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**THÈSE DE DOCTORAT DE
L'UNIVERSITÉ PARIS I PANTHÉON-SORBONNE**

Mention

Mathématiques Appliquées

Présentée par

Nikolaos PNEVMATIKOS

Pour obtenir le grade de

Docteur de l'Université Paris I Panthéon-Sorbonne

**Contributions à la théorie des jeux:
valeur asymptotique des jeux dépendant de la fréquence et
décompositions des jeux finis**

Soutenue le 01/07/2016 devant le jury composé de:

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“L’Université Paris 1 - Panthéon Sorbonne n’entend donner aucune approbation, ni improbation aux opinions émises dans cette thèse; elles doivent être considérées comme propres à leur auteur.”

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*...et c'est pour cela qui n'existent pas des limites.
Les seules limites de l'homme sont ceux de sa pensée, de son idiosyncrasie et de ses
choix...*

Résumé

Les problèmes abordés et les résultats obtenus dans cette thèse se divisent en deux parties. La première concerne l'étude de la valeur asymptotique de jeux répétés à somme nulle ayant un aspect stochastique. La deuxième porte sur les structures algébriques de l'espace des jeux finis; précisément, nous étudions de décompositions adéquates d'un jeu fini arbitraire en une somme de jeux distingués.

Partie I. Valeur asymptotique des jeux dépendant de la fréquence

Cette partie concerne les jeux à somme nulle dépendant de la fréquence de répétition de chaque profil d'actions (jeux- FD par suite) ; précisément nous étudions le comportement asymptotique de leur valeur. Il s'agit de jeux dynamiques qui évoluent à temps discret. Cette partie est divisée en deux chapitres.

Dans le premier chapitre, nous présentons de résultats obtenus dans [Contou-Carrère \[2011\]](#) concernant un problème de la décision dépendant de la fréquence ; il s'agit d'un cas particulier de notre modèle.

Dans le deuxième chapitre, nous étudions les jeux- FD à somme nulle à deux joueurs dont le paiement d'étape est additive au sens qu'elle est constituée d'une partie qui dépend d'actions courantes des joueurs et d'une partie qui dépend d'une variable laquelle exprime le nombre de fois que chaque profil d'actions a été choisi jusqu'à cet étape. Si l'on considère cette variable comme un état, le jeu peut s'interpréter comme un jeu stochastique. Nous introduisons un jeu différentiel associé au jeu- FD dont la valeur se ramène à une équation de Hamilton-Jacobi-Bellman-Isaacs. La fonction de paiement présente une irrégularité à l'origine. En affrontant ce problème, nous prouvons l'existence de la valeur du jeu différentiel sur l'intervalle du temps continu $[0, 1]$. Les résultats de la théorie des jeux différentiels sur les schémas d'approximation de la solution de l'équation Hamilton-Jacobi-Bellman-Isaacs impliquent la convergence des valeurs discrètes vers la valeur du jeu différentiel. Ceci nous permet de prouver que la valeur du jeu- FD converge vers la valeur du jeu continu qui débute à l'état initial $\mathbf{0}$. En d'autres termes, via l'analyse de la solution de l'équation de Hamilton-Jacobi-Bellman-Isaacs d'un jeu différentiel adéquat, nous prouvons la convergence de la valeur du jeu discret original.

Partie II. Décompositions des jeux finis

Dans cette partie, l'objectif fondamental est la décomposition de l'espace des jeux finis en sous espaces de jeux adéquats et plus faciles à étudier vu que leurs équilibres sont distingués. Cette partie est divisée en deux chapitres.

Dans le premier chapitre, afin d'établir une décomposition adéquate de l'espace des jeux finis, nous introduisons la classe des jeux uniformes et celle des jeux non-interactifs qui interviennent dans la décomposition de tout jeu fini arbitraire. La première classe se réfère aux jeux finis qui admettent le profil uniformément mixte comme équilibre de Nash et la deuxième est un sous espace de l'espace des jeux potentiels admettant un équilibre en stratégies dominantes. Alors, nous introduisons

des opérateurs de projection appropriés qui conduisent à une décomposition canonique de tout jeu arbitraire fini en trois composantes; la première se réfère aux jeux uniformes avec constante zéro, la deuxième aux jeux non-interactifs de somme totale zéro et la troisième aux jeux dont le paiement de chaque joueur est constant sur les profils d'actions. Lorsque l'espace des jeux finis est muni du produit scalaire standard, la décomposition qui en découle est orthogonale et dans ce cas, étant donné un jeu fini arbitraire, nous fournissons des expressions explicites des jeux uniformes et non-interactifs qui sont les plus proches au jeu donné. Alors, nous caractérisons les équilibres approximatifs d'un jeu donné par les équilibres uniformément mixtes et en stratégies dominantes lesquels apparaissent sur ses composantes.

Dans le deuxième chapitre, nous introduisons sur l'espace des jeux finis une famille de produits scalaires et nous définissons la classe des jeux harmoniques relativement au produit scalaire choisi dans cette famille. Inspiré de la décomposition de Helmholtz-Hodge et de la procédure d'association d'un graphe à chaque jeu fini introduite dans [Candogan et al. \[2011\]](#), nous établissons une décomposition orthogonale de l'espace des jeux finis, par rapport au produit scalaire choisi, en les sous espaces des jeux potentiels, des jeux harmoniques et des jeux non-stratégiques. Les composantes des jeux finis qui résultent de cette décomposition possèdent des équilibres distingués plus souples. Nous montrons qu'un profil complètement mixte lié au produit scalaire choisi, est un équilibre de Nash dans chaque jeu harmonique et que génériquement les jeux harmoniques n'admettent pas d'équilibres pures. De plus, nous généralisons leurs résultats concernant les jeux harmoniques matriciels en montrant que les équilibres corrélés et mixtes coïncident dans cette classe des jeux et l'équilibre est unique lorsque les joueurs possèdent le même nombre d'actions. Notre approche peut se généraliser aux jeux dénombrables.

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Part I

Asymptotic value in frequency dependent games

Chapter 1

Introduction Partie I

L'objet de cette partie de la thèse est l'étude de la valeur asymptotique du jeu dépendant de la fréquence (Jeux-*FD*). On peut considérer ce jeu comme un jeu stochastique à transition déterministe. Nous présentons alors les outils utilisés dans ce cadre.

1.1 Jeux stochastiques à transition déterministe

Un jeu stochastique de transition déterministe à somme nulle et à deux joueurs est défini par $(\mathcal{Z}, I, J, q, g)$ où :

- \mathcal{Z} est l'ensemble d'états.
- I et J sont des ensembles finis représentant les actions respectives des joueurs.
- $q : \mathcal{Z} \times I \times J \rightarrow \mathcal{Z}$ est la fonction de transition.
- $g : \mathcal{Z} \times I \times J \rightarrow \mathbb{R}$ est la fonction de paiement.

Un tel jeu, d'état initial $z_0 \in \mathcal{Z}$, se déroule de la manière suivante : à chaque étape $t \geq 0$, les joueurs, ayant connaissance de l'histoire passée $h_t = (z_0, i_0, j_0, \dots, i_{t-1}, j_{t-1}, z_t)$, choisissent simultanément, de façon indépendante l'un à l'autre, une action dans leurs ensembles d'actions respectifs, $i_t \in I$ et $j_t \in J$. Le paiement à l'étape t qui reçoit le joueur 1 de la part du joueur 2 est $g_t := g(z_t, i_t, j_t)$. L'état z_{t+1} à l'étape $t+1$ est défini par $q(z_t, i_t, j_t)$ et ensuite le triplet (z_{t+1}, i_t, j_t) est annoncé aux joueurs.

L'ensemble des histoires possibles, avant l'étape t , est noté $\mathbf{H}_t := (\mathcal{Z} \times I \times J)^{t-1} \times \mathcal{Z}$ et l'ensemble des toutes les histoires finies est $\mathbf{H} = \cup_{t \geq 1} \mathbf{H}_t$. Dans un tel jeu, on parle de *stratégies de comportement* pour le joueur 1 (resp. le joueur 2), s'exprimant par une application $\sigma : \mathbf{H} \rightarrow \Delta(I)$ (resp. $\tau : \mathbf{H} \rightarrow \Delta(J)$). Les ensembles de stratégies de comportement des joueurs 1 et 2 sont notés respectivement Σ et T .

Un état initial z_0 et un couple de stratégies de comportement (σ, τ) induisent naturellement une mesure de probabilité sur l'ensemble des histoires finies \mathbf{H} . Le théorème d'extension de Kolmogorov assure que cette mesure de probabilité s'étend de manière unique à l'ensemble des histoires infinies $\mathbf{H}_\infty := (\mathcal{Z} \times I \times J)^{\mathbb{N}^*}$ qui est

muni de la tribu produit. Cette mesure de probabilité est notée $\mathbb{P}_{\sigma,\tau}^{z_0}$ et l'espérance correspondante est notée $\mathbb{E}_{\sigma,\tau}^{z_0}$. Avec cette mesure, on peut définir le paiement de ce jeu. Il existe deux manières classiques d'agréger les paiements d'étapes : Étant donné $N \in \mathbb{N}^*$, le jeu répété N fois est le jeu $\Gamma_N(z_0)$ dont l'ensemble des stratégies du joueur 1 est Σ et du joueur 2 est T et la fonction de paiement $\gamma_N(z_0) : \Sigma \times T \rightarrow \mathbb{R}$ est définie par :

$$\gamma_N(z_0, \sigma, \tau) = \mathbb{E}_{\sigma,\tau}^{z_0} \left(\frac{1}{N} \sum_{t=0}^{N-1} g_t \right).$$

Dans ce jeu, le joueur 1 cherche à maximiser et le joueur 2 à minimiser $\gamma_N(z_0)$ en adoptant chacun une stratégie de comportement adéquate.

Étant donné $\lambda \in (0, 1]$, le jeu escompté de taux λ est le jeu $\Gamma_\lambda(z_0)$ dont l'ensemble des stratégies du joueur 1 est Σ et du joueur 2 est T et la fonction de paiement $\gamma_\lambda(z_0) : \Sigma \times T \rightarrow \mathbb{R}$ est définie par :

$$\gamma_\lambda(z_0, \sigma, \tau) = \mathbb{E}_{\sigma,\tau}^{z_0} \left(\sum_{t \geq 0} \lambda(1-\lambda)^t g_t \right).$$

Dans ce jeu, le joueur 1 cherche à maximiser et le joueur 2 à minimiser $\gamma_\lambda(z_0)$ en adoptant chacun une stratégie de comportement adéquate.

Le théorème de Kuhn (1953) (cf. [Sorin \[2002\]](#)) affirme que dans le cas fini¹, pour toute stratégie de comportement σ du joueur 1, il existe une stratégie mixte u telle que, pour toute stratégie de comportement τ du joueur 2, les triplets (z_0, σ, τ) et (z_0, u, τ) induisent les mêmes distributions de probabilité sur \mathbf{H}_∞ ; de même en inversant les rôles des joueurs 1 et 2. Ainsi, toute stratégie de comportement peut être vue comme une stratégie mixte et réciproquement.

Le théorème de minmax de [von Neumann \[1928\]](#) assure l'existence d'une valeur $\mathbf{V}_N(z_0)$ pour le jeu $\Gamma_N(z_0)$ et d'une valeur $\mathbf{V}_\lambda(z_0)$ pour le jeu $\Gamma_\lambda(z_0)$ et ces valeurs sont :

$$\begin{aligned} \mathbf{V}_N(z_0) &= \max_{\sigma \in \Sigma} \min_{\tau \in T} \gamma_N(z_0, \sigma, \tau) = \min_{\tau \in T} \max_{\sigma \in \Sigma} \gamma_N(z_0, \sigma, \tau) \\ \mathbf{V}_\lambda(z_0) &= \max_{\sigma \in \Sigma} \min_{\tau \in T} \gamma_\lambda(z_0, \sigma, \tau) = \min_{\tau \in T} \max_{\sigma \in \Sigma} \gamma_\lambda(z_0, \sigma, \tau) \end{aligned}$$

Une stratégie d'un joueur, telle qu'elle est définie précédemment, peut dépendre des toutes les actions et tous les états passés. Le calcul de la valeur et la détermination de stratégies optimales semblent être très difficiles vu la complexité des ensembles des stratégies. Cependant, comme nous allons le voir, une telle étude est faisable grâce au théorème suivant de Shapley (cf. [Shapley \[1953\]](#)). Puisque la transition est à support fini, pour $f \in \mathbb{R}^{\mathcal{Z}}$ et $(z, u, v) \in \mathcal{Z} \times \Delta(I) \times \Delta(J)$, on pose :

$$\mathbb{E}_{u,v}^z(f) = \sum_{(i,j) \in I \times J} u(i)v(j)f(q(z, i, j))$$

¹Pour un jeu d'espace d'états et d'ensembles d'actions infinis, cf. [Aumann \[1961\]](#)

qui est bien définie car I et J sont finis. Aussi, on pose:

$$g(z, u, v) = \sum_{(i,j) \in I \times J} u(i)v(j)g(z, i, j).$$

Théorème 1.1.1. *Étant donnés $n \in \mathbb{N}^*$ et $\lambda \in (0, 1]$, les équations suivantes - dites équations de Shapley - sont vérifiées :*

$$\begin{aligned} n \mathbf{V}_n(z) &= \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} (g(z, u, v) + (n-1)\mathbb{E}_{u,v}^z(\mathbf{V}_{n-1})) \\ \lambda \mathbf{V}_\lambda(z) &= \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} (\lambda g(z, u, v) + (1-\lambda)\mathbb{E}_{u,v}^z(\mathbf{V}_\lambda)) \end{aligned}$$

Les équations de Shapley indiquent que si aujourd'hui l'état est z alors les joueurs savent qu'ils pourront jouer demain de manière optimale dans la suite du jeu.

Dans la littérature des jeux stochastiques, une partie importante se réfère au déroulement du jeu en temps long et ceci pour comprendre comment les valeurs et les stratégies optimales se comportent lorsque la durée du jeu tend à l'infini. Pour le jeu escompté cela correspond au cas où λ tend vers 0, alors que pour le jeu répété à N étapes il correspond au cas où N tend vers l'infini.

Définition 1.1.2. Un jeu stochastique a une *valeur asymptotique* si \mathbf{V}_N converges lorsque N tend vers l'infini.

Le résultat suivant est dû à [Bewley and Kohlberg \[1976\]](#) :

Théorème 1.1.3. *Tout jeu stochastique fini admet une valeur asymptotique.*

L'étude de la valeur asymptotique offre aux joueurs un objectif de long terme. Lorsque celle-ci existe, on peut affirmer que deux joueurs rationnels, jouant un jeu dynamique pendant une longue durée, obtiendront approximativement l'un la valeur asymptotique et l'autre son opposé. Pour autant, les stratégies optimales peuvent dépendre fortement du taux d'escompte et du nombre de répétitions.

Dans notre étude, nous nous intéressons à la valeur asymptotique d'une classe de jeux dynamiques qui peuvent être interprétés comme de jeux stochastiques avec un espace d'états dénombrable et à transition déterministe où les ensembles d'actions sont finis. Nous étudions le jeu répété à N étapes.

1.1.1 Jeux- FD (dépendant de la fréquence)

Les jeux- FD sont des jeux dynamiques où les gains dépendent à chaque étape des actions joués mais aussi des fréquences relatives aux actions jouées dans le passé. En d'autres termes, il s'agit des jeux répétés dont le paiement à chaque étape dépend des actions actuelles et d'une certaine moyenne des actions antérieures. Ces jeux sont introduits par [Brenner and Witt \[2003\]](#). Dans le cas de deux joueurs, ce modèle est déjà traité par [Joosten et al. \[2003\]](#), mais la notion d'équilibre qui est utilisée par ces auteurs est faible puisqu'elle n'intègre pas le caractère uniforme.

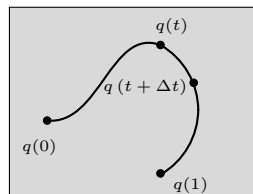
Un jeu de ce type permet de modéliser une situation où le choix d'une action génère des externalités qui s'accumulent au cours du temps. C'est le cas, par exemple, des modèles de pollution où chaque joueur dispose de deux actions, l'une qui détériore l'environnement et l'autre qui le préserve ; l'action polluante procure au joueur le plus grand paiement immédiat, mais l'utilisation répétée de cette action conduit à la réduction de son paiement à l'horizon du temps. La question de la préservation de l'environnement pour les générations futures est essentielle dans la pensée économique depuis l'apparition du concept de développement durable. Ceci conduit à considérer le déroulement des jeux sur un horizon lointain afin de répondre aux besoins des générations présentes aussi bien que des générations futures. Mais, comment déterminer un horizon temporel éloigné alors que l'on ne sait pas évaluer exactement les dommages potentiels des activités humaines sur l'environnement ? L'enjeu consiste à attribuer des poids adéquats afin de réussir un arbitrage entre le présent et le futur. Lorsque un facteur d'escompte intervient dans l'évaluation des décisions, les générations futures situées sur un horizon temporel très éloigné ne sont prises en compte qu'imperceptiblement par rapport aux générations présentes. L'option d'obtention d'un gain immédiat favorise la satisfaction des besoins des générations présentes et défavorise les besoins des générations futures. Dans certains modèles, un taux d'actualisation variable est envisagé et ceci conduit à la dégression de la valeur du taux d'actualisation dans le temps proche afin d'accorder un poids plus grand au temps éloigné qui concerne les générations futures.

1.2 Jeux différentiels sur $[0, 1]$

Les jeux différentiels sont des jeux dynamiques en temps continu ; ils sont introduits dans Isaacs [1965]. Dans un tel jeu les joueurs agissent continûment et le temps prend ses valeurs sur l'intervalle $[0, 1]$ de la droite réelle. Aux jeux en temps continu les états successifs définissent une trajectoire continue et lisse dans l'espace d'états \mathbb{R}^k , i.e., entre deux instants rapprochés les décisions des joueurs ont un effet infinitésimal sur l'évolution des états. L'évolution sur l'espace d'états est régie par une équation différentielle de la forme $\dot{q} = f(q, u, v)$ où (u, v) désigne le profil des stratégies mixtes dépendant du temps. Lorsqu'on fixe les conditions initiales $(t, q_0) \in [0, 1] \times \mathbb{R}^k$, on est devant un problème de Cauchy :

$$\begin{cases} \dot{q} = f(q, u, v) \\ q(t) = q_0, \quad q_0 \in \mathbb{R}^k. \end{cases}$$

Au cas déterministe une trajectoire lisse dans l'espace d'états est présentée ci-dessous:



On considère les ensembles des contrôles \mathcal{U} et \mathcal{V} , supposés mesurables au sens de Lebesgue, définis sur $[0, 1]$ à valeurs respectivement dans $\Delta(I)$ et $\Delta(J)$. Ainsi,

la fonction de paiement est $g : \mathbb{R}^k \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$; le profil des stratégies mixtes à l'instant $s \in [t, 1)$ est noté (u_s, v_s) . Si les joueurs choisissent les contrôles $(\tilde{u}, \tilde{v}) \in \mathcal{U} \times \mathcal{V}$, alors le paiement que le joueur 1 a intérêt à maximiser et le joueur 2 à minimiser dans le jeu différentiel, de conditions initiales $(t, q_0) \in [0, 1] \times \mathbb{R}^k$, est donné par :

$$G(q_0, \tilde{u}, \tilde{v}) = \int_t^1 g(q_s, u_s, v_s) ds.$$

1.2.1 Stratégies et valeurs en temps continu

Aux jeux en temps continu, la formalisation du fait que chaque joueur connaît les contrôles choisis antérieurement par son adversaire pose un problème en ce qui concerne la définition des stratégies afin que le processus induit (q_s, u_s, v_s) soit bien défini. Afin de répondre à cette question nous adoptons dans notre étude le concept des *stratégies non-anticipatives* introduit dans [Varaiya \[1967\]](#), [Roxin \[1969\]](#) et [Elliott and Kalton \[1972b\]](#).

Définition 1.2.1. Une application $\alpha : \mathcal{V} \rightarrow \mathcal{U}$ définie une stratégie non-anticipative pour le joueur 1 si, pour tout $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$ tels que $\tilde{v}_1(s) = \tilde{v}_2(s)$ sur $[0, t]$, on a $\alpha[\tilde{v}_1](s) = \alpha[\tilde{v}_2](s)$ sur $[0, t]$, pour tout $t \in [0, 1]$. De façon symétrique, on définit la stratégie du joueur 2, $\beta : \mathcal{U} \rightarrow \mathcal{V}$. Les ensembles des stratégies non-anticipatives des deux joueurs sont notés respectivement \mathcal{A} et \mathcal{B} .

Un couple $(\alpha, \tilde{v}) \in \mathcal{A} \times \mathcal{V}$ ou $(\tilde{u}, \beta) \in \mathcal{U} \times \mathcal{B}$, induit une paire de contrôles et donc une dynamique bien définie sur \mathbb{R}^k . Les stratégies non-anticipatives reflètent le fait qu'aucun parmi les deux joueurs ne peut deviner d'avance le comportement futur de son adversaire. Dès que les stratégies sont bien définies, on est prêt à présenter la valeur *inférieure* et la *valeur supérieure* du jeu continu en tenant compte quel est le joueur qui choisit en premier son contrôle ; c'est à dire, pour tout $(t, q_0) \in [0, T] \times \mathbb{R}^k$:

$$W^-(t, q_0) = \sup_{\alpha \in \mathcal{A}} \inf_{\tilde{v} \in \mathcal{V}} G(q_0, \alpha[\tilde{v}], \tilde{v}) \quad \text{et} \quad W^+(t, q_0) = \inf_{\beta \in \mathcal{B}} \sup_{\tilde{u} \in \mathcal{U}} G(q_0, \tilde{u}, \beta[\tilde{u}]).$$

Les valeurs d'un jeu différentiel ainsi définies sont aussi données dans [Bardi and Capuzzo-Dolcetta \[2008\]](#) et [Souganidis \[1999\]](#). Le concept de la *stratégie non-anticipative avec délai* est défini un peu plus tard, (cf. [Cardaliaguet \[2000\]](#)), afin d'étudier les jeux sous forme normale ; un couple de stratégies non-anticipatives avec délai induit en fait une paire de contrôles unique et ainsi on peut établir que la trajectoire représentant l'évolution de l'état est bien définie dans \mathbb{R}^k . Plusieurs exemples, plutôt pratiques, sont élaborés où chaque joueur connaît l'état courant du jeu et éventuellement garde une trace de l'histoire antérieure à cet état. Dans le but de modélisation d'une telle situation sont introduites les *stratégies feedback*, (cf. [Bardi and Capuzzo-Dolcetta \[2008\]](#)). Aussi, quelques notions de stratégies plus raffinées (*random strategies*) apparaissent dans la littérature des jeux différentiels à information incomplète ou encore des jeux pour lesquels on ne peut pas établir immédiatement l'existence de la valeur en utilisant des outils suffisamment efficaces (cf. [Cardaliaguet \[2000\]](#)).

1.2.2 Existence et schémas d'approximation de la valeur

Dans jeux différentiels, la valeur du jeu est celle de la fonction de paiement lorsque les deux joueurs font de leurs mieux, c'est à dire, ils utilisent leurs stratégies optimales ; pourtant, ces stratégies optimales n'existent pas toujours. Isaacs [1965], afin d'affronter les problèmes qui surgissent dans la théorie des jeux différentiels, a introduit la procédure de la dérivation heuristique en tenant compte que la valeur - lorsqu'elle est lisse - doit satisfaire l'équation:

$$\begin{cases} \frac{\partial W}{\partial t}(t, q) + \mathcal{H}(\nabla_q W(t, q), q) = 0, & (t, q) \in [0, 1] \times \mathbb{R}^k, \\ W(1, q) = \phi(q), & q \in \mathbb{R}^k, \end{cases} \quad (1.2.1)$$

où $\phi : \mathbb{R}^k \rightarrow \mathbb{R}$ est bornée et lipschitzienne et pour tout $(\xi, q) \in \mathbb{R}^k \times \mathbb{R}^k$ on a :

$$\mathcal{H}(\xi, q) = \mathcal{H}^-(\xi, q) = \max_u \min_v \langle f(q, u, v), \xi \rangle + g(q, u, v)$$

ou

$$\mathcal{H}(\xi, q) = \mathcal{H}^+(\xi, q) = \min_v \max_u \langle f(q, u, v), \xi \rangle + g(q, u, v).$$

Ces équations s'appellent *équations de Hamilton-Jacobi-Bellman-Isaacs* et nous les noterons (HJBI). En général on a :

$$\mathcal{H}^+(\xi, q) \leq \mathcal{H}^-(\xi, q), \quad \forall (\xi, q) \in \mathbb{R}^k \times \mathbb{R}^k.$$

On dit que la *condition d'Isaacs* est satisfaite lorsque :

$$\mathcal{H}(\xi, q) = \mathcal{H}^+(\xi, q) = \mathcal{H}^-(\xi, q), \quad \forall (\xi, q) \in \mathbb{R}^k \times \mathbb{R}^k. \quad (1.2.2)$$

Dans Fleming [1961], Fleming [1964], les jeux différentiels sont étudiés par discrétisation du temps avec un pas uniforme et les valeurs inférieures et supérieures sont définies en tenant compte du joueur qui joue en premier ; l'auteur établit l'existence de la valeur pour les jeux à horizon fini. Puis, Friedman [1970] a introduit une notion de stratégie qui change un nombre fini d'instantans dans le temps et prouve l'existence de la valeur lorsque les contrôles des joueurs agissent séparément sur la dynamique et le paiement. Elliott and Kalton [1972b] ont étendu cette approche lorsque les stratégies changent un nombre non nécessairement fini de fois.

Les équations (HJBI) n'admettent pas en général une solution globale et lisse ; aussi cette solution même si elle existe n'est pas forcément unique. Pour les équations (HJBI) de premier ordre, une notion plus faible de solution, celle de *solution de viscosité* est introduite dans Crandall and Lions [1983]. Lorsque la dynamique est déterministe, la relation² de la valeur avec les solutions de viscosité est établie dans Evans and Souganidis [1984] ; ces auteurs prouvent que, sous certaines conditions de régularité des fonctions de paiement et de la dynamique et sous la condition d'Isaacs (1.2.2), la valeur du jeu différentiel existe et elle est la solution unique de l'équation (HJBI) dans l'espace des fonctions bornées et continues sur $[0, 1] \times \mathbb{R}^k$ satisfaisant en plus une condition au bord (1.2.1).

²La connection entre la valeur introduite dans Fleming [1961] avec la notion de viscosité pour les jeux à un horizon fini apparaît dans Souganidis [1985]. En ce qui concerne la valeur dans le sens définie dans Friedman [1970], le lien avec les solutions de viscosité est établi à Barron et al. [1984].

L'existence et l'unicité des solutions de viscosité ouvrent le chemin pour l'étude de la convergence des approximations discrètes. En général, ces approximations ne convergent pas lorsque la fonction valeur est discontinue. Les résultats sur l'existence de la valeur, sa caractérisation comme solution d'une équation (HJBI) et la convergence des approximations discrètes à cette valeur pour les jeux à horizon fini, sont présentés d'une façon simplifiée dans [Souganidis \[1999\]](#). Les résultats sur la convergence des approximations discrètes pour les jeux poursuite-évasion établis dans [Bardi and Soravia \[1991\]](#) sont étendus par [Bardi and Capuzzo-Dolcetta \[2008\]](#), sous ces mêmes conditions de régularité, aux jeux à horizon infini.

1.3 Interconnection entre jeux répétés et différentiels

Selon [Sorin \[2002\]](#), la valeur asymptotique des jeux dynamiques discrets devrait exister sous certaines conditions de régularité portant sur les fonctions de paiement et de transition. Un jeu répété à n -étapes peut être vu alternativement comme un jeu joué sur l'intervalle $[0, 1]$ où le paiement total est donné par l'intégrale de la fonction de paiement sur $[0, 1]$ et où les joueurs choisissent leurs actions seulement sur les nœuds de la partition uniforme de l'intervalle du temps, i.e., $\frac{k}{n}$ où k varie entre 0 et n . Lorsque n tend à l'infini, ce jeu peut se voir comme un jeu hypothétique qui est discrétisé un certain nombre de fois sur l'intervalle du temps continu $[0, 1]$ et alors la valeur devrait converger vers la valeur de ce *jeu limite*.

