exists}(g, r) : Ext(g) \wedge (\forall x : A) C(x, g(x))$	12
13	$R^{\exists}(g, r) / ?$	1 2	$r : Ext(g) \wedge (\forall x : A) C(x, g(x))$	14
15	$L?$		$L^{\wedge}(r) : Ext(g)$	16
17	$L^{\wedge}(r) / ?$		$s : Ext(g)$	18
19	$L^{Ext(g)}(s) : a =_A b <a : A, b : A>$	1 8	$w / R^{Ext(g)}(s) : g(a) =_{Ba} g(b)$	36
21	$R^{\rightarrow}(t) : (\exists f : (\forall x : A) Bx) (\forall x : A) C(x, f(x))$		$H_1$ $L^{\rightarrow}(t) : (\forall x : A) (\exists !y : Bx) C(x, y)$	20
23	... $u / R^{\exists}(g, u) : (\forall x : A) C(x, g(x))$		$H_2$ ... (copy-cat 7-14)	22
25	$v / R^{\forall}(u) : C(a, g(a))$		23 $a / L^{\forall}(u) : A$	24
27	$v / R^{\forall}(u) : C(b, g(b))$		23 $b / L^{\forall}(u) : A$	26
29	$v : Ext(C(b, g(b)))$		27, $H_3$ $?Ext(C)$	28
31	$v : C(a, g(b))$		29 $v : a =_A b$	30
33	$g : (\forall x : A) (\forall y : Bx) (\forall z : Bx) C(x, y) \wedge C(x, z) \rightarrow y =_{Bx} z$	5	- ? $\exists!$	32
35	$w / R^{\rightarrow}(g) : g(a) =_{Ba} g(b)$		33 $v / L^{\rightarrow}(g) : C(a, g(a)) \wedge C(a, g(b)) (a : A, g(a), g(b) : Ba)$	34

**Description:**

**Move 3:** After setting the thesis and establishing the repetition ranks **O** launches an attack on material implication.



**Move 4:** **P** launches a counterattack and asks for the play object that corresponds to  $L^{\rightarrow}(p)$ .

**Moves 5, 6:** **O** responds to the challenge of 4 . **P** posits the right side of the material implication.

**Moves 7, 8:** **O** asks for the play object that corresponds to  $R^{\rightarrow}(p)$ . **P** responds to the challenge by choosing the pair  $(v, r)$  where  $v$  is the play object chosen to substitute the variable  $f$  and  $r$  the play object for the right side of the existential.

**Move 9:** **O** has here the choice to ask for the left or the right side of the existential. The present play describes the development of the play triggered by the right choice.

**Moves 10-18:** follow from a straightforward application of the dialogical rules.

**Move 19:** **O** launches an attack to the extensionality of  $g$ , choosing the element  $a$  and  $b$  from  $A$  and positing its extensional equality in  $A$ .

**Moves 20-23:** **P** forces **O** to posit  $u/ R^{\exists}(g,u): (\forall x : A) C(x, g(x))$ . In fact, as mentioned above, the moves are a copy-cat of the moves -7-14 that force **P** to posit  $r/ R^{\exists}(g,r): (\forall x : A) C(x, g(x))$ . In other words, **P** forces **O** to concede by the intensional axiom of choice ( $H_1$ ) that there is a function  $g$  such that  $(\forall x : A) C(x, g(x))$ .

**Moves 24-26:** **P** forces to concede both  $C(a, g(a))$  and  $C(b, g(b))$ , choosing precisely  $a, b$  (the elements of  $A$  chosen by **O** before).

**Moves 28-31:** **P** forces **O** to also concede  $C(a, g(b))$  by making use of the extensionality of  $C$  ( $H_3$ ).

**Moves 32-35:** because of moves 25 and 27 **P** has all what he needs to apply unicity and force **O** to concede that  $g(a)$  and  $g(b)$  are extensionally the same function in  $Ba$ . We did not display here the possible (unsuccessful) counterattack of **O** on the conjunction  $C(a, g(a)) \wedge C(A, g(b))$  since this posit is composed by copy-cat moves of **P** from moves 25 and 27 of **O**.

**Move 36:** the just conceded extensional equality  $g(a) =_{Ba} g(b)$ , can be now posited by **P** as an answer to the challenge of **O** launched at move 19, and win!

### *III 4. Conclusion: The antirealist rejoinder of Hintikka's take on the axiom of choice*

According to Hintikka, his preferred formulation of the axiom of choice, namely:  $\forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x,f(x))$ <sup>65</sup>, is perfectly acceptable for the constructivists. Let us recall, that the GTS reading of its truth amounts to the existence of a winning strategy for Eloise in the game  $G(\forall x \exists y C(x,y))$ . The latter amounts to finding a "witness individual"  $y$ , dependent upon  $x$ , such that  $C(x,y)$  is true – notice how close this formulation is with Martin-Löfs' "informal" description of the proof of the axiom quoted in the preceding section. In other words, the existence of a winning strategy for the game provides *a proof that the proposition  $S(x,y)$  is true in the model* (Hintikka, 1996a, ch 2). Hintikka claims that it is the GTS reading that makes the axiom of choice acceptable for the constructivists:

*Moreover, the rules of semantical games should likewise be acceptable to a constructivist. In order to verify an existential sentence  $\exists x S[x]$  I have to find an individual  $b$  such that I can (win in the game played with)  $S[b]$ . What could be a more constructivistic requirement than that? Likewise, in the verification game  $G(S1 \vee S2)$  connected with a disjunction  $(S1 \vee S2)$ , the verifier must choose  $S1$  or  $S2$  such that the game connected with it (i.e.,  $G(S1)$  or  $G(S2)$ ) can be won by the verifier. Again, there does not seem to be anything here to alienate a constructivist. (Hintikka, 1996a: 212)*

---

<sup>65</sup> It is left implicit that  $\forall x$  quantifies over some set  $A$ ,  $\exists y$  quantifies over some set  $B$ , and  $\exists f$ , over the set  $((\forall x : A) Bx)$ .

As mentioned above Hintikka is right, this is acceptable for the constructivists, but the reason is the underlying intensionality of the choice function. Hintikka, would like to have a logic that renders classical mathematics. However, this is only possible if we assume extensionality. To have both extensionality and the constructive axiom of choice – without assuming unicity of the function – seems not to be possible.

Hintikka's intention was to offer a realist foundation of mathematics at a first - order level in a way that all classical mathematics can be comprised and that it can still be acceptable for a constructivist. Preceding analysis shows that it amounts to having the cake and eating it too. As pointed out in (Sundhom, 2013), under the constructivist reading, IF logic is granted to be a first - order logic, but in that case not all of classical mathematics can be saved. Moreover, if we comply with the constructive reading of IF logic, it makes the whole endeavour somewhat superfluous. Then we can use a version of constructive logic without going into the whole trouble with a non- axiomatisable logic.

We want to remind the reader of the point made earlier in section *I.7*. Hintikka argues that the axiom of choice should already be acceptable for a constructivist under the GTS reading. However, if one wants more, one can make a restriction on players' strategies to the *recursive ones* (even if Hintikka himself is not in favour of such a restriction). The previous analysis confirms what was said in *I.7*: to obtain a constructivist reading of GTS in general and of the axiom of choice in particular, it is not the recursivity of function that is needed. Game theoretic approach acceptable for constructivist must have:

1. The constructive rule for negation, where  $\neg A$  is understood as  $A \rightarrow \perp$ ;

2. The means to express the force sentences have in the language use (assertions, questions...);
3. The intensional reading of the choice function in the formulation of the axiom of choice.

Finally, let us recall once more one of the fundamental features of the dialogical approach to CTT: that all the actions that constitute the meaning are rendered explicit in the object – language. The game theoretical reading of the axiom of choice stresses one of the more salient characteristics of the CTT language: the judgement that a proposition is true can be expressed at the language level. The existence of a winning strategy is a part of a first-order language indeed – as pursued by the introduction of the truth predicate in IF logic. Moreover, this explicit theory of meaning is neutral in relation to the classical or constructive logic. However, the proof of the axiom of choice is constructive and its game theoretical interpretation is antirealist after all!

## ***IV. Anaphora***<sup>66</sup>

In the present chapter we will develop one last point from chapter *I* where we presented Hintikka's account of IF logic and GTS. We mentioned there that, beside the use of IF logic for the foundation of mathematics, Hintikka and his associates think that IF logic and GTS are more suitable for a formalisation of natural languages. One of Hintikka's most cherished example of a language phenomenon that can be treated in a satisfactory manner with IF logic is anaphora.

Hintikka's and associates' work on anaphora, based on GTS, constitutes a landmark in the field and it triggered many valuable contributions and discussions. In this section we will compare the GTS approach to the problem of anaphora with a solution for anaphora in the dialogical approach to CTT. In our opinion, the GTS approach has considerable advantages over other existing theories in dealing with anaphora. However, we will show that the extension of the dialogical framework discussed in the preceding sections contains both the contentual (first-order) features of CTT and the interactive aspects of GTS.

### ***IV. 1. The GTS approach to anaphora***

The issue is to find a satisfactory account of anaphoric expressions occurring in the sentences such as:

1. *If Michal smiles he is happy.*
2. *If a man smiles he is happy.*

---

<sup>66</sup> This chapter will be published for the most part in (Rahman, Clerbout, Jovanovic, forthcoming).

3. *Every man that smiles is happy;*

and of more problematic examples such as the famous donkey sentence:

4. *Every man who owns a donkey beats it.*

The first sentence apparently is not problematic. The Pronoun “he” has a strict interpretation (Michal), so it can be treated as a singular term. The issue is to provide a satisfactory semantic analysis of pronouns "he" or "it" which becomes more challenging when there is an interplay between pronouns and indefinites (such as in 2).

We mentioned priviously that in (Hintikka, 1997a) the first game theoretical analysis of anaphora was presented that stressed the distinction between priority and binding scope.<sup>67</sup> According to this analysis, if a quantifier is understood as a logical expression then we are speaking of its *priority scope* in relation to the rest of the sentence, but if it is understood as the antecedent for anaphoric pronoun that appears in the rest of the sentence then we are speaking of its *binding scope*. It is a pity, according to the author, that those two different moments are expressed by the same syntactic expression. At first glance, it is appealing to interpret an anaphoric pronoun as a variable available for a quantification. But Hintikka contests this view:

[...] *they do not behave like bound variables. An anaphoric pronoun does not receive their reference by sharing it with the quantifier phrase that is its "head", anymore than a definite description does. An anaphoric pronoun is*

---

<sup>67</sup> GTS was used before in the context of the analysis of natural language. Indeed, already in 1985 Hintikka and Kulas, made use of GTS in order to provide semantics of definite descriptions.

*assigned a reference in a semantical game through a strategic choice of a value from the choice set by one of the players. When the member of the choice set whose selection is a part of the winning strategy of the player in question happens to be introduced to the choice set by a quantifier phrase, that phrase could perhaps be called the head of the pronoun. But, as was pointed out, the origin of the members of the choice set does not matter at all in the semantical rules for anaphoric pronouns. (Hintikka, 1997a: 530)*

The GTS approach to anaphora was further developed by Sandu and his associates (Sandu 1997, Sandu, Jacot, 2012). Once again, players' strategies, introduced at the semantic level, are Skolem functions that show a player which disjunct / conjunct or which individual in the model to choose every time it is the player's turn to play. The Skolem form of a formula can be always obtained from the formula in the prenex normal form by replacing systematically every existential quantifier by an appropriate Skolem function, the argument of which is a variable bound by a universal quantifier in the scope of which that existential quantifier lays. Now, the idea is that indefinites can be represented as Skolem terms (Sandu, Jacot, 2012). We can first look at the first part of the donkey- sentence:

*5. Every man owns a donkey.*

In a game played for this sentence, the Falsifier first chooses an individual that satisfies the predicate of being a man. Then it is the Verifier's task to find a donkey owned by that individual, in order to win the game. The game can be represented as a tree with branches that show all possible outcomes of the game (for any individual chosen by falsifier). The Verifier's strategy is a function  $f$  that for any individual  $a$ ,

chosen by the Falsifier, gives as a result  $f(a)$ , that is, a donkey owned by a. Sentence 5 is formalised in the following manner:

$$6. \forall x (Man(x) \rightarrow Donkey(f(x)) \wedge Own(x, f(x))).$$

We can now turn to the anaphoric pronoun in the sentence 4. The solution for the problematic anaphora is found with the Skolem term. The pronoun "it" is the copy of the Skolem term in the antecedent. The formalisation of sentence 4 is:

$$7. \forall x (Man(x) \wedge Donkey(f(x)) \wedge Own(x, f(x)) \rightarrow Beats(x, f(x))).$$

In (Sandu and Jacot, 2012, p. 620) it is claimed that Skolem terms are very useful semantic tools for the interpretation of anaphora because they keep track of the entire history of a game. All the variables bound by the quantifiers superior to the indefinites are found as the arguments of each Skolem term. This solution combine at once the quasi - referential view on quantifiers, which is appealing when an anaphoric pronoun appears in a sentence, and the idea of semantic dependency, which is needed both when there is a nesting of indefinites and where there is an interplay of indefinites with quantifiers. In this way we can find a solution to some complicated examples of anaphora.

However, there are some more complicated cases of dependency that involve Henkin's branching quantifiers. In order to deal with those, GTS is combined with IF logic. We presented earlier the typical example of branching quantifiers:



$$\left. \begin{array}{l} \forall z \exists u \\ \forall x \exists y \end{array} \right\} S(x,y,z,u)$$

and we explained that it cannot be expressed with one linearly disposed expression of classical first-order language. However this can be done in IF with the following formalisation:

$$8. \forall x \forall z (\exists y / \forall z) (\exists u / \forall x) S(x, y, z, u)$$

The slash signs indicate that  $\exists y$  is independent of  $\forall z$  and that  $\exists u$  is independent of  $\forall x$ . Recall that IF logic captures exactly the expressive power of Henkin's branching quantifiers at the first - order level.

A classical example of a natural language sentence that involves branching quantifiers from (Hintikka, 1973: 344) is:

9. *Some relative of each villager and some relative of each townsman hate each other.*

This sentence is expressed by the existential second order sentence

$$10. \exists f \exists g \forall x \forall z ((Villager(x) \wedge Townsman(z)) \rightarrow Relative(x, f(x)) \wedge Relative(z, g(z)) \wedge Hate(f(x), g(z))),$$

which can be translated into first - order IF language like in the formalization 8.

According to Hintikka, IF allows us to take into account different patterns of dependency among logical expressions that can appear in a sentence, and it is thus more appropriate for the translation of natural languages. He states in (Hintikka, 1997a: 523):

*...this second-order statement expresses the logical form of the given natural-language sentence. It is equivalent to an IF first-order sentence, which can also be considered as the translation of the given natural language sentence into logical notation.*

However, IF approach is necessarily a model-theoretic one (since IF logic has no proof system). We repeat our point: there might be some other reasons that might force us to accept a model theory – this must be discussed independently, but neither the semantic analysis of anaphora nor that of branching quantifiers does.

#### ***IV.2 The dialogical approach to anaphora***

In what follows we will give a dialogical account of anaphora making use of CTT. We will argue that the GTS approach, that puts emphases on expressing the dependence relations in terms of choices resulting from interaction, is indeed a good way to deal with anaphora. Moreover, the "outside- in" semantics seems to be much more promising in the analysis of a natural language than the classical "inside- out" semantics. However, the dialogical approach provides a first - order solution that does not require other devices but the ones of constructive or even classical logic, when formulated within a suitably adapted CTT frame. In fact, Hintikka's remark on the

*binding scope* touches a crucial point in the semantics of anaphora. However, on our view, this point involves dependencies between play objects in general and choices for the substitution of instructions (occurring at the object - language level) in particular. Our approach is closer to the recent Skolem - term framework developed by Sandu and Jacot than to the original analysis of Hintikka, although, the dialogical framework can also deal with the more complicated cases involving branching quantifiers without making use of IF logic.

In the dialogical approach, pronouns are considered in their anaphoric role. Moreover, in the dialogical approach linked with CTT framework, we are also taking account of the dependence upon a context – that dependence has itself an anaphoric structure. In (Ranta, 1994: 78) the *pronominalisation rule for inference* is introduced, where the dependence of anaphoric pronouns on the context is rendered explicit<sup>68</sup>. The context is understood as the assumption that the picked object is of a given type (e.g. the assumption that  $x$  is of type  $A$ ). Thus the pronouns (and more generally the indexicals) dependence upon a context is understood as a reference to any object of the appropriate type.

For example, Ranta’s inferential rules for the pronoun *he* deploy the identity mapping on the set of *man* and *she* as identity mapping on the set of *woman* as resulting from the contextual dependence:

a: man  
 -----  
 he(a): man

And the rule of a substitution is:

a: man  
 -----

---

<sup>68</sup> “Context” should be understood in a technical sense of CTT.

$$he(a) = a: man^{69}$$

In the previous rule the instance given by the context is a constant expression, but it can also be a variable – in that case the context would be an open assumption. General inference rule in (Ranta 1994: 80) is the following:

$$\frac{a(x_1, \dots, x_n) : man (x_1 : A_1, \dots, x_n : A_n (x_1, \dots, x_{n-1}))}{\begin{array}{l} he(a(x_1, \dots, x_n)) : man (x_1 : A_1, \dots, x_n : A_n (x_1, \dots, x_{n-1})), \\ he(a(x_1, \dots, x_n)) = a(x_1, \dots, x_n) : man (x_1 : A_1, \dots, x_n : A_n (x_1, \dots, x_{n-1})) \end{array}}$$

In (Ranta, 1994: 80) one more rule is introduced for the treatment of pronouns, the *sugaring rule*:

$$he(a)(a : A) \triangleright he: A$$

This rule allows the use of *he* alone in the context A, without caring further the argument a.

Let us see now how those rules work in the dialogical framework. If player X posits that *he (a): man*, then his opponent can challenge this posit by asking him to show that *a* is of the type *man*. Once again we make use of the instructions which allows taking variables into the consideration as well.

Posit	Challenge	Defence
X $he(I^{pron}): man$	Y $?I^{pron}: man$	X $I^{pron}: man$

The resolution of the instruction in  $I^{pron}: man$  allows the defender to introduce explicitly an identity within the set *man*:

<sup>69</sup> In (Ranta, 1994, p. 78) those two rules are united in one rule with two conclusions.

Posit	Challenge	Defence
$X I^{\text{pron}} : \text{man}$	$Y I^{\text{pron,man}}/? :$	$X \text{he}(I^{\text{pron}}) = I^{\text{pron}} : \text{man}$

Sugaring rule goes as follows:

Posit	Challenge	Defence
$X \text{he}(a) = a : \text{man}$ ... $X \phi[\text{he}(a)]$	$Y ?_{a/\text{he}(a)}$	$X \phi[a]$

Sugaring rule allows the challenger to substitute  $he(a)$  for  $a$  at any place where  $h(a)$  occurs, given that the identity  $he(a) = a : \text{man}$  is once established and given that the suitable play – object has been already substituted for the instruction  $I^{\text{pron}}$ . Similar rules can be formulated for *she*. The case of *it* requires more care, since its type might vary from context to context. Anyway some variations are possible for all pronouns (e.g. *she: ship*).

Let us come back to our example of the happy man:

*If a man smiles he is happy.*

The idea is that in order to obtain the interpretation for the pronoun *he*, we first formalize the first part “*A man smiles*” as:

$(\exists x : \text{man}) \text{smiles}(x),$

and then we consider the sentence “*he is happy*” in the context

$z: (\exists x: \text{man}) \text{smiles}(x)$ .<sup>70</sup>

This analysis yields the following formalization:

$(\forall z: (\exists x: \text{man}) \text{smiles}(x)) (\text{happy} (\text{he} (L^\forall(z))))$

In the dialogical framework the left part of the universal given above consists of the set of all man that smile and the right part claims that an object chosen from that set is happy. The left part of the universal is  $L^\forall(z)$ , that is, the set of all man who smile. Its role is to pick up one of the individuals of the set of smiley men (the set  $(\exists x: \text{man}) \text{smiles}(x)$ ). Let's see now the dialogical game for this sentence. We don't make use of the instruction  $P^{pron}$  now, we assume that the pronoun has picked an instruction already. Also we will leave out the moves involving the choice of repetition ranks for the sake of simplicity:

	<b>O</b>			<b>P</b>	
				$(\forall z: (\exists x: \text{man}) \text{smiles}(x)) (\text{happy} (\text{he} (L^\exists(L^\forall(z))))$	0
1	$L^\forall(z): (\exists x: \text{man}) \text{smiles}(x)$	0	1	$R^\forall(z): \text{happy} (\text{he}((L^\forall(z))))$	12
3	$a : (Ex : \text{man}) \text{smiles}(x)$			$L^\forall(z)/?$	2
5	$L^E(a) : \text{man}$		3	$?L$	4
7	$R^E(a) : \text{smiles}(L^E(a))$		3	$?R$	6
9	$a_1 : \text{man}$		5	$L^E(a)/?$	8
11	$a_2 : \text{smiles}(a_1)$		7	$R^E(a)/?$	10
13	$?L^\forall(z)/? \dots$	12		$R^\forall(z) : \text{happy} (\text{he}(a_1))$	14
15	$? a_1: \text{man}$	14		$a_1: \text{man}$	16
17	$I^{he,man}/?:$	16		$R^\forall(z) : \text{happy} (\text{he}(a_1) = a_1)$	18
19	$? a/he(a)$	18		$R^\forall(z) : \text{happy} (a_1)$	20

<sup>70</sup> In (Ranta, 1994, p. 79) the example is "If a man walks he talks"

21	$?R^{\forall}(z)/$			<b>P</b> loses unless he can force <b>O</b> to concede that there is a play object $b$ for <i>happy</i> ( $a_1$ ), such that it allows <b>P</b> to choose $b$ for $R^{\forall}(z)$ while responding to the challenge of move 21 on move 12	
----	--------------------	--	--	--	--

**Description:**

**Move 0:** **P** states the thesis.

**Move 1:** **O** challenges the universal by positing an arbitrary man that smiles, that is  $z$ :  $(\exists x$ : man) smiles( $x$ ).