L'étude de la valeur asymptotique d'un jeu à temps discret, vue comme la valeur d'un jeu à temps continu avec une durée fixée, est apparue d'abord à Big Match à information incomplète d'un seul côté dans [Sorin \[1984\]](#). L'étude d'approchabilité faible pour les jeux répétés à paiements vectoriels, introduits dans [Blackwell et al. \[1956\]](#) est élaborée via un jeu différentiel dans [Vieille \[1992\]](#). La même dynamique est apparue dans un jeu différentiel considéré par [Laraki \[2002\]](#) qui a démontré l'existence de la valeur asymptotique pour les jeux à information incomplète d'un seul côté ; il s'agit d'une classe de jeux qui a été étudiée tout d'abord par [Aumann et al. \[1995\]](#). [Sorin \[2011\]](#) a ensuite étudié l'interconnection entre jeux répétés et différentiels pour une plus vaste catégorie de jeux. En fait, en considérant le jeu différentiel joué sur $[0, 1]$, il a mis en clair que les mêmes outils mathématiques conduisent à la preuve de l'existence de la valeur asymptotique du jeu à n -étapes et du jeu λ -escompté. Ensuite, [Cardaliaguet et al. \[2012\]](#) ont transféré aux jeux discrets les schémas numériques utilisés pour l'approximation de la valeur du jeu différentiel via les arguments des solutions de viscosité dans [Barles and Souganidis \[1991\]](#). Précisément, ils prouvent la convergence asymptotique de la valeur dans les jeux absorbés, les jeux de splitting et d'information incomplète. Ils considèrent dans leur travail les points d'accumulation des suites des fonctions qui satisfont une équation récursive adéquate, laquelle n'est pas valable en temps continu, et ils utilisent un principe ad-hoc de comparaison, tiré de [Barles and Souganidis \[1991\]](#), pour démontrer l'unicité du point d'accumulation et alors la convergence. [Sorin and Vigerat \[2013a\]](#), en utilisant des principes de comparaison, ont établi de nouvelles preuves en ce qui concerne l'existence de la valeur asymptotique des jeux absorbés, des jeux récursifs et des jeux à information incomplète des deux côtés.

1.4 Résultats de la partie I

L'objectif que nous avons fixé dans cette partie de la thèse est l'élaboration d'une approche différentielle afin d'étudier la valeur asymptotique d'un jeu- FD à somme nulle à deux joueurs. Dans le modèle que nous élaborons, la fonction de paiement est additive au sens qu'elle est constituée de deux parties, la partie courante et la partie externe. La première ne dépend que des stratégies courantes des joueurs et la deuxième que de l'état courant. Nous expliciterons ce modèle et l'approche différentielle utilisée basée sur l'équation (HJBI), puis nous donnerons notre résultat.

Soit $\mathcal{Z} = \mathbb{N}^{I \times J}$ l'espace d'états où I et J sont respectivement les ensembles finis d'actions des deux joueurs et soit $A = [a]_{ij} \in \mathbb{R}^{I \times J}$.

À l'étape 0, un état initial $z_0 \in \mathcal{Z}$ est annoncé publiquement et les joueurs choisissent simultanément leurs actions ; le couple d'actions choisi (i_0, j_0) est annoncé aux joueurs. Le paiement à l'étape $t = 0$ est donné par

$$g_0 := g(z_0, i_0, j_0) = a_{i_0 j_0} + h(z_0), \quad h(z_0) := \begin{cases} \left\langle H, \frac{z_0}{\|z_0\|_1} \right\rangle, & z_0 \neq \mathbf{0} \\ 0, & z_0 = \mathbf{0}, \end{cases}$$

avec $H \in \mathbb{R}^{I \times J}$ et $\langle \cdot, \cdot \rangle$ est le produit scalaire standard dans $\mathbb{R}^{I \times J}$. Notons $(e_{ij})_{ij}$ la base canonique de $\mathbb{R}^{I \times J}$. L'état à l'étape $t = 1$ est défini par $z_1 = z_0 + e_{i_0 j_0}$.

Si à l'étape t l'état est z_t , les deux joueurs choisissent leurs actions (i_t, j_t) en sachant l'histoire passée $h_t = (z_0, i_0, j_0, \dots, i_{t-1}, j_{t-1}, z_t)$ et le joueur 2 paye $g_t \in \mathbb{R}$ au joueur 1. L'état à l'étape $t + 1$ est $z_{t+1} = z_t + e_{i_t j_t}$. Les joueurs connaissent ce déroulement.

Soit \mathbf{H}_t l'ensemble des toutes les histoires jusqu' à l'étape t et $\mathbf{H} = \cup_{t \geq 0} \mathbf{H}_t$ l'ensemble de toutes les histoires du jeu. Les joueurs sont autorisés à jouer des stratégies mixtes $u \in \Delta(I)$ et $v \in \Delta(J)$ respectivement. Une stratégie de comportement du joueur 1, $\sigma \in \Sigma$, est une fonction de \mathbf{H} dans $\Delta(I)$. De même, une stratégie $\tau \in T$ du joueur 2 est une fonction de \mathbf{H} dans $\Delta(J)$.

Étant donné $z \in \mathcal{Z}$, chaque profil des stratégies (σ, τ) induit une distribution de probabilité sur \mathbf{H} et l'espérance associée est notée $\mathbb{E}_{\sigma, \tau}^z$. On s'intéresse au jeu répété en un nombre fini des fois $n \in \mathbb{N}^*$, noté $\Gamma_n(z)$, avec paiement :

$$\mathbb{E}_{\sigma, \tau}^z \left(\frac{1}{n} \sum_{t=0}^{n-1} g_t \right).$$

Ce jeu a une valeur en stratégies mixtes selon le théorème de minimax ([von Neumann \[1928\]](#)). Puisque le jeu- FD est à mémoire parfaite, on peut étendre l'existence de la valeur en stratégies de comportement. La valeur de ce jeu est :

$$\mathbf{V}(z) = \sup_{\sigma} \inf_{\tau} \mathbb{E}_{\sigma, \tau}^z \left(\frac{1}{n} \sum_{t=0}^{n-1} g_t \right).$$

D'après [Shapley \[1953\]](#), on peut obtenir une formule récursive pour les valeurs inférieures et supérieures de ce jeu à n étapes ; c'est à dire, le jeu répété à n étapes peut se décomposer à un jeu en un coup et un jeu de longueur $n - 1$, qui commence par une nouvelle valeur de la variable de l'état. Le fait que la transition dans l'espace d'états est affine permet la représentation explicite de la valeur du jeu à n étapes. La question qui surgit concerne l'existence ou non de sa limite et la réponse ne peut pas venir directement de l'expression explicite de la valeur.

Nous élaborons alors une approche particulière afin d'étudier l'existence de la valeur asymptotique. Pour chaque longueur n , on définit d'abord l'espace quotient $\mathcal{Q}_n = \{(z/n) : z \in \mathcal{Z}\}$ et la partition uniforme $\mathcal{I}_n = \{0, (1/n), \dots, 1\}$ de $[0, 1]$. Ensuite, on introduit un jeu auxiliaire en temps discret qui se joue sur $\mathcal{I}_n \times \mathcal{Q}_n$ et sa valeur existe et elle se trouve en relation avec la valeur \mathbf{V}_n du jeu initial. Puis, on définit un jeu différentiel joué sur $[0, 1] \times \mathbb{R}^{I \times J}$, comme la version naturelle en temps continu du jeu auxiliaire. Précisément, étant donné $(t, q) \in [0, 1] \times \mathbb{R}_+^{I \times J}$, on définit un jeu différentiel à somme nulle, noté $\mathcal{G}(t, q)$, qui commence à l'instant t avec état initial q . Son espace d'états est $\mathcal{Q} = \mathbb{R}_+^{I \times J}$; notons $\mathcal{Q}^* = \mathcal{Q} \setminus \{\mathbf{0}\}$. L'intervalle du temps de ce jeu est $[t, 1]$. Le joueur 1 utilise un contrôle mesurable au sens de Lebesgue $\tilde{u} : [t, 1] \rightarrow \Delta(I)$ et son espace des contrôles est noté \mathcal{U}_t . De même, le joueur 2 utilise un contrôle mesurable au sens de Lebesgue $\tilde{v} : [t, 1] \rightarrow \Delta(J)$ et son espace des contrôles est noté \mathcal{V}_t . Si le joueur 1 choisit $\tilde{u} \in \mathcal{U}_t$ et le joueur 2 utilise $\tilde{v} \in \mathcal{V}_t$, alors la dynamique dans l'espace d'états est définie comme suit :

$$\begin{cases} \dot{q} = \tilde{u}_s \otimes \tilde{v}_s, & s \in (t, 1), \\ q(t) = q. \end{cases} \quad (1.4.1)$$

Le paiement que le joueur 1 reçoit à l'instant $s \in [t, 1]$ de la part du joueur 2 est donné par $g : \mathcal{Q} \times U \times V \rightarrow \mathbb{R}$ qui est définie par :

$$g_s := g(q_s, \tilde{u}_s, \tilde{v}_s) = h(q_s) + \langle \tilde{u}_s \otimes \tilde{v}_s, A \rangle, \quad h(q_s) = \begin{cases} \left\langle H, \frac{q_s}{|q_s|} \right\rangle, & q \neq \mathbf{0} \\ 0, & q_s = \mathbf{0}. \end{cases} \quad (1.4.2)$$

Les valeurs inférieures et supérieures de $\mathcal{G}(t, q)$ sont définies à partir du concept des stratégies non-anticipatives :

$$W^-(t, q) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} \int_t^1 g_s ds \quad \text{et} \quad W^+(t, q) = \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} \int_t^1 g_s ds. \quad (1.4.3)$$

La fonction g est bornée par $\|H\|_\infty + \|A\|_\infty$ et puisque $q : [t, 1] \rightarrow \mathcal{Q}$ est une fonction différentiable, (cf. (1.4.1)), on a que g est différentiable sur \mathcal{Q}^* . Cependant, lorsque $q \rightarrow \mathbf{0}$, le paiement total qui intervient dans (1.4.3) devient une intégrale impropre et donc on est en présence d'une irrégularité. D'après la section 1.2.2, vu que la condition d'Isaacs est satisfaite (cf. (1.2.2)), on peut obtenir l'existence de la valeur pour le jeu $\mathcal{G}(t, q)$ pour tout $(t, q) \in [0, 1] \times \mathcal{Q}^*$. L'obstacle à surmonter pour prouver l'existence de la valeur pour tout $(t, q) \in [0, 1] \times \mathcal{Q}$ vient du fait que la fonction de

païement n'est pas localement lipschitzienne à l'état $q = \mathbf{0}$ (cf. (1.4.2)) et alors il s'ensuit que les outils forts de la théorie des jeux différentiels deviennent impuissants.

La fonction $W : [0, 1] \times \mathcal{Q}^* \rightarrow \mathbb{R}$ est la solution unique de l'équation (HJBI) dans l'espace des fonctions bornées, continues, définies sur $[0, 1] \times \mathcal{Q}^*$:

$$\begin{cases} \frac{\partial W}{\partial t}(t, q) + \mathcal{H}(\nabla_q W(t, q), q) = 0, & (t, q) \in [0, 1) \times \mathcal{Q}^*, \\ W^-(1, q) = 0, & q \in \mathcal{Q}^*, \end{cases}$$

où, \mathcal{H} est l'hamiltonien défini comme dans la Section 1.2.2.

Nous montrons l'existence de la valeur du jeu différentiel $\mathcal{G}(0, \mathbf{0})$ qui débute à l'instant $t = 0$ avec état initial $q = \mathbf{0}$ malgré son irrégularité à l'état $q = \mathbf{0}$; on note cette valeur $W(\mathbf{0})$. On énonce ce résultat :

Théorème 1.4.1. *Le jeu $\mathcal{G}(0, \mathbf{0})$ admet une valeur.*

Les propriétés de la valeur du jeu à n -étapes, venant de la transition affine dans \mathcal{Z} , permettent d'établir la coïncidence entre la fonction valeur du jeu auxiliaire et la valeur correspondante à la discrétisation uniforme de $\mathcal{G}(0, \mathbf{0})$. Précisément, nous montrons que ces deux fonctions satisfont la même formule récursive et ont la même valeur à l'instant final $t = 1$.

Les schémas d'approximation de la solution de l'équation (HJBI), (cf. [Bardi and Capuzzo-Dolcetta \[2008\]](#)), impliquent la convergence des valeurs discrètes vers les valeurs du jeu en temps continu, lorsque le pas de la discrétisation tend vers zéro et ceci permet d'établir l'existence de $\lim_n \mathbf{V}_n$. On peut alors énoncer le résultat principal qui est prouvé dans cette partie de la Thèse :

Théorème 1.4.2. *Pour tout $z \in \mathcal{Z}$, lorsque n tend à l'infini, la valeur du jeu-FD, $\mathbf{V}_n(z)$, converge vers la valeur du jeu différentiel, $W(\mathbf{0})$, joué sur $[0, 1]$, d'état initial $q = \mathbf{0}$.*

Chapter 2

A frequency dependent control problem

Abstract. In this chapter, we present results from [Contou-Carrère \[2011\]](#). The author studied a particular case of our *FD*-game with one decision maker and proves that this control problem does not admit a uniform value while establishes existence of the asymptotic value. It is further shown that convergence is not uniform. Consequently, it is natural to expect that our *FD*-game considered in Chapter 3 does not admit either a uniform value and if one can establish existence of the asymptotic value, then convergence must be pointwise.

Let $I = \{\mathcal{L}, \mathcal{N}\}$ be the set of actions, with \mathcal{L} denoting a littering action and \mathcal{N} a non-littering one. The state space, denoted by \mathcal{Z} , is the set of mappings on I with positive integer values. For each $z \in \mathcal{Z} \setminus \{\mathbf{0}\}$, $\tilde{z}(i) = z(i) / \|z\|_1$ for all $i \in I$ and by convention, $\tilde{\mathbf{0}} = \mathbf{0}$. The stage payoff function g is defined for all $(z, i) \in \mathcal{Z} \times I$:

$$g(z, i) = \begin{cases} a - c \cdot \tilde{z}(\mathcal{L}), & \text{if } i = \mathcal{L} \\ -b - c \cdot \tilde{z}(\mathcal{L}), & \text{if } i = \mathcal{N}, \end{cases}$$

where a, b and c are real coefficients such that $a + b > 0$ and $c > 0$. We will use the notation $\delta^t := \delta_{\mathcal{L}}(i_t)$, where $\delta_{\mathcal{L}}(i_t) = 1$ if $i_t = \mathcal{L}$ and $\delta_{\mathcal{L}}(i_t) = 0$ if $i_t = \mathcal{N}$.

For all stage $t \geq 1$, let $\mathbf{H}_t = I^{t-1}$ denote the cartesian product of I , $t - 1$ times ($\mathbf{H}_1 = \{\emptyset\}$). A *pure strategy* for the decision maker is an element $\sigma = (\sigma_t)_{t \geq 1}$, such that for each stage $t \geq 1$, σ_t is a mapping from \mathbf{H}_t to I . Let Σ be the set of pure strategies. Each pure strategy σ induces, in a natural way, a unique play. Let $i_t(\sigma)$ denote the action played according σ at stage t . The player is assumed to have perfect recall. The decision maker aims at maximizing his average payoff, given by:

$$\gamma_N(z_0) = \frac{1}{N} \sum_{t=1}^N g(z_t, i_t).$$

Without loss of generality, we restrict ourselves to pure strategies since any average payoff that can be achieved by a mixed strategy can be induced by a pure strategy, and can also be achieved by a behavioral strategy.

The value for the N -long problem with initial state z_0 is given by:

$$\mathbf{V}_N(z_0) = \max_{\sigma \in \Sigma} \gamma_N(z_0, \sigma)$$

A strategy σ^* such that $\mathbf{V}_N(z_0) = \gamma_N(z_0, \sigma^*)$ is an optimal strategy in the N -long game starting at initial state z_0 .

Definition 2.0.1. $\mathbf{V}(z_0)$ is a value of the control problem with initial state z_0 if for all $\varepsilon > 0$, there exist $N_\varepsilon \in \mathbb{N}^*$ and σ_ε such that for all $N \geq N_\varepsilon$:

- $\gamma_N(z_0, \sigma_\varepsilon) \geq \mathbf{V}(z_0) - \varepsilon$
- $\forall \sigma \in \Sigma, \gamma_N(z_0, \sigma) \leq \mathbf{V}(z_0) + \varepsilon$.

Definition 2.0.2. $\mathbf{V}(\cdot)$ is a *uniform value* of the game if for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}^*$ such that for all $z_0 \in \mathcal{Z}$, there exists σ_ε such that for all $N \geq N_\varepsilon$,

- $\gamma_N(z_0, \sigma_\varepsilon) \geq \mathbf{V}(z_0) - \varepsilon$
- $\forall \sigma \in \Sigma, \gamma_N(z_0, \sigma) \leq \mathbf{V}(z_0) + \varepsilon$.

In the sequel of this section, we present the existing results, when the initial state is assumed to be equal to zero and z_0 will be omitted from the notation.

2.1 Asymptotic value in the FD -control problem

We first present the results on the existence of the asymptotic value.

Proposition 2.1.1. *Given $N \in \mathbb{N}^*$, for every pure strategy σ*

$$\gamma_N(\sigma) = \frac{1}{N} \sum_{t=1}^N (a + b - \Lambda_N(t)c)\delta^t - b,$$

where $\Lambda_N(t) = \frac{1}{t} + \dots + \frac{1}{N-1}$ for $t = 1, \dots, N-1$ and $\Lambda_N(N) = 0$.

Proof. Consider a pure strategy σ . The payoff induced at stage t is

$$g_t = a\delta^t - b(1 - \delta^t) - c\tilde{z}_t(\mathcal{L})$$

It follows that

$$\gamma_N(\sigma) = \frac{a+b}{N} \sum_{t=1}^N \delta^t - \frac{c}{N} \sum_{t=1}^N \tilde{z}_t(\mathcal{L}) - b,$$

where $\sum_{t=1}^N \tilde{z}_t(\mathcal{L}) = \frac{\delta^1}{1} + \frac{\delta^1 + \delta^2}{2} + \dots + \frac{\delta^1 + \dots + \delta^{N-1}}{N-1}$. After rearranging the terms, we get

$$\sum_{t=1}^N \tilde{z}_t(\mathcal{L}) = \sum_{t=1}^N \Lambda_N(t)\delta^t,$$

where $\Lambda_N(t) = \frac{1}{t} + \dots + \frac{1}{N-1}$ and $\Lambda_N(N) = 0$. Therefore,

$$\gamma_N(\sigma) = \frac{1}{N} \sum_{t=1}^N (a + b - \Lambda_N(t)c) \delta^t - b.$$

Notice that for all $t \in \{1, \dots, N\}$, $\Lambda_N(t+1) < \Lambda_N(t)$ and $\lim_{N \rightarrow +\infty} \Lambda_N(1) = +\infty$. Hence, for any sufficiently long N , there exists a unique $t^* \in \mathbb{N}^*$, such that

$$\Lambda_N(t^*) < \frac{a+b}{c} \leq \Lambda_N(t^* - 1).$$

□

Proposition 2.1.2. *The pure strategy σ^* defined by*

$$i_t(\sigma^*) = \begin{cases} \mathcal{N}, & \text{if } 1 \leq t \leq t^* - 1, \\ \mathcal{L}, & \text{if } t^* \leq t \leq N \end{cases}$$

is an optimal strategy of the N -long problem and its value is:

$$\mathbf{V}_N = \frac{N - (t^* - 1)}{N} a - \frac{t^* - 1}{N} b - \frac{1}{N} \sum_{k=1}^{N-t^*} \frac{k}{t^* + k - 1} c.$$

Proof. By definition of σ^* , at each stage t , we have:

$$\delta^t = \begin{cases} 0, & \text{if } a + b - \Lambda_N(t)c \leq 0, \\ 1, & \text{if } a + b - \Lambda_N(t)c > 0. \end{cases}$$

It follows that for every other pure strategy σ , one has $\gamma_N(\sigma) \leq \gamma_N(\sigma^*)$.

By definition of σ^* , $\tilde{z}_t(\mathcal{L})$ is equal to zero from stage $t = 1$ up to $t = t^*$, and $\tilde{z}_{t^*+k}(\mathcal{L}) = \frac{k}{t^*+k-1}$ for each integer $k \in \{1, \dots, N - t^*\}$. Therefore,

$$\mathbf{V}_N = \frac{N - (t^* - 1)}{N} a - \frac{t^* - 1}{N} b - \frac{1}{N} \sum_{k=1}^{N-t^*} \frac{k}{t^* + k - 1} c.$$

□

Notation: In the sequel we put $\lambda = e^{-\frac{a+b}{c}}$.

Proposition 2.1.3. *For all $N \in \mathbb{N}^*$,*

$$\lambda N \leq t^* \leq \lambda N - \lambda + 2$$

and therefore,

$$\lim_{N \rightarrow +\infty} \frac{t^*}{N} = \lambda$$

Proof. For all $N \in \mathbb{N}^*$,

$$\sum_{k=t^*}^{N-1} \frac{1}{k} \leq \frac{a+b}{c} \leq \sum_{t^*-1}^{N-1} \frac{1}{k}$$

For all $k \in \mathbb{N}^*$,

$$\frac{1}{k+1} \leq \int_k^{k+1} \frac{dx}{x} \leq \frac{1}{k}.$$

Therefore,

$$\begin{aligned} \bullet \quad & \sum_{k=t^*-2}^{N-2} \frac{1}{k+1} \leq \sum_{k=t^*-2}^{N-2} \int_k^{k+1} \frac{dx}{x} \quad \text{and} \quad \sum_{t^*-1}^{N-1} \frac{1}{k} \leq \int_{t^*-2}^{N-1} \frac{dx}{x}, \\ \bullet \quad & \sum_{k=t^*}^{N-1} \int_k^{k+1} \frac{dx}{x} \leq \sum_{k=t^*}^{N-1} \frac{1}{k} \quad \text{and} \quad \int_{t^*}^N \frac{dx}{x} \leq \sum_{k=t^*}^{N-1} \frac{1}{k}. \end{aligned}$$

Hence, $\int_{t^*}^N \frac{dx}{x} \leq \frac{a+b}{c} \leq \int_{t^*-2}^{N-1} \frac{dx}{x}$ and

$$\log\left(\frac{N}{t^*}\right) \leq \frac{a+b}{c} \leq \log\left(\frac{N-1}{t^*-2}\right).$$

Therefore, $\lambda N \leq t^* \leq \lambda N - \lambda + 2$. □

From Proposition 2.1.3, it follows that the sequences $\tilde{z}_N(\mathcal{N})$ and $\tilde{z}_N(\mathcal{L})$ induced by σ^* in the N -long problem converge to λ and $1 - \lambda$ respectively. Consequently, the sequence \mathbf{V}_N converges as N tends to infinity.

Next Lemma will be useful for the proof of the main Theorem of this section.

Lemma 2.1.4. *It holds true that*

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^{N-t^*} \frac{k}{t^* + k - 1} = 1 - \lambda + \lambda \log(\lambda).$$

Proof. Consider $N \in \mathbb{N}^*$ and let $S_N = \frac{1}{N} \sum_{k=1}^{N-t^*} \frac{k}{t^* + k - 1}$. The change of variables $\tilde{k} = t^* + k - 1$ implies:

$$S_N = \frac{1}{N} \sum_{\tilde{k}=t^*}^{N-1} \frac{\tilde{k} + 1 - t^*}{\tilde{k}} = \frac{N - t^*}{N} + \frac{1 - t^*}{N} \sum_{\tilde{k}=t^*}^{N-1} \frac{1}{\tilde{k}}.$$

The sequence $\left(\log(N) - \sum_{k=1}^N \frac{1}{k} \right)_{N \in \mathbb{N}^*}$ converges to the Euler constant γ_E . Denote by $\theta : \mathbb{R} \rightarrow \mathbb{R}$ the mapping such that $\lim_{x \rightarrow +\infty} \theta(x) = 0$ and

$$\log(N) - \sum_{k=1}^N \frac{1}{k} = \gamma_E + \theta(N).$$

It follows:

$$S_N = \frac{N - t^*}{N} + \frac{1 - t^*}{N} \log \left(\frac{N - 1}{t^* - 1} \right) + \eta_N,$$

where $\eta_N = \frac{1-t^*}{N} (\theta(N-1) - \theta(t^*-1))$ and thus, $\lim_{N \rightarrow +\infty} \eta_N = 0$.

By Proposition 2.1.3, $\lim_{N \rightarrow +\infty} \left(\frac{t^*}{N} \right) = \lambda$ and $\lim_{N \rightarrow +\infty} \log \left(\frac{N-1}{t^*-1} \right) = -\log(\lambda)$. Therefore,

$$\lim_{N \rightarrow +\infty} S_N = 1 - \lambda + \lambda \log(\lambda),$$

concluding the proof of the Lemma. \square

Theorem 2.1.5. *It holds true that*

$$\lim_{N \rightarrow +\infty} \mathbf{V}_N = a - (1 - \lambda)c.$$

Proof. By definition of σ^* , $\tilde{z}_t(\mathcal{L}) = 0$ for $t = 1, \dots, t^*$ and $\tilde{z}_{t^*+k}(\mathcal{L}) = \frac{k}{t^*+k-1}$ for all $k \in \mathbb{N}^*$, such that $k \leq N - t^*$. Hence, for all $N \in \mathbb{N}^*$,

$$\mathbf{V}_N = \frac{N - (t^* - 1)}{N} a - \frac{t^* - 1}{N} b - \frac{1}{N} \sum_{k=1}^{N-t^*} \frac{k}{t^* + k - 1} c.$$

By Lemma 2.1.4, we have:

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^{N-t^*} \frac{k}{t^* + k - 1} = 1 - \lambda + \lambda \log(\lambda).$$

It follows that:

$$\lim_{N \rightarrow +\infty} \mathbf{V}_N = (1 - \lambda)a - \lambda b - (1 - \lambda + \lambda \log(\lambda))c.$$

Since $\log(\lambda) = -\frac{a+b}{c}$, we get $\lim_{N \rightarrow +\infty} \mathbf{V}_N = a - (1 - \lambda)c$. \square

Notation. In the sequel of this section, we denote $\mathbf{V}^* := \lim_{N \rightarrow +\infty} \mathbf{V}_N$.

We further present the preceding results for any $z_0 \in \mathcal{Z}$, since we will need the expressions in the sequel. The following proposition concerns the optimal strategy and the value in the game $\Gamma_N(z_0)$.

Proposition 2.1.6. *The pure strategy σ^* defined by*

$$i_t(\sigma^*) = \begin{cases} \mathcal{N}, & 1 \leq t \leq t_{z_0}^* - 1 \\ \mathcal{L}, & t_{z_0}^* \leq t \leq N \end{cases}$$

is an optimal strategy in the FD-control problem

$$\mathbf{V}_N(z_0) = a - \frac{t_{z_0}^* - 1}{N} (a + b) - \frac{1}{N} \sum_{t=1}^{t_{z_0}^*} \frac{z_0(\mathcal{L})}{|z_0| + t - 1} c - \frac{1}{N} \sum_{k=1}^{N-t_{z_0}^*} \frac{z_0(\mathcal{L}) + k}{|z_0| + t_{z_0}^* + k - 1} c$$

Remark 2.1.7. From the definition of the FD -control problem, we deduce that for all $z_0 \in \mathcal{Z}$,

$$\lim_{N \rightarrow +\infty} \mathbf{V}_N(z_0) = \mathbf{V}^*.$$

For a proof of the presented above results, see [Contou-Carrère \[2011\]](#) (Chapter 1).

Notation. In the sequel we set: $\Lambda_{z_0, N}(t) = \frac{1}{|z_0|+t} + \dots + \frac{1}{|z_0|+N-1}$ and $\Lambda_{z_0, N}(N) = 0$.

Theorem 2.1.8. *Given $z_0 \in \mathcal{Z}$, the sequence $\mathbf{V}_N(z_0)$ does not uniformly converge to \mathbf{V}^* as N tends to infinity.*

Proof. Assume by contradiction that for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}^*$, such that for all $N \geq N_\varepsilon$, for all $z_0 \in \mathcal{Z}$,

$$|\mathbf{V}_N(z_0) - \mathbf{V}^*| \leq \varepsilon.$$

Consider $\varepsilon > 0$ and an integer $N \geq N_\varepsilon$. Then, consider $z_0 \in \mathcal{Z}$, such that $\Lambda_{z_0, N}(1) < \frac{a+b}{c}$. The optimal strategy in the FD -control problem starting at initial state z_0 , is to play the action \mathcal{L} at each stage, and its value is equal to $\mathbf{V}_N(z_0) = a - \frac{c}{N} \sum_{t=1}^N \frac{z_0(\mathcal{L})+t-1}{|z_0|+t-1}$. If $z_0(\mathcal{L}) = |z_0|$ then, $\mathbf{V}_N(z_0) = a - c$. It follows: $\mathbf{V}_N(z_0) - \mathbf{V}^* = (a - c) - (a - (1 - \lambda)c) = \lambda c$. A contradiction is obtained for $\varepsilon = \frac{\lambda c}{2}$. \square

2.2 Uniform value in the FD -control problem

In this paragraph, we present the results concerning the existence of the uniform value in the FD -control problem; namely, [Contou-Carrère \[2011\]](#) proves that the FD -control problem does not admit a uniform value.