**Move 2:** **P** counterattacks by asking who that man is.

**Move 3:** **O** responds by choosing some play object.

**Move 4:** Since  $a$  is an existential it is constituted by two parts: **P** starts by asking for its left part.

**Move 5:** **O** answers that  $L(a)$  is a man.

**Move 6:** **P** challenges now the right part of the existential.

**Move 7:** **O** responds to the attack.

**Move 8:** **P** asks **O** to resolve the instruction occurring in the expression brought forward in move 5

**Move 9:** **O** responds by choosing  $a_1$ .

**Move 10:** **P** asks **O** to resolve the instruction occurring in the expression brought forward in move 7.

**Move 11:** **O** responds by choosing  $a_2$ .

**Move 12:** **P** answers now the challenge of move 1

**Move 13:** **O** asks **P** to resolve the instruction occurring in the expression brought forward in move 12.

**Move 14:** **P** chooses  $a_1$

**Move 15:** **O** challenges the pronoun *he*

**Move 16:** **P** can answer  $a_1$ : *man*, since **O** conceded it before (namely in move 9)

**Move 17:** **O** forces **P** to bring forward the identity underlying the pronoun *he*

**Move 18:** **P** brings forward the required identity

*Move 19: O forces P to make use of the identity brought forward in move 18 and apply it to drop the pronoun occurring in 14*

*Move 20: P drops the pronoun and this yields  $R^V(z) : \text{happy}(a_1)$*

*Move 21: O asks to resolve the instruction occurring in the last move. Since it is an elementary expression and O did not concede it before P cannot has no move to play and lost the play.*

P has a winning strategy if in a given context  $a_1$  is a man who smiles and is happy and that stands for every choice of man that O can make. Of course, this sentence is not valid. We could develop a material dialogue by introducing the initial concessions of Opponent (premisses) and thus check if there is a winning strategy. The winning strategy for Proponent then amounts to an inference from the materially given premisses. Remember that to design a material dialog the model is not introduced in a way Hintikka does it. The dialogical approach is not the model - theoretic one.

Let us consider now the famous donkey - sentence. We follow the analysis from (Sundholm, 1986), which constitutes a landmark in the application of CTT to the natural language. In order to keep the focus on the interdependence of choices we skip the pronouns *he* and *it* and we replace them with the corresponding instructions.

*Every man who owns a donkey beats it.*

As in the example above, first we formalize the first part of the sentence “man who owns a donkey” and we consider that sentence in the context, so we obtain:

$z: (\exists x: \text{man}) (\exists y: \text{donkey})(x \text{ owns } y).$



Since the existential is in fact the set that  $z$  is an element of, it is more convenient to use the set-separation notation:

$$z: \{x : \text{man} \mid (\exists y: \text{donkey})(x \text{ owns } y)\}.$$

We take the left part of  $z$  to pick up a man (that owns a donkey). The right part of  $z$  is the owned donkey (that is beaten). Put all together it yields the following formulation:

$$p : (\forall z: \{x : \text{man} \mid (\exists y: \text{donkey})(x \text{ owns } y)\}) (L^{\{\dots\}}(z) \text{ beats } L^{\exists}(R^{\{\dots\}}(z)))$$

Or more briefly:

$$p : (\forall z: \{x : M \mid (\exists y: D) Oxy\}) B(L^{\{\dots\}}(z), L^{\exists}(R^{\{\dots\}}(z)))$$

Let us run now the play for a material dialog. We will ignore again the choice of repetition ranks. Since it is a material dialogue, we know by the formation rules how the sets are composed and we also know that  $m$  is a man,  $d$  is a donkey and that  $p'$  is a play object for the proposition  $m \text{ owns } d$ .

	<b>O</b>				<b>P</b>	
I	$M: \text{set}$					
II	$D: \text{set}$					
III	$Oxy: \text{set } (x:M, y:D)$					
IV	$Bx,y: \text{set}$					
V	$!p: (\forall z: \{x : M \mid (\exists y: D) Oxy\}) B(L^{\{\dots\}}(z), L^{\exists}(R^{\{\dots\}}(z)))$					
VI	$!m: M$					
VII	$!d: D$					
VIII	$!p': Omd$					
					$!B(m,d)$	$0$
1	$n: = \dots$				$m: = \dots$	$2$
3	$? \text{ play-object}$	$(o)$			$!q: B(m,d)$	$30$
25	$!R^{\forall}(p): B(L^{\{\dots\}}(z), L^{\exists}(R^{\{\dots\}}(z)))$			V	$!L^{\forall}(p): \{x : M \mid (\exists y: D) Oxy\}$	$4$

5	$L^{\forall}(p)/? \dots$	(4)		$!z: \{x : M \mid (\exists y: D) Oxy\}$	6
7	? L	(6)		$!L^{\{\dots\}}(z): M$	8
9	$L^{\{\dots\}}(z)?$	(8)		$!m: M$	10
11	? R	(6)		$!R^{\{\dots\}}(z): (\exists y: D)$ <i>Omy</i>	12
13	$R^{\{\dots\}}(z)?$	(12)		$L^{\exists}(R^{\{\dots\}}(z)), R^{\exists}(R^{\{\dots\}}(z)):$ $(\exists y: D) Omy$	14
15	$L^{\exists}(R^{\{\dots\}}(z))/?, R^{\exists}(R^{\{\dots\}}(z))/?$	(14)		$!(d,p'): (\exists y: D) Omy$	16
17	? L	(16)		$!L^{\exists}(d,p'): D$	18
19	$L^{\exists}(R^{\{\dots\}}(z))/?,$	(18)		$!d: D$	20
21	? R	(16)		$!R^{\exists}(d,p'): Omd$	22
23	$R^{\exists}(R^{\{\dots\}}(z))/?$	(22)		$!p': Omd$	24
27	$!R^{\forall}(p): B(m,d)$		(25)	$L^{\{\dots\}}(z)/m, L^{\exists}(R^{\{\dots\}}(z))/d$	26
	$!q: B(m,d)$		(27)	$R^{\forall}(p)/?$	28

*Description:*

**Moves I- VIII:** These moves are **O**'s initial concessions. Moves I- IV deal with formation of expressions. After that the Opponent concedes the donkey sentence and atomic expressions related to the sets  $M$ ,  $D$  and  $Oxy$ .

**Moves 0- 3:** The Proponent posits the thesis. The players chose their repetition ranks in moves 1 and 2. The actual value they chose does not really matter for the point we want to illustrate here, thus we simply assume that they are enough for this play and live them unspecified. Now, when **P** posited the thesis he did not specified the play- objects so **O** asks for it in move 3.

**Move 4:** **P** chooses to launch a counter- attack by challenging the donkey sentence which **O** conceded at V. The rules allow him to answer directly to the challenge, but then he would not be able to win.

**Move 5 – 24:** The dialog then proceeds in a straightforward way with respect to the rules introduced earlier. More precisely, this dialog displays the case where **O** chooses to challenge **P**'s posits as much as she can before answering **P**'s challenge 4.

Notice that the Opponent cannot challenge the Proponent's atomic expressions posited at moves 10, 20 and 24: since **O** made the same posits in her initial concessions VI- VIII, the modified formal rule SR3 forbids her to challenge them.

**Move 25:** When there is nothing left for her to challenge, **O** comes back to the last unanswered challenge by **P** which was move 4 and make the relevant defence according to the particle rule for universal quantification.

**Moves 26- 27:** The resolution for instructions  $L^{\{\dots\}}(z)$  and  $L^{\exists}(R^{\{\dots\}}(z))$  has been carried out during the dialog with moves 9- 10 and 23- 24. Thus the Proponent can use the established substitution to challenge move 25 according to the structural rule SR4.2. The Opponent defends by performing the requested substitutions.

**Moves 28- 30:** The Proponent then asks the play - objects for which the instruction  $R^{\forall}(z)$  stands. When she answers, the Opponent posits exactly what **P** needs to defend against **O**'s challenge 3. Notice that at this point this is the last unanswered challenge by **O**, therefore **P** is allowed to answer it in accordance to the structural rule SR1i. He does so with his move 30.

Since **O** made the same posit the rule SR3 forbids her to challenge it. She then has no further possible move, and the Proponent wins this dialog.

### VI. 3. Branching quantifiers

In the dialogical framework more difficult examples involving branching quantifiers can also be handled in a satisfactory manner. Let us repeat ones again the example:

$$\left. \begin{array}{l} \forall x \exists y \\ \\ \forall z \exists u \end{array} \right\} S(x,y,z,u)$$

Skolem form of this example is:

$$\exists f \exists g \forall x \forall z S(x, f(x), z, g(z))$$

In (Sundholm, 2013a) the CTT analysis of branching quantifiers gives the following presentation:

$$(\exists f \in (\Pi x \in D)D)(\exists g \in (\Pi x \in D)D)(\forall x \in D)(\forall z \in D)A[ap(f, x)/y, ap(g, z)/u].$$

with the explanation:

*The quantifiers in this Skolemziation are with respect to elements  $f$  and  $g$  of the function set  $(\Pi x \in D)D$ , but do not quantify over a function type. The function sets, like all sets of Constructive Type Theory, are predicative and generated from below. Their elements correspond to Russell's entities of lowest level, that is, to individuals.*

The dialogical development is straightforward, if we recall that the application of a function follows the following rule:

<b>Posit</b>	<b>Challenge</b>	<b>Defence</b>
$\mathbf{X} \ ! \ p : \varphi [f(k_1)]$	$\mathbf{Y} \ f(k_1)/?$	$\mathbf{X} \ p : \varphi [k_2 / f(k_1)]$ $\langle \varphi [f(k_1)] = \varphi [k_2 / f(k_1)] : \text{set} \rangle$

Moreover, in the dialogical framework sentences involving branching quantifiers can be handled at the first - order level in such a way that all the objections rose against branching quantifiers can be now answered in a satisfactory manner. The objections relying on the fact that semantics of branching quantification necessarily involves quantification over functions has been already dismissed in **I.6.2.** by Sundholm's arguments. But let us recall Patton's objection from the same section: Patton said that branching quantification actually shouldn't be considered

quantification at all because the necessary condition for a quantifier is to have instances. Notice that the treatment of branching quantifiers in the dialogical approach allows having instances in question! There are very good reasons to state that branching quantifiers are of the first - order level after all – but this interpretation is supported by the dialogical framework.

Finally, recall Ranta's remark, mentioned in *I.7.* that Hintikka's argument against constructivist proof – theoretical semantics is its alleged incapacity of providing a satisfactory interpretation of branching quantification. It is clearly dismissed by the account of branching quantifiers in the dialogical approach linked with CTT: the interpretation of branching quantifiers does not push us toward model theory after all.

#### ***IV. 4. Final remarks***

As already mentioned, Hintikka's remark on *binding scope* touches a crucial point in the semantics of anaphora, that naturally leads to the game - theoretical interpretation. However, on our view, this point involves dependencies between play - objects in general and the choices for a substitution of instructions in particular. It was necessary a framework such as the one of CTT to make the point about these dependences explicit. The dialogical approach implements these dependencies within a game - theoretical analysis. Indeed, without such an approach choice dependencies cannot be expressed at the object - language level. Beside, in dialogues those dependencies can be expressed without the possibility of some non – desirable phenomena, such as *signalling*. This is one more reason to favourite the dialogical over the GTS approach.

One other advantage of the dialogical approach is that the meaning of anaphoric expressions is obtained at the play - level and not through the existence of a winning strategy for a player, as in GTS. That is an advantage because it shows how we can understand the meaning of an anaphoric pronoun without knowing how to win the game: it is enough that one understands all the steps Proponent is committed to in a dialogical game.

## *Conclusions*

It is time to sum up briefly the results of this thesis. We examined Hintikka's Independent Friendly Logic and the main uses Hintikka makes of it: one as the medium for mathematical reasoning and the other, as a mean of formalisation and analysis of natural languages. We presented it in detail, trying to be faithful to Hintikka's own line of thoughts. The properties of IF logic has been discussed as well as the advantages of this approach: the possibility to take account of (in)dependency relations among variables; GTS account of two different notions of scope of quantifiers; the "outside – in" direction in approaching the meaning, which turns out to be advantageous over the traditional "inside - out" approach; the usefulness of game – theoretic reasoning in mathematics; expressiveness of IF language which allows formulating branching quantifiers at the first - order level, as well as defining the truth predicate in the language itself. We defended Hintikka's standing of the first - order character of IF logic against some objections on this point.

However, the weak points have also been discussed: first and foremost, the lack of a full axiomatisation for IF logic and second, the problem of signalling, the problematic phenomenon related to the possibility of imperfect information in a game. As for the first, we argued that, while it is certainly true that there are many other domains where it is not validity we are interested in (like when it comes to the analysis of natural languages), it is hard to argue the same when it comes to mathematical reasoning. Moreover, without entering into a deep discussion concerning the advantages of a proof – conditional semantics over a model – theoretic one, it has been argued that neither the semantics of branching quantifiers not that of anaphora really force us to adopt a model - theoretic point of view. Hintikka's

argument against a proof – conditional approach to meaning - that it cannot deliver a satisfactory account of branching quantifiers - has been refuted.

In the continue, another game – theoretically oriented semantics has been presented, naimely, the dialogical logic linked with Martin Löf's Constructive Type Theory. It was shown that in this framework dependency relations can be accounted of, but without using more means than constructive logic and dialogical approach to meaning have to offer. Thus, we used it first to analyse Hintikka's take on the axiom of choice and then to analyse the GTS account of anaphora.

We contrasted Per Martin- Löf's analysis of the axiom of choice with Hintikka's standing on this axiom. Hintikka claims that GTS for IF logic justifies Zermelo's axiom of choice at the first - order level in a way perfectly acceptable for the constructivists. We used Martin- Löf's result to develop the following points:

- 1) Hintikka preferred version of the axiom of choice is indeed acceptable for the constructivists and its meaning does not involve a higher - order logic.
- 2) However, the version acceptable for the constructivists is based on the intensional take on functions, while extensionality is the essential part of the classical understanding of Zermelo's axiom and this is the real reason behind the constructivists' rejection of it. Moreover, we showed that the restriction to recursive functions that Hintikka proposed to yield a constructive reading of GTS is actually not the relevant one. It is the intensionality of the choice function that is needed.
- 3) More generally, the dependence and independence features that motivate IF logic, can be formulated within the CTT frame without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference.



We concluded pointing out that recent developments in the dialogical logic show that the CTT approach to meaning in general and to the axiom of choice in particular is very natural to game theoretical approaches where (standard) meta - logical features are explicitly displayed at the object - language level. In some way, this justifies, although in a different manner, Hintikka's appeal for the use of a game - theoretical semantics in the context of the foundations of mathematics.

Moreover, one of the fundamental features of the dialogical approach to CTT is that all the actions that constitute the meaning are rendered explicit in the object - language. The game theoretical reading of the axiom of choice stresses one of the more salient characteristics of CTT language: the judgement that a proposition is true can be expressed at the language level. The existence of a winning strategy is part of a first-order language indeed - as pursued by the introduction of a truth predicate in IF logic. Moreover, this explicit theory of meaning is neutral in relation to classical or constructive logic. However, the proof of the axiom of choice is constructive but its game theoretical interpretation is antirealist.

In the last section, we analysed the GTS account of anaphora. We argued that the GTS approach has considerable advantages over other existing theories in dealing with anaphora. However, we tried to show that the extension of the dialogical framework contains both the contentual (first - order) features of CTT and the interactive aspects of GTS.

Hintikka's remark on *binding scope* touches the crucial point in the semantics of anaphora that naturally leads to the game - theoretical interpretation. However, on our view, this point involves the dependences between play objects in general and the choices for a substitution of instructions in particular. It was necessary a framework

such as the one of CTT to make the point of these dependences explicit. The dialogical approach implements these dependences within a game - theoretical analysis. Indeed, without such an approach choice dependences cannot be expressed at the object - language level. Beside, in the dialogues those dependences can be expressed without the possibility of some non – desirable phenomena, such as *signalling*. This is one more reason for favourite the dialogical over the GTS approach.

One other advantage of the dialogical approach is that the meaning of anaphoric expressions is obtained at the play- level and not through the existence of a winning strategy for a player, as in GTS. That is an advantage because it shows how we can understand the meaning of anaphoric pronoun without knowing how to win the game: it is enough that one understands all the steps Proponent is committed to in a dialogical game.

Finally, we would like to make one more remark concerning the dialogical approach to CTT. The CTT - dialogues prove to be a very powerful tool, both for the analysis of a natural language and potentially, for the foundations of mathematics. Its subtle approach to meaning, as well as the fully interpreted language, offers a number of possibilities for future application. CTT has been already shown to be very fruitful for the foundational purposes in mathematics. In a dialogical game – theoretical frame it can be even more interesting for this purpose. It is a challenging idea for a future work in this domain.

## *Appendix*

### *Dialogues for first - order logic*<sup>71</sup>

We define the vocabulary for first - order logic in the same manner we did before in section *I.1.1.* of this thesis. Validity of a first - order formula is proved through a dialogical game for the formula between two players: *Proponent* who defends the formula and *Opponent* who tries to falsify it. Therefore, we need one more set of additional signs,

$S_7 = \{P, Q\}$ , where P and O correspond to players in the dialogical game (Proponent and Opponent, respectively).

$S_8 = \{?, !\}$  is another set of additional signs where ? and ! are *force symbols*, indicating that the move in the game is an *attack* or a *defence*.

$S_9 = \{X, Y\}$  is a set of *players* in a dialogical game for a formula that can assume the roles of Opponent or Proponent in the game.

Let us present a few definitions.

- *Definition: Dialogically signed expressions* are strings  $\langle P - ! - \varphi \rangle$ ;  $\langle O - ! - \varphi \rangle$ ;  $\langle P - ? - \varphi \rangle$  and  $\langle O - ? - \varphi \rangle$  indicating who is making a move for an expression  $\varphi$  and if it is an attack or a defence.

- *Definition: The dialogical game* is a sequence of dialogically signed expressions.

- *Definition: The play* is a sequence of dialogical games.

---

<sup>71</sup> Based on (Rahman, Tulenheimo, 2006).

Dialogical games can be seen as a special case of plays: all dialogical games are plays but not the vice versa. As we will show later, certain rule<sup>72</sup> for dialogues allows splitting of a game into two branches. Each of the branches will be a separate dialogical game, but taken together they will form a play.

*-Definition: Dialogue* is a set of plays.

*-Definition:* The thesis of the dialog  $D(\varphi)$  is  $\varphi$ , that is, the formula asserted by the initial Proponent in the dialogue.

*-Definition:* We will denote  $\Delta[n]$  a member of the sequence  $\Delta$  of the dialogue<sup>73</sup> that has a position  $n$ .

As mentioned above, a dialogical game for a first - order formula is played by two players, Proponent who tries to defend the formula and Opponent who tries to challenge it. The game is *finite* i.e. after a finite number of steps one of the players wins while other loses. The course of the game is determined with two groups of rules: *particle rules* and *structural rules*.

### ***-Particle rules***

To describe a dialogical game for a first - order formula we start off with *particle rules* that provide the so - called *local semantics*. They show how the game is played locally for every logical constant in the formula. Those rules indicate how every logical constant can be attacked and defended and whose turn is to play in every

---

<sup>72</sup> See Branching rule for dialogical games.

<sup>73</sup> which is a dialogically signed expression.

step of the game. Thus, particle rules yield local semantics without taking into account the context in which the argumentation is taking place. The context - dependence will enter into the game with structural rules.

Atomic formulas have no particle rule corresponding to them in a formal dialogue.

Assertion	Attack	Defence
$X - ! - \varphi \wedge \psi$	$Y - ? - L$ or $Y - ? - R$ The choice is made by the player in the role of Opponent	$X - ! - \varphi$ or $X - ! - \psi$ respectively
$X - ! - \varphi \vee \psi$	$Y - ? - \vee$	$Y - ? - L$ or $Y - ? - R$ The choice is made by the player in the role of Proponent
$X - ! - \varphi \rightarrow \psi$	$Y - ! - \varphi$	$X - ! - \psi$
$X - ! - \neg \varphi$	$Y - ! - \varphi$	The defence is not possible. The only possible move is counterattack.
$X - ! - \forall x \varphi$	$Y - ? - \forall x / a$ Y chooses any a available to him	$X - ! - \varphi [x/a]$ for a chosen by Y
$X - ! - \exists x \varphi$	$Y - ? - \exists$	$X - ! - \varphi [x/a]$ X chooses any a available to him

" $\varphi [x/a]$ " in the diagram means that every occurrence of x in the formula  $\varphi$  is substituted by a. "L" and "R" stand for *left* and *right* part of the conjunction or disjunction.

Particle rules describe the course of the dialogical game, or how we can pass from one *state of the dialogue* to another.

*-Definition:* Let  $\varphi$  be a first - order formula, and  $\{a_0, a_1, \dots\}$  a countable set of individual constants. Then the *state of dialog*  $S$  is defined by specification of the following components:

- 1)  $\psi$  - a subformula of  $\varphi$
- 2)  $\langle X- Y - e \rangle$  - a dialogically signed expression where  $X$  stands for either P or O i.e. the roles of players in the game;  $Y$  stands for ? or ! i.e. it indicates if the move is an attack or a defence and  $e = \psi$ .
- 3)  $\sigma$ :  $\text{Free} [\psi] \rightarrow \{ a_0, a_1, \dots \}$  - a function from free variables in  $B$  to individual constants.
- 4)  $\rho$ :  $\{P, O\} \rightarrow \{?, !\}$  - a function assigning the roles to players.

If we take that we are in the state  $S_1 = \langle \psi_1, X_1, Y_1, e_1, \sigma_1 \rangle$  of the dialog  $D$  ( $\varphi$ ) then particle rules show how the dialog can evolve from the state  $S_1$  to the state  $S_2 = \langle \psi_2, X_2, Y_2, e_2, \sigma_2 \rangle$ .  $S_2$  is reached when the player  $X_1$  makes the move with the force  $Y_1$  in accordance with the appropriate particle rule. Depending on the  $Y_1$  the move is either the attack (?) or the defence (!).