We first provide a sketch of the proof. Assume by contradiction that $\Gamma(\mathbf{0})$ admits a value. Consider $\varepsilon > 0$ and σ_ε be an ε -optimal strategy. There exists $N_\varepsilon \in \mathbb{N}^*$, such that for all $N \geq N_\varepsilon$, $|\gamma_N(\sigma_N) - \gamma_N(\sigma_\varepsilon)| \leq \varepsilon$. The strategy σ_N starts to play the non littering action \mathcal{N} up to stage $t^*(N) - 1$, abandons it and plays the littering action from stage $t^*(N)$ up to stage N . For all $N \in \mathbb{N}^*$, we define $t_1^*(N)$ as the first stage t such that $|a + b - \Lambda_N(t)c| > \frac{a+b}{2}$, so that for each stage t such that $t_1^*(N) \leq t \leq N$, one has $|a + b - \Lambda_N(t)c| > \frac{a+b}{2}$. Consider a length \tilde{N} , such that $N < t^*(\tilde{N})$, which implies that at each stage $t \in \{t_1^*(N), t_1^*(N) + 1, \dots, N\}$, we have $i_t(\sigma^N) \neq i_t(\sigma^{\tilde{N}})$. We can choose \tilde{N} , such that for each stage $t \in \{t_1^*(N), t_1^*(N) + 1, \dots, N\}$, we have: $|a + b - \Lambda_N(t)c| > \frac{a+b}{2}$. The ε -closeness between on the one side $\gamma_N(\sigma^N)$ and $\gamma_N(\sigma^\varepsilon)$ and on the other side $\gamma_{\tilde{N}}(\sigma_{\tilde{N}})$ and $\gamma_{\tilde{N}}(\sigma_\varepsilon)$ lead to a contradiction for appropriate values for N and \tilde{N} .

Denote by $c(t, N) = a + b - \Lambda_N(t)c$ for every stage $t \in \{1, \dots, N\}$ and further put $K_{1, N} = \{t \in \{1, \dots, N\} : i_t(\sigma_\varepsilon) \neq i_t(\sigma^N)\}$.

Lemma 2.2.1. *For all $N \in \mathbb{N}^*$,*

$$N(\gamma_N(\sigma_N) - \gamma_N(\sigma_\varepsilon)) = \sum_{t \in K_{1,N}} |c(t, N)|$$

Proof. For all $N \in \mathbb{N}^*$,

$$N(\gamma_N(\sigma_N) - \gamma_N(\sigma_\varepsilon)) = \sum_{t \in K_{1,N}} c(t, N)(\delta_{\mathcal{L}}(i_t(\sigma_N)) - \delta_{\mathcal{L}}(i_t(\sigma_\varepsilon)))$$

For each $t \in K_{1,N}$: if $c(t, N) > 0$ then, $i_t(\sigma_N) = \mathcal{L}$ and $i_t(\sigma_\varepsilon) = \mathcal{N}$; if $c(t, N) < 0$ then, $i_t(\sigma_N) = \mathcal{N}$ and $i_t(\sigma_\varepsilon) = \mathcal{L}$ that completes the proof of the Lemma.

Since $\log\left(\frac{\tilde{N}}{N}\right) \geq -2\log(\lambda)$, it follows that $|c(N, \tilde{N})| \geq \frac{a+b}{2}$. \square

Lemma 2.2.2. *There exists $N_1 \in \mathbb{N}^*$, such that for all $N \geq N_1$, we have:*

$$|c(N, N_0)| \geq \frac{a+b}{2},$$

where $N_0 = \lfloor \frac{N}{\lambda^2} \rfloor + 1$.

Proof. There exists a mapping $\theta : \mathbb{R} \rightarrow \mathbb{R}$, such that $\lim_{x \rightarrow +\infty} \theta(x) = 0$ and for all $N \in \mathbb{N}^*$, $1 + \dots + \frac{1}{N} = \gamma_E + \log(N) + \theta(N)$, where γ_E stands for the Euler constant. It follows:

$$\Lambda_{N_0}(N) = \log\left(\frac{N_0}{N}\right) + \frac{1}{N} - \frac{1}{N_0} + \theta(N_0) - \theta(N).$$

Since $N < t^*(N_0)$ one gets $c(N, N_0) = a + b - \Lambda_{N_0}(N)c < 0$. Therefore,

$$|c(N, N_0)| = -(a+b) \log\left(\frac{N_0}{N}\right)c + c\left(\frac{1}{N} - \frac{1}{N_0} + \theta(N_0) - \theta(N)\right)$$

Consider $N_1 \in \mathbb{N}^*$, such that for all $N \geq N_1$, we have:

$$\left| \frac{1}{N_0} - \frac{1}{N} + \theta(N_0) - \theta(N) \right| \leq \frac{a+b}{2c},$$

ending the proof of the Lemma. \square

Theorem 2.2.3. $\Gamma(\mathbf{0})$ has no value.

Proof. Assume by contradiction that $\Gamma(\mathbf{0})$ has a value. Consider $\varepsilon > 0$ and let σ_ε be an ε -strategy. There exists $N_\varepsilon \in \mathbb{N}^*$, such that for all integer $N \geq N_\varepsilon$,

$$|\gamma_N(\sigma_N) - \gamma_N(\sigma_\varepsilon)| \leq \varepsilon. \quad (2.2.1)$$

Consider an integer $N \geq N_\varepsilon$. By (2.2.1) and Lemma 2.2.1, we have: $\sum_{t \in K_{1,N}} |c(t, N)| \leq \varepsilon N$. Since $c(1, N) < 0$ and $c(N, N) = a + b$, there exists an integer $t_1^*(N)$, such that

$$c(t_1^*(N) - 1, N) < \frac{a+b}{2} < c(t_1^*(N), N).$$

Thus, for all $t \in \{t_1^*(N), \dots, N\}$,

$$c(t, N) \geq \frac{a+b}{2}.$$

Notice that $t^*(N) < t_1^*(N)$. Adapting the proof for Proposition 2.1.3, we obtain:

$$\sqrt{\lambda}N \leq t_1^*(N) \leq \sqrt{\lambda}N - \sqrt{\lambda} + 2.$$

For all $N \in \mathbb{N}^*$, and stage $\tilde{t} \in \{1, \dots, N\}$, let us denote by

$$\begin{aligned} K_{\tilde{t}, N} &= \{t \in \{\tilde{t}, \dots, N\} : i_t(\sigma_N) \neq i_t(\sigma_\varepsilon)\}, \\ \tilde{K}_{\tilde{t}, N} &= \{t \in \{\tilde{t}, \dots, N\} : i_t(\sigma_N) = i_t(\sigma_\varepsilon)\}. \end{aligned}$$

Let $\kappa_{\tilde{t}, N}$ and $\tilde{\kappa}_{\tilde{t}, N}$ be the cardinality of $K_{\tilde{t}, N}$ and $\tilde{K}_{\tilde{t}, N}$ respectively. Note that $\tilde{\kappa}_{\tilde{t}, N} = (N - \tilde{t}) - \kappa_{\tilde{t}, N} + 1$. We then have:

$$\kappa_{t_1^*, N} \frac{a+b}{2} \leq \sum_{t \in K_{t_1^*, N}} |c(t, N)| \leq \sum_{t \in K_{1, N}} |c(t, N)| \leq \varepsilon N.$$

It follows:

$$\tilde{\kappa}_{t_1^*, N} \geq (1 - \sqrt{\lambda})N - (2 - \sqrt{\lambda}) - \frac{2\varepsilon}{a+b}N + 1.$$

Let us define $N_0 = \lfloor \frac{N}{\lambda^2} \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the floor function. Since $N_0 \geq N$, we also have $N_0 \geq N_\varepsilon$. By (2.2.1), it follows:

$$\sum_{t \in K_{1, N_0}} |c(t, N_0)| \leq N_0 \varepsilon.$$

From Proposition 2.1.3,

$$\lambda N_0 \leq t^*(N_0) \leq \lambda N_0 - \lambda + 2.$$

Since $N \leq t^*(N_0)$, we get $i_t(\sigma_N) = i_t(\sigma_{N_0})$ for all $t \in \{t_1^*(N), \dots, N\}$. Therefore,

$$\tilde{K}_{t_1^*, N} \subset \{t_1^*(N), \dots, N\} \cap K_{1, N_0}.$$

Due to $|c(N, N_0)| \leq |c(N-1, N_0)| \leq \dots \leq |c(t_1^*(N), N_0)|$, one gets:

$$\tilde{\kappa}_{t_1^*, N} |c(N, N_0)| \leq \sum_{t \in K_{1, N_0}} |c(t, N_0)| \leq N_0 \varepsilon$$

and therefore,

$$\tilde{\kappa}_{t_1^*, N} |c(N, N_0)| \leq N_0 \varepsilon. \quad (2.2.2)$$

Lemma 2.2.2 and (2.2.2) imply $\tilde{\kappa}_{t_1^*, N} \leq \frac{2}{a+b} N_0 \varepsilon$ for all integer $N \geq N_1$, where N_1 satisfies Lemma 2.2.2. Therefore,

$$\tilde{\kappa}_{t_1^*, N} \leq \frac{2}{a+b} \left(\frac{N}{\lambda^2} + 1 \right) \varepsilon. \quad (2.2.3)$$

By (2.2.2) and (2.2.3), we get:

$$\frac{2\varepsilon}{a+b} - \frac{2-\sqrt{\lambda}}{N} + 1 - \sqrt{\lambda} \leq \frac{2}{a+b} \left(\frac{1}{\lambda^2} + 1 \right) \varepsilon.$$

Then, whenever N tends to infinity, it follows $1 - \sqrt{\lambda} \leq \frac{2}{a+b} \left(2 + \frac{1}{\lambda^2} \right) \varepsilon$. We obtain a contradiction when $\varepsilon = \frac{\lambda^2(1-\sqrt{\lambda})}{2\lambda^2+1} \frac{a+b}{4}$. \square

Chapter 3

Asymptotic value in frequency dependent games. A differential approach

Abstract¹. We introduce a frequency-dependent zero-sum game. We define a differential game as the natural continuous time version of an auxiliary game related to the original game and we provide its corresponding Hamilton-Jacobi-Bellman-Isaacs equation. Although the differential game presents an irregularity at the origin, we prove existence of the value on the time interval $[0, 1]$. The discrete value corresponding to the uniform discretization of the differential game and the value function of the discrete time auxiliary game satisfy the same recursive formula and share the same terminal condition and thus, they coincide. Results from the theory of differential games on the approximation schemes of the solution of the Hamilton-Jacobi-Bellman-Isaacs equation, imply the convergence of the discrete values to the value of the differential game. We conclude that the limit of V_n exists and it coincides with the value of the associated continuous time game.

3.1 Introduction

Frequency-dependent games (*FD-games*) are a particular class of repeated games in which stage payoffs depend on the passed actions. They have been introduced by [Brenner and Witt \[2003\]](#). Such games consist in the repetition at discrete moments in time, of a one-shot game in which the stage payoff functions vary over time depending on the choices of the players at the current stage, as well as on the relative frequencies whereby all actions were chosen by the players at previous stages; this average of past actions is the state of the game. Alternatively, a *FD-game* can be interpreted as a stochastic game with a countable state space in which each state can be visited at most one time. Stage payoffs may be frequency-dependent over time because of several reasons. The actions undertaken by the players at each stage may generate externalities, which accumulate as the game unfolds. For instance, payoffs

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may change due to, learning, habit formation, addiction, or satiation. An extensive review of this class of games can be found in [Joosten et al. \[2003\]](#). The authors analyse in particular non zero-sum *FD*-games and derive several *folk theorem* like results; however the equilibrium notion used from them, is weak because it does not deal with the uniform meaning.

Some aspects of *FD*-games have been studied in [Contou-Carrère \[2011\]](#) like the model presented in [Chapter 2](#). Recall that this control problem does not admit uniform value while convergence of the N -stage value must be pointwise. In this paper, for any $N \in \mathbb{N}^*$, we define an N -stage, two player zero sum *FD*-game with finite action sets I and J respectively. As we mentioned in [Chapter 1](#), this game can be seen a stochastic game with a countable state space. Player 1 maximizes and Player 2 minimizes the average payoff on the first N stages. Furthermore, this game is played under *perfect-recall* meaning that both players know the current state, as well as the entire history, i.e., the state visited and action pair played at each of the preceding stages. The previously described *FD*-control problem becomes a particular case of the introduced game and thus, as it is natural to expect, the uniform value does not exist and if one can establish existence of the asymptotic value, then convergence must be pointwise too.

Historical review on the existence of asymptotic value

In finite stochastic games, existence and characterization of the values for fixed $N \in \mathbb{N}^*$ or discounted factor $\lambda \in (0, 1]$ is due to [Shapley \[1953\]](#) and relies on the von Neumann's minmax theorem [von Neumann \[1928\]](#), as well as on Banach's fixed point theorem.

One of the main questions in the theory of zero-sum repeated games is related to the asymptotic behavior of the values as both players grow more and more patients. In the context of a finite number of states and of players' actions, existence of the asymptotic value was first established for recursive games by [Everett \[1957\]](#) and absorbing games by [Kohlberg \[1974\]](#), then in general using Tarski-Seidenberg theorem by [Bewley and Kohlberg \[1976\]](#). Existence of the limit value is further assured for repeated games with incomplete information on both sides by [Mertens and Zamir \[1971\]](#) and with lack of information on one side by [Aumann et al. \[1995\]](#). An existence result is also given in some finite stochastic games with lack of information on one side; for instance, in [Rosenberg \[2000\]](#) for zero-sum absorbing games and in [Renault \[2006\]](#) and [Gensbittel and Renault \[2015\]](#) for Markov chain games. Let us mention that existence of the limit value was conjectured to hold true in general in [Mertens et al. \[1986\]](#), however, [Ziliotto \[2013\]](#) provides counter-examples of zero-sum repeated games, in which there is no asymptotic value. Recently, [Sorin and Vigerel \[2013b\]](#), furnish a manner to construct zero-sum stochastic games with finite state space having oscillating discounted values.

Differential games and approximation schemes of the value

Formalizing the fact that players play in continuous time and each one knows the previous control of his opponent induces an issue about how to define strategies

in such a way that the induced process (q_s, u_s, v_s) is also well defined. In this work, we will use the concept of *non-anticipative strategies* introduced by [Varaiya \[1967\]](#), [Roxin \[1969\]](#) and [Elliott and Kalton \[1972b\]](#). Non-anticipative strategies reflect the fact that no player can guess in advance the future behaviour of the other player. This was the initial procedure to define values in [Bardi and Capuzzo-Dolcetta \[2008\]](#) and [Souganidis \[1999\]](#).

Since strategies are defined, we can define the *lower* and *upper values* of the continuous time game depending on which player chooses his control first; Concerning the lower value function, Player 1 has an advantage because he chooses after Player 2 and in the upper value function the opposite occurs. When this information advantage is irrelevant, i.e. both functions coincide, we say that the differential game has a *value*. The value of the differential game should be the value of the payoff function when both players they do their best, i.e., they use their optimal strategies, which however, do not exist in general. The theory of two-player zero-sum differential games started from [Isaacs \[1965\]](#). A main contribution of this early work is the heuristic derivation of the fact that the value of the continuous time game $W : [0, 1] \times \mathbb{R}^k \rightarrow \mathbb{R}$ must satisfy, whenever smooth a *Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation*.

[Fleming \[1961\]](#), [Fleming \[1964\]](#) study differential games by discretizing in time using a uniform-step and defining lower and upper values depending on whether Player 1 or Player 2 moves first. The authors establish existence of the value for finite-horizon games. Later, [Friedman \[1970\]](#) introduces a notion of non-anticipative strategy that varies at only finitely many instants of the time interval. Then, it is proved existence of the value in differential games that players' controls act separately on the dynamics and running payoff functions f and g . [Elliott and Kalton \[1972b\]](#) provide a similar approach but, strategies may change at any instant of time instead of varying only at the division points; namely, players are allowed to use non-anticipative strategies earlier presented. Then, existence of the value under some conditions was proved by [Elliott and Kalton \[1972a\]](#). In the two latter approaches, the authors study differential games by looking directly at the ordinary differential equation according to the state variable evolves instead of discretizing.

Let us mention that (HJBI) equations, do not have in general globally defined smooth solutions. Furthermore, solution may not be unique. A solution concept corresponding to a *weaker* notion, namely the *viscosity solution* is firstly introduced by [Crandall and Lions \[1983\]](#) for fully non-linear, first order (HJBI) equations. In this paper, we deal with deterministic two-player zero-sum finite horizon differential games and we define values with respect to the definition of [Elliott and Kalton \[1972b\]](#). Their relationship with viscosity solutions was first properly mentioned by [Evans and Souganidis \[1984\]](#), where it is proved that as soon as some regularity conditions are satisfied by the payoff and dynamics functions and if further the Isaacs condition holds, i.e., lower and upper hamiltonians coincide, the value of the differential game exists² and it is characterized as the unique solution, in the space of bounded, continuous functions defined on $[0, 1] \times \mathbb{R}^k$, of the associated (HJBI)

²Existence of the value follows from the standard comparison and uniqueness theorems for viscosity solutions presented in [Crandall and Lions \[1983\]](#).

equation with a boundary condition.

Hence, viscosity solutions turn to exist and be unique under very general hypotheses. This fact qualifies them to play a leading role in the study of the convergence of discrete approximations. These approximations do not converge in general if the value function is discontinuous. The results of the existence of the value, its characterization as a solution of a (HJBI) equation and the convergence of the discrete approximation schemes to this value, under some regularity conditions, has been shown for finite horizon differential games with deterministic dynamics in [Souganidis \[1999\]](#). [Bardi and Capuzzo-Dolcetta \[2008\]](#) adapt to infinite-horizon games the convergence results on discrete approximations obtained by [Bardi and Soravia \[1991\]](#) for pursuit-evasion games.

Links between repeated and differential games

According to [Sorin \[2002\]](#), the asymptotic value of a game should exist when the game is regular enough. Such a belief derives its rationale from the fact that a N -stage game can be alternatively viewed as a game played in the time interval $[0, 1]$, where the total payoff is given by the integral of the payoff function over $[0, 1]$ and players only move on the nodes of the uniform partition of the time interval, i.e., $\frac{k}{N}$, with k varying between 0 and N . As N goes to infinity, this game can therefore be regarded as some time discretizations of a hypothetic continuous-time game played over $[0, 1]$ and its value should thus converge to the value of this *limit game*.

The idea of studying the asymptotic value of a discrete-time repeated game as the value of a game in continuous time with fixed duration first appears in Big Match with lack of information on one side, by [Sorin \[1984\]](#). [Vieille \[1992\]](#) introduces a differential game to study weak approachability for repeated games with vector payoffs as developed by [Blackwell et al. \[1956\]](#). The same dynamics appears in the differential game proposed by [Laraki \[2002\]](#), who proves existence of the asymptotic value in N -stage and λ -discounted repeated games with incomplete information on one side (a class of games that were initially studied by [Aumann et al. \[1995\]](#)).

An extensive study of the interfaces between repeated and differential games can be found in [Sorin \[2011\]](#), who underlines the key insight that considering the associated differential game over $[0, 1]$, the same mathematical tools are needed in order to prove existence of asymptotic value in both N -stage and λ -discounted repeated games. Building on this, [Cardaliaguet et al. \[2012\]](#) achieve a transposition to discrete time games of the numerical schemes used to approximate the value function of differential games via viscosity solution arguments, presented in [Barles and Souganidis \[1991\]](#). The authors prove uniform convergence of the value in absorbing, splitting and incomplete information games. More precisely, they consider accumulation points of sequences of functions satisfying an adapted recursive equation, which is not available in continuous time and they use an ad-hoc comparison principle to prove uniqueness and thus, convergence. The main difference in differential games lies in the manner that the state variable evolves in the state space; namely, since in the continuous time, dynamics is given in continuous time, the limit game is well defined. The resulting question concerns the existence of its value. Recently, [Sorin](#)

and Vigerál [2013a], using comparison principles, provide new proofs of existence of the asymptotic value in absorbing, recursive and incomplete information on both sides games.

Main results

In this work, we introduce a two-player zero-sum *FD*-game with action sets of the players denoted by I and J respectively. The state space is given by $\mathcal{Z} = \mathbb{N}^{I \times J}$ and it is endowed with the norm $\|\cdot\|_1$. Let $A = [a_{ij}]_{ij}$ a real coefficient matrix. The payoff function is additive in the sense that it consists of two parts: the current and the externality payoff functions. The former depends only on the current actions of the players while the latter, only on the state variable. For any $(z, i, j) \in \mathcal{Z} \times I \times J$, the stage payoff is given by:

$$g(z, i, j) = \left\langle H, \frac{z}{\|z\|_1} \right\rangle + a_{ij}.$$

If at stage t the state is z , then at stage $t + 1$ the transition in \mathcal{Z} is given by $z \rightsquigarrow z + e_{ij}$, where e_{ij} is the standard basis vector in $\mathbb{R}^{I \times J}$. Therefore, if z_t stands for the current state at stage t , it must satisfy $\|z_t\|_1 = \|z_0\|_1 + t$, where z_0 is the initial state of the game. For any z_0 , existence of the value of the finite game with initial state z_0 , $\mathbf{V}_N(z_0)$ follows from von Neumann [1928].

From Shapley [1953], it follows that the N -stage value of $\Gamma_N(z_0)$ satisfies a recursive formula; the N -stage game can be decomposed into a one stage game and a game of length $N - 1$, starting with a new value of the state variable. Inspired by the fact that transition in the state space is affine, we investigate useful properties of the N -stage value that allow us to represent it explicitly. Despite this, it is technically difficult to prove existence or not of its limit. This fact leads us to develop a novel approach in order to study the existence of the asymptotic value.

What we do is to define for any length $N \in \mathbb{N}^*$ the quotient state space and the uniform partition of $[0, 1]$. Then, we introduce an auxiliary discrete time game played on the quotient state space times the uniform partition. The value of this game is well-defined and related to the value of the original *FD*-game. Such a discrete-time game allows us to define an associated differential game played over $[0, 1] \times \mathbb{R}_+^{I \times J}$ as its natural continuous time version. More precisely, given $(t, q) \in [0, 1] \times \mathbb{R}_+^{I \times J}$, we define a differential zero sum game, denoted by $\mathcal{G}(t, q)$, which starts at time t with initial state q . Player 1 uses a measurable control $\tilde{u} : [t, 1] \rightarrow \Delta(I)$ and similarly Player 2 uses a measurable control $\tilde{v} : [t, 1] \rightarrow \Delta(J)$. We define lower and upper values of $\mathcal{G}(t, q)$, using the concept of non-anticipative strategies. Whenever $q \rightarrow \mathbf{0}$, the total payoff turns to be a generalized integral. Due to this irregularity, since the Isaacs condition is well-established, we deduce existence of the value in $\mathcal{G}(t, q)$ for all $t \in [0, 1] \times \mathbb{R}^{I \times J} \setminus \{\mathbf{0}\}$, denoted by $W(t, q)$. Mainly, the reason is that the payoff function is not locally Lipschitz at $q = \mathbf{0}$ leading to the violation of the regularity conditions which are required in order to use the powerful tools of the theory of differential games. Despite that irregularity at state $q = \mathbf{0}$, we prove existence of the value in the differential game $\mathcal{G}(0, \mathbf{0})$, starting at time $t = 0$ and initial state

$q = \mathbf{0}$, $\mathcal{G}(0, \mathbf{0})$ and we denote its value by $W(\mathbf{0})$; namely, we prove the following theorem:

Theorem 3.1.1. *The game $\mathcal{G}(\mathbf{0})$ has a value, $W(\mathbf{0}) = \lim_{q \rightarrow \mathbf{0}} W(0, q)$.*

Properties of the N -stage value, which come of the affine transition in \mathcal{Z} , lead us to prove coincidence between the value function of the auxiliary game and the discrete value corresponding to the uniform discretization of $\mathcal{G}(0, \mathbf{0})$. Precisely, we show that both functions satisfy the same recursive formula and share the same value at final time $t = 1$.

For any $(t, q) \in [0, 1] \times \mathbb{R}^{I \times J} \setminus \{\mathbf{0}\}$, results on the approximation schemes of the solution of the (HJBI) equation in [Bardi and Capuzzo-Dolcetta \[2008\]](#) imply that the discrete values converge to the values of the continuous time game, when the mesh of the discretization tends to zero. We use this tool to conclude that asymptotic value in the FD -game exists; namely, we prove the following theorem:

Theorem 3.1.2. *For all $z \in \mathcal{Z}$, as N tends to infinity, the value of the FD -game $V_N(z)$ converges to the value of its associated differential game played over $[0, 1]$, starting at initial state $\mathbf{0}$, $W(\mathbf{0})$.*

Comments

The dynamics according to which, the state variable evolves in the associated continuous time FD -game, is similar to this one in [Laraki \[2002\]](#) and [Vieille \[1992\]](#). Contrary to [Laraki \[2002\]](#), the existence of the value is not immediate in our setting. Furthermore, the state space of the initial FD -game does not involve any stationarity properties in the sense that each state can be visited at most one time during the game-play. Since the auxiliary discrete time game is defined on the quotient state space, stationarity property is now established. This stands in contrast to [Laraki \[2002\]](#), where the dual game and the associated differential game are endowed with the same state space.

Structure of the paper

The remainder of the paper is organized as follows: In [Section 3.2](#), we give the description of a two-player zero-sum FD -game and we provide properties of the n -stage value function, which will be useful, in the sequel. In [Section 3.3](#), we heuristically derive a partial differential equation (PDE) and we introduce the associated differential game as a natural continuous time version of an auxiliary game played over the cartesian product of the uniform partition of the time interval $[0, 1]$ with the quotient state space. In [Section 3.4](#), we prove existence of the value in the differential game played over $[0, 1]$ starting at initial state $q = \mathbf{0}$. In [Section 3.5](#), we conclude by identifying the value of the continuous time game, as the limit value of the N -stage FD -game. [Section 3.6](#) deals with perspectives and future work. Appendix can be found in [Section 3.7](#).

3.2 The game $\Gamma_N(z_0)$

3.2.1 Definitions

Let I, J be finite sets and denote the space of real matrices with $|I|$ rows and $|J|$ columns by $\mathcal{M}^{I \times J}$. Let $A = [a_{ij}]_{ij} \in \mathcal{M}^{I \times J}$. We introduce the following zero-sum FD -game form:

- Player 1 (the maximizer) and Player 2 (the minimizer).
- We define a state space $\mathcal{Z} = \mathbb{N}^{I \times J}$, where $z = (z(i, j))_{ij} \in \mathcal{Z}$, $z(i, j)$ is a function defined on $I \times J$ with positive integer values and $\mathbf{0} = (0)_{ij}$ stands for the zero-vector in \mathcal{Z} . \mathcal{Z}^* denotes the set $\mathcal{Z} \setminus \{\mathbf{0}\}$. The state space is endowed with the norm $\|\cdot\|_1$, which for notational convenience will be denoted to $|\cdot|$, i.e., for each $z \in \mathcal{Z}$, $|z| = \sum_{i,j} z(i, j)$.

- Let $H \in \mathcal{M}^{I \times J}$. For each $z \in \mathcal{Z}$, we define

$$h(z) := \begin{cases} \left\langle H, \frac{z}{|z|} \right\rangle, & z \neq \mathbf{0} \\ 0, & z = \mathbf{0}, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical inner product in $\mathbb{R}^{I \times J}$.

- At stage 0, a state $z_0 \in \mathcal{Z}$ is publicly announced and each player chooses a move $i_0 \in I$ and $j_0 \in J$ respectively. Then, the couple (i_0, j_0) is announced to both players. The stage payoff is:

$$g(z_0, i_0, j_0) = a_{i_0 j_0} + h(z_0).$$

A new state z_1 is determined as follows: $z_1 = z_0 + e_{i_0 j_0}$, where $(e_{ij})_{ij}$ stands for the canonical basis in $\mathbb{R}^{I \times J}$.

- Note that, due to the nature of the transition in the state space, announcing the selected moves publicly also reveals the state variable to the players. Therefore, we will denote by $\mathbf{H}_t = \mathcal{Z} \times (I \times J)^t$ the set of histories at stage t and $\mathbf{H} = \cup_{t \geq 0} \mathbf{H}_t$ will stand for the set of all histories. The notation $h_t \in \mathbf{H}_t$ stands for a certain history of length t .
- At stage t , knowing the past history $h_t = (z_0, i_0, j_0, \dots, i_{t-1}, j_{t-1})$, both players choose simultaneously and independently a move, i.e., $i_t \in I$ and $j_t \in J$. Then, the new history $h_{t+1} = (h_t, i_t, j_t)$ is told to both. The stage payoff is:

$$g_t := g(z_t, i_t, j_t) = a_{i_t j_t} + h(z_t).$$

A new state z_{t+1} is determined as follows: $z_{t+1} = z_t + e_{i_t j_t}$. For all $(i, j) \in I \times J$,

$$z_t(i, j) = z_0(i, j) + \sum_{\kappa=0}^{t-1} \delta_{i,j}(i_\kappa, j_\kappa),$$

where $\delta_{i,j}(i_\kappa, j_\kappa) = 1$ if $(i_\kappa, j_\kappa) = (i, j)$ and $\delta_{i,j}(i_\kappa, j_\kappa) = 0$ otherwise.

- At stage $N - 1$, the players choose their last actions of the game and the payoff is realized between them at the last stage N of the game.
- The description of the game is publicly known.

$\Delta(I)$ and $\Delta(J)$ are the sets of mixed moves of Player 1 and Player 2 respectively. A behavioral strategy for Player 1 is a family of maps $\sigma = (\sigma_t)_{t \geq 0}$, such that $\sigma_t : \mathbf{H}_t \rightarrow \Delta(I)$. Similarly, a behavioral strategy for Player 2 is a family of maps $\tau = (\tau_t)_{t \geq 0}$, where $\tau_t : \mathbf{H}_t \rightarrow \Delta(J)$. Σ and T denote the sets of behavioral strategies of Player 1 and Player 2, respectively. Given $z_0 \in \mathcal{Z}$, each strategy profile (σ, τ) induces a unique probability distribution $\mathbb{P}_{\sigma, \tau}^{z_0}$ on the set $\mathcal{Z} \times (I \times J)^\infty$ of plays (endowed with the σ -field generated by the cylinders). $\mathbb{E}_{\sigma, \tau}^{z_0}$ stands for the corresponding expectation.

3.2.2 The value of the finite FD -game $\Gamma_N(z_0)$

Given $N \in \mathbb{N}^*$, we will be interested in the finite N -stage game of initial state $z_0 \in \mathcal{Z}$. In this game, the expected average payoff of Player 1 is $\gamma_N(z_0, \sigma, \tau) = \mathbb{E}_{\sigma, \tau}^{z_0} \left(\frac{1}{N} \sum_{t=0}^{N-1} g_t \right)$ and the lower value is defined as:

$$\mathbf{V}_N^-(z_0) = \sup_{\sigma \in \Sigma} \inf_{\tau \in T} \gamma_N(z_0, \sigma, \tau)$$

Symmetrically, we define the upper value, i.e.,

$$\mathbf{V}_N^+(z_0) = \inf_{\tau \in T} \sup_{\sigma \in \Sigma} \gamma_N(z_0, \sigma, \tau)$$

Given $z_0 \in \mathcal{Z}$, if $\mathbf{V}_N^-(z_0) = \mathbf{V}_N^+(z_0)$, then the game $\Gamma_N(z_0)$ admit a *value* denoted by $\mathbf{V}_N(z_0)$.

Proposition 3.2.1. *Given $n \in \mathbb{N}^*$ and a state $z \in \mathcal{Z}$, the game $\Gamma_n(z)$ has a value, $\mathbf{V}_n(z)$. Moreover, the value of the game satisfies the following recursive formula:*

$$(n + 1) \mathbf{V}_{n+1}(z) = h(z) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j (a_{ij} + n \mathbf{V}_n(z + e_{ij})) \right) \quad (3.2.1)$$

Proof. Existence of the value in $\Gamma_n(z)$ in mixed strategies follows from the minmax theorem of von Neumann [1928] and since the game is played under perfect-recall by Kuhn's theorem the existence of the value is extended over the set of behavioral strategies.

For in $\Gamma_n(z)$, the players know the current history of the game and the transition of the state variable is deterministic we can obtain a simple recursive structure. The *Shapley operator* (Shapley [1953]) implies that

$$\mathbf{V}_{n+1}(z) = \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \mathbb{E}_{u \otimes v} \left(\frac{1}{n+1} g(i^*, j^*, z) + \frac{n}{n+1} \mathbf{V}_n(z + e_{i^* j^*}) \right)$$

where (i^*, j^*) are random variables defined on $I \times J$. Since the payoff function h does not depend on the selected mixed moves (u, v) one gets

$$(n+1)\mathbf{V}_{n+1}(z) = h(z) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \mathbb{E}_{u \otimes v}(a_{i^*j^*} + n\mathbf{V}_n(z + e_{i^*j^*})),$$

which concludes the proof. \square

3.2.3 Properties of the value function

Notation. Given $t \in \mathbb{N}$, let Π_t denote the subset of the state space \mathcal{Z} defined as follows:

$$\Pi_t = \{z \in \mathcal{Z} : |z| = t\}.$$

We further denote the max min operator by **val**.

Proposition 3.2.2. *For all $n \in \mathbb{N}^*$, for all $t \in \mathbb{N}^*$, there exist $K_{n,t} \in \mathcal{M}^{I \times J}$ and $c_{n,t} \in \mathbb{R}$, such that for all $z \in \Pi_t$*

$$n\mathbf{V}_n(z) = \langle K_{n,t}, z \rangle + c_{n,t}.$$

Proof. We proceed by induction on the variable n :

- For $n = 1$, the argument is true too; if $z = \mathbf{0}$, then the result is trivial, so assume that $z \in \mathcal{Z}^*$ then by (3.2.1) one has:

$$\mathbf{V}_1(z) = \left\langle H, \frac{z}{|z|} \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j a_{ij} \right), \quad z \in \Pi_t,$$

where $t = |z| \in \mathbb{N}^*$. It follows that $\mathbf{V}_1(z) = \langle K_{1,t}, z \rangle + c_{1,t}$, where $K_{1,t} = \frac{H}{t}$ and $c_{1,t} = \mathbf{val}(A)$.

- Assume the result is true for $n = m$ and we want to establish it for $n = m + 1$. Since the result holds for $n = m$, for all $t \in \mathbb{N}^*$, there exist $K_{m,t} \in \mathcal{M}^{I \times J}$ and $c_{m,t} \in \mathbb{R}$, such that:

$$m\mathbf{V}_m(z) = \langle K_{m,t}, z \rangle + c_{m,t}, \quad z \in \Pi_t.$$

- For $n = m + 1$, for all $z \in \Pi_t$ using (3.2.1), we obtain:

$$\begin{aligned} (m+1)\mathbf{V}_{m+1}(z) &= \left\langle H, \frac{z}{t} \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \left(a_{ij} + \langle K_{m,t+1}, z + e_{ij} \rangle + c_{m,t+1} \right) \right) \\ &= \left\langle \frac{H}{t} + K_{m,t+1}, z \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \left(a_{ij} + \langle K_{m,t+1}, e_{ij} \rangle + c_{m,t+1} \right) \right) \\ &= \left\langle \frac{H}{t} + K_{m,t+1}, z \right\rangle + \mathbf{val} \left(A + K_{m,t+1} \right) + c_{m,t+1}, \end{aligned}$$

where the former equality follows from the induction hypothesis, the middle one from the inner product properties and the fact that $\sum_{ij} u_i v_j = 1$, and the latter one from the **val** operator properties. Hence,

$$(m+1)V_{m+1}(z) = \langle K_{m+1,t}, z \rangle + c_{m+1,t},$$

where $K_{m+1,t} = \frac{H}{t} + K_{m,t+1}$ and $c_{m+1,t} = \mathbf{val}\left(A + K_{m,t+1}\right) + c_{m,t+1}$, concluding the proof. □

To summarize, the recurrent sequences $(K_{n,t})_{(n,t) \in \mathbb{N} \times \mathbb{N}^*}$ and $(c_{n,t})_{n,t \in \mathbb{N}}$ are defined respectively by

$$\begin{cases} K_{n+1,t} = \frac{H}{t} + K_{n,t+1}, & n, t \geq 1, \\ K_{1,t} = \frac{H}{t}, & n = 0, t \geq 1. \end{cases} \quad (3.2.2)$$

and,

$$\begin{cases} c_{n+1,t} = \mathbf{val}(A + K_{n,t+1}) + c_{n,t+1}, & n \geq 1, t \geq 0 \\ c_{1,t} = \mathbf{val}(A), & n = 0, t \geq 0 \end{cases} \quad (3.2.3)$$

Notation. In the sequel, for any $(a, p) \in \mathbb{R}_+^* \times \mathbb{N}^*$, we let

$$\Lambda_p(a) = \frac{1}{a} + \frac{1}{a+1} \dots + \frac{1}{a+p-1} = \sum_{k=0}^{p-1} \frac{1}{a+k}.$$

Corollary 3.2.3. *The general terms of the sequences $(K_{n,t})_{(n,t) \in \mathbb{N} \times \mathbb{N}^*}$ and $(c_{n,t})_{n,t \in \mathbb{N}}$ are given by:*

$$K_{n+1,t} = \Lambda_{n+1}(t)H, \quad c_{n+1,t} = \sum_{k=1}^n \mathbf{val}(A + \Lambda_{n+1-k}(t+k)H).$$

Proof. For all $(n, t) \in \mathbb{N} \times \mathbb{N}^*$, by (3.2.2) we have:

$$\begin{aligned} K_{n+1,t} &= \frac{H}{t} + K_{n,t+1} \\ K_{n,t+1} &= \frac{H}{t+1} + K_{n-1,t+2} \\ K_{n-1,t+2} &= \frac{H}{t+2} + K_{n-2,t+3} \\ &\vdots \\ K_{2,n+t-1} &= \frac{H}{n+t-1} + K_{1,n+t} \end{aligned}$$

$$K_{1,n+t} = \frac{H}{n+t}$$

$$K_{0,n+t+1} = 0.$$

Recursively, it follows that:

$$K_{n,t} = \Lambda_n(t)H.$$

One also has:

$$\begin{aligned} c_{n+1,t} &= \mathbf{val}(A + K_{n,t+1}) + c_{n,t+1} \\ c_{n,t+1} &= \mathbf{val}(A + K_{n-1,t+2}) + c_{n-1,t+2} \\ c_{n-1,t+2} &= \mathbf{val}(A + K_{n-2,t+3}) + c_{n-2,t+3} \\ &\vdots \\ c_{2,n+t-1} &= \mathbf{val}(A + K_{1,n+t}) + c_{1,n+t} \\ c_{1,n+t} &= \mathbf{val}(A) \\ c_{0,n+t+1} &= 0. \end{aligned}$$

and adding up the terms of each side of the equations, we obtain:

$$c_{n,t} = \sum_{k=1}^{n-1} \mathbf{val}(A + \Lambda_{n-k}(t+k)H),$$

which concludes the proof. \square

Given $z_0 = \mathbf{0}$ and $N \in \mathbb{N}^*$, in the game $\Gamma_N(\mathbf{0})$ the recursive formula in (3.2.1) involves the coefficients: $\{K_{n,N-n} | n \in \mathbb{N}\}$.

Corollary 3.2.4. *In $\Gamma_N(\mathbf{0})$, let $n \in \mathbb{N}$ be such that whenever N goes to infinity, one has $\frac{n}{N} \rightarrow 1 - \rho$, for some $\rho \in (0, 1)$. Then, $K_{n,N-n}$ converges in \mathbb{R} .*

Proof. From Corollary 3.2.3, we have:

$$\Lambda_n(t) = \sum_{k=0}^{n-1} \frac{1}{t+k}.$$

Consequently,

$$\begin{aligned} K_{n,t} &= H \left(\log(n+t-1) - \log(t) + o\left(\frac{1}{n+t-1}\right) + o\left(\frac{1}{t}\right) \right) \\ &= -H \left(\log\left(\frac{t}{n+t-1}\right) + o\left(\frac{1}{n+t-1}\right) \right) \\ &= -H \left(\log\left(\frac{\frac{t}{n}}{1 + \frac{t}{n} - \frac{1}{n}}\right) + o\left(\frac{1}{n+t-1}\right) \right). \end{aligned}$$

In $\Gamma_N(\mathbf{0})$, we have $t = N - n$ and thus, $\frac{t}{n} = \frac{N}{n} - 1$. By assumption, we get $\frac{N}{n} \rightarrow \frac{1}{1-\rho}$, where $\rho \in (0, 1)$. Hence, $\log\left(\frac{\frac{t}{n}}{1+\frac{t}{n}-1}\right) \rightarrow \log(\rho)$ as $N \rightarrow \infty$. Therefore,

$$\lim_{N \rightarrow \infty} K_{n, N-n} = -H \log(\rho).$$

The result follows since $H \in \mathcal{M}^{I \times J}$ and $\rho \in (0, 1)$. \square

It is important to stress that studying the convergence $(c_{n,t})_{n,t \in \mathbb{N}}$ is technically difficult. This obstacle motivates us to investigate the issue of the existence of the asymptotic value by means of a differential approach obtained by introducing an appropriate continuous time game. To illustrate this, in Section 3.3.1 and Section 3.3.2, moving from the recursive formula obtained in Proposition 3.2.1, we heuristically derive a Partial Differential Equation (PDE), which is identical to the (HJBI) equation associated to the differential game.

3.3 A differential approach

For any $N \in \mathbb{N}^*$, we define the quotient state space:

$$\mathcal{Q}_N := \left\{ q \mid q = \frac{z}{N}, z \in \mathcal{Z} \right\}.$$

We further consider the uniform partition of $[0, 1]$, i.e., $\mathcal{I}_N := \left\{ 0, \frac{1}{N}, \dots, 1 \right\}$.

3.3.1 The auxiliary game and the associated function Ψ_N

We introduce an auxiliary game which admits the same game-play as $\Gamma_N(z_0)$, but is played over $\mathcal{I}_N \times \mathcal{Q}_N$. We define its value Φ_N as follows:

$$\Phi_N(s, q) := \mathbf{V}_n(z), \quad (3.3.1)$$

where $z = Nq$ and $n = Ns$. By (3.2.1), it further satisfies the following recursive formula:

$$N \left(s + \frac{1}{N} \right) \Phi_N \left(s + \frac{1}{N}, q \right) = h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(a_{ij} + Ns \Phi_N \left(s, q + \frac{e_{ij}}{N} \right) \right) \right) \quad (3.3.2)$$

Moreover, we define the function $\Psi_N : \mathcal{I}_N \times \mathcal{Q}_N \rightarrow \mathbb{R}$, such that

$$\Psi_N(t, q) := s \Phi_N(s, q), \quad (3.3.3)$$

where $s + t = 1$. Such function Ψ_N satisfies the recursive formula:

$$N \Psi_N(t, q) = h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(a_{ij} + N \Psi_N \left(t + \frac{1}{N}, q + \frac{e_{ij}}{N} \right) \right) \right) \quad (3.3.4)$$

3.3.2 Heuristic derivation of the associated (PDE) equation

By (3.3.4), it equivalently holds true:

$$0 = h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(a_{ij} + N \left(\Psi_N \left(t + \frac{1}{N}, q + \frac{e_{ij}}{N} \right) - \Psi_N(t, q) \right) \right) \right). \quad (3.3.5)$$

The variation of the function Ψ_N in both variables can be expressed using the sum:

$$\left(\Psi_N \left(t + \frac{1}{N}, q + \frac{e_{ij}}{N} \right) - \Psi_N \left(t, q + \frac{e_{ij}}{N} \right) \right) + \left(\Psi_N \left(t, q + \frac{e_{ij}}{N} \right) - \Psi_N(t, q) \right).$$

Letting N tend to infinity, the sequence $(\mathcal{I}_N \times \mathcal{Q}_N)_{N \in \mathbb{N}^*}$ yields $[0, 1] \times \mathbb{R}_+^{I \times J}$. Let us assume that $\Psi_N(t, q)$ converges to some $\Psi(t, q)$. Then, from (3.3.5) we get:

$$0 = h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \sum_{i,j} u_i v_j \left(a_{ij} + \frac{\partial \Psi}{\partial t}(t, q) + \frac{\partial \Psi}{\partial q_{ij}}(t, q) \right).$$

Hence, one may heuristically derive that there exists a continuous and bounded function $\Psi : [0, 1] \times \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$, which satisfies the following (PDE) with boundary condition (3.3.3)

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t, q) + h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \sum_{i,j} u_i v_j \left(a_{ij} + \frac{\partial \Psi}{\partial q_{ij}}(t, q) \right) = 0, & (t, q) \in [0, 1] \times \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}, \\ \Psi(1, q) = 0, & q \in \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}. \end{cases} \quad (3.3.6)$$

3.3.3 The associated continuous and discretized games

The family of differential games $(\mathcal{G}(t, q))_{t, q}$

Given $(t, q) \in [0, 1] \times \mathbb{R}_+^{I \times J}$, we define a differential zero-sum game, denoted by $\mathcal{G}(t, q)$ starting at time t with initial state q . It consists of:

- The state space is $\mathcal{Q} = \mathbb{R}_+^{I \times J}$.
- The time interval of the game $T = [t, 1]$.
- Player 1 uses a measurable control $\tilde{u} : [t, 1] \rightarrow \Delta(I)$ and his control space is \mathcal{U}_t .
- Player 2 uses a measurable control $\tilde{v} : [t, 1] \rightarrow \Delta(J)$ and his control space is \mathcal{V}_t .
- If Player 1 uses \tilde{u} and Player 2 uses \tilde{v} , then the dynamics in the state space is defined as follows:

$$\begin{cases} \frac{dq}{dt}(s) = \tilde{u}(s) \otimes \tilde{v}(s), & s \in (t, 1), \\ q(t) = q. \end{cases} \quad (3.3.7)$$

Clearly, the dynamics is driven by a bounded, continuous function, which is Lipschitz³ in q and thus (3.3.7) admits a unique solution. We denote the evaluation at time $s > t$ of such solution by $\mathbf{q}[t, q, \tilde{u}, \tilde{v}](s)$ in order to indicate the dependence on the initial conditions (t, q) and on the control processes (\tilde{u}_s) and (\tilde{v}_s) .

- The running-payoff at time $s \in [t, 1]$ that Player 1 receives from Player 2 is given by $g : \mathcal{Q} \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ and defined as:

$$g(q, u, v) = h(q) + \langle u \otimes v, A \rangle \quad (3.3.8)$$

where,

$$h(q) := \begin{cases} \left\langle H, \frac{q}{|q|} \right\rangle, & q \neq \mathbf{0} \\ 0, & q = \mathbf{0}. \end{cases}$$

It is easy to see that g is bounded by $\|H\|_\infty + \|A\|_\infty$ and since $q : [t, 1] \rightarrow \mathcal{Q}$ is a differentiable function of time (see (3.3.7)), g is differentiable on $\mathcal{Q} \setminus \{\mathbf{0}\}$.

For all $t \in [0, 1)$ and $q \in \mathcal{Q}$, let us define:

- The payoff associated to the pair of controls $(\tilde{u}, \tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$ that Player 2 pays to Player 1 at time 1 is given by:

$$G(t, q, \tilde{u}, \tilde{v}) = \int_t^1 g(q_s, \tilde{u}_s, \tilde{v}_s) ds. \quad (3.3.9)$$

We allow the players to update their controls using non-anticipative strategies. A non-anticipative strategy for Player 1 is a map $\alpha : \mathcal{V}_t \rightarrow \mathcal{U}_t$ such that for any time $\tilde{t} > t$,

$$\tilde{v}_1(s) = \tilde{v}_2(s) \quad \forall s \in [t, \tilde{t}] \quad \Rightarrow \quad \alpha[\tilde{v}_1(s)] = \alpha[\tilde{v}_2(s)] \quad \forall s \in [t, \tilde{t}].$$

The definition of non-anticipative strategies for Player 2 is analogous. Denote by \mathcal{A}_t and \mathcal{B}_t the sets of non-anticipative strategies of the players respectively. We next define the lower and upper value functions:

$$W^-(t, q) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} G(t, q, \alpha[\tilde{v}], \tilde{v})$$

$$W^+(t, q) := \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} G(t, q, \tilde{u}, \beta[\tilde{u}]).$$

When both functions coincide, we say that the game $\mathcal{G}(t, q)$ has a *value*.

³The Lipschitz constant is 0 since the dynamics function varies constantly in the state variable q .

The Hamilton-Jacobi-Bellman-Isaacs equation

In this section, we present the Dynamic Programming Principle (DPP) for lower and upper values. We further provide the lower and upper Hamilton-Jacobi-Bellman-Isaacs equations (HJBI) that are associated to $\mathcal{G}(t, q)$. Finally, we show that the (PDE) established in (3.3.6) is identical to the (HJBI) equation associated to the differential game.

We first define the lower and upper hamiltonian functions of the game $\mathcal{G}(t, q)$ by $\mathcal{H}^\pm : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$, where

$$\mathcal{H}^-(\xi, q) = h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \langle u \otimes v, A + \xi \rangle \quad (3.3.10)$$

$$\mathcal{H}^+(\xi, q) = h(q) + \min_{v \in \Delta(J)} \max_{u \in \Delta(I)} \langle u \otimes v, A + \xi \rangle.$$

Notation. In the sequel, \mathcal{Q}^* stands for $\mathcal{Q} \setminus \{0\}$.

Given $(t, q) \in [0, 1] \times \mathcal{Q}^*$, since: (i) $\Delta(I)$ and $\Delta(J)$ are compact sets; (ii) the dynamics in the state space (3.3.7) and the running payoff (3.3.8) are bounded, continuous in all their variables and Lipschitz in the state variable q functions, by Elliott and Kalton [1974] and Evans and Souganidis [1984] the lower and upper values of minorant and majorant games respectively are characterized by means of the (DPP). Namely, for all $(t, q) \in [0, 1] \times \mathcal{Q}^*$ and all $\delta \in (0, 1 - t)$, we have:

$$W^-(t, q) = \left\langle H, \int_t^{t+\delta} \frac{q(s)}{|q| + s} ds \right\rangle + \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} \left\{ \left\langle \int_t^{t+\delta} \alpha[\tilde{v}(s)] \otimes \tilde{v}(s) ds, A \right\rangle + W^* \right\}, \quad (3.3.11)$$

where $W^* := W^-(t + \delta, q(t + \delta))$ with $q(t + \delta) = q + \int_t^{t+\delta} \alpha[\tilde{v}(s)] \otimes \tilde{v}(s) ds$.

In a symmetric way, a characterization is obtained for the upper value W^+ as follows:

$$W^+(t, q) = \left\langle H, \int_t^{t+\delta} \frac{q(s)}{|q| + s} ds \right\rangle + \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} \left\{ \left\langle \int_t^{t+\delta} \tilde{u}(s) \otimes \beta[\tilde{u}(s)] ds, A \right\rangle + W^* \right\},$$

where $W^* := W^+(t + \delta, q(t + \delta))$ with $q(t + \delta) = q + \int_t^{t+\delta} \tilde{u}(s) \otimes \beta[\tilde{u}(s)] ds$.

Furthermore under the preceding assumptions, by Evans and Souganidis [1984], $W^-(t, q)$ is the unique solution in the space of real-valued, bounded, continuous functions defined over $[0, 1] \times \mathcal{Q}^*$ of the following (HJBI) equation:

$$\begin{cases} \frac{\partial W^-}{\partial t}(t, q) + \mathcal{H}^-(\nabla_q W^-(t, q), q) = 0, & (t, q) \in [0, 1] \times \mathcal{Q}^*, \\ W^-(1, q) = 0, & q \in \mathcal{Q}^*. \end{cases} \quad (3.3.12)$$

where, \mathcal{H}^- is the lower hamiltonian defined previously.

Likewise, the upper value W^+ is the unique solution in the space of real-valued, bounded, continuous functions defined over $[0, 1] \times \mathcal{Q}^*$ of the symmetric equation:

$$\begin{cases} \frac{\partial W^+}{\partial t}(t, q) + \mathcal{H}^+(\nabla_q W^+(t, q), q) = 0, & (t, q) \in [0, 1] \times \mathcal{Q}^*, \\ W^+(1, q) = 0, & q \in \mathcal{Q}^*, \end{cases} \quad (3.3.13)$$

where \mathcal{H}^+ is the upper hamiltonian presented earlier.

Moreover, from the minmax theorem in [von Neumann \[1928\]](#), it clearly follows that the Isaacs condition, i.e., $\mathcal{H}^- = \mathcal{H}^+$ holds true (see [\(3.3.10\)](#)). Hence, by [Evans and Souganidis \[1984\]](#) the differential game $\mathcal{G}(t, q)$, starting at time $t \in [0, 1]$ with initial state $q \in \mathcal{Q}^*$ admits a value, i.e., $W(t, q) := W^-(t, q) = W^+(t, q)$. Consequently, one can identify the (PDE) obtained in [\(3.3.6\)](#) with the (HJBI) equations of [\(3.3.12\)](#) and [\(3.3.13\)](#).

We are not allowed to use these powerful results on the differential game played over $[0, 1] \times \mathcal{Q}$ since the payoff function is not Lipschitz at $q = \mathbf{0}$. In [Section 3.4](#), we prove existence of the value in the continuous time game played over $[0, 1] \times \mathcal{Q}$.

The family of discretized games $(\mathcal{G}_{\mathcal{P}}(t_0, q_0))_{t_0, q_0}$

In [Section 3.3.3](#), we prove a coincidence result between the function Ψ_N and the value of the associated differential game in which players are allowed to choose their actions only on the nodes of the uniform partition of the time interval. To that point, we next introduce some finite subdivisions of $[0, 1]$ and we define a family of discretized games played on them.

Notations. Let us consider the following subdivisions of the time interval $[0, 1]$:

- For all $t_0 \in [0, 1)$, \mathcal{P} stands for any finite subdivision of $[t_0, 1]$ and $m_{\mathcal{P}}$ denotes the number of intervals of the subdivision \mathcal{P} .
- Given $N \in \mathbb{N}^*$, $\mathcal{P}_N = (t_k^N)_{0 \leq k \leq N}$, where $t_k^N := \frac{k}{N}$ stands for the uniform subdivision of $[0, 1]$ in N intervals. We will also use the notation $\mathcal{P}_N = (t_n^N)_{0 \leq n \leq N}$ for $n = N - k$.

Fix \mathcal{P} and let $\pi_k := t_{k+1} - t_k$ for $k \in \{0, \dots, h_{\mathcal{P}} - 1\}$ be the k -th increment of it. We denote by $|\mathcal{P}|$ the mesh of the subdivision \mathcal{P} , i.e., $|\mathcal{P}| = \sup_k |\pi_k|$.

Given \mathcal{P} , for all $(t_0, q_0) \in [0, 1) \times \mathcal{Q}$ we associate to $\mathcal{G}(t_0, q_0)$ a discrete time game adapted to the subdivision \mathcal{P} denoted by $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$. Such discrete time game starts at time t_0 , has initial state $q_0 \in \mathcal{Q}$ and is repeated $h_{\mathcal{P}}$ times. At time t_k with $k = 0, \dots, h_{\mathcal{P}}$, both players observe the current state q_k and choose simultaneously and independently actions u_k and v_k in $\Delta(I)$ and $\Delta(J)$ respectively. The control sets are denoted by $\Delta(I)^{h_{\mathcal{P}}}$ and $\Delta(J)^{h_{\mathcal{P}}}$, indicating that players now choose piecewise constant functions defined over the $h_{\mathcal{P}}$ -times cartesian product of

their corresponding mixed strategy sets. We will use the notation $\hat{u} = (u_k)_{k=0}^{h_{\mathcal{D}}}$ and $\hat{v} = (v_k)_{k=0}^{h_{\mathcal{D}}}$. The state evolves according to:

$$(\tilde{\mathcal{J}}) \quad \begin{cases} q_{k+1} = q_k + \pi_k u_k \otimes v_k, & k \in \{0, \dots, h_{\mathcal{D}}\}, \\ q_0 = q. \end{cases}$$

At stage k , the expected payoff that Player 1 receives from Player 2 is given by:

$$g(q_k, u_k, v_k) = h(q_k) + u_k A v_k \quad (3.3.14)$$

and given $(\hat{u}, \hat{v}) \in \Delta(I)^{h_{\mathcal{D}}} \times \Delta(J)^{h_{\mathcal{D}}}$, the total payoff of the game is

$$G_{\mathcal{D}}(q_0, \hat{u}, \hat{v}) = \sum_{k=0}^{h_{\mathcal{D}}} \pi_k g(q_k, u_k, v_k). \quad (3.3.15)$$

The *minorant game* $\mathcal{G}_{\mathcal{D}}^{-}(t_0, q_0)$ is similar to $\mathcal{G}_{\mathcal{D}}(t_0, q_0)$, except that Player 2 announces his move to Player 1 at each stage before Player 1 chooses his move. In the *majorant game* $\mathcal{G}_{\mathcal{D}}^{+}(t_0, q_0)$, the advantage in the information pattern is reversed and goes to Player 2 because Player 1 must commit himself first to each move. The notation $W_{\mathcal{D}}^{-}(t_0, q_0)$ and $W_{\mathcal{D}}^{+}(t_0, q_0)$ stands for the lower and upper values of $\mathcal{G}_{\mathcal{D}}(t_0, q_0)$ respectively.