*-Definition:* A pair of the attack and the defence forms a *round* in a dialogic game. The round is opened by an attack and closed with a defence. Now we can state particle rules.

**PR $\wedge$  (particle rule of conjunction):**

If  $e = \psi$ , where  $\psi$  is the subformula of  $\varphi$  of the form  $\alpha \wedge \beta$ , and  $Y = !$ , then the Opponent  $Y$  choses to attack the left or the right part of the conjunction, so the attack is  $\langle Y - ? - L \rangle$  or  $\langle Y - ? - R \rangle$ . The state of dialog  $S_2$  is either  $\langle \alpha, X, !, \alpha, \sigma \rangle$  or  $\langle \beta, X, !, \beta, \sigma \rangle$ , respectively.

Intuitively, for Opponent to falsify the conjunction it is enough to show that one of the conjuncts can be refuted. Thus, Opponent decides which conjunct he finds easier to challenge. In response, Proponent of the conjunction must be able to defend challenged conjunct.

**PR $\vee$  (particle rule of disjunction):**

If  $e = \psi$ , where  $\psi$  is the subformula of  $\phi$  of the form  $\alpha \vee \beta$ , and  $Y = !$ , then the attack consists in Opponent's question  $? - \vee$ , and the state of dialog  $S_2$  is either  $\langle \alpha, X, !, \alpha, \sigma \rangle$  or  $\langle \beta, X, !, \beta, \sigma \rangle$ , depending on the choice of Proponent who is defending the disjunction.

Intuitively, for the disjunction to be defended it is enough to show that one of disjuncts can be defended, so Opponent is making a challenge to Proponent to show that is the case. It is then up to Proponent to choose which disjunct he wants to defend.

**PR $\rightarrow$  (particle rule of implication):**

If  $e = \psi$ , where  $\psi$  is the subformula of  $\phi$  of the form  $\alpha \rightarrow \beta$ , and  $Y = !$ , the attack of the Opponent in  $S_2$  consists in defending the antecedent of the conditional,  $S_2 = \langle \alpha, Y, !, \alpha, \sigma \rangle$ . Then in the state  $S_3$  it is Proponent who must defend the consequent, so  $S_3 = \langle \beta, X, !, \beta, \sigma \rangle$ , or else, Proponent can challenge Opponent's defence of  $\alpha$  to yield the state  $S_4$  of the dialog,  $S_4 = \langle \alpha, X, ?, \dots, \sigma \rangle$ .<sup>74</sup>

Intuitively, to show that implication is valide Proponent must be able either to defend both antecedent and consequent either to show that antecedent can be refuted. So the attack of Opponent consists in defending the antecedent thus challenging

---

<sup>74</sup> In  $S_4$ ,  $\alpha$  is the formula in question and fourth component is determined by a logical form of  $\alpha$  i.e. the relevant logical constant in  $\alpha$  is attacked.

Proponent to show that he can defend the consequent. Proponent can do that or he can try to refute the antecedent by attacking antecedent in the next state of dialogue.

**PR $\neg$  (particle rule of negation):**

If  $e = \psi$ , where  $\psi$  is the subformula of  $\phi$  of the form  $\neg \alpha$ , and  $Y = !$ , then the attack of the Opponent consists in his defending of  $\alpha$  in the state  $S_2$  of the dialog.

$S_2 = \langle \alpha, Y, !, \alpha, \sigma \rangle$ . Proponent doesn't have a proper defence against this attack, but he can counterattack formula  $\alpha$  in the next state of dialog  $S_3 = \langle \alpha, X, ?, \dots, \sigma \rangle$ .<sup>75</sup>

The rule of negation amounts to switching roles of players in the game. To challenge the negation means to charge ourselves of defending the non - negated formula. The only response from the part of a defender of the initial negated formula is to try to refute the formula without negation.

**PR $\forall$  (particle rule of universal quantifier):**

If  $e = \psi$ , where  $\psi$  is the subformula of  $\phi$  of the form  $\forall x \alpha x$ , and  $Y = !$ , the attack of Opponent is the question  $? \forall x/a_i$  where  $a_i$  is chosen by him. In his response Proponent must defend the formula where  $x$  is substituted by  $a_i$ , so the state of dialog  $S_2$  is

$S_2 = \langle \alpha x, X, !, \alpha x, \sigma[x/a_i] \rangle$ . As usual,  $\sigma[x/a_i]$  is the function replacing all the occurrences of  $x$  bound by the universal quantifier with  $a_i$ .

This rule is rather obvious. To defend a formula  $\forall x \alpha x$  Proponent must be able to defend  $\alpha x$  for any individual constant that can be substituted for  $x$ , so it is up to Opponent to choose the constant. Proponent has to defend the new formula with the constant substituted for  $x$ .

---

<sup>75</sup> See the footnote 6.



### **PR $\exists$ (particle rule of existential quantifier):**

If  $e = \psi$ , where  $\psi$  is the subformula of  $\varphi$  of the form  $\exists x \alpha x$ , and  $Y = !$ , then the attack of the Opponent is the question ?-  $\exists x$  and the state  $S_2 = \langle \alpha x, X, !, \alpha x, \sigma[x/a_i] \rangle$ , only this time it is Proponent who is choosing the individual constant for substitution.

The rule for the existential quantifier is the dual of the previous rule. To defend formula  $\exists x \alpha x$  for Proponent it is enough to show that  $\alpha x$  can be defended for one individual constant substituted for  $x$ , so it is this time up to Proponent to choose the individual constant.

### **-Structural rules**

Those rules organise the entire dialog. The meaning of a formula is determined from *outside - in*, meaning that the dialogue starts with entire initial formula, the thesis, defended by initial Proponent. Through the rest of the dialog the Proponent try to give an argumentation in favour of the thesis, so to speak, while Opponent challenges it. That is done through *moves* made by players in accordance with particle rules.

Formal dialogues serve for testing logical validity of a formula. The formula is proven to be valid if Proponent succeeds in defending it against any attack of the Opponent, that is, if there is a winning strategy for Proponent. This notion of validity captures standard notion of validity as *the truth in every model*.

A dialog can be given a tree representation. The root of the tree is the theses together with a set of other premises (that can be empty). The branches are sequences of dialogically signed expressions generated by the course of the dialog. Branches can only be made by the choices of Opponent - by his attack against a conjunction,

his defence of a disjunction or his reaction against an attack on implication. Moves of Proponent cannot generate a branch, as will be clarified soon.

We can proceed now with presentation of structural rules. Let the language be  $L$  and  $\varphi$  a first - order formula. The thesis of the dialog  $D(\varphi)$  is  $\varphi$ . Recall that we denote  $\Delta[n]$  a member of the sequence  $\Delta$  of the dialog that has the position  $n$ .

**SR 0 (Starting rule):**

a) The first sequence of the dialog,  $\Delta[0]$  is the thesis  $\varphi$  stated by Proponent together with a set of other initial premises, stated by Opponent. Thus, we will have dialogically signed expressions  $\langle P - ! - \varphi \rangle$ ,  $(\langle O - ! - h_1 \rangle \dots \langle O - ! - h_n \rangle)$  for the set of initial hypotheses  $\{h_1 \dots h_n\}$ . When proving the validity of a first - order formula, the set of additional premises is empty.

b) After the theses stated in  $\Delta[0]$  the players are making moves one after another. Every move is a reaction to an earlier move of the adversary. At even positions it is Proponent who makes the move and in odd positions it is Opponent. Therefore, in the positions  $\Delta[2n]$  the sequence  $\Delta$  will be of the form  $\langle P - Y - \psi \rangle$ , where  $\psi$  is a subformula of  $\varphi$  and  $Y \in \{?, !\}$  while in the positions  $\Delta[2n+1]$  the sequence  $\Delta$  will be of the form  $\langle O - Y - \psi \rangle$ .

**SR 1.C (Classical round closing rule):**

A player can attack any complex formula asserted by his opponent within the appropriate particle rule and he can defend themselves against any attack. If needed, he is allowed to repeat his earlier defences.

**SR 1.I (Intuitionistic round closing rule):**

A players can attack any complex formula asserted by his opponent within the appropriate particle rule and he can defend themselves against *the last attack of his opponent that still has not been answered* . In other words, repeating of earlier defences is not allowed and only the latest attack can be answered if it hasn't already been answered before.

A defence can be delayed as long as it is possible to attack the opponent instead.

**SR 2 (Branching rule for dialogical games):**

As previously mentioned, only certain moves of Opponent can produce branches in the game tree, namely, his attack on a conjunction, his defence of a disjunction and his reaction on Proponent's attack against an implication. In those cases sequence  $\Delta$  splits into two sequences of the game  $\Delta_1$  and  $\Delta_2$ . No move of Proponent can create branching nor Opponent's moves for the quantifiers. Each branch of a game is a separate dialogical game.

Let  $D(\varphi)$  be a dialogue for a first - order formula  $\varphi$  and let  $\Delta$  be such that it is Opponent's turn to move.<sup>76</sup>

1) If  $m$  is a natural number such that  $m < n$ ,  $\Delta[m] = \langle O - ! - \alpha \vee \beta \rangle$ , and

$\Delta[n] = \langle P - ? - \vee \rangle$ , then  $O$  can split  $\Delta$  in two sequences,  $\Delta_1$  and  $\Delta_2$ :

$\Delta_1[n+1] = \langle O - ! - \alpha \rangle$ ;  $\Delta_2[n+1] = \langle O - ! - \beta \rangle$ .

2) If  $\Delta[n] = \langle P - ! - \alpha \wedge \beta \rangle$ , then  $O$  can split  $\Delta$  in two sequences  $\Delta_1$  and  $\Delta_2$ :

$\Delta_1[n+1] = \langle O - ? - L \rangle$ ;  $\Delta_2[n+1] = \langle O - ? - R \rangle$ .

3) If  $m$  is a natural number such that  $m < n$ ,  $\Delta[m] = \langle O - ! - \alpha \rightarrow \beta \rangle$ , and

$\Delta[n] = \langle P - ! - \alpha \rangle$ , then  $O$  can split  $\Delta$  in two sequences,  $\Delta_1$  and  $\Delta_2$ :

---

<sup>76</sup> Meaning that the number of the position of  $\Delta$  in the game is odd.

$$\Delta_1[n+1] = \langle O - ? - \dots \rangle^{77}; \Delta_2[n+1] = \langle O - ! - \beta \rangle.$$

**SR 3 (Shifting rule):**

With previous branching rule we saw how the play of a dialog can split into two dialogical games with a choice of Opponent. Shifting rule says that if O has chosen to split  $\Delta$  into two sequences,  $\Delta_1$  and  $\Delta_2$ , and further he lost the play  $\Delta_1$ , he can come back and try to win the play  $\Delta_2$ . Notice that Opponent cannot switch to alternative play  $\Delta_2$  unless he has already lost the play  $\Delta_1$ . Let us present the example:

$$\Delta[n] = \langle O - ! - \alpha \vee \beta \rangle$$

$$\Delta[n+1] = \langle P - ? - \vee \rangle$$

$$\Delta[n+2] = \langle O - ! - \alpha \rangle.$$

Suppose that Opponent lost this game i.e. he can't succeed in defending  $\alpha$ . Shifting rule allows him to come back and defend the disjunct  $\beta$  this time.

$$\Delta'[n] = \langle O - ! - \alpha \vee \beta \rangle$$

$$\Delta'[n+1] = \langle P - ? - \vee \rangle$$

$$\Delta'[n+2] = \langle O - ! - \alpha \rangle.$$

The difference between a *dialogical game* and a *play* can now be explained. Both sequences  $(\Delta[n], \Delta[n+1], \Delta[n+2])$  and  $(\Delta'[n], \Delta'[n+1], \Delta'[n+2])$  are *plays that are dialogical games*. Put together the sequence  $(\Delta[n], \Delta[n+1], \Delta[n+2], \Delta'[n], \Delta'[n+1], \Delta'[n+2])$  is a *play which is not a dialogical game*. We can say that the play is the order of the dialogical games in the dialog.