Following [Friedman \[1970\]](#), we characterize⁴ $W_{\mathcal{D}}^{-}$ and $W_{\mathcal{D}}^{+}$ by means of discrete versions of the lower and upper (HJBI) equations [\(3.3.12\)](#), [\(3.3.13\)](#):

- $W_{\mathcal{D}}^{+}(t_k, q_k) = \pi_k h(q) + \min_{v \in \Delta(J)} \max_{u \in \Delta(I)} \{ \langle \pi_k u \otimes v, A \rangle + W_{\mathcal{D}}^{+}(t_{k+1}, q_k + \pi_k u \otimes v) \},$
- $W_{\mathcal{D}}^{-}(t_k, q_k) = \pi_k h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \{ \langle \pi_k u \otimes v, A \rangle + W_{\mathcal{D}}^{-}(t_{k+1}, q_k + \pi_k u \otimes v) \},$
- $W_{\mathcal{D}}^{\pm}(1, q) = 0.$

We will refer to the first two equations above as the *discrete Dynamic Programming Principle* that will be abbreviated to (discrete DPP).

Coincidence of Ψ_N and $W_{\mathcal{D}_N}$ over \mathcal{P}_N

We define $\Psi_N : \mathcal{P}_N \times \mathcal{Q} \rightarrow \mathbb{R}$ by the following recursive formula and terminal condition:

$$\begin{cases} \Psi_N(t_k^N, q) = \frac{h(q)}{N} + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N(t_{k+1}^N, q + \frac{e_{ij}}{N}) \right) \right), & 0 \leq k \leq N-1, q \in \mathcal{Q}, \\ \Psi_N(1, q) = 0, & k = N, q \in \mathcal{Q}. \end{cases} \quad (3.3.16)$$

⁴We refer to [Bardi and Capuzzo-Dolcetta \[2008\]](#) for the characterization of the discrete values.

Proposition 3.3.1. *Given $N \in \mathbb{N}^*$, for all $n \in \{0, \dots, N\}$ and all $q \in \mathcal{Q}$, there exist $K_{n,|q|} \in \mathcal{M}^{I \times J}$ and $c_{n,|q|} \in \mathbb{R}$, such that*

$$\Psi_N(t_n^N, q) = \langle K_{n,|q|}, q \rangle + c_{n,|q|}.$$

Proof. It suffices to prove the result for all $q \in \mathcal{Q}^*$ since for $q = \mathbf{0}$ it is trivial. We proceed by induction on the variable n :

- For $n = 0$, $\Psi_N(1, q) = 0$ for any $q \in \mathcal{Q}^*$ and thus, for $n = 1$,

$$\Psi_N\left(1 - \frac{1}{N}, q\right) = \left\langle \frac{H}{N|q|}, q \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \frac{a_{ij}}{N} \right),$$

for any $q \in \mathcal{Q}^*$, which clearly proves the property with the coefficients to be given by,

$$K_{1,|q|} = \frac{H}{N|q|}, \quad c_{1,|q|} = \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \frac{a_{ij}}{N} \right).$$

- Assume the result is true for $n = m$, i.e., for all $q \in \mathcal{Q}^*$, there exist $K_{m,|q|} \in \mathcal{M}^{I \times J}$ and $c_{m,|q|} \in \mathbb{R}$, such that

$$\Psi\left(1 - \frac{m}{N}, q\right) = \langle K_{m,|q|}, q \rangle + c_{m,|q|},$$

- For $n = m + 1$, for all $q \in \mathcal{Q}^*$, we get:

$$\begin{aligned} \Psi\left(1 - \frac{m+1}{N}, q\right) &= \left\langle \frac{H}{N}, \frac{q}{|q|} \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle K_{m,|q|+1}, q + \frac{e_{ij}}{N} \right\rangle + c_{m,|q|+1} \right) \right) \\ &= \left\langle \frac{H}{N|q|} + K_{m,|q|+1}, q \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle K_{m,|q|+1}, \frac{e_{ij}}{N} \right\rangle \right) \right) + c_{m,|q|+1}. \end{aligned}$$

and thus,

$$\Psi\left(1 - \frac{m+1}{N}, q\right) = \langle K_{m+1,|q|}, q \rangle + c_{m+1,|q|}$$

where,

$$K_{m+1,|q|} = \frac{H}{N|q|} + K_{m,|q|+1}, \quad c_{m+1,|q|} = \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle K_{m,|q|+1}, \frac{e_{ij}}{N} \right\rangle \right) \right) + c_{m,|q|+1}, \quad (3.3.17)$$

which proves the result since $(K_{n,|q|})_{0 \leq n \leq N}$ and $(c_{n,|q|})_{0 \leq n \leq N}$ do not depend on q .

□

Corollary 3.3.2. *Given $q \in \mathcal{Q}^*$, the general terms of the sequences $(K_{n,|q|+m})_{n,m \in \mathbb{N}}$ and $(c_{n,|q|+m})_{n,m \in \mathbb{N}}$ are given by:*

$$K_{n,|q|} = \Lambda_n(N|q|)H, \quad c_{n,|q|} = \sum_{k=1}^{n-1} \mathbf{val} \left(\frac{A}{N} + \Lambda_{n-k}(N|q| + k)H \right).$$

Proof. By Proposition 3.3.1, we get the recurrent formulas defined for all $q \in \mathcal{Q}^*$ and for all $k \in \{0, \dots, N-1\}$ by

$$K_{n+1,|q|} = \frac{H}{N|q|} + K_{n,|q|+1}, \quad c_{n+1,|q|} = \mathbf{val} \left(\frac{A}{N} + K_{n,|q|+1} \right) + c_{n,|q|+1}$$

with terminal values $K_{0,|q|+n+1} = 0$ and $c_{0,|q|+n+1} = 0$.

For all $(n, |q|) \in \mathbb{N} \times \mathbb{R}_+^*$, by (3.3.17) we have:

$$\begin{aligned} K_{n+1,|q|} &= \frac{H}{N|q|} + K_{n,|q|+1} \\ K_{n,|q|+1} &= \frac{H}{N|q|+1} + K_{n-1,|q|+2} \\ &\vdots \\ K_{1,|q|+n} &= \frac{H}{N|q|+n} \\ K_{0,|q|+n+1} &= 0. \end{aligned}$$

Recursively, it follows:

$$K_{n,|q|} = \Lambda_n(N|q|)H.$$

One also has:

$$\begin{aligned} c_{n+1,|q|} &= \mathbf{val} \left(\frac{A}{N} + K_{n,|q|+1} \right) + c_{n,|q|+1} \\ c_{n,|q|+1} &= \mathbf{val} \left(\frac{A}{N} + K_{n-1,|q|+2} \right) + c_{n-1,|q|+2} \\ &\vdots \\ c_{1,|q|+n} &= \mathbf{val} \left(\frac{A}{N} \right) \\ c_{0,|q|+n+1} &= 0. \end{aligned}$$

and adding up the terms of each side of the equations, we finally obtain:

$$c_{n,|q|} = \sum_{k=1}^{n-1} \mathbf{val} \left(\frac{A}{N} + \Lambda_{n-k}(N|q| + k)H \right)$$

that concludes the proof. \square

Notation. Given $N \in \mathbb{N}^*$ and $q_0 \in \mathcal{Q}$, for all $t \in \mathcal{P}_N$, we define the subset of \mathcal{Q} :

$$\mathcal{Q}_N(t, q_0) = \{q \in \mathcal{Q} : |q| = |q_0| + t\}.$$

From the minmax theorem in von Neumann [1928], we get that for all $(t, q) \in \mathcal{P}_N \times \mathcal{Q}$, the game $\mathcal{G}_{\mathcal{P}_N}(t, q)$ admits a value, denoted by $W_{\mathcal{P}_N}(t, q)$.

Proposition 3.3.3. *Given $N \in \mathbb{N}^*$ and $q_0 \in \mathcal{Q}$, for all $t \in \mathcal{P}_N$ and all $q \in \mathcal{Q}_N(t, q_0)$,*

$$\Psi_N(t, q) = W_{\mathcal{P}_N}(t, q).$$

Proof. Both functions share the same terminal condition, i.e $\Psi_N(1, q) = W_{\mathcal{P}_N}(1, q) = 0$, for all $q \in \mathcal{Q}$, (see (3.3.3), and characterization of $W_{\mathcal{P}_N}$ in terms of (discrete DPP)). Thus, it suffices to prove that Ψ_N and $W_{\mathcal{P}_N}$ satisfy the same recursive formula. To that point, fix $q_0 \in \mathcal{Q}$ and time $t = \frac{k}{N}$, where $k \in \{0, \dots, N-1\}$. From the (discrete (DPP)), it follows that for all $q \in \mathcal{Q}_N(\frac{k}{N}, q_0)$,

$$W_{\mathcal{P}_N}\left(\frac{k}{N}, q\right) = \frac{1}{N}h(q) + \max_{u \in U} \min_{v \in V} \left(\left\langle \frac{u \otimes v}{N}, A \right\rangle + W_{\mathcal{P}_N}\left(\frac{k+1}{N}, q + \frac{u \otimes v}{N}\right) \right) \quad (3.3.18)$$

By (3.3.16), for any $k \in \{0, \dots, N-1\}$,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{1}{N}h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N\left(\frac{k+1}{N}, q + \frac{e_{ij}}{N}\right) \right) \right),$$

where $q \in \mathcal{Q}_N(\frac{k}{N}, q_0)$. Equivalently:

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{1}{N}h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \sum_{ij} u_i v_j \Psi_n\left(\frac{k+1}{N}, q + \frac{e_{ij}}{N}\right) \right).$$

By Lemma 3.3.1, it follows that

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{1}{N}h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \Psi_n\left(\frac{k+1}{N}, \sum_{ij} u_i v_j \left(q + \frac{e_{ij}}{N} \right) \right) \right).$$

Hence,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{1}{N}h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \Psi_n\left(\frac{k+1}{N}, q + \frac{\sum_{ij} u_i v_j e_{ij}}{N}\right) \right),$$

which implies,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{1}{N}h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\left\langle \frac{u \otimes v}{N}, A \right\rangle + \Psi_N\left(\frac{k+1}{N}, q + \frac{u \otimes v}{N}\right) \right).$$

This in view of (3.3.18), proves that $\Psi_N = W_{\mathcal{P}_N}$. \square

3.4 The differential game over $[0, 1] \times \mathcal{Q}$

In this section, we prove existence of the value in the differential game played over $[0, 1] \times \mathcal{Q}$.

3.4.1 Properties on the value function

We first prove a lemma that will be useful in the sequel.

Lemma 3.4.1. *Let $t \in [0, 1]$, $q, \tilde{q} \in \mathcal{Q}^*$ and $(\tilde{u}, \tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$. Denote by $q(\cdot)$ and $\tilde{q}(\cdot)$ the trajectories with initial conditions $q(t) = q$ and $\tilde{q}(t) = \tilde{q}$ obtained from (3.3.7). Then, for all $s \in [t, 1]$, one has:*

$$\left| \frac{q(s)}{|q(s)|} - \frac{\tilde{q}(s)}{|\tilde{q}(s)|} \right| \leq 2 \frac{|q - \tilde{q}|}{|q(s)|}.$$

Proof. The left term of the inequality is equal to:

$$\begin{aligned} \left| \frac{q(s)|\tilde{q}(s)| - \tilde{q}(s)|q(s)|}{|q(s)||\tilde{q}(s)|} \right| &= \left| \frac{|\tilde{q}(s)|(q(s) - \tilde{q}(s))}{|q(s)||\tilde{q}(s)|} + \frac{\tilde{q}(s)(|\tilde{q}(s)| - |q(s)|)}{|q(s)||\tilde{q}(s)|} \right| \\ &= \left| \frac{q(s) - \tilde{q}(s)}{|q(s)|} + \frac{\tilde{q}(s)(|\tilde{q}(s)| - |q(s)|)}{|q(s)||\tilde{q}(s)|} \right| \end{aligned}$$

Hence,

$$\begin{aligned} \left| \frac{q(s)}{|q(s)|} - \frac{\tilde{q}(s)}{|\tilde{q}(s)|} \right| &\leq \frac{|q(s) - \tilde{q}(s)|}{|q(s)|} + \frac{|\tilde{q}(s)| \left| |\tilde{q}(s)| - |q(s)| \right|}{|q(s)||\tilde{q}(s)|} \\ &\leq \frac{|q(s) - \tilde{q}(s)|}{|q(s)|} + \frac{|\tilde{q}(s) - q(s)|}{|q(s)|}. \end{aligned}$$

Since the controls of the players depend only on the time variable, by the ordinary differential equation (3.3.7), for all $s \in [t, 1]$,

$$|q(s) - \tilde{q}(s)| = |q - \tilde{q}|. \quad (3.4.1)$$

Then, it follows that,

$$\left| \frac{q(s)}{|q(s)|} - \frac{\tilde{q}(s)}{|\tilde{q}(s)|} \right| \leq 2 \frac{|q - \tilde{q}|}{|q(s)|}.$$

□

Corollary 3.4.2. *Let $t \in [0, 1]$, $(q, \tilde{q}) \in \mathcal{Q}^* \times \mathcal{Q}$ and $(\tilde{u}, \tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$. Denote by $q(\cdot)$ and $\tilde{q}(\cdot)$ the trajectories with initial conditions $q(t) = q$ and $\tilde{q}(t) = \tilde{q}$ obtained from (3.3.7). Then, for all $s \in [t, 1]$, one has:*

$$|h(q(s)) - h(\tilde{q}(s))| \leq \|H\|_\infty 2 \frac{|q - \tilde{q}|}{|q(s)|}$$

Proof. The result clearly follows from Lemma 3.4.1 and definition of the function h . □

Since for all $t, \tilde{t} \in [0, 1]$ and all $q \in \mathcal{Q}^*$, the value in the game $\mathcal{G}(t, q)$ exists, i.e., $W(t, q) := W^-(t, q) = W^+(t, q)$, we establish next result on the value W ; we obtain an upper bound of the value function with respect to the variation of the state variable.

Lemma 3.4.3. *For all $t \in [0, 1]$ and all $q, \tilde{q} \in \mathcal{Q}^*$, it holds true:*

$$|W(t, q) - W(t, \tilde{q})| \leq 2 \|H\|_\infty \left| \log \left(\frac{1 + |q| - t}{|q|} \right) \right| |q - \tilde{q}|$$

Proof. Fix $t \in [0, 1]$, $q, \tilde{q} \in \mathcal{Q}^*$ and $(\tilde{u}, \tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$. To simplify notations, let us put $G(q) := G(t, q, \tilde{u}, \tilde{v})$ and $G(\tilde{q}) := G(t, \tilde{q}, \tilde{u}, \tilde{v})$. Then,

$$|G(q) - G(\tilde{q})| = \left| \int_t^1 (h(q(s)) - h(\tilde{q}(s))) ds \right|$$

For all $s \in [t, 1]$, it holds true that $|q(s)| = |q| + s - t$. By Corollary 3.4.2,

$$\begin{aligned} |G(q) - G(\tilde{q})| &\leq \|H\|_\infty \left| \int_t^1 2 \frac{|q - \tilde{q}|}{|q| + s - t} ds \right| \leq 2 \|H\|_\infty |q - \tilde{q}| \left| \int_t^1 \frac{ds}{|q| + s - t} \right| \\ &\leq 2 \|H\|_\infty \left| \left(\log(1 + |q| - t) - \log(|q|) \right) \right| |q - \tilde{q}| \\ &\leq 2 \|H\|_\infty \left| \log \left(\frac{1 + |q| - t}{|q|} \right) \right| |q - \tilde{q}|. \end{aligned}$$

Thus, in value terms, we have:

$$\begin{aligned} W(t, q) &= \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} G(t, q, \alpha[\tilde{v}], \tilde{v}) \\ &\leq \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} G(t, \tilde{q}, \alpha[\tilde{v}], \tilde{v}) + 2 \|H\|_\infty \left| \log \left(\frac{1 + |q| - t}{|q|} \right) \right| |q - \tilde{q}| \\ &\leq W(t, \tilde{q}) + 2 \|H\|_\infty \left| \log \left(\frac{1 + |q| - t}{|q|} \right) \right| |q - \tilde{q}|. \end{aligned}$$

The symmetric inequality follows exchanging q and \tilde{q} completing the proof. \square

Corollary 3.4.4. *Fix $\eta \in (0, \frac{1}{4})$. Then, for all $t \geq 0$ and all $q, \tilde{q} \in \mathcal{Q}^*$ such that, $t, |q|, |\tilde{q}| \leq \eta$,*

$$|W(t, q) - W(t, \tilde{q})| \leq 2 \|H\|_\infty \left(\log(2) \max(|q|, |\tilde{q}|) + \min(|q| \log(|q|), |\tilde{q}| \log(|\tilde{q}|)) \right)$$

Proof. Fix $\eta \in (0, \frac{1}{4})$. For all $t \geq 0$ and all $q \in \mathcal{Q}^*$, such that $t, |q| < \eta$, we have $|\log(1 + |q| - t)| < \log(2)$ and in view of Lemma 3.4.3, for all $q, \tilde{q} \in \mathcal{Q}^*$

$$\begin{aligned} |W(t, q) - W(t, \tilde{q})| &\leq 2 \|H\|_\infty |\log(2) - \log(|q|)| |q - \tilde{q}| \\ &\leq 2 \|H\|_\infty (\log(2) + |\log(|q|)|) |q - \tilde{q}| \end{aligned}$$

Due to the symmetry of the upper bound obtained in Lemma 3.4.3 with respect to q and \tilde{q} , exchanging them, it implies:

$$|W(t, q) - W(t, \tilde{q})| \leq 2 \|H\|_\infty (\log(2) + |\log(|\tilde{q}|)|) |q - \tilde{q}|$$

Hence, for all $q, \tilde{q} \in \mathcal{Q}^*$,

$$|W(t, q) - W(t, \tilde{q})| \leq 2 \|H\|_\infty \left(\log(2) \max(|q|, |\tilde{q}|) + \min(|q| \log(|q|), |\tilde{q}| \log(|\tilde{q}|)) \right),$$

which concludes the proof. \square

3.4.2 Existence of the value over $[0, 1] \times \mathcal{Q}$

In this section, we extend the results on the differential game $\mathcal{G}(t, q)$ over the set $[0, 1] \times \mathcal{Q}$. Namely, we show that $W(0, q)$ has a limit as q tends to $\mathbf{0}$ and we further establish that such limit is the value of the game starting at $(0, \mathbf{0})$, which therefore exists.

Lemma 3.4.5. *For any $\varepsilon > 0$, there exists $\eta > 0$, such that for all $t \in [0, \frac{1}{4}]$ and $q, \tilde{q} \in \mathcal{Q}^*$, if $|q|, |\tilde{q}| \leq \eta$, we then have:*

$$|W(t, q) - W(t, \tilde{q})| < \varepsilon.$$

Proof. We only prove the claim for $|q| > |\tilde{q}|$ since the opposite case is deduced in a similar way. For any $\varepsilon > 0$, we choose $\eta \in (0, \frac{1}{4})$, such that for all ρ with $0 < \rho < \eta$, one has:

$$\begin{cases} \rho \log(\rho) < \frac{\varepsilon}{4\|H\|_\infty}, \\ \rho < \frac{\varepsilon}{4\log(2)\|H\|_\infty}. \end{cases}$$

Fix $t \geq 0$, such that $t < \eta$. Then, for all $q, \tilde{q} \in \mathcal{Q}^*$ such that $|q|, |\tilde{q}| < \eta$, using Corollary 3.4.4 it follows that:

$$\begin{aligned} |W(t, q) - W(t, \tilde{q})| &< 2 \|H\|_\infty (\log(2)|q| + |q| \log(|q|)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

\square

Proposition 3.4.6. *There exists $\ell \in \mathbb{R}$, such that for all $\varepsilon > 0$, there exists $\eta > 0$, such that for all $q \in \mathcal{Q}^*$, if $|q| \leq \eta$, we then have:*

$$|W(0, q) - \ell| < \varepsilon$$

Proof. The result follows from Lemma 3.4.5. \square

Notation. In the sequel, we denote by:

- $\mathcal{G}(\mathbf{0})$, the differential game starting at time $t = 0$ and initial state $q = \mathbf{0}$ and by $W^\pm(\mathbf{0})$ the corresponding lower and upper values.
- $G_{[t, t']}(\cdot)$ the payoff that Player 1 receives over the time interval $[t, t']$.

Next, we prove existence of the value in $\mathcal{G}(\mathbf{0})$ and we further show that $W(\mathbf{0}) = \ell$. In doing so, we prove that lower and upper values are continuous functions at $(0, \mathbf{0})$ and their images at $(0, \mathbf{0})$ are both equal to ℓ . The idea of the proof lies in the consideration of ε -optimal strategies in the game $\mathcal{G}(0, q)$, which will be some ε times-optimal in the game $\mathcal{G}(\mathbf{0})$.

Theorem 3.4.7. *The game $\mathcal{G}(\mathbf{0})$ has a value, $W(\mathbf{0}) = \ell$.*

Proof. For any $\varepsilon > 0$, we choose $\eta > 0$, such that for all ρ with $0 < \rho < \eta$, one has:

$$\begin{cases} \rho |\log(\rho)| < \frac{\varepsilon}{8\|H\|_\infty}, \\ \rho < \min\left(\frac{\varepsilon}{10\log(2)\|H\|_\infty}, \frac{\varepsilon}{40\|H\|_\infty}\right) \end{cases}$$

In view of Proposition 3.4.6, we also require η to be such that for all $q \in \mathcal{Q}^*$ with $|q| \leq \eta$,

$$|W(0, q) - \ell| < \frac{\varepsilon}{4}. \quad (3.4.2)$$

Let $\alpha_q \in \mathcal{A}_0$ an $\frac{\varepsilon}{4}$ -optimal strategy for Player 1 in $\mathcal{G}(0, q)$. Then, for all $\tilde{v} \in \mathcal{V}_0$, Player 1 guarantees,

$$G(0, q, \alpha_q, \tilde{v}) \geq W(0, q) - \frac{\varepsilon}{4}.$$

We define $\tilde{q} := \mathbf{q}[0, q, \alpha_q, \tilde{v}](\eta)$ and $\hat{q} := \mathbf{q}[0, \mathbf{0}, \alpha_q, \tilde{v}](\eta)$. It follows that $|\tilde{q}| = |q| + \eta$, $|\hat{q}| = \eta$ and $|\tilde{q} - \hat{q}| = |q|$.

$$\begin{aligned} |G_{[0, \eta]}(0, q, \alpha_q, \tilde{v}) - G_{[0, \eta]}(0, \mathbf{0}, \alpha_q, \tilde{v})| &\leq \left| \left\langle H, \int_0^\eta \left(\frac{q(s)}{|q(s)|} - \frac{q_0(s)}{|q_0(s)|} \right) ds \right\rangle \right| \\ &\leq \|H\|_\infty \int_0^\eta \left(\left| \frac{q(s)}{|q(s)|} \right| + \left| \frac{q_0(s)}{|q_0(s)|} \right| \right) ds \\ &\leq 2\eta \|H\|_\infty \\ &\leq \frac{\varepsilon}{20}. \end{aligned}$$

Moreover, since $|\tilde{q}| > |\hat{q}|$ using Corollary 3.4.4, we obtain that

$$\begin{aligned} |G_{[\eta,1]}(\eta, \tilde{q}, \alpha_q, \tilde{v}) - G_{[\eta,1]}(\eta, \hat{q}, \alpha_q, \tilde{v})| &\leq 2 \|H\|_\infty \left(\log(2) + |\log(|\tilde{q}|)| \right) |\tilde{q} - \hat{q}| \\ &\leq 2 \|H\|_\infty \left(\log(2) + |\log(|\tilde{q}|)| \right) |q| \\ &\leq 2 \|H\|_\infty \eta \log(2) + 2 \|H\|_\infty \eta |\log(\eta)| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{4} \\ &< \frac{9\varepsilon}{20}, \end{aligned}$$

where the second inequality holds because $|\tilde{q} - \hat{q}| = |q|$ and the middle one follows from the fact that $x \rightarrow |\log(x)|$ is a decreasing function over $(0, 1]$. Hence, we take:

$$|G(0, q, \alpha_q, \tilde{v}) - G(0, \mathbf{0}, \alpha_q, \tilde{v})| < \frac{\varepsilon}{2},$$

and so, in $\mathcal{G}(\mathbf{0})$ Player 1 guarantees:

$$G(0, \mathbf{0}, \alpha_q, \tilde{v}) \geq G(0, q, \alpha_q, \tilde{v}) - \frac{\varepsilon}{2} \geq W(0, q) - \frac{\varepsilon}{4} - \frac{\varepsilon}{2},$$

where the right inequality is established since α_q is an $\frac{\varepsilon}{4}$ -optimal strategy in $\mathcal{G}(0, q)$. Therefore, using (3.4.2) we get that in $\mathcal{G}(\mathbf{0})$, for all $\tilde{v} \in \mathcal{V}_0$,

$$\begin{aligned} G(0, \mathbf{0}, \alpha_q, \tilde{v}) - \ell &\geq W(0, q) - \ell - \frac{3\varepsilon}{4} \\ &> -\frac{\varepsilon}{4} - \frac{3\varepsilon}{4} \\ &> -\varepsilon. \end{aligned}$$

Following similar arguments, we can further prove that for all $q \in \mathcal{Q}^*$ Player 2 chooses an $\frac{\varepsilon}{4}$ -optimal strategy in $\mathcal{G}(0, q)$, denoted by $\beta_q \in \mathcal{B}_0$, such that in $\mathcal{G}(\mathbf{0})$, for all $\tilde{u} \in \mathcal{U}_0$,

$$G(0, \mathbf{0}, \tilde{u}, \beta_q) < \ell + \varepsilon.$$

that concludes the proof. \square

3.5 Existence of the limit value in $\Gamma_N(z)$

In this section, we provide the main result of the paper. We prove that $\lim_{N \rightarrow +\infty} \mathbf{V}_N(z)$ exists and it is independent of z . We first prove some useful lemmas on the value of the original game and the associated function Ψ_N .

Lemma 3.5.1. *If for some $z_0 \in \mathcal{Z}$, there exists $\ell \in \mathbb{R}$, such that $\lim_{N \rightarrow +\infty} \mathbf{V}_N(z_0) = \ell$, then for all $z \in \mathcal{Z}$, $\lim_{N \rightarrow +\infty} \mathbf{V}_N(z) = \ell$.*

Proof. Given $N \in \mathbb{N}^*$, fix $z \in \mathcal{Z}^*$ and let us consider the games $\Gamma_N(z)$ and $\Gamma_N(\mathbf{0})$. Fix a pair of behavioral strategies (σ, τ) in $\Sigma \times T$. The notation $(z_t)_{t \geq 0}$ stands for the process in \mathcal{Z} , induced by the initial condition $z_0 = \mathbf{0}$. Then, it follows that

$$\begin{aligned} |\gamma_N(\mathbf{0}, \sigma, \tau) - \gamma_N(z, \sigma, \tau)| &= \frac{1}{N} \left| \sum_{t=0}^{N-1} h(z_t) - h(z + z_t) \right| = \frac{1}{N} \left| \left\langle H, \frac{z}{|z|} \right\rangle + \sum_{t=1}^{N-1} \left\langle H, \frac{z_t}{t} - \frac{z + z_t}{|z| + t} \right\rangle \right| \\ &= \frac{1}{N} \left| \left\langle H, \frac{z}{|z|} \right\rangle + \sum_{t=1}^{N-1} \left\langle H, \frac{z_t(|z| + t) - t(z + z_t)}{t(|z| + t)} \right\rangle \right| \\ &= \frac{1}{N} \left| \left\langle H, \frac{z}{|z|} \right\rangle + \sum_{t=1}^{N-1} \left\langle H, \frac{z_t|z| - tz}{t(|z| + t)} \right\rangle \right|. \end{aligned}$$

Hence,

$$\begin{aligned} |\gamma_N(\mathbf{0}, \sigma, \tau) - \gamma_N(z, \sigma, \tau)| &\leq \frac{1}{N} \left(\|H\|_\infty + \sum_{t=1}^{N-1} \|H\|_\infty \frac{|z_t||z| + tz|}{t(|z| + t)} \right) \\ &\leq \frac{\|H\|_\infty}{N} \left(1 + \sum_{t=1}^{N-1} \frac{|z_t||z| + t|z|}{t(|z| + t)} \right) \\ &\leq \|H\|_\infty \left(\frac{1}{N} + \frac{2}{N} |z| \sum_{t=1}^{N-1} \frac{1}{|z| + t} \right) \\ &\leq \|H\|_\infty \left(\frac{1}{N} + 2|z| \frac{\log\left(\frac{|z|+N-1}{|z|+1}\right)}{N} + o\left(\frac{1}{|z| + N - 1}\right) + o\left(\frac{1}{|z| + 1}\right) \right), \end{aligned}$$

which since $\frac{\log\left(\frac{|z|+N-1}{|z|+1}\right)}{N} \xrightarrow{N \rightarrow +\infty} 0$, for any $\varepsilon > 0$ and for all $z \in \mathcal{Z}^*$ and $(\sigma, \tau) \in \Sigma \times T$, there exists $N_{\varepsilon, z} \in \mathbb{N}^*$, such that for all $N \geq N_{\varepsilon, z}$,

$$|\gamma_N(\mathbf{0}, \sigma, \tau) - \gamma_N(z, \sigma, \tau)| < \varepsilon.$$

□

Lemma 3.5.2. *Given $N \in \mathbb{N}^*$, for all $q, \tilde{q} \in \mathcal{Q}^*$, we have:*

$$|\Psi_N(0, q) - \Psi_N(0, \tilde{q})| \leq 2 \|H\|_\infty |q - \tilde{q}| \sum_{k=0}^{N-1} \frac{1}{N|q| + k}.$$

Proof. We prove the result by induction on the variable n :

- For $n = 0$, the result is trivial since $\Psi(1, q) = 0$ for all $q \in \mathcal{Q}$.
- For $n = 1$, using (3.3.16) we get:

$$\begin{aligned} \left| \Psi\left(1 - \frac{1}{N}, q\right) - \Psi\left(1 - \frac{1}{N}, \tilde{q}\right) \right| &= \frac{1}{N} \left| \left\langle H, \frac{q}{|q|} - \frac{\tilde{q}}{|\tilde{q}|} \right\rangle \right| \\ &\leq \frac{1}{N} \|H\|_\infty \left| \frac{q}{|q|} - \frac{\tilde{q}}{|\tilde{q}|} \right| \\ &\leq \frac{2 \|H\|_\infty}{N|q|} |q - \tilde{q}|, \end{aligned}$$

where we used the Corollary 3.4.2.

- Assume the result is true for $n = m$, i.e.,

$$\left| \Psi\left(1 - \frac{m}{N}, q\right) - \Psi\left(1 - \frac{m}{N}, \tilde{q}\right) \right| \leq 2 \|H\|_\infty |q - \tilde{q}| \sum_{k=0}^{m-1} \frac{1}{N|q| + k}.$$

- We prove the result for $n = m + 1$.

$$\begin{aligned} \left| \Psi\left(1 - \frac{m+1}{N}, q\right) - \Psi\left(1 - \frac{m+1}{N}, \tilde{q}\right) \right| &\leq \frac{1}{N} \left| \left\langle H, \frac{q}{|q|} - \frac{\tilde{q}}{|\tilde{q}|} \right\rangle \right| \\ &\quad + \left| \mathbf{val}\left(\Psi_N\left(1 - \frac{m}{N}, q + \frac{e_{ij}}{N}\right)\right) - \mathbf{val}\left(\Psi_N\left(1 - \frac{m}{N}, \tilde{q} + \frac{e_{ij}}{N}\right)\right) \right|. \end{aligned}$$

By the Lipschitz property of the \mathbf{val} operator, we get:

$$\begin{aligned} \left| \Psi\left(1 - \frac{m+1}{N}, q\right) - \Psi\left(1 - \frac{m+1}{N}, \tilde{q}\right) \right| &\leq \frac{1}{N} \left| \left\langle H, \frac{q}{|q|} - \frac{\tilde{q}}{|\tilde{q}|} \right\rangle \right| \\ &\quad + \sup_{i,j} \left| \left(\Psi_N\left(1 - \frac{m}{N}, q + \frac{e_{ij}}{N}\right) \right) - \left(\Psi_N\left(1 - \frac{m}{N}, \tilde{q} + \frac{e_{ij}}{N}\right) \right) \right|. \end{aligned}$$

By Corollary 3.4.2 and the induction hypothesis, it follows that:

$$\left| \Psi\left(1 - \frac{m+1}{N}, q\right) - \Psi\left(1 - \frac{m+1}{N}, \tilde{q}\right) \right| \leq \frac{2 \|H\|_\infty}{N|q|} |q - \tilde{q}| + 2 \|H\|_\infty |q - \tilde{q}| \sum_{k=0}^{m-1} \frac{1}{N|q| + \frac{1}{N} + k}.$$

Therefore,

$$\begin{aligned} \left| \Psi\left(1 - \frac{m+1}{N}, q\right) - \Psi\left(1 - \frac{m+1}{N}, \tilde{q}\right) \right| &\leq \frac{2 \|H\|_\infty}{N|q|} |q - \tilde{q}| + 2 \|H\|_\infty |q - \tilde{q}| \sum_{k=1}^m \frac{1}{N|q| + k} \\ &\leq 2 \|H\|_\infty |q - \tilde{q}| \left(\frac{1}{N|q|} + \sum_{k=1}^m \frac{1}{N|q| + k} \right) \\ &\leq 2 \|H\|_\infty |q - \tilde{q}| \sum_{k=0}^m \frac{1}{N|q| + k}, \end{aligned}$$

concluding the proof. \square

Lemma 3.5.3. *For any $\varepsilon > 0$, for all $q, \tilde{q} \in \mathcal{Q}^*$, there exists $\tilde{N} \in \mathbb{N}^*$, such that for all $N \geq \tilde{N}$,*

$$\left| \Psi_N(0, q) - \Psi_N(0, \tilde{q}) \right| \leq 2 \|H\|_\infty \log\left(\frac{|q| + 1}{|q|}\right) |q - \tilde{q}| + \varepsilon$$

Proof. Fix $q, \tilde{q} \in \mathcal{Q}^*$. By Lemma 3.5.2, we have:

$$\left| \Psi_N(0, q_0) - \Psi_N\left(0, \frac{e_{i_1 j_1}}{N}\right) \right| \leq 2 \|H\|_\infty |q - \tilde{q}| \sum_{k=0}^{N-1} \frac{1}{N|q| + k}.$$

Since for all $N \in \mathbb{N}^*$, $\lfloor N|q| \rfloor < N|q| < \lfloor N|q| \rfloor + 1$,

$$\sum_{k=0}^{N-1} \frac{1}{\lfloor N|q| \rfloor + 1 + k} \leq \sum_{k=0}^{N-1} \frac{1}{N|q| + k} \leq \sum_{k=0}^{N-1} \frac{1}{\lfloor N|q| \rfloor + k}.$$

and thus,

$$\log \left(\frac{\lfloor N|q| \rfloor + N}{\lfloor N|q| \rfloor + 1} \right) + o \left(\frac{1}{\lfloor N|q| \rfloor + N} \right) \leq \sum_{k=0}^{N-1} \frac{1}{N|q| + k} \leq \log \left(\frac{\lfloor N|q| \rfloor + N - 1}{\lfloor N|q| \rfloor} \right) + o \left(\frac{1}{\lfloor N|q| \rfloor + N - 1} \right).$$

The above inequality can be equivalently written as:

$$\log \left(\frac{\frac{\lfloor N|q| \rfloor}{N} + 1}{\frac{\lfloor N|q| \rfloor}{N} + \frac{1}{N}} \right) + o \left(\frac{\frac{1}{N}}{\frac{\lfloor N|q| \rfloor}{N} + 1} \right) \leq \sum_{k=0}^{N-1} \frac{1}{N|q| + k} \leq \log \left(\frac{\frac{\lfloor N|q| \rfloor}{N} + 1 - \frac{1}{N}}{\frac{\lfloor N|q| \rfloor}{N}} \right) + o \left(\frac{\frac{1}{N}}{\frac{\lfloor N|q| \rfloor}{N} + 1 - \frac{1}{N}} \right)$$

Since the sequence $\frac{\lfloor N|q| \rfloor}{N}$ converges to $|q|$, there exists $N_2 \in \mathbb{N}^*$, such that for all $N \geq N_2$,

$$\left| \log \left(\frac{\frac{\lfloor N|q| \rfloor}{N} + 1}{\frac{\lfloor N|q| \rfloor}{N} + \frac{1}{N}} \right) - \log \left(\frac{|q| + 1}{|q|} \right) \right| < \varepsilon.$$

Likewise, there exists $N_3 \in \mathbb{N}^*$, such that for all $N \geq N_3$,

$$\left| \log \left(\frac{\frac{\lfloor N|q| \rfloor}{N} + 1 - \frac{1}{N}}{\frac{\lfloor N|q| \rfloor}{N}} \right) - \log \left(\frac{|q| + 1}{|q|} \right) \right| < \varepsilon.$$

We therefore deduce that there exists $\tilde{N} := \max\{N_2, N_3\}$, such that for every integer $N \geq \tilde{N}$,

$$\left| \Psi_N(0, q) - \Psi_N(0, \tilde{q}) \right| \leq 2 \|H\|_\infty |q - \tilde{q}| \log \left(\frac{|q| + 1}{|q|} \right) + \varepsilon.$$

□

Theorem 3.5.4. *For any $\varepsilon > 0$ and all $z \in \mathcal{Z}$, there exists $N_{\varepsilon, z}$ such that for all $N \geq N_{\varepsilon, z}$,*

$$|V_N(z) - W(\mathbf{0})| < \varepsilon$$

Proof. For any $\varepsilon > 0$, we choose $\eta > 0$, such that

$$\begin{cases} \eta |\log(\eta)| < \frac{\varepsilon}{36 \|H\|_\infty}, \\ \eta < \frac{\varepsilon}{36 \log(2) \|H\|_\infty}, \end{cases}$$

and in view of Proposition 3.4.6, we also require η to be such that for all $q_0 \in \mathcal{Q}^*$ with $|q_0| = \eta$,

$$|W(0, q_0) - \ell| < \frac{\varepsilon}{3}.$$

Assumptions on the strategy sets and on the dynamics and running payoff functions of Theorem 3.7.1 in the appendix are established in $\mathcal{G}(0, q_0)$. Therefore, there exists

$\delta > 0$, such that for all $|\mathcal{P}| < \delta$, the value $W_{\mathcal{P}}$ converges uniformly on every compact set of \mathcal{Q} to W , as the mesh of the discretization $|\mathcal{P}|$ tends to 0. Fix $N_0 = \lfloor \frac{1}{\delta} \rfloor + 1$ and associate to $\mathcal{G}(0, q_0)$, for all $N \geq N_0$, a discrete time game adapted to the subdivision \mathcal{P}_N , denoted by $\mathcal{G}_{\mathcal{P}_N}(0, q_0)$. Then,

$$\left| W_{\mathcal{P}_N}(0, q_0) - W(0, q_0) \right| < \frac{\varepsilon}{3}.$$

By Proposition 3.3.3, $W_{\mathcal{P}_N}(0, q_0) = \Psi_N(0, q_0)$.

Notice that $|\mathcal{P}_N| = \frac{1}{N}$. Then, fix $N_1 := \lfloor \frac{1}{\eta} \rfloor + 1$. By Lemma 3.5.2, there exists $N_2 \in \mathbb{N}^*$, such that for every integer $N \geq \max\{N_1, N_2\}$,

$$\begin{aligned} \left| \Psi_N(0, q_0) - \Psi_N\left(0, \frac{e_{i_1 j_1}}{N}\right) \right| &\leq 2 \|H\|_{\infty} \eta \log\left(\frac{\eta+1}{\eta}\right) + \frac{2\varepsilon}{9} \\ &\leq 2 \|H\|_{\infty} \eta (\log(2) + |\log(\eta)|) + \frac{2\varepsilon}{9} \\ &< \frac{\varepsilon}{18} + \frac{\varepsilon}{18} + \frac{2\varepsilon}{9} \\ &< \frac{\varepsilon}{3} \end{aligned}$$

Therefore, there exists $\tilde{N} := \max\{N_0, N_1, N_2\}$, such that for every integer $N \geq \tilde{N}$,

$$\begin{aligned} \left| \Psi_N\left(0, \frac{e_{i_1 j_1}}{N}\right) - \ell \right| &\leq \left| \Psi_N\left(0, \frac{e_{i_1 j_1}}{N}\right) - \Psi_N(0, q_0) \right| + \left| \Psi_N(0, q_0) - W(0, q_0) \right| + \left| W(0, q_0) - \ell \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \varepsilon. \end{aligned}$$

By (3.2.1) and (3.3.3), for $(t, q) = (0, \frac{e_{i_1 j_1}}{N})$, we get:

$$\Psi_N\left(0, \frac{e_{i_1 j_1}}{N}\right) = \mathbf{V}_N(z_1),$$

where $z_1 = e_{i_1 j_1}$. Theorem 3.4.7 establishes that $\ell = W(\mathbf{0})$. Hence, for all $\varepsilon > 0$, there exists $\tilde{N} := \max\{N_0, N_1, N_2\}$, such that for all $N \geq \tilde{N}$,

$$\left| \mathbf{V}_N(z_1) - W(\mathbf{0}) \right| < \varepsilon.$$

Using Lemma 3.5.1 we conclude that for any given $z \in \mathcal{Z}$, $\mathbf{V}_N(z)$ converges to the value of the associated differential game, $W(\mathbf{0})$ as N tends to infinity. \square

3.6 Conclusion and perspectives

In this paper, we study the existence of the asymptotic value in a dynamic game with frequency-dependent payoffs. Although it is possible to express explicitly the

n -stage value, it seems technically difficult to study its asymptotic behavior using the form obtained. This leads us to develop a differential approach to study the asymptotic behavior of the value. We proceed as follows. We first highlight and prove a decisive property of the value function in the introduced FD -game that allows us to prove coincidence between the value of the auxiliary game and the value corresponding to the uniformly discretized differential game. More precisely, if we endow the state space \mathcal{Z} with the norm $\|\cdot\|_1$, the current time is distinctly involved in the norm of the state variable. We then show that for each fixed time, the value function is affine in the normalized state variable over the subset of \mathcal{Z} consisting of the states having norm equal to that fixed time. This property helps us to throw a bridge across the Shapley and Friedman ((DPP)) equations. In the former, the transition in the state space is realized under a certain probability while in the latter one the state variable of tomorrow is defined as the average of all possible states. We associate a continuous-time game, in which an irregularity occurs at initial state $q = \mathbf{0}$. This fact does not allow us to use immediately the powerful tools of the theory of differential games (see [Bardi and Capuzzo-Dolcetta \[2008\]](#)) in order to obtain existence of its value. Despite that irregularity, we prove existence of the value in the game starting at time $t = 0$ with initial state $q = \mathbf{0}$ and we then prove that the value of the original FD -game converges, as the number of stages tends to infinity, to the value of its associated differential game played over $[0, 1]$.

In the introduced model, the uniform value does not exist as established by [Contou-Carrère \[2011\]](#), who studied a FD -control problem that is a particular case of our model. The authors also show that convergence of the n -stage value is non-uniform. Consequently, as soon as the existence of the asymptotic value is established, it follows that convergence must be pointwise in the introduced FD -game. Consequently, we cannot use the results of [Ziliotto \[2015\]](#) in order to provide an answer on the convergence of the λ -discounted value as λ tends to 0. A first extension of our result may concern the λ -discounted value of the FD -game. Does it converge as λ tends to 0, to the same limit $W(\mathbf{0})$? That is, do we have $\lim_{n \rightarrow \infty} \mathbf{V}_n = \lim_{\lambda \rightarrow 0} \mathbf{V}_\lambda = W(\mathbf{0})$ in this class of games? A second possible extension would be to obtain an existence result in a FD -game with a payoff function that is assumed to be bilinear in the mixed action variables and affine in the state one, but non-additive in the sense the payoff function is in the model studied.

3.7 Appendix

In this section, we present the result on the approximation schemes of the value of a differential game as it is provided in [Bardi and Capuzzo-Dolcetta \[2008\]](#):

Theorem 3.7.1. *Assume that strategy spaces are compact sets, the running payoff and dynamics functions are bounded and continuous in all their variables and Lipschitz in the state variable $q \in \mathbb{R}^k$. Then, the value functions $W_{\mathcal{P}}^-$ and $W_{\mathcal{P}}^+$ converge uniformly on every compact set of \mathbb{R}^k as $|\mathcal{P}| \rightarrow 0^+$ to the lower and the upper value of the differential game W^- and W^+ respectively.*

Part II

Decompositions of finite games

Chapter 4

Introduction Partie II

Les résultats obtenus et présentés dans cette partie de la thèse ont comme objectif la décomposition de l'espace des jeux finis en somme directe orthogonale, pour un produit scalaire choisi, de sous espaces de jeux adéquats qui possèdent des équilibres distingués. Afin de présenter ces résultats décrivons auparavant brièvement les théorèmes classiques et les approches usuelles qui constituent la base de cette théorie.

4.1 Équilibres de Nash et équivalence stratégique

Un *jeu fini* est défini par un triplet $\mathbf{g} = (N, (S^i)_{i \in N}, (g^i)_{i \in N})$ où $N = \{1, \dots, n\}$ est l'ensemble des joueurs, S^i est un ensemble appelé *ensemble des stratégies* du joueur $i \in N$ et $g^i : S \rightarrow \mathbb{R}$ est la *fonction de paiement* du joueur i définie sur l'ensemble des *profils des stratégies* $S = \prod_{i \in N} S^i$ qui mesure le gain ou la perte que ce joueur tire de son interaction avec les autres joueurs. Un profil de stratégies est noté $\mathbf{s} = (s^i, s^{-i}) \in S$ où s^{-i} désigne les stratégies adoptées par l'ensemble des joueurs omis le joueur $i \in N$. À ces jeux, chaque joueur intervient une seule fois et ceci dans l'ignorance des décisions prises par les autres joueurs.

Un *équilibre de Nash* est un profil de stratégies pour lequel aucun des joueurs ne regrette d'avoir joué comme il l'a fait, étant donné la manière dont les autres joueurs ont joué. La notion de ϵ -équilibre de Nash est introduite dans [Nash et al. \[1950\]](#) comme concept de solution d'un jeu à somme non-nulle de la façon suivante: Étant donné $\epsilon > 0$, un profil de stratégies $\mathbf{s} = (s^i, s^{-i}) \in S$ est un ϵ -équilibre de Nash, si pour chaque $i \in N$ et tout $t^i \in S^i$,

$$g^i(\mathbf{s}) \geq g^i(t^i, s^{-i}) - \epsilon.$$

Lorsque $\epsilon = 0$, on dit qu'il s'agit d'un équilibre de Nash.

Dans la suite, le nombre des joueurs et les ensembles des leurs stratégies sont supposés finis et dans ce cas, on parlera d'*actions* (stratégies pures) plutôt que de stratégies.

En identifiant chaque jeu à sa fonction de paiement $\mathbf{g} = (g^i)_{i \in N}$, on constate que la classe des jeux finis à n joueurs, noté \mathcal{G} , est un espace vectoriel isomorphe à $\mathbb{R}^{n|S|}$.

Vu que l'équilibre de Nash est défini en termes des différences aux paiements, on peut introduire sur l'espace \mathcal{G} une relation d'équivalence de sorte que chaque classe d'équivalence soit constituée des jeux qui ont des équilibres identiques.

Définition 4.1.1. Les jeux finis $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{G}$ sont *stratégiquement équivalents* ($\mathbf{g}_1 \sim \mathbf{g}_2$), si pour chaque $i \in N$ et tout $s^{-i} \in S^{-i}$:

$$g_1^i(s^i, s^{-i}) - g_1^i(t^i, s^{-i}) = g_2^i(s^i, s^{-i}) - g_2^i(t^i, s^{-i}), \quad \forall s^i, t^i \in S^i.$$

Dans la littérature plusieurs notions d'équivalence sont employées, cf. [Moulin and Vial \[1978\]](#), [Morris and Ui \[2004\]](#), mais l'équivalence que nous adoptons, introduite par [Candogan et al. \[2011\]](#)¹, est l'adéquate pour les objectifs de notre étude. Dans ce but, les représentants convenables sont définis par la normalisation suivante:

Définition 4.1.2. Un jeu fini \mathbf{g} est *normalisé* si pour chaque $i \in N$ et tout $s^{-i} \in S^{-i}$,

$$\sum_{s^i \in S^i} g^i(s^i, s^{-i}) = 0.$$

L'espace des jeux normalisés est un sous espace vectoriel de l'espace des jeux, noté \mathcal{NO} .

Les jeux pour lesquels le paiement de chaque joueur dépend seulement des actions choisies par les autres joueurs, constituent une classe particulière de l'espace \mathcal{G} appelée *classe des jeux non-stratégiques*.

Définition 4.1.3. Un jeu fini \mathbf{g} est *non-stratégique* si, pour chaque $i \in N$, il existe une fonction $\ell^i : S^{-i} \rightarrow \mathbb{R}$ telle que $\ell^i(s^{-i}) = g^i(s^i, s^{-i})$.

L'espace des jeux non-stratégiques est un sous espace vectoriel de l'espace des jeux, noté \mathcal{NS} . Nous verrons que le sous espace des jeux non-stratégiques est le complément orthogonal du sous espace des jeux normalisés pour le produit scalaire standard dans l'espace \mathcal{G} .

Un jeu fini n'admet pas toujours un équilibre de Nash, mais l'existence d'un tel équilibre est assurée lorsque les joueurs procèdent de manière aléatoire, cf. [Nash et al. \[1950\]](#). Procéder de manière aléatoire signifie que chaque joueur attribue à ses actions une probabilité et dans ce cas on parle de *stratégie mixte*. Sur l'ensemble d'actions S^i de chaque joueur $i \in N$ est alors défini le simplexe des probabilités $\mathbf{P}(S^i)$ et dans ce cas les fonctions de paiements g^i sont étendues multi-linéairement sur $\mathbf{P}(S^i)$ et on parle de l'*extention mixte* du jeu. Le résultat fondamental concernant les jeux à somme non-nulle, dû à [Nash et al. \[1950\]](#), est :

Théorème 4.1.4. *Etant donné un jeu fini \mathbf{g} , son extention mixte admet un équilibre de Nash.*

¹Les jeux qui partagent la même correspondance de la meilleure réponse sont stratégiquement équivalents. La notion d'équivalence qui paraît dans [Candogan et al. \[2011\]](#) implique celles qui paraissent dans [Moulin and Vial \[1978\]](#), [Morris and Ui \[2004\]](#)

4.2 Décomposition des jeux statiques et équilibres approximatifs

Dans la littérature de la théorie des jeux apparaît depuis bien longtemps la question de la décomposition d'un jeu à des jeux plus simples dont l'analyse des équilibres s'avère plus souple. On trouve cette idée sous forme primitive dans [Von Neumann and Morgenstern \[1945\]](#) qui ont procédé à la décomposition d'un jeu avec un grand nombre des joueurs à des jeux ayant un petit nombre des joueurs. Depuis, plusieurs types de décomposition ont été proposées ayant leurs propres objectifs². Une décomposition de l'espace des jeux finis est établie dans [Sandholm \[2010\]](#) ; dans ce contexte, à chaque sous ensemble de joueurs est associé un jeu composant qui permet d'explorer sous une approche alternative si un jeu donné est ou non potentiel.

Les jeux potentiels, introduits dans [Monderer and Shapley \[1996\]](#), présentent un intérêt particulier car il existe une fonction qui exprime les déviations unilatérales des joueurs et alligne les paiements de tous les joueurs. Pour ces jeux, l'existence d'un équilibre de Nash en stratégies pures est établie.

Définition 4.2.1. Un jeu fini \mathbf{g} est un *jeu de potentiel* s'il existe une fonction $\varphi : S \rightarrow \mathbb{R}$, appelée *fonction potentielle*, telle que pour tout $i \in N$, $s^i, t^i \in S^i$ et $s^{-i} \in S^{-i}$ on a :

$$\varphi(s^i, s^{-i}) - \varphi(t^i, s^{-i}) = g^i(s^i, s^{-i}) - g^i(t^i, s^{-i}).$$

L'espace des jeux potentiels est un sous-espace vectoriel de l'espace des jeux, noté \mathcal{P} ; il contient un sous espace particulier de jeux définis comme suit :

Définition 4.2.2. Un jeu fini est *d'intérêt commun* si les paiements des joueurs sont identiques à chaque profil d'actions. Cet espace de jeux est noté \mathcal{CS} .

Dans [Kalai and Kalai \[2010\]](#), un jeu fini à deux joueurs se décompose en un jeu à somme nulle et un d'intérêt commun. Dans cette décomposition, le paiement sur chaque profil d'actions de la composante coopérative est défini comme la moyenne des paiements des joueurs et celui dans la composante compétitive comme la moitié de la différence des paiements des joueurs ; dans la première composante les intérêts des joueurs sont alignés et dans la deuxième sont conflictuels. Dans [Jessie and Saari \[2013\]](#) est proposée une décomposition des jeux, lorsque chaque joueur possède deux actions, à une composante *stratégique* et une de *comportement* ; dans ce travail, certains concepts de solution sont proposés issus de cette décomposition. Dans [Hwanga and Rey-Belleb \[2014\]](#) sont proposées certaines décompositions à des jeux potentiels et jeux qui sont stratégiquement équivalents à des jeux à somme-nulle ; Les auteurs établissent que chaque jeu potentiel qui est équivalent à un jeu à somme-nulle possède un équilibre en stratégies dominantes. Rappelons qu'un profil d'actions $\mathbf{s} = (s^1, \dots, s^n) \in S$ est un *équilibre en stratégies dominantes* si, pour tout joueur $i \in N$, on a :

$$g^i(s^i, t^{-i}) \geq g^i(t^i, t^{-i}), \quad \forall \mathbf{t} = (t^i, t^{-i}) \in S.$$

²[Kleinberg and Weiss \[1986\]](#), en mettant en lumière les propriétés des structures algébriques de l'espace des jeux coopératifs ont établi une décomposition d'un jeu coopératif à des jeux dont la structure est plus simple. Dans cette procédure, la valeur de Shapley admet une décomposition orthogonale à son noyau et son complémentaire. Cette approche ne simplifie pas forcément l'étude d'un jeu coopératif mais elle fournit des expressions alternatives de la valeur de Shapley.

Un des objectifs recherchés via une décomposition adéquate de l'espace des jeux est la caractérisation des équilibres approximatifs d'un jeu donné³. Notons que l'espace $\mathbb{R}^{|S|}$ est muni usuellement du produit scalaire standard :

$$\langle f_1, f_2 \rangle_0 = \sum_{\mathbf{s} \in S} f_1(\mathbf{s}) f_2(\mathbf{s}) \quad (4.2.1)$$

qui induit naturellement sur l'espace des jeux \mathcal{G} le produit scalaire :

$$\langle \mathbf{g}_1, \mathbf{g}_2 \rangle_{\mathcal{G}} = \sum_{i \in N} h_i \langle g_1^i, g_2^i \rangle_0. \quad (4.2.2)$$

Ainsi, sur l'espace des jeux \mathcal{G} est induite la norme :

$$\|\mathbf{g}\|_{\mathcal{G}}^2 = \langle \mathbf{g}, \mathbf{g} \rangle_{\mathcal{G}}, \quad (4.2.3)$$

et la quantification de la distance entre deux jeux finis est évaluée comme suit :

$$d(\mathbf{g}_1, \mathbf{g}_2) = \|\mathbf{g}_1 - \mathbf{g}_2\|_{\mathcal{G}}. \quad (4.2.4)$$

4.3 Quelques résultats de la Partie II

Dans un premier temps, la question qui nous intéresse dans cette partie de la thèse concerne la caractérisation des équilibres approximatifs d'un jeu fini arbitraire. Dans ce but, nous introduisons la classe des *jeux uniformes* et celle des *jeux non-interactifs*. Les jeux uniformes sont les jeux qui admettent le profil des stratégies uniformément mixtes comme équilibre de Nash. Les jeux non-interactifs constituent un sous espace vectoriel des jeux potentiels et se caractérisent par le fait que le paiement de chaque joueur dépend seulement de son action propre. Nous montrons tout d'abord qu'un tel jeu admet toujours un équilibre en stratégies dominantes avec un paiement unique.

Définition 4.3.1. Un jeu fini s'appelle *non-interactif* si, pour tout $i \in N$, il existe une fonction $\lambda^i : S^i \rightarrow \mathbb{R}$ telle que $\lambda^i(s^i) = g^i(s^i, s^{-i})$. L'espace de ces jeux est noté $\mathcal{N}\mathcal{I}$.

Définition 4.3.2. La *stratégie uniformément mixte* du joueur $i \in N$ est la stratégie mixte $x^i \in \mathbf{P}(S^i)$ selon que le joueur i choisit équiprobablement chaque action $s^i \in S^i$. Si tous les joueurs choisissent de stratégies uniformément mixtes alors le profil induit par ces stratégies s'appelle *profil de stratégies uniformes*.