---

<sup>77</sup>  $\alpha$  is the formula in question and third component is determined by a logical form of  $\alpha$  i.e. the relevant logical constant in  $\alpha$  is attacked.

**SR 4a (Winning rule for dialogical games):**

A dialogical game is *closed* if there is some atomic formula asserted in two positions, once by player X and once by player Y.

Let  $D(\varphi)$  be the dialogue for the first - order formula  $\varphi$  and let  $\psi$  be the subformula of  $\varphi$  such that  $\psi$  is atomic. Let  $\{\Delta[n_1] \dots \Delta[n_i]\}$  be the set of sequences that form the dialogical game. The dialogical game is closed if there are two sequences  $\Delta[n_j], \Delta[n_k] \in \{\Delta[n_1] \dots \Delta[n_i]\}$  such that  $\Delta[n_j] = \langle O - ! - \psi \rangle$  and  $\Delta[n_k] = \langle P - ! - \psi \rangle$ .

A dialogical game is *opened* if it is not closed.

A dialogical game is *finished* either if it is closed or if it is opened but the rules don't allow any more moves. Finished dialogical game cannot be further continued.

If a dialogical game is closed Proponent of the thesis is a winner. If the game is opened Proponent loses.

**SR 4b (Winning rule of plays):**

A play is *closed* if all the dialogical games in it are closed. If the play consists of the games  $\Delta_1 \dots \Delta_n, \Delta$ , the play is *finished* if all games  $\Delta_1 \dots \Delta_n$  are closed and  $\Delta$  is finished.

If the play is closed Proponent wins. Otherwise, he loses.

**SR 5 (Formal use of atomic formulas):**

a) Proponent is not allowed to introduce atomic formulas. Any such formula must be asserted first by Opponent.

b) Atomic formulas cannot be attacked.

The explanation of this rule is rather simple. Proponent wins the game if there is an atomic formula stated both by himself and by Opponent, thus, it is not in his favour to introduce new atomic formulas. On the other hand, from the Opponent's

perspective, the goal is not to have some atomic formula repeated, so he will try to introduce as many atomic formulas as possible.

Before we present the last structural rule we need some more definitions.

*-Definition: Strict repetition of an attack is*

a) an attack against some assertion of the opponent in a game even if that assertion has already been attacked earlier with the same dialogically signed expression or

b) an attack against a universal quantifier stated by the opponent with a new constant even if the universal quantifier has already been attacked earlier with a constant that was new at the moment of the attack.

From careful reading of a) it should be clear that in the case of attack against a conjunction moves  $\langle Y - ? - L \rangle$  and  $\langle Y - ? - R \rangle$  do not count as a strict repetition in a sense described in the definition.

*-Definition: Strict repetition of a defence is*

a) a defence against some attack of the opponent in a game even if that attack has already been answered earlier with the same dialogically signed expression or

b) a defence of an existential quantifier attacked by the opponent in a game with a new constant even if the existential quantifier has already been defended earlier with a constant that was new in the moment of that attack.

Again, it should be noted that in the case of a defence of a disjunction, moves  $\langle Y - ? - L \rangle$  and  $\langle Y - ? - R \rangle$  do not count as a strict repetition in a sense described in the definition. Finally we can state the last rule.

### **SR 6.C (Classical "no delaying tactics" rule)**

It is forbidden to use strict repetition in a game. This version of the rule is for classical first - order logic and obviously, should be use only with SR 1.C rule.

**SR 6.I (Intuitionistic "no delaying tactics" rule):**

It is forbidden to use strict repetition in a game, except in the case where Opponent has introduced a new atomic formula that can now be used by Proponent. Then Proponent is allowed to perform a strict repetition of his move.

Of course, this rule is appropriate for intuitionistic first - order logic and should only be used with SR 1.I rule.

*-Definition:* The first - order sentence  $\varphi$  is *classically valid* if all the plays of the classical dialog  $D(\varphi)$  are closed.

*-Definition:* The first - order sentence  $\varphi$  is *intuitionistically valid* if all the plays of the intuitionistic dialog  $D(\varphi)$  are closed.

For the dialogical concept of validity it has been proved that it coincides with standard concepts of validity for classical and intuitionistic first - order logic. First such a proof had been provided by Kuno Lorenz.<sup>78</sup> It was since developed by number of authors as Stegmüller, Haas, Felscher, Krabbe, Rahman, who established the equivalence between other systems of proofs, as sequent calculi and tableaux system and the dialogues.

We will proceed to give two simple examples of classical and intuitionistic dialogues to illustrate how the game works.

*-Example:* Let us observe a classical dialogue for the propositional formula  $p \vee \neg p$ , where  $p$  is an atomic sentence.

---

<sup>78</sup> Can be found in collection of papers by (Lorenzen, Lorenz, 1978).

	Opponent		Proponent	
			$p \vee \neg p$	0
1	$? - \vee$	0	$\neg p$	2
3	$p$	2		
(1)	$(? - \vee)$	(0)	$p$	4

In the columns on the side of each player there are the numbers of the move in the dialogue. If the move counts as an attack we note in the most inner columns the number of the move that is attacked. When writing the moves of players we made some abbreviations. When a formula is asserted instead of writing "P - ! -  $p \vee \neg p$ " for example, we simply write the formula " $p \vee \neg p$ ".

In the first row Proponent is asserting the thesis  $p \vee \neg p$ , it counts as the move number 0. In the first move Opponent is attacking the disjunction according to the rule **PR $\vee$** . According to the rule, it is Proponent who has a choice which disjunct he wants to defend, so he chooses  $\neg p$  in the move 2. In the move 3, Opponent is attacking the negation by asserting the non - negated formula  $p$ , according to the rule **PR $\neg$** . The rule for negation stated that there is no defence available to Proponent so the next row in his column is empty. However, structural rule **SR 1.C** allows to Proponent to repeat his earlier defence against an attack even if it is not the last non-answered attack. Therefore, in the move 4 Proponent is responding once more to the Opponent's attack against the disjunction from the move 1, but this time he chooses the left disjunct  $p$ . Opponent is not really repeating his attack, so we note it with the brackets, just to make clear to what move this defence corresponds. The atomic formula  $p$  is now repeated both by Opponent and Proponent, so Proponent wins the dialogue. The law of the excluded middle holds for classical first - order logic.

We will see now the example of intuitionistic dialog for the same formula.



	Opponent		Proponent	
			$p \vee \neg p$	0
1	$? - \vee$	0	$\neg p$	2
3	$p$	2		

As previously, Proponent responds to Opponent's attack on disjunction choosing the right disjunct. In the following step, he doesn't have a proper defence against Opponent's attack against negation. Only this time the dialog ends there. According to the structural rule **SR 1.I** Proponent doesn't have the right to repeat the defense against the previous attack of Opponent. Remember that, according to this rule, only the last attack that hasn't still be answered can be defended. This time the dialog is opened and Proponent loses the game. The law of the excluded middle is not valid in intuitionistic logic.

## ***Bibliography***

Jc. Beall, M. Glanzberg (2014). "Liar Paradox", in Edward N. Zalta (ed.); *The Stanford Encyclopedia of Philosophy* (Spring 2014 Edition), URL = <http://plato.stanford.edu/archives/fall2014/entries/liar-paradox/>.

J. Bell (2009). *The Axiom of Choice*. London: College Publications, 2009.

E. Bishop (1967). *Foundations of constructive mathematics*. New York, Toronto, London: McGraw- Hill, 1967.

A. Blass, Y. Gurevich (1986). "Henkin quantifiers and complete problems". *Annals of pure and applied logic*, 1986, vol. 32, pp. 1-16.

A. Blass (1992). "A game semantics for linear logic". *Annals of Pure and Applied Logic*, 1992. vol. 56, pp.183–220.

X. Caicedo, F. Dechesne, T. M. V. Janssen (2009). "Equivalence and Quantifier Rules for Logic with Imperfect Information". *Logic Journal of the IGPL*, 2009, vol. 17(1), pp. 91–129.

P. Cameron, W. Hodges (2001). "Some Combinatorics of Imperfect Information". *Journal of Symbolic Logic*, 2001, vol. 66, pp. 673–684.

N. Clerbout, M.H. Gorisse, S. Rahman (2011). "Context-sensitivity in Jain philosophy: A dialogical study of Siddharsiganis *Commentary on the Handbook of Logic*". *Journal of Philosophical Logic*, 2011, vol. 40(5), pp. 633–662.

N. Clerbout, S. Rahman (2013). "On Dialogues, Predication and Elementary Sentences". *Revista de Humanidades de Valparaiso*, 2013, vol. 2, pp. 7 – 47.

N. Clerbout (2013a). "First-order dialogical games and tableaux". *Journal of Philosophical Logic*, 2013, doi: 10.1007/s10992-013-9289-z. URL <http://dx.doi.org/10.1007/s10992-013-9289-z>, pp. 1–17.

N. Clerbout (2013b). *Etude sur quelques sémantiques dialogiques: Concepts fondamentaux et éléments de métathéorie*. PhD thesis, Lille 3 / Leiden: Universities of Lille and Leiden, 2013.

N. Clerbout, S. Rahman (2014). *Linking Game-Theoretical Approaches with Constructive Type Theory: Dialogical Strategies as CTT-Demonstrations*. to appear.

R. Cook, S. Shapiro (1998). "Hintikka's Revolution: The Principles of Mathematics Revisited". *The British Journal for the Philosophy of Science*, 1998, vol. 49 (2), pp. 309 – 316.

- Th. Coquand (2014). "Recursive functions and Constructive Mathematics", in J. Dubucs, M. Bourdeau (eds.); *Constructivity and Computability in Historical and Philosophical Perspective*. Dordrech: Springer, 2014.
- F. Dechesne (2005). *Game, Set, Maths: formal investigations into logic with imperfect information*. PhD thesis, Tilburg: Tilburg University, 2005.
- R. Dedekind (1996). "What are Numbers and What should they Be? ". *Journal of Symbolic Logic*, 1996, Vol. 61 (2), pp. 688-689.
- R. Diaconescu (1975). "Axiom of choice and complementation". *Proc. Amer. Math. Soc.*, 1975, vol. 51, pp.176–178.
- M. Ekland, D. Kolak (2002). "Is Hintikka's Logic First Order?". *Synthese*, 2002, vol. 131 (3), pp. 371 - 388.
- H. Enderton (1970). "Finite partially ordered quantifiers". *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 1970, vol. 16, pp. 393-397.
- S. Feferman (2006). "What kind of logic is 'Independence Friendly' logic?", in R. E. Auxier, L. E. Hahn (eds.); *The Philosophy of Jaakko Hintikka*. Library of Living Philosophers, Open Court, 2006, vol. 30, pp. 453-469.
- W. Felscher (1985). "Dialogues as a foundation for intuitionistic logic" in D. Gabbay and F. Guentner (eds); *Handbook of Philosophical Logic*. Dordrecht: Kluwer, 1985, vol. 3, pp. 341–372.
- W. Felscher (1994). "Review of Jean E. Rubin 'Mathematical logic: applications and theory'". *The Journal of Symbolic Logic*, 1994, vol. 59, pp. 670-671. doi:10.2307/2275418.
- V. Fiutek, H. Rucket, S. Rahman (2010). "A dialogical semantics for Bonannos system of belief revision", in P. Bour and al. (eds); *Constructions*. London: College Publications, 2010, pp. 315–334.
- M. Fontaine (2013). *Argumentation et engagement ontologique. Etre, c'est être choisi*. London: College Publications, 2013.
- D. Gale, F. Stewart (1953). "Infinite games with perfect information". *Ann. of Mathematical Studies*, 1953, vol.28, pp. 245 – 266.
- N.Gierasimczuk, J.Szymanik (2009). "Branching quantification v. two-way quantification". *Journal of Semantics*, 2009, vol. 26 (4), pp. 367 – 392.
- J. Ginzburg (2012). *The Interactive Stance: Meaning for Conversation*. Oxford: Oxford University Press, 2012.
- J.Y. Girard (1999). "On the meaning of logical rules I : syntax vs. semantics", in U. Berger, H. Schwichtenberg (eds); *Computational Logic*. Verlag, Heidelberg: Springer, 1999, pp. 215–272.

- N. D. Goodman, J. Myhill (1978). "Choice implies excluded middle". *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 1978, vol. 24.
- J. Granström (2011). *Treatise on Intuitionistic Type Theory*. Dordrecht: Springer, 2011.
- M. Hand (1993). "A defence of Branching Quantification". *Synthese*, 1993, vol. 95, pp. 419 – 432.
- A. Heyting (1962). "After Thirty Years". *Logic, methodology and philosophy of science*. Stanford: Stanford University Press, 1962, vol. 289, (pp. 194–197).
- L. Henkin (1961). "Some remarks on infinitely long formulas". *Infinitistic Methods, Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 2-9 September, 1959*, 1961, pp. 167–183.
- J. Hintikka (1962). *Knowledge and Belief*. Ithaca, N.Y., Cornell University Press.
- J. Hintikka (1968). "Language-Games for Quantifiers". *American Philosophical Quarterly Monograph Series 2: Studies in Logical Theory*. Oxford: Basil Blackwell, pp. 46–72.
- J. Hintikka (1973). *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic*. Oxford: Clarendon Press, 1973.
- J. Hintikka (1979). "Language Games", in Saarinen (ed.); *Game Theoretical Semantics*. Dordrecht: D. Reidel, 1979, pp. 1 – 26.
- J. Hintikka, J. Kulas (1985). *Anaphora and Definite Descriptions, Two Applications of Game- Theoretical Semantics*. Dordrecht, Boston, Lancaster: D. Reidel publishing company, 1985.
- J. Hintikka, G. Sandu (1989). "Informational Independence as a Semantical Phenomenon", in J. E. Fenstad, I. T. Frolov, R. Hilpinen (eds.); *Logic, Methodology and Philosophy of Science*. Amsterdam: Elsevier, 1989, vol. 8, pp. 571–589.
- J. Hintikka (1991). *Defining Truth, the Whole Truth and Nothing but the Truth*, Reports from the Department of Philosophy, No. 2, Helsinki: University of Helsinki, 1991.
- J. Hintikka (1996a). *The Principles of Mathematics Revisited*. Cambridge University Press, 1996.
- J. Hintikka (1996b). *Lingua Universalis vs. Calculus Ratiocinator: An Ultimate Presupposition of Twentieth-Century Philosophy*. Dordrecht: Kluwer, 1996.
- J. Hintikka (1997a) "No Scope for Scope? ". *Linguistics and Philosophy*, 1997, vol. 20, pp. 515–544.

- J. Hintikka (1997b). "Hilbert Vindicated?". *Synthese*, 1997, vol. 110, No. 1, pp. 15-36.
- J. Hintikka, G. Sandu (1997). "Game-theoretical semantics", in J. van Benthem and A. ter Meulen (eds.); *Handbook of Logic and Language*. Amsterdam: Elsevier, 1997. pp. 361–410.
- J. Hintikka (1999). *Inquiry as Inquiry: A Logic of Scientific Discovery*. Dordrecht: Springer, 1999.
- J. Hintikka, I. Halonen, A. Mutanen (1999). "Interrogative Logic as a General Theory of Reasoning", in J. Hintikka, *Inquiry as Inquiry: A Logic of Scientific Discovery*. Dordrecht: Springer, 1999, pp. 47–90.
- J. Hintikka (2000). "Game Theoretical Semantics as a Challenge to Proof Theory". *Nordic Journal of Philosophical logic*, vol. 4, No 2, pp. 127 – 141.
- J. Hintikka (2001). "Intuitionistic Logic as Epistemic Logic". *Synthese*, 2001, vol. 127 (1-2), pp. 7 – 19.
- J. Hintikka (2002a). "Hyperclassical Logic (A.K.A. IF Logic) and Its Implications for Logical Theory". *The Bulletin of Symbolic Logic*, 2002, vol. 8, No. 3, pp. 404-423.
- J. Hintikka (2002b). "Negation in logic and in natural language". *Linguistics and Philosophy*, 2002, vol. 25, pp. 585 – 600.
- J. Hintikka (2006a). "Reply to Wilfrid Hodges", in R. E. Auxier, L. E. Hahn (eds.); *The Philosophy of Jaakko Hintikka*. Library of Living Philosophers, Open Court, 2006, vol. 30, pp. 535 – 540.
- J. Hintikka (2006b). "A Proof of Nominalism: An Exercise in Successful Reduction in Logic". *From ontos verlag: Publications of the Austrian Ludwig Wittgenstein Society*, New Series, 2006, vol. 1.
- J. Hintikka (2006c). "Truth, Negation and Other Basic Notions of Logic", in van Benthem, Heinzman, M. Rebusi, H. Visser (eds.); *The Age of Alternative Logics*. Springer, 2006.
- J. Hintikka (2008). "Reforming Logic (and Set Theory)". [http://people.bu.edu/hintikka/Papers\\_files/Hintikka.Reforming%20logic.and%20set%20theory.0408.pdf](http://people.bu.edu/hintikka/Papers_files/Hintikka.Reforming%20logic.and%20set%20theory.0408.pdf) (8 july, 2014, 10:55 am)
- J. Hintikka (2011). "What is the Axiomatic Method?". *Synthese*, 2011, vol. 183, pp. 69 – 85.
- W. Hodges (1997a). "Compositional semantics for a language of imperfect information". *Logic Journal of the IGPL*, 1997, vol. 5, pp. 539–563.
- W. Hodges (1997b). "Some strange quantifiers", in J. Mycielski, G. Rozenberg, A. Salomaa, (eds); *Structures in Logic and Computer Science: A Selection of Essays in*

*Honor of A. Ehrenfeucht*. Lecture Notes in Computer Science, Verlag: Springer, 1997, pp 51–65.

W. Hodges (2001). "Dialogue foundations. a sceptical look". *Proceedings of the Aristotelian Society*, 2001, vol. LXXV, pp. 17–32.

W. Hodges (2013). "Logic and Games", in Edward N. Zalta (ed.); *The Stanford Encyclopedia of Philosophy*. (Spring 2013 Edition), URL = <<http://plato.stanford.edu/archives/spr2013/entries/logic-games/>>.

T. Janssen (2002). "Independent choices and the interpretation of IF-logic". *Journal of Logic, Language and Information*, 2002, vol. 11, pp. 367–387.

T. Janssen, F. Dechesne (2006). "Signalling in IF Games: A Tricky Business". *The Age of Alternative Logics*, 2006, part IV, pp. 221-241.

R. Jovanović (2013). "Hintikka's Take on the Axiom of Choice and the Constructivist Challenge". *Revista de Humanidades de Valparaiso*, 2013. No 2, pp. 135 – 152.

W. Kamlah, P. Lorenzen (1972). *Logische Propädeutik*. Stuttgart, Weimar: Metzler, second edition, 1972.

W. Kamlah, P. Lorenzen (1984). *Logical Propaedeutic*. Lanham Md. University Press of America, English translation of Kamlah and Lorenzen (1972) by H. Robinson.

L. Keiff (2007). *Le Pluralisme Dialogique: Approches dynamiques de l'argumentation formelle*. PhD thesis, Lille 3, 2007.

L. Keiff (2009). "Dialogical Logic", in Edward N. Zalta (ed.); *Stanford Encyclopedia of Philosophy* (Summer 2011 Edition), <http://plato.stanford.edu/entries/logic-dialogical/>.

A. Lecomte, M. Quatrini (2010). "Pour une étude du langage via l'interaction: Dialogues et sémantique en Ludique". *Mathématiques et Sciences Humaines*, 2010, vol. 189, pp. 37–67.

A. Lecomte, M. Quatrini (2011). "Figures of dialogue: A view from Ludics". *Synthese*, 2011, vol.183(1), pp. 59–85.

A. Lecomte (2011). *Meaning, Logic and Ludics*. London: Imperial College Press, 2011.

A. Lecomte, S. Troncon (eds); (2011). *Ludics, Dialogues and Interaction: PRELUDE Project 2006–2009. Revised Selected Papers*. Berlin, Heidelberg: Springer, 2011.

K. Lorenz (1970). *Elemente der Sprachkritik Eine Alternative zum Dogmatismus und Skeptizismus in der Analytischen Philosophie*. Frankfurt: Suhrkamp Verlag, 1970.

K. Lorenz (2001). "Basic objectives of dialogue logic in historical perspective". *Synthese*, 2011, vol. 127 (1-2), pp. 255–263.