Définition 4.3.3. Un *jeu uniforme* est un jeu fini qui admet le profil de stratégies uniformes comme équilibre de Nash. L'espace des jeux uniformes est noté \mathcal{U} .

Les jeux uniformes sont caractérisés par le fait que, pour chaque action fixé d'un joueur, la somme de ses paiements sur les profils d'actions formés par ses adversaires est constante. Lorsque cette constante est la même pour tous les joueurs et si elle est

³Une telle étude des équilibres approximatifs est élaborée dans Candogan et al. [2011].

égale à zéro alors on parle de jeux uniformes de constante nulle et leur sous-espace est noté \mathcal{U}_0 . Les stratégies uniformément mixtes peuvent être considérées comme modèle de *rationalité bornée* selon Rubinstein [1998]. Notons que dans la classe des jeux uniformes sont incorporés les *jeux des groupes* introduits dans Capraro and Scarsini [2013].

Dans notre travail, afin d'établir une décomposition adéquate de l'espace des jeux finis et étudier les équilibres approximatifs, nous considérons le sous-espace de jeux uniformes de constante nulle, le sous-espace des jeux constants noté \mathcal{C} , (i.e., le paiement de chaque joueur est le même sur chaque profil d'actions), et le sous-espace des jeux non-interactifs normalisés. Si X et Y sont des sous-espaces de l'espace des jeux \mathcal{G} , notons $X \oplus Y$ leur somme orthogonale pour le produit scalaire introduit par (5.2.2) sur l'espace \mathcal{G} . Notre résultat s'énonce ainsi :

Théorème 4.3.4. *L'espace des jeux est la somme orthogonale, pour le produit scalaire introduit par (5.2.2), des sous-espaces des jeux uniformes de constante nulle, des jeux constants et des jeux non-interactifs normalisés :*

$$\mathcal{G} = \mathcal{U}_0 \oplus \mathcal{C} \oplus (\mathcal{N}\mathcal{I} \cap \mathcal{N}\mathcal{O})$$

Une conséquence de ce théorème est la suivante :

Corollaire 4.3.5. *Étant donné un jeu fini \mathbf{g} tout équilibre en stratégies dominantes qui paraît dans la composante non-interactive est un ϵ -équilibre de \mathbf{g} pour*

$$\epsilon \leq \max_{i \in N} (2/\sqrt{h_i}) \|\mathbf{g}_{\mathcal{U}_0}\|_{\mathcal{G}}$$

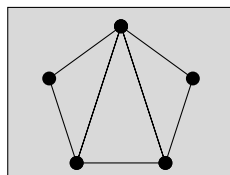
et le profil uniformément mixte est un $\tilde{\epsilon}$ -équilibre de \mathbf{g} , pour

$$\tilde{\epsilon} \leq \max_{i, s^i} \{\lambda^i(s^i)\}$$

où $\lambda^i(s^i)$ est le paiement du joueur i pour son action s^i dans la composante non-interactive.

4.4 Sur la décomposition de Helmholtz-Hodge pour les graphes

La décomposition d'un champ de vecteurs comme somme du *gradient d'un champ scalaire* et du *rotationnel d'un champ vectoriel* introduite par Helmholtz-Hodge, est un outil mathématique très puissant sous l'aspect de la théorie de Hodge. Les applications de cette théorie sont nombreuses, cf. Bartholdi et al. [2012] pour les espaces métriques et Jiang et al. [2011] pour le classement statistique. Nous nous intéressons à l'utilisation de cette théorie dans le cas des graphes non-orientés, telle qu'elle est adaptée aux jeux finis dans Candogan et al. [2011].



Soit $G = (V, E)$ un graphe non-orienté où V et E désignent respectivement les ensembles des noeuds et des arrêtes. L'ensemble des triangles de ce graphe est noté $T = \{(u, v, w) : (u, v), (v, w), (w, u) \in E\}$. L'ensemble des fonctions réelles définies sur les noeuds du graphe, $f : V \rightarrow \mathbb{R}$ est noté C_0 . L'ensemble des flôts sur ce graphe, noté C_1 , est constitué des formes bilinéaires antisymétriques $X : V \times V \rightarrow \mathbb{R}$ représentées par les matrices antisymétriques via le choix d'une base ; pour tout $u, v \in V$ on a $X(u, v) = -X(v, u)$ et $X(u, v) = 0$ si $(u, v) \notin E$. L'ensemble de tous les flôts triangulaires noté C_2 , est constitué des fonctions : $\Psi : V \times V \times V \rightarrow \mathbb{R}$ définies par

$$\Psi(u_\theta, v_\theta, w_\theta) = \begin{cases} \text{sign}(\theta)\Psi(u, v, w), & \text{si } (u, v, w) \in T \\ 0, & \text{sinon} \end{cases}$$

où θ est une permutation de trois éléments. On constate que C_0, C_1 et C_2 possèdent une structure naturelle d'espace vectoriel pour les opérations usuelles de l'addition et de la multiplication par un scalaire. Considérons la fonction indicatrice $W : V \times V \rightarrow \{0, 1\}$,

$$W(u, v) = \begin{cases} 1, & \text{si } (u, v) \in E \\ 0, & \text{sinon.} \end{cases}$$

L'opérateur gradient $\delta_0 : C_0 \rightarrow C_1$ est défini pour tout $f \in C_0$ par

$$(\delta_0 f)(u, v) = W(u, v)(f(v) - f(u)), \quad u, v \in V. \quad (4.4.1)$$

L'opérateur curl $\delta_1 : C_1 \rightarrow C_2$ est défini pour tout $X \in C_1$ par

$$(\delta_1 X)(u, v, w) = \begin{cases} X(u, v) + X(v, w) + X(w, u), & \text{si } (u, v, w) \in T \\ 0, & \text{sinon.} \end{cases} \quad (4.4.2)$$

Les espaces vectoriels C_0, C_1, C_2 connectés par les opérateurs δ_0 et δ_1 , vu que $\delta_1 \circ \delta_0 = 0$, donnent naissance à un *complexe de co-chaînes* :

$$0 \rightarrow C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2 \rightarrow 0.$$

Pour $k \in \{0, 1\}$, étant donné un produit scalaire dans $\langle \cdot, \cdot \rangle_k$ dans C_k , on définit les opérateurs adjoints respectifs aux opérateurs gradient et curl, notés δ_0^* et δ_1^* :

$$\langle \delta_k f_k, g_{k+1} \rangle_{k+1} = \langle f_k, \delta_k^* g_{k+1} \rangle_k.$$

Les espaces vectoriels C_0, C_1, C_2 , connectés par les opérateurs δ_0^* et δ_1^* , vu que $\delta_0^* \circ \delta_1^* = 0$, donnent naissance à un *complexe des chaînes* :

$$0 \leftarrow C_0 \xleftarrow{\delta_0^*} C_1 \xleftarrow{\delta_1^*} C_2 \leftarrow 0.$$

La *laplacienne* k -dimensionnelle est l'opérateur $\Delta_k : C_k \rightarrow C_k$ défini par :

$$\Delta_k = \delta_k^* \circ \delta_k + \delta_{k-1} \circ \delta_{k-1}^*.$$

La décomposition de Helmholtz-Hodge sur les graphes non-orientés s'exprime ainsi :

Théorème 4.4.1. *L'espace des flôts C_1 sur un graphe non-orienté admet la décomposition orthogonale suivante :*

$$C_1 = \text{Im}(\delta_0) \oplus \text{Ker}(\Delta_1) \oplus \text{Im}(\delta_0^*).$$

De plus,

$$\text{Ker}(\Delta_1) = \text{Ker}(\delta_1) \cap \text{Ker}(\delta_0^*). \quad (4.4.3)$$

L'équation de Laplace est définie par $\Delta_0 f = 0$ et les fonctions qui vérifient cette équation s'appellent *fonctions harmoniques*. Les *flôts harmoniques* $X \in C_1$ sont définis par l'équation $\Delta_1 X = 0$ qui est connue dans la littérature comme *équation de Helmholtz*. Étant donné $X \in C_1$, s'il existe $f \in C_0$ telle que $X = \delta_0 f$ alors f est une *fonction de potentielle* pour X qui s'appelle *flôt potentiel*. Les flôts qui vérifient l'équation $\delta_1 X = 0$ s'appellent *flôt localement consistants*. Or $\delta_1 \circ \delta_0 = 0$, les flôts potentiels sont localement consistants ; mais, les flôts localement consistants ne sont pas toujours potentiels vu que la topologie du graphe n'est pas triviale. Ainsi, les éléments du $\text{ker}(\delta_1)$ sont les flôts localement consistants qui peuvent être ou ne pas être potentiels.

4.5 Décomposition des jeux finis et leurs graphes associés

À chaque jeu fini est associé un graphe⁴ dont les sommets correspondent aux profils d'actions et dont les arrêtes relient les couples des profils d'actions entre lesquels a lieu une déviation unilatérale. Cette procédure est développée dans [Candogan et al. \[2011\]](#).

Désignons le graphe d'un jeu arbitraire par $\Gamma = (S, A)$ où S sont ses sommets et A ses arrêtes. On peut ainsi définir un flôt sur ce graphe en attribuant un nombre réel à chaque arrête qui est égal à la différence des paiements du joueur qui dévie entre les couples d'actions correspondants. Étant donné un chemin sur ce graphe, on appelle *flôt total* sur ce chemin, la somme des valeurs sur les arrêtes parcourues par le chemin. Le graphe d'un jeu et son flôt contiennent et préservent toute l'information nécessaire pour la détermination des équilibres de Nash. Aussi, la représentation du flôt sur le graphe d'un jeu fini permet d'établir les caractéristiques fondamentales des préférences des joueurs et en particulier il permet la caractérisation suivante des jeux de potentiel dû à [Monderer and Shapley \[1996\]](#) :

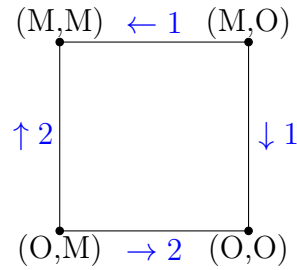
Théorème 4.5.1. *Un jeu fini est de potentiel si et seulement si son flôt total sur chaque chemin de longueur 4 de son graphe associé est égal à zéro.*

Exemple 4.5.2. (La bataille des sexes) Un couple décide de faire une sortie. L'homme (joueur 1) préfère aller à un match de foot et la femme (joueur 2) préfère aller à l'Opéra. Pour chacun, sortir avec son conjoint est plus important que le lieu où ils vont aller. Les deux joueurs doivent choisir le Match de foot ou l'Opéra. Les paiements sont indiqués dans la case qui correspond au couple d'actions choisies.

⁴Des idées relatives à propos du graphe associé à un jeu fini sont donnés dans [Goemans et al. \[2005\]](#) et [Christodoulou et al. \[2012\]](#), pour étudier différents concepts des solutions dans la théorie des jeux.

	M	O
M	$(2, 1)$	$(0, 0)$
O	$(0, 0)$	$(1, 2)$

Dans ce jeu, il y a deux équilibres de Nash et les profils d'actions (M, M) et (O, O) . Il s'agit d'un jeu de potentiel dont la fonction de potentiel est définie par $\varphi(\mathbf{s}) = 3$ si $\mathbf{s} = (M, M)$ et $\mathbf{s} = (O, O)$, $\varphi(\mathbf{s}) = 2$ si $\mathbf{s} = (O, M)$ et $\varphi(\mathbf{s}) = 1$ si $\mathbf{s} = (M, O)$. Si les joueurs sont autorisés d'utiliser de manière aléatoire leurs actions, alors un nouveau équilibre de Nash fait son apparition dont le profil des stratégies mixtes est donné par les vecteurs de probabilité $(\frac{2}{3}, \frac{1}{3})$ et $(\frac{1}{3}, \frac{2}{3})$. Le graphe et le flôt associé à ce jeu sont présentés ci-dessous et puisque il s'agit d'un jeu de potentiel le flôt total est zéro :



Aux graphes associés aux jeux finis, les seuls triangles qui apparaissent sont dûs aux déviations successives du même joueur. Vu la définition (4.4.2) de l'opérateur curl, les flôts sur les graphes associés aux jeux appartiennent au $\text{Ker}(\delta_1)$; ainsi vu (4.4.3) les flôts harmoniques associés à ces graphes sont contenus au $\text{Ker}(\delta_0^*)$. Bien évidemment, la définition suivante qui caractérise les jeux harmoniques dépend du produit scalaire par lequel est muni l'espace C_0 .

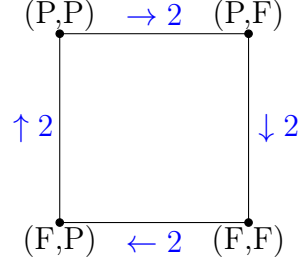
Définition 4.5.3. Un flôt X sur le graphe associé à un jeu fini s'appelle *harmonique* si $X \in \text{Ker}(\delta_0^*)$. Les jeux dont le flôt est harmonique s'appellent *jeux harmoniques*.

Si C_0 est muni du produit scalaire standard, cette définition s'interprète par le fait qu'un jeu fini est harmonique si et seulement si à tout noeud de son graphe la somme des flôts entrants et des flôts sortants est égale à zéro. Dans ce cas, les jeux harmoniques sont de jeux à somme nulle lorsque les joueurs possèdent le même nombre d'actions, cf. Candogan et al. [2011]. L'espace des jeux harmoniques est un sous espace vectoriel des jeux, noté \mathcal{H} ; il contient de jeux célèbres comme *Pile ou Face* et *Pierre-Feuille-Ciseaux*.

Exemple 4.5.4. (Version du jeu Pile ou Face) Deux joueurs, le premier cherchant à attraper le second, ont le choix entre aller à la **C**averne ou à la **F**orêt ; autrement dit, chacun des joueurs choisit un élément dans l'ensemble $I = J = \{M, C\}$. Si le premier joueur attrape le second alors il gagne un point ; sinon il perd un point et réciproquement. Ce jeu est représenté par la matrice ci-dessous :

	C	F
C	$+1$	-1
F	-1	$+1$

Dans ce jeu, le joueur 1 cherche à maximiser son paiement et le joueur 2 à le minimiser. À chaque profil d'actions, il y a un joueur qui a intérêt à dévier de son action choisie. L'unique équilibre de Nash est en stratégies mixtes et il est défini par les vecteurs de probabilité $(\frac{1}{2}, \frac{1}{2})$ et $(\frac{1}{2}, \frac{1}{2})$. Si l'espace C_0 est muni du produit scalaire standard alors ce jeu est harmonique, cela résulte de son graphe et de son flôt représentés ci-dessous :



Inspirés de la décomposition de Helmholtz-Hodge, Candogan et al. [2011] fournissent une décomposition de l'espace des jeux finis en somme directe orthogonale pour le produit scalaire standard dont les composantes des jeux potentiels, harmoniques et non-stratégiques :

Théorème 4.5.5. *L'espace des jeux est la somme directe orthogonale pour le produit scalaire défini par (5.2.2) :*

$$\mathcal{G} = (\mathcal{P} \cap \mathcal{NO}) \oplus (\mathcal{H} \cap \mathcal{NO}) \oplus \mathcal{NS}.$$

Notons que le sous espace des jeux potentiels résulte de la somme du sous espace des jeux potentiels normalisés et de sous espace des jeux non-stratégiques. Du théorème précédent découle la caractérisation des équilibres approximatifs d'un jeu fini via les équilibres purs de la composante potentielle qui apparaît dans cette décomposition :

Corollaire 4.5.6. *Soit un jeu fini \mathbf{g} et \mathbf{g}_P sa composante potentielle. Tout $\tilde{\epsilon}$ -équilibre de \mathbf{g}_P est un ϵ -équilibre de \mathbf{g} pour un certain $\epsilon \leq \max_{i \in N} \frac{2\|\mathbf{g} - \mathbf{g}_P\|_{\mathcal{G}}}{\sqrt{h_i}}$ où h_i est le nombre d'actions du joueur $i \in N$.*

4.6 Résultats de la Partie II

Dans cette partie, nous introduisons sur l'espace des jeux finis \mathcal{G} une famille de produits scalaires. Étant donné un jeu fini, nous choisissons dans cette famille un produit scalaire approprié dans le but d'établir une décomposition orthogonale avantageuse de l'espace \mathcal{G} qui permettra l'obtention d'une *bonne* approximation des équilibres du jeu donné via les équilibres distincts de ses composantes qui résultent de sa projection sur les sous espaces de cette décomposition. La procédure que nous suivons pour obtenir cet objectif est basée sur la décomposition de Helmholtz-Hodge

pour les graphes non-orientés sous l'aspect d'une famille des produits scalaires liée à la famille de produits scalaires introduite sur l'espace des jeux.

Soit un graphe non-orienté $G = (V, E)$ et une fonction $\mu : V \rightarrow \mathbb{R}$. Nous introduisons une famille de produits scalaires sur les espaces C_0, C_1, C_2 , indexée par μ qui est définie comme suit :

$$\langle f_1, f_2 \rangle_0 = \sum_{v \in V} \mu(v) f_1(v) f_2(v), \quad (4.6.1)$$

$$\langle X, Y \rangle_1 = \frac{1}{2} \sum_{(u,v) \in E} \mu(u) \mu(v) X(u, v) Y(u, v), \quad (4.6.2)$$

$$\langle \Psi, Z \rangle_2 = \frac{1}{6} \sum_{(u,v,w) \in T} \mu(u) \mu(v) \mu(w) \Psi(u, v, w) Z(u, v, w). \quad (4.6.3)$$

Étant donnés les produits scalaires $\langle \cdot, \cdot \rangle_k$ sur $C_k, k \in \{0, 1\}$, vu les définitions données dans ((4.4.1), (4.4.2)) l'adjoint de $\delta_k : C_k \rightarrow C_{k+1}$, noté $\delta_k^* : C_{k+1} \rightarrow C_k$, est l'opérateur linéaire qui est défini par la relation :

$$\langle \delta_k f_k, F_{k+1} \rangle_{k+1} = \langle f_k, \delta_k^* F_{k+1} \rangle_k, \quad \forall f_k \in C_k, F_{k+1} \in C_{k+1}. \quad (4.6.4)$$

Alors, les adjoints respectifs des δ_0 et δ_1 sont donnés par :

$$\delta_0^* X(u) = - \sum_{v:(u,v) \in E} \mu(v) X(u, v) \quad \text{et} \quad \delta_1^* \Psi(u, v) = \sum_{w \in V:(u,v,w) \in T} \mu(w) \Psi(u, v, w).$$

Théorème 4.6.1. *L'espace des flôts C_1 admet une décomposition orthogonale pour le produit scalaire introduit par (4.6.2) :*

$$C_1 = \text{Im}(\delta_0) \oplus \ker(\Delta_1) \oplus \text{Im}(\delta_1^*), \quad (4.6.5)$$

où

$$\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*).$$

Rappelons l'interprétation des composantes de la décomposition de Helmholtz-Hodge :

- $\text{Im}(\delta_0)$: flôts potentiels.
- $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$: flôts harmoniques qui ne sont pas potentiels mais localement consistants. Notons que $\ker(\delta_1)$ est constitué des flôts localement consistants qui peuvent être ou ne pas être potentiels et $\ker(\delta_0^*)$ est constitué des flôts qui ne sont pas potentiels et qui peuvent être ou ne pas être localement consistants.
- $\text{Im}(\delta_1^*)$: flôts qui ne sont pas localement consistants (car $\text{Im}(\delta_1^*) = \ker(\delta_1)^\perp$).

Ceci étant, nous passons dorénavant aux graphes associés aux jeux finis. Étant donné un jeu fini, associons à chaque joueur $i \in N$ la fonction $\mu^i : S^i \rightarrow \mathbb{R}$ et notons que pour tout $\mathbf{s} \in S$ on a : $\mu(\mathbf{s}) = \prod_{i \in N} \mu^i(s^i)$.

Afin de fixer les représentants adéquats des jeux stratégiquement équivalents relativement à μ , nous introduisons les *jeux μ -normalisés* comme suit :

Définition 4.6.2. Un jeu fini \mathbf{g} est *μ -normalisé* si, pour chaque $i \in N$ et tout $s^{-i} \in S^{-i}$, on a :

$$\sum_{s^i \in S^i} \mu^i(s^i) g^i(s^i, s^{-i}) = 0.$$

L'espace de jeux μ -normalisés est un sous espace vectoriel de l'espace des jeux \mathcal{G} , noté \mathcal{NO} .

Théorème 4.6.3. *L'espace des jeux finis est la somme directe de jeux μ -normalisés et de jeux non-stratégiques :*

$$\mathcal{G} = \mathcal{NO} \oplus \mathcal{NS}.$$

L'opérateur gradient partiel associé au joueur $i \in N$ est noté $\delta_0^i : C_0 \rightarrow C_1$ et est défini, pour tout $\mathbf{s}, \mathbf{t} \in S$ comme suit :

$$\delta_0^i(g^i)(\mathbf{s}, \mathbf{t}) = W^i(\mathbf{s}, \mathbf{t})(g^i(\mathbf{t}) - g^i(\mathbf{s})). \quad (4.6.6)$$

L'opérateur gradient δ_0 , cf. (4.4.1), satisfait $\delta_0 = \sum_{i \in N} \delta_0^i$. On associe au jeu \mathbf{g} l'opérateur gradient commun $D : C_0^n \rightarrow C_1$ défini comme suit :

$$D(\mathbf{g}) = \sum_{i \in N} \delta_0^i g^i. \quad (4.6.7)$$

Dans ce contexte, vu la Définition 4.1.1, on a : $\mathbf{g}_1 \sim \mathbf{g}_2$ si et seulement si $D(\mathbf{g}_1) = D(\mathbf{g}_2)$.

Les espaces vectoriels C_0^n et C_1, C_2 , connectés par les opérateurs D et δ_1 , constituent un complexe de co-chaînes :

$$0 \rightarrow C_0^n \xrightarrow{D} C_1 \xrightarrow{\delta_1} C_2 \rightarrow 0.$$

Afin de construire la famille de produits scalaires qui assurent l'orthogonalité de la décomposition de l'espace des jeux finis, notons pour chaque $i \in N$ et tout $s^{-i} \in S^{-i}$,

$$\lambda^i(s^{-i}) = \sqrt{\mu^{-i}(s^{-i})\mu^i(S^i)} \quad \text{et} \quad \mu^i(S^i) = \sum_{s^i \in S^i} \mu^i(s^i).$$

Pour chaque $i \in N$, on a un automorphisme sur C_0 défini par l'opérateur $M^i : C_0 \rightarrow C_0$ où

$$(M^i f)(\mathbf{s}) = \lambda^i(s^{-i})f(\mathbf{s}), \quad (4.6.8)$$

Il en résulte l'opérateur $M : C_0^n \rightarrow C_0^n$ qui est défini par : $M\mathbf{g} = (M^1 g^1, \dots, M^n g^n)$. Les espaces vectoriels C_0^n et C_1, C_2 connectés par les opérateurs D^* et δ_1^* constituent un complexe de chaînes :

$$0 \leftarrow C_0^n \xleftarrow{D^*} C_1 \xleftarrow{\delta_1^*} C_2 \leftarrow 0.$$

On définit maintenant sur l'espace \mathcal{G} le produit scalaire :

$$\langle \mathbf{g}_1, \mathbf{g}_2 \rangle_{\mathcal{G}} = \sum_{i \in N} \langle M^i g_1^i, M^i g_2^i \rangle_0, \quad (4.6.9)$$

où dans le membre de droite intervient le produit scalaire qui est défini sur C_0 par (4.4.1).

Définition 4.6.4. Un jeu fini \mathbf{g} s'appelle *jeu harmonique* si

$$D(\mathbf{g}) \in \text{Ker}(\delta_0^*).$$

Nous énonçons maintenant les résultats principaux qui répondent aux problèmes posés dans cette partie de la thèse :

Théorème 4.6.5. *Si \mathbf{g} un jeu harmonique, le profil complètement mixte $\eta = (\eta^i)_{i \in N}$ qui est défini pour chaque $i \in N$ et tout $s^i \in S^i$, comme suit est un équilibre de Nash dans \mathbf{g} :*

$$\eta^i(s^i) = \frac{(\mu^i(s^i))^2}{\sum_{s^i \in S^i} (\mu^i(s^i))^2}. \quad (4.6.10)$$

Théorème 4.6.6. *L'espace des jeux finis muni du produit scalaire introduit dans (4.6.9) est la somme directe orthogonale des sous espaces des jeux potentiels μ -normalisés, des jeux harmoniques μ -normalisés et des jeux non-stratégiques :*

$$\mathcal{G} = (\mathcal{N} \mathcal{O} \cap \mathcal{P}) \oplus (\mathcal{N} \mathcal{O} \cap \mathcal{H}) \oplus \mathcal{N} \mathcal{S}. \quad (4.6.11)$$

Dans la suite en parlant des jeux matriciels nous utilisons les notations pour les ensembles d'actions respectifs des deux joueurs $I = \{1, \dots, m\}$ et $J = \{1, \dots, n\}$ et un tel jeu sera souvent noté (A, B) où $A, B \in \mathbb{R}^{m \times n}$. Étant donné un jeu matriciel $\mathbf{g} = (g^1, g^2)$, d'après [Aumann \[1987\]](#), une distribution de probabilité $\mathbf{z} \in \mathbf{P}(I \times J)$ est un équilibre corrélé si,

$$\sum_{j \in J} (g^1(i, j) - g^1(i_0, j))z(i, j) \geq 0, \quad \forall i, i_0 \in I$$

et,

$$\sum_{i \in I} (g^2(i, j) - g^2(i, j_0)) z(i, j_0) \geq 0, \quad \forall j, j_0 \in J.$$

On en déduit que chaque équilibre de Nash mixte est un équilibre corrélé où la distribution correspondante $(\mathbf{x}, \mathbf{y}) \in \mathbf{P}(I) \times \mathbf{P}(J) \subset \mathbf{P}(I \times J)$ est la distribution produit, i.e., pour tout $(i, j) \in I \times J$, on a $z(i, j) = x(i)y(j)$. De plus, les équilibres corrélés sont indépendants de la composante non-stratégique.

Théorème 4.6.7. *Soit (A, B) un jeu matriciel harmonique et $\mathbf{z} \in \mathbf{P}(I \times J)$ un équilibre corrélé dans ce jeu. Génériquement, la famille des stratégies mixtes $(x_{i_0})_{i_0 \in I}$ et $(y_{i_0})_{i_0 \in I}$, définies respectivement pour tout $i \in I$ par $x_{i_0}(i) = \frac{Z_{i_0}}{\alpha_i}$ pour un certain $\alpha_i \in \mathbb{R}$ et pour tout $j \in J$ par $y_{i_0}(j) = \frac{z(i_0, j)}{Z_{i_0}}$ où $Z_{i_0} = \sum_{j \in J} z(i_0, j)$, sont des équilibres mixtes dans ce jeu.*

La procédure que nous avons suivie s'offre pour affronter ce même problème dans le contexte plus étendu des jeux dénombrables, i.e., des jeux qui ont des espaces d'actions dénombrables. Or, la procédure des Candogan et al. [2011], basée sur une extension naturelle du produit scalaire standard sur l'espace des jeux finis, ceci induit le profil des stratégies uniformes comme un équilibre de Nash dans la composante harmonique. Mais, la mesure uniforme n'est pas σ -additive et alors il n'est pas possible d'étendre leur approche aux jeux dénombrables. Cependant, en travaillant sur l'espace des suites de carré sommables et en suivant notre approche, les résultats sur la décomposition peuvent être généralisés aux jeux dénombrables.

Chapter 5

Uniformity and games decomposition

Abstract¹. We introduce the classes of uniform and non-interactive games. We study appropriate projection operators over the space of games in order to propose a novel canonical direct sum decomposition of an arbitrary game into three components, which we refer to as the uniform with zero-constant, the non-interactive total-sum zero and the constant components. With respect to the standard inner product, we show that orthogonality holds between the components and we provide explicit expressions for the closest uniform and non-interactive games to a given game. We then characterize the set of its approximate equilibria in terms of the uniformly mixed and dominant strategies equilibria profiles of its closest uniform and non-interactive games respectively.

5.1 Introduction

Several approaches to decompose a game into simpler games which admit more tractable equilibrium analysis have been developed so far in the literature. The study of components with distinct equilibrium properties allows to gain insights on the static and dynamic features of an arbitrary game.