- K. Lorenz (2010a). *Logic, Language and Method: On Polarities in Human Experience*. Berlin, New York: De Gruyter, 2010.
- K. Lorenz (2010b). *Philosophische Variationen: Gesammelte Aufsätze unter Einschluss gemeinsam mit Jürgen Mittelstraß geschriebener Arbeiten zu Platon und Leibni*. Berlin, New York: De Gruyter, 2010.
- P. Lorenzen, O. Schwemmer (1975). *Konstruktive Logik, Ethik und Wissenschaftstheorie*. Mannheim: Bibliographisches Institut, second edition, 1975.
- P. Lorenzen, K. Lorenz (1978). *Dialogische Logik*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1978.
- P. Lorenzen (1995). *Einführung in die operative Logik und Mathematik*. Berlin: Springer, 1995.
- S. Magnier (2013). *Approche dialogique de la dynamique épistémique et de la condition juridique*. London: College Publications, 2013.
- A. Mann, G. Sandu, M. Sevenster (2011). *Independence-Friendly Logic: A Game-Theoretic Approach*. London Mathematical Society Lecture Note Series: 386, 2011.
- M. Marion (2006). "Hintikka on Wittgenstein : From Language-Games to Game Semantics", in T. Aho, A.V. Pietarinen (eds); *Truth and Games, Essays in Honour of Gabriel Sandu*. Acta Philosophica Fennica, 2006, vol. 78, pp. 223-242.
- M. Marion (2009). "Why Play Logical Games ?", O. Majer, A.V. Pietarinen, T. Tulenheimo (eds.); *Logic and Games, Foundational Perspectives*. Dordrecht: Springer, 2009, pp. 3-26.
- P. Martin-Löf (1984). *Intuitionistic Type Theory. Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980*. Naples: Bibliopolis, 1984.
- P. Martin-Löf (2006). "100 years of Zermelo's axiom of choice: what was the problem with it?", *The Computer Journal*, 2006, vol 49 (3), pp. 345–350.
- J. von Neumann, O. Morgenstern (1944). *The Theory of Games and Economic Behavior*. Princeton: Princeton University Press, 1944.
- M. Pantasar (2009). *Truth, proof and Gödelian arguments : a defence of Tarskian truth in mathematics*. PHD thesis, Helsinki: University of Helsinki, Department of philosophy, 2009.
- T.E.Patton (1991). "On the Ontology of Branching Quantifiers". *Journal of Philosophical Logic*, 1991, vol. 20, pp. 205 – 223.
- C. S. Peirce (1898). *Reasoning and the Logic of Things: The Cambridge Conferences Lectures of 1898*, in Kenneth Laine Ketner (ed.); Cambridge Mass: Harvard University Press, 1992.

- A. Popek (2012). "Logical dialogues from Middle Ages", in C. Barés and al. (eds); *Logic of Knowledge. Theory and Applications*. London: College Publications, 2012, pp. 223–244.
- D. Prawitz (1979). "Proofs and the meaning and completeness of the logical constants", in J. Hintikka et al. (eds.); *Essays on Mathematical and Philosophical Logic*. Dordrecht: Reidel, 1979, pp. 25–40.
- D. Prawitz (2012). "Truth and Proof in Intuitionism", in P. Dybjer, S. Lindström, G. Sundholm, E. Palmgren (eds.); *Epistemology versus Ontology: Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf*. Dordrecht: Springer, 2012, pp. 45-68.
- G. Primiero (2008). "Constructive modalities for information", talk given at the *Young Researchers Days in Logic, Philosophy and History of Science*. Brussels, 1-2 September 2008.
- G. Primiero (2012). "A contextual type theory with judgmental modalities for reasoning from open assumptions". *Logique et Analyse*, 2012, vol. 220, pp. 579–600.
- S. Rahman (1993). *Über Dialoge, Protologische Kategorien und andere Seltenheiten*. Frankfurt, Paris, New York: P. Lang, 1993.
- S. Rahman, L. Keiff (2005). "On how to be a dialogician", in D. Vanderveken (ed.); *Logic, Thought, and Action*. Dordrecht: Kluwer, 2005, pp. 359–408.
- S. Rahman, T. Tulenheimo (2006). "From Games to Dialogues and Back: Towards a General Frame for Validity", in O. Majer and al. (eds); *Games: Unifying Logic, Language and Philosophy*. Dordrecht: Springer, 2006, pp. 153–208.
- S. Rahman, L. Keiff (2010). "La Dialectique entre logique et retorique". *Revue de Métaphysique et de Morale*, 2010, vol. 66(2), pp. 149–178.
- S. Rahman (2012). "Negation in the Logic of first degree entailment and *tonk*: A dialogical study", in S. Rahman, G. Primiero, M. Marion (eds); *The Realism/Antirealism Debate in the Age of Alternative Logics*. Dordrecht: Springer, 2012, pp. 213–250.
- S. Rahman, N. Clerbout (2013). "Constructive Type Theory and the Dialogical Approach to Meaning". *The Baltic International Yearbook of Cognition, Logic and Communication*, 2013, vol. 8: Games, Game Theory and Game Semantics, pp. 1-72. DOI: 10.4148/1944-3676.1077. Also online in: [www.thebalticyearbook.org](http://www.thebalticyearbook.org).
- S. Rahman, N. Clerbout (2014). *Linking Games and Constructive Type Theory: Dialogical Strategies, CTT-Demonstrations and the Axiom of choice*. Springer, forthcoming.
- S. Rahman, N. Clerbout, Z. McConaughy (2014). "On play-objects in dialogical games. Towards a Dialogical approach to Constructive Type Theory", in P. Allo (ed.); *Tribute to J.P. van Bendegem*. College Publications, to appear.



- S. Rahman, N. Clerbout, R. Jovanović (forthcoming). "Knowledge and its Game Theoretical Foundations: The Challenges of the Dialogical Approach to Constructive Type Theory". Springer, forthcoming.
- A. Ranta (1988). "Propositions as games as types" *Synthese*, 1988, vol. 76, pp. 377–395.
- A. Ranta (1991). "Constructing possible worlds". *Theoria*, 1991, vol. 57(1-2), pp. 77–99.
- A. Ranta (1994). *Type-Theoretical Grammar*. Oxford: Clarendon Press, 1994.
- S. Read (2008). "Harmony and modality", in C. Dégrémont et al. (eds.); *Dialogues, Logics and Other Strange Things: Essays in Honour of Shahid Rahman*. London: College Publications, 2008, pp. 285–303.
- S. Read (2010). "General Elimination Harmony and the Meaning of the Logical Constants". *Journal of Philosophical Logic*, 2010, vol. 39 (5), pp. 557–576.
- J. Redmond (2010). *Logique dynamique de la fiction: Pour une approche dialogique*. London: College Publications, 2010.
- J. Redmond, M. Fontaine (2011). *How to Play Dialogues: An Introduction to Dialogical Logic*. London: College Publications, 2011.
- H. Rückert (2011a). *Dialogues as a Dynamic Framework for Logic*. London: College Publications, 2011.
- H. Rückert (2011b). "The Conception of Validity in Dialogical Logic", Talk at the workshop *Proofs and Dialogues*. Tübingen, 2011.
- P. de Rouilhan, S. Bozon (2006). "The Truth of IF: Has Hintikka Really Exorcised Tarski's Curse?". *Auxier & Hahn*, 2006, pp. 683–705.
- E. Saarinen (1978). "Dialogue semantics versus game-theoretical semantics". *Proceedings of the Biennial Meeting of the Philosophy of Science Association Vol. 2: Symposia and Invited Papers*. Chicago: University of Chicago Press, 1978, pp. 41–59.
- G. Sandu (1991). *Studies in Game-Theoretical Logics and Semantics*. Ph.D. thesis, Helsinki: University of Helsinki, 1991.
- G. Sandu (1997). "On the theory of anaphora: dynamic predicate logic vs. game-theoretical semantics". *Linguistics and Philosophy*, 1997, vol. 20, pp.147-174.
- G. Sandu, J. Jacot (2012). "Quantification and Anaphora in Natural Language". *Prospect for meaning*, Berlin, New York: Walter de Gruyter Inc, 2012.

- P. Schroeder-Heister (2008). "Lorenzen's operative justification of intuitionistic logic", in M. van Atten et al. (eds.); *One Hundred Years of Intuitionism (1907–2007)*. Basel: Birkhäuser, 2008, pp. 214–240.
- T. Skolem (1955). "A critical remark on foundational research". *Norske Vid. Selsk. Forh., Trondheim*, 1955, vol 28, pp.100–105.
- G. Sundholm (1986). "Proof-theory and meaning", in D. Gabbay, F. Guentner (eds.); *Handbook of Philosophical Logic*, vol. 3, Dordrecht: Reidel, 1986, pp. 471–506.
- G. Sundholm (1997). "Implicit epistemic aspects of constructive logic". *Journal of Logic, Language, and Information*, 1997, vol. 6 (2), pp. 191-212.
- G. Sundholm (1998). "Inference versus Consequence", in T. Childers (ed.), *The Logica Yearbook 1997*, Prague:Filosofia, pp. 26–36.
- G. Sundholm (2001). "A Plea for Logical Atavism", in O. Majer (ed.); *The Logica Yearbook 2000*. Prague: Filosofia, 2001, pp. 151-162.
- G. Sundholm (2009). "A Century of Judgment and Inference: 1837–1936", in L. Haaparanta (ed.); *The Development of Modern Logic*. Oxford: Oxford University Press, 2009, pp. 263–317.
- G. Sundholm (2013a). "Independence Friendly Language is First - order after all?". *Logique et Analyse*, 2013.
- G. Sundholm (2013b). "Inference and Consequence in an Interpreted Language". Talk at the Workshop *Proof theory and Consequence in an Interpreted Language*. Groningen, December 3-5, 2013.
- W. Tait (1994). "The law of excluded middle and the axiom of choice", in A. George (ed.); *Mathematics and Mind*, New York: Oxford University Press, 1994, pp. 45–70.
- A. Tarski (1935). translation by J.H. Woodger (1983). "The Concept of Truth in Formalized Languages", in J. Corcoran (ed.); *Logic, Semantics, Metamathematics*. Hackett, 1935.
- N. Tennant (1998). "Game Some People Would Have All of Us Play". *Philosophia Mathematica*, 1998, vol. 6 (3), pp. 90 - 115.
- T. Tulenheimo (2009). "Independence Friendly Logic", in Edward N. Zalta (ed.); *The Stanford Encyclopedia of Philosophy (Summer 2009 Edition)*, URL = <<http://plato.stanford.edu/archives/sum2009/entries/logic-if/>>.
- T. Tulenheimo (2011). "On Some Logic Games in Their Philosophical Context", in A. Lecomte, S. Tronçon (eds.); *Ludics, Dialogues and Interaction: PRELUDE Project 2006–2009, Revised Selected Papers*. Berlin, Heidelberg: Springer, 2011, pp. 88–113.
- W. Walkoe (1970). "Finite partially ordered quantification". *Journal of symbolic logic*, 1970, vol. 35, pp. 535-555.

- J. Väänänen (2001). "Second-Order Logic and Foundations of Mathematics". *Bulletin of Symbolic Logic*, 2001, vol. 7, pp. 504–520.
- J. Van Benthem (1996). *Exploring Logical Dynamics*. Stanford Calif., Cambridge: CSLI Publications and Cambridge University Press, 1996.
- J. Van Benthem (2001a). "Correspondence Theory", in D. Gabbay, F. Guentner (eds.); *Handbook of Philosophical Logic*. Volume 2, Dordrecht: Reidel, 2001, pp. 167–247, Reprint with addenda in second edition, pp. 325–408.
- J. Van Benthem (2001b) "Logic in Games". Lecture notes, unpublished.
- J. Van Benthem(2011). *Logical Dynamics of Information and Interaction*. Cambridge: Cambridge University Press, 2011.
- H. Van Ditmarsch, W. van der Hoek, B. Kooi (2007). *Dynamic Epistemic Logic*. Berlin: Springer, 2011.
- E. Zermelo (1908). "Untersuchungen über die Grundlagen der Mengenlehre, I". *Mathematische Annalen*, 1908, vol. 65, pp. 261 – 281.