The class of uniform games

The mixed strategy according to which each action is selected with equal probability is called *uniform strategy* and the profile it induces, a *uniformly mixed strategy profile*. *Uniform games* are strategic-form finite games in which the uniformly mixed strategy profile appears as a Nash equilibrium. In other words, each player's sum of payoffs over the strategy profiles chosen by the rest of the players is constant. Well-known games such as Matching Pennies and Rock-Paper-Scissors are uniform games. Since playing a uniform strategy is arguably the simplest way of mixing pure strategies, uniformly mixed strategy equilibria can be viewed as somewhat lying in between pure and mixed Nash equilibria. Uniform strategies are also easier to implement and may therefore be seen as a model of *bounded rationality* (see [Rubinstein \[1998\]](#)). *Group games*, introduced by [Capraro and Scarsini \[2013\]](#) belong

¹Ce chapitre est issu de l'article "Uniformity and games decomposition" en collaboration avec Joseph Abdou et Marco Scarsini.

to the class of uniform games. Likewise, *harmonic games* due to Candogan et al. [2011] are uniform.

The class of potential games and the non-interactive subspace

Potential games became a tractable research topic since the seminal paper of Monderer and Shapley [1996] due to their desirable equilibrium properties, i.e., existence of a pure strategy Nash equilibrium. The structure of these games and the convergence of simple player dynamics to a Nash equilibrium (see Neyman [1997]) qualified them to play an important role in game-theoretic analysis. Consequently, such games found numerous applications in various control and resource allocation problems. They can be regarded as games in which the interests of the players are aligned with a global potential function.

We introduce a particular subclass of potential games, the *non-interactive games*. In these games, each player's payoff depends only on his selected strategy. This class forms a subspace of potential games having a dominant strategy equilibrium with unique payoff, which results when each player plays the action that maximizes his payoff. Therefore in this class of games players do not interact with each other and every decision is taken ignoring the opponents' moves.

Historical review on the decomposition of games

The idea of decomposing a game appears first in Von Neumann and Morgenstern [1945], where games with large number of players are decomposed into games with fewer players.

In the context of non-cooperative game theory, a decomposition for games in normal form appeared in Sandholm [2010]. The authors define a component game for each subset of players and obtains a decomposition of normal form games suggesting alternative ways to verify whether a game is potential or not. Kalai and Kalai [2010] propose a decomposition into zero-sum and common interest components for a given two-player game. Hence, the authors highlight cooperation-related issues that emerge in games where players strategically interact with each other. More precisely, in the cooperative component the payoff is defined at each strategy profile as the average of both players' payoffs while in the competitive one as the difference of players' payoffs divided by two. In general, the common interest component is a potential game which can be used to approximate a given game. However, this approximation needs not be the closest potential game to a given game. In Candogan et al. [2011], the authors provide example in which even though the original game is a potential game, the zero-sum and common interest game decomposition may lead to a potential game which is much farther away than the closest potential game. The key point lies in the fact that the Candogan et al. [2011] decomposition clearly separates the strategic and non strategic components and further identifies components such as potential and harmonic components. To that point the authors propose a novel flow representation of the payoff structure in finite games which enables to identify the fundamental characteristics in preferences that lead to potential

games. Their main result relies on the *Helmholtz flow-decomposition*², a well-known mathematical tool of vector calculus that allows one to represent a vector field as the sum of a *divergence-free* vector field and a *curl-free* field. The authors provide a direct orthogonal sum decomposition of an arbitrary game into a *potential*, an *harmonic* and a *non-strategic* components, each with distinct equilibrium properties. For instance, harmonic games do not have pure Nash equilibria and belong to the uniform class of games.

Jessie and Saari [2013] provide a strategic and behavioral decomposition of games in which all players have two actions. Their goal is to highlight that certain solution concepts are either determined by a game's strategic part, or influenced by its behavioral component. Hwanga and Rey-Belletb [2014] provide decomposition results of an arbitrary finite game by identifying components such as potential games and games that are strategically equivalent to zero-sum games and they end up proving an alternative proof of the characterization for potential games appeared in Monderer and Shapley [1996]. The authors prove analogous results in games with continuous strategy sets.

Main results

We provide a novel direct sum decomposition of an arbitrary game into three components, which we refer to as the *uniform with zero constant*, the *non-interactive total sum zero* and the *constant* components. The former component refers to games in which each player's sum of payoffs over the strategy profiles chosen by the rest of the players amounts to zero. The second class refers to non-interactive games, in which each player's sum of payoffs over all strategy profiles is equal to zero. The latter component describes games in which each player's payoff is constant over all strategy profiles. Under the standard inner product, these components are orthogonal and, using the induced distance of a given game to its closest uniform and non interactive games, we characterize the approximate equilibrium set of an arbitrary game in terms of the dominant strategy equilibria with unique payoff of its closest non-interactive game and the uniformly mixed strategy equilibrium of its closest uniform game. A dominant strategy equilibrium profile of the corresponding non-interactive game turns out to be an ε -equilibrium of the original game, for some ε whose upper bound is specified. Similarly, the uniformly mixed strategy equilibrium profile of the projected game onto the uniform component is an $\tilde{\varepsilon}$ -equilibrium of the arbitrary game for some $\tilde{\varepsilon}$ which is bounded by a constant that is determined as well.

Structure of the paper

The remainder of the paper is organized as follows. In Section 5.2 we present the relevant definitions and notations. In Section 5.3 we introduce the classes of uniform and non-interactive games and present the main results of this work. Section 5.4

²An application of the Helmholtz decomposition in the context of Statistical ranking is given in Jiang et al. [2011].

deals with examples in which the proposed decomposition is useful. Section 5.5 concludes the paper and outlines future perspectives.

5.2 Basic definitions and notations

A *finite game* consists of:

- a finite set of players, denoted by $N = \{1, \dots, n\}$.
- a finite set of strategies S^i , for every $i \in N$. The joint strategy space is $S = \prod_{i \in N} S^i$.
- a payoff function of each player $i \in N$, denoted by $g^i : S \rightarrow \mathbb{R}$.

Accordingly, a finite game is given by the triplet $(N, (S^i)_{i \in N}, (g^i)_{i \in N})$, which for notational convenience will often be abbreviated to \mathbf{g} , where $\mathbf{g} = (g^i)_{i \in N}$. We use the notation $s^i \in S^i$ for an action of player i . An action profile is given by $\mathbf{s} = (s^i)_{i \in N}$. A collection of actions for all players but the i -th one, is denoted by $s^{-i} \in S^{-i}$. We use $h_i = |S^i|$ for the cardinality of the action space of player i , and $|S| = \prod_{i \in N} h_i$ for the overall cardinality of the action space. Given N and S , every game is uniquely defined by its set of payoffs functions. The payoff function of each player can be viewed as an element of $\mathbb{R}^{|S|}$, i.e., $g^i \in \mathbb{R}^{|S|}$ for all $i \in N$. Hence, the space of games with set of players N and joint strategy space S can be identified as $\mathcal{G} \cong \mathbb{R}^{n|S|}$. We denote the *player-specific space of games* by \mathcal{G}^i and so $\mathcal{G} = \prod_{i \in N} \mathcal{G}^i$. For any subspace $X \subseteq \mathcal{G}$, we have $X = \prod_{i \in N} X^i$.

The basic solution concept of a game is the one of *Nash equilibrium*. A strategy profile $\mathbf{s} = (s^1, \dots, s^n)$ is an ϵ -equilibrium if for all $i \in N$ and all $t^i \in S^i$,

$$g^i(s^i, s^{-i}) \geq g^i(t^i, s^{-i}) - \epsilon.$$

A pure Nash equilibrium is said to be an ϵ -equilibrium with $\epsilon = 0$.

Games in which each player's payoff depends only on the actions selected by the rest of the players are called *non-strategic games*.

Definition 5.2.1. A *non-strategic game* is a finite game in which for all $i \in N$, there exists a function $\ell^i : S^{-i} \rightarrow \mathbb{R}$, such that $\ell^i(s^{-i}) = g^i(s^i, s^{-i})$. The space of non-strategic games will be denoted by \mathcal{NS} .

Actual payoffs are not required for the identification of Nash equilibria, as long as the payoff differences are well-defined. Taking advantage of this fact, we introduce an equivalence relation in the space of games, such that each equivalence class consists of games sharing identical payoff differences and have therefore identical equilibrium sets. This is the approach adopted by Candogan et al. [2011] and Hwanga and Rey-Belletb [2014].

Definition 5.2.2. The games \mathbf{g}_1 and \mathbf{g}_2 are *strategically equivalent* ($\mathbf{g}_1 \sim \mathbf{g}_2$), if for all $i \in N$ and all $s^{-i} \in S^{-i}$,

$$g_1^i(s^i, s^{-i}) - g_1^i(t^i, s^{-i}) = g_2^i(s^i, s^{-i}) - g_2^i(t^i, s^{-i}),$$

for any $s^i, t^i \in S^i$.

Proposition 5.2.3. *If $\mathbf{g}_1 \sim \mathbf{g}_2$ then the game defined as $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$ is a non-strategic game.*

Proof. By Definition 5.2.2, for all $i \in N$, all $s^{-i} \in S^{-i}$ and all $s^i, t^i \in S^i$

$$g_1^i(s^i, s^{-i}) - g_2^i(s^i, s^{-i}) = g_1^i(t^i, s^{-i}) - g_2^i(t^i, s^{-i}),$$

proving the result. □

A common way to fix a representative for strategically equivalent games is by means of normalization.

Definition 5.2.4. A finite game \mathbf{g} is *normalized*, if for all $i \in N$ and all $s^{-i} \in S^{-i}$,

$$\sum_{s^i \in S^i} g^i(s^i, s^{-i}) = 0.$$

The space of normalized games will be denoted by \mathcal{NO} .

In normalized games, each player's expected payoff, when playing all of his actions with equal probability (uniform strategy), is equal to zero.

The standard inner product in $\mathbb{R}^{|S|}$ is given by:

$$\langle f_1, f_2 \rangle_0 = \sum_{\mathbf{s} \in S} f_1(\mathbf{s}) f_2(\mathbf{s}). \quad (5.2.1)$$

We will use a natural extension of the standard inner product on \mathcal{G} , which was also adopted by Candogan et al. [2011] and Hwanga and Rey-Belletb [2014]:

$$\langle \mathbf{g}_1, \mathbf{g}_2 \rangle_{\mathcal{G}} = \sum_{i \in N} h_i \langle g_1^i, g_2^i \rangle_0, \quad (5.2.2)$$

where the inner product in the right-hand side is the inner product on $\mathbb{R}^{|S|}$, defined in (5.2.1). The inner product in (5.2.2) also induces a norm that will help us quantifying the distance between games. The norm on \mathcal{G} is defined as follows:

$$\|\mathbf{g}\|_{\mathcal{G}}^2 = \langle \mathbf{g}, \mathbf{g} \rangle_{\mathcal{G}}. \quad (5.2.3)$$

For any $X, Y \subset \mathcal{G}$, the notation $X \oplus Y$ stands for the direct orthogonal sum of X and Y .

5.3 Decomposition of the games space

We next provide an orthogonal decomposition of the space of games with respect to the inner product introduced in (5.2.2).

5.3.1 First Decomposition

Given a game \mathbf{g} we introduce, for each player i , the linear operator $\Pi^i : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, defined for all $\mathbf{s} = (s^i, s^{-i}) \in S$ by:

$$\Pi^i g^i(s^i, s^{-i}) = g^i(s^i, s^{-i}) - \frac{1}{h_i} \sum_{s^i \in S^i} g^i(s^i, s^{-i}), \quad (5.3.1)$$

and we further define $\Pi : \mathbb{R}^{n|S|} \rightarrow \mathbb{R}^{n|S|}$, such that $\Pi \mathbf{g} = (\Pi^1 g^1, \dots, \Pi^n g^n)$.

Moreover, we introduce for each player i the linear operator $\Lambda^i : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, defined for all $\mathbf{s} \in S$ as follows:

$$\Lambda^i g^i(s^i, s^{-i}) = \frac{1}{h_i} \sum_{s^i \in S^i} g^i(s^i, s^{-i}). \quad (5.3.2)$$

Similarly, we define $\Lambda : \mathbb{R}^{n|S|} \rightarrow \mathbb{R}^{n|S|}$, such that $\Lambda \mathbf{g} = (\Lambda^1 g^1, \dots, \Lambda^n g^n)$. The notation I stands for the identity operator and we clearly have $I = \Pi + \Lambda$.

Proposition 5.3.1. *The space of games is the direct orthogonal sum, with respect to the inner product introduced in (5.2.2), of the normalized and non-strategic games, i.e.,*

$$\mathcal{G} = \mathcal{NO} \oplus \mathcal{NS}, \quad (5.3.3)$$

where $\mathcal{NO} = \text{Im}(\Pi) = \text{Ker}(\Lambda)$ and $\mathcal{NS} = \text{Ker}(\Pi) = \text{Im}(\Lambda)$.

Proof. We work on the operators Π^i , Λ^i and we equivalently obtain the result on Π and Λ using their definitions. It is easy to check that $\Pi^i \circ \Pi^i = \Pi^i$ and $\Lambda^i \circ \Lambda^i = \Lambda^i$. Therefore, the operators Π^i and Λ^i are projections. It follows that $\mathcal{G}^i = \text{Im}(\Pi^i) \oplus \text{Ker}(\Pi^i)$ and since Λ^i is the complementary projection operator of Π^i , we get $\mathcal{G}^i = \text{Im}(\Pi^i) \oplus \text{Im}(\Lambda^i)$.

Let \mathbf{g} be a normalized game. Since by (5.3.1), for each player i , $\Pi^i g^i = g^i$, each Π^i is the projection operator onto the normalized payoffs of player i . Conversely, let us consider game \mathbf{g} such that for all i , $g^i \in \text{Im}(\Pi^i)$. Using (5.3.1), we get $\sum_{s^i \in S^i} g^i(s^i, s^{-i}) = 0$ and by definition of Π , it follows that $\text{Im}(\Pi) = \mathcal{NO}$.

To prove the argument concerning the non-strategic component, let us fix a game \mathbf{g} such that for all i , $g^i \in \text{Im}(\Lambda^i)$. By (5.3.2), for all $i \in N$ and all $s^{-i} \in S^{-i}$, the payoff of player i for his selected strategy s^i is defined as:

$$g^i(s^i, s^{-i}) = \frac{1}{h_i} \sum_{s^i \in S^i} g^i(s^i, s^{-i})$$

and by Definition 5.2.1, it is immediate that $\mathbf{g} \in \mathcal{NS}$. Next, fix $\mathbf{g} \in \mathcal{NS}$. Once more, Definition 5.2.1 implies that for every selected collection of strategies but the i -th one, the payoff of player i is given by $g^i(s^i, s^{-i}) = (1/h_i) \sum_{s^i \in S^i} g^i(s^i, s^{-i})$ so we obtain that $\Lambda^i g^i = g^i$. Hence, $\text{Im}(\Lambda) = \mathcal{NS}$.

In order to show orthogonality between the components, let us consider $\mathbf{g}_0 \in \mathcal{NO}$ and $\mathbf{g}_{\mathcal{NS}} \in \mathcal{NS}$. It suffices to prove

$$\langle \mathbf{g}_0, \mathbf{g}_{\mathcal{NS}} \rangle_{\mathcal{G}} = \sum_{i \in N} h_i \langle g_0^i, g_{\mathcal{NS}}^i \rangle_0 = 0.$$

In doing so, notice that, using the standard inner product in (5.2.1), it follows that:

$$\langle g_0^i, g_{\mathcal{N}, \mathcal{S}}^i \rangle = \sum_{\mathbf{s} \in S} g_0^i(\mathbf{s}) g_{\mathcal{N}, \mathcal{S}}^i(\mathbf{s}) = \sum_{s^{-i} \in S^{-i}} \sum_{s^i \in S^i} g_0^i(s^i, s^{-i}) g_{\mathcal{N}, \mathcal{S}}^i(s^i, s^{-i})$$

and by Definition 5.2.1, for all $s^{-i} \in S^{-i}$, $\sum_{s^i \in S^i} g_{\mathcal{N}, \mathcal{S}}^i(s^i, s^{-i}) = h_i \ell(s^{-i})$ for some $\ell : S^{-i} \rightarrow \mathbb{R}$. Hence,

$$\sum_{s^i \in S^i} g_0^i(s^i, s^{-i}) g_{\mathcal{N}, \mathcal{S}}^i(s^i, s^{-i}) = (h_i \ell(s^{-i})) \sum_{s^i \in S^i} g_0^i(s^i, s^{-i}) = 0,$$

where the latter equality follows by Definition 5.2.3. \square

5.3.2 Uniform and non-interactive classes of games

In this section, we introduce the classes of uniform and non interactive games and we then characterize their equilibrium sets.

Uniform games

We first define the uniformly mixed strategy profile and the class of uniform games.

Definition 5.3.2. The *uniformly mixed strategy* of player $i \in N$ is the mixed strategy that selects with equal probability $\frac{1}{h_i}$ each action $s^i \in S^i$. The induced strategy profile defines the *uniformly mixed strategy profile*.

Definition 5.3.3. A finite game is *uniform* if and only if the uniformly mixed strategy profile is a Nash equilibrium. The class of uniform games is denoted by \mathcal{U} .

Clearly, uniform games constitute a subspace of games. An aftereffect of Definition 5.3.2, is that in uniform games the sum of each player's payoffs, fixing any of his actions, over the action profiles consisted of the rest of the players, is constant. Namely, for all $i \in N$, there exists constant $c^i \in \mathbb{R}$, such that for all $s^i \in S^i$,

$$\sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) = c^i. \quad (5.3.4)$$

Definition 5.2.1 and (5.3.4) imply that the class of non-strategic games forms a subclass of uniform games. Next proposition states that uniform games are stable by projection onto the subspace of normalized games.

Lemma 5.3.4. *Given a game \mathbf{g}_1 and a uniform game \mathbf{g}_2 such that $\mathbf{g}_1 \sim \mathbf{g}_2$. Then, \mathbf{g}_1 is a uniform game.*

Proof. If $\mathbf{g}_1 \sim \mathbf{g}_2$ then, for all players $i \in N$ and all $(\mathbf{s}, \mathbf{t}) \in S \times S$, such that $\mathbf{s} = (s^i, s^{-i})$ and $\mathbf{t} = (t^i, s^{-i})$ with $s^i \neq t^i$, next holds true:

$$g_1^i(\mathbf{t}) - g_1^i(\mathbf{s}) = g_2^i(\mathbf{t}) - g_2^i(\mathbf{s}).$$

Considering the sum over $s^{-i} \in S^{-i}$ to both sides of the above equation and since \mathbf{g}_2 is uniform by assumption, due to (5.3.4) it follows:

$$\sum_{s^{-i} \in S^{-i}} (g_1^i(t^i, s^{-i}) - g_1^i(s^i, s^{-i})) = c^i - c^i = 0.$$

Therefore, for all players $i \in N$ and all $s^i, t^i \in S^i$, one has:

$$\sum_{s^{-i} \in S^{-i}} g_1^i(s^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} g_1^i(t^i, s^{-i}),$$

that concludes the proof. \square

Non-interactive games

In this paragraph, we introduce the non-interactive games and we show that they constitute a subspace of potential games. Then, we show that non-interactive games admit a dominant strategy equilibrium with unique payoff.

Definition 5.3.5. A finite game \mathbf{g} is *potential* if there exists a function $\varphi : S \rightarrow \mathbb{R}$ such that, for all $i \in N$, all $s^i, t^i \in S^i$ and all $s^{-i} \in S^{-i}$.

$$\varphi(s^i, s^{-i}) - \varphi(t^i, s^{-i}) = g^i(s^i, s^{-i}) - g^i(t^i, s^{-i}).$$

The function φ is referred to as a *potential function* of the game. The space of potential games will be denoted by \mathcal{P} .

Since players are interested of maximizing the potential function, it then follows:

Theorem 5.3.6. *Potential games always admit a pure Nash equilibrium.*

In the sequel, we define the class of non-interactive games, and we show that they constitute a subspace of potential games.

Definition 5.3.7. A *non-interactive game* is a finite game in which for all $i \in N$, there exists a function $\lambda^i : S^i \rightarrow \mathbb{R}$, such that $\lambda^i(s^i) = g^i(s^i, s^{-i})$. That class of games will be denoted by $\mathcal{N}\mathcal{I}$.

Lemma 5.3.8. *The class of non-interactive games consists a subspace of potential games with a potential function of such a game to be given by $\varphi : S \rightarrow \mathbb{R}$ defined as*

$$\varphi(\mathbf{s}) = \sum_{i \in N} \lambda^i(s^i)$$

Proof. Let us consider a non interactive game $\mathbf{g}_{\mathcal{N}\mathcal{I}}$. By Definition 5.3.7, for all $i \in N$ and all $s^i, t^i \in S^i$, such that $s^i \neq t^i$ it holds:

$$g^i(s^i, s^{-i}) - g^i(t^i, s^{-i}) = \lambda^i(s^i) - \lambda^i(t^i).$$

According to Definition 5.3.5, non-interactive games constitute a subspace of potential games and it is easy to see that a potential function in this subclass of games is defined by $\varphi : S \rightarrow \mathbb{R}$, such that $\varphi(s^i, s^{-i}) = \sum_{i \in N} \lambda^i(s^i)$. \square

An action profile $\mathbf{s} = (s^1, \dots, s^n) \in S$ is a *dominant strategy equilibrium*, if for every player $i \in N$,

$$g^i(s^i, t^{-i}) \geq g^i(t^i, t^{-i}),$$

for all $\mathbf{t} = (t^i, t^{-i}) \in S$.

Proposition 5.3.9. *Every non-interactive game admits a dominant strategy equilibrium with unique payoff.*

Proof. Let us consider a non-interactive game \mathbf{g} . From Lemma 5.3.8, \mathbf{g} is a potential game and thus, from Theorem 5.3.6, it has a pure Nash equilibrium, denoted by $\mathbf{s} = (s^1, \dots, s^i, \dots, s^n)$. By contradiction, let us assume that \mathbf{s} is not a dominant strategy equilibrium. Then, there exists $i \in N$ and $\mathbf{t} = (t^i, t^{-i}) \in S$, such that $g^i(t^i, t^{-i}) > g^i(s^i, t^{-i})$. However, since \mathbf{g} is a non-interactive game, for all $i \in N$, there exists $\lambda^i : S^i \rightarrow \mathbb{R}$, such that

$$\lambda^i(s^i) = g^i(s^i, s^{-i}) = g^i(s^i, t^{-i}) < g^i(t^i, t^{-i}) = g^i(t^i, s^{-i}) \leq g^i(s^i, s^{-i}) = \lambda^i(s^i),$$

where the right inequality follows, since \mathbf{s} is a pure Nash equilibrium. \square

Second Decomposition

The subspace of uniform games with zero-constant, i.e., the same constant $c^i = 0$ for all $i \in N$, in (5.3.4), will be denoted by \mathcal{U}_0 .

Given a game \mathbf{g} , we introduce for each player i , the linear operator $\Sigma^i : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, defined for all $\mathbf{s} = (s^i, s^{-i}) \in S$ as follows:

$$\Sigma^i g^i(s^i, s^{-i}) = u = g^i(s^i, s^{-i}) - \frac{1}{\prod_{j \neq i} h_j} \sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) \quad (5.3.5)$$

We also define $\Sigma : \mathbb{R}^{n|S|} \rightarrow \mathbb{R}^{n|S|}$, such that $\Sigma \mathbf{g} = (\Sigma^1 g^1, \dots, \Sigma^n g^n)$.

We further introduce for each player i , the linear operator $\Theta^i : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, defined for all $\mathbf{s} \in S$ as follows:

$$\Theta^i g^i(s^i, s^{-i}) = \frac{1}{\prod_{j \neq i} h_j} \sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) \quad (5.3.6)$$

In like manner, we have $\Theta : \mathbb{R}^{n|S|} \rightarrow \mathbb{R}^{n|S|}$, such that $\Theta \mathbf{g} = (\Theta^1 g^1, \dots, \Theta^n g^n)$ and it clearly holds true: $I = \Sigma + \Theta$.

Proposition 5.3.10. *The space of games is the direct orthogonal sum, with respect to the inner product introduced in (5.2.2), of the uniform games with zero-constant and non-interactive games, i.e.,*

$$\mathcal{G} = \mathcal{U}_0 \oplus \mathcal{N}\mathcal{I},$$

where $\mathcal{U}_0 = \text{Im}(\Sigma) = \text{Ker}(\Theta)$ and $\mathcal{N}\mathcal{I} = \text{Ker}(\Sigma) = \text{Im}(\Theta)$.

Proof. Likewise, we work on the player-specific operators Σ^i, Θ^i and we equivalently conclude the result on Σ and Θ , using their definitions. It is easy to check that $\Sigma^i \circ \Sigma^i = \Sigma^i$ and $\Theta^i \circ \Theta^i = \Theta^i$. Therefore, Θ^i and Σ^i are projection operators. It clearly follows that $\mathcal{G}^i = \text{Im}(\Sigma^i) \oplus \text{Ker}(\Sigma^i)$ and since Θ^i is the complementary projection operator of Σ^i , we get that $\mathcal{G}^i = \text{Im}(\Sigma^i) \oplus \text{Im}(\Theta^i)$.

From (5.3.5), it is immediate that in a given game $\mathbf{g} \in \mathcal{U}_0$, for each player i , it holds $\Sigma_i g^i = g^i$. Thus, Σ^i is the projection operator onto the uniformized with zero constant payoffs of player i . Conversely, let us consider \mathbf{g} , such that for all i , $g^i \in \text{Im}(\Sigma^i)$. Since by (5.3.5), for all $s^i \in S^i$, $\sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) = 0$, using the definition of Σ , it follows $\text{Im}(\Sigma) = \mathcal{U}_0$.

Concerning the non interactive component, let us fix \mathbf{g} , such that for all i , $g^i \in \text{Im}(\Theta^i)$. By (5.3.6) equivalently, for all $i \in N$ and all $s^i \in S^i$, the payoff of player i , for the selected strategy profile by the rest of players s^{-i} , is defined as follows:

$$g^i(s^i, s^{-i}) = \frac{1}{\prod_{j \neq i} h_j} \sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}).$$

Definition 5.3.7 implies that $\mathbf{g} \in \mathcal{N}\mathcal{I}$. Now, let us assume that $\mathbf{g} \in \mathcal{N}\mathcal{I}$. Likewise, from Definition 5.3.7 we have $g^i(s^i, s^{-i}) = (1 / \prod_{j \neq i} h_j) \sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i})$, which is equivalent to $\Theta^i g^i = g^i$ that concludes this part of the proof.

In order to show orthogonality between the components, let us consider $\mathbf{g}_0 \in \mathcal{U}_0$ and $\mathbf{g}_{\mathcal{N}\mathcal{I}} \in \mathcal{N}\mathcal{I}$. It suffices to prove

$$\langle \mathbf{g}_0, \mathbf{g}_{\mathcal{N}\mathcal{I}} \rangle_{\mathcal{G}} = \sum_{i \in N} h_i \langle g_0^i, g_{\mathcal{N}\mathcal{I}}^i \rangle = 0.$$

In doing so, notice that using the standard inner product in (5.2.1) we get,

$$\langle g_0^i, g_{\mathcal{N}\mathcal{I}}^i \rangle = \sum_{\mathbf{s} \in S} g_0^i(\mathbf{s}) g_{\mathcal{N}\mathcal{I}}^i(\mathbf{s}) = \sum_{s^i \in S^i} \sum_{s^{-i} \in S^{-i}} g_0^i(s^i, s^{-i}) g_{\mathcal{N}\mathcal{I}}^i(s^i, s^{-i}).$$

From Definition 5.3.7, we have for all $s^i \in S^i$, $\sum_{s^{-i} \in S^{-i}} g_{\mathcal{N}\mathcal{I}}^i(s^i, s^{-i}) = \lambda^i(s^i) \prod_{j \neq i} h_j$.

Hence,

$$\sum_{s^{-i} \in S^{-i}} g_0^i(s^i, s^{-i}) g_{\mathcal{N}\mathcal{I}}^i(s^i, s^{-i}) = \left(\lambda^i(s^i) \prod_{j \neq i} h_j \right) \sum_{s^{-i} \in S^{-i}} g_0^i(s^i, s^{-i}) = 0,$$

where the right equality follows from definition of the space \mathcal{U}_0 . \square

Third decomposition

Next, we define two classes of games that will appear as components in the presented decomposition result of this paragraph according to which an arbitrary game can be decomposed into the components of a total-sum zero and constant games.

Definition 5.3.11. A finite game \mathbf{g} is a *total-sum zero game*, if for all $i \in N$,

$$\sum_{\mathbf{s} \in S} g^i(\mathbf{s}) = 0.$$

The space of total-sum zero games will be denoted by \mathcal{G}_* .

Definition 5.3.12. A finite game \mathbf{g} is a *constant game*, if for all $i \in N$, there exists c^i , such that for all $\mathbf{s} \in S$,

$$g^i(\mathbf{s}) = c^i.$$

The space of constant games will be denoted by \mathcal{C} .

Constant games is a subspace of non-interactive and of non-strategic games and thus, of uniform games too. It is easy to see that $\mathcal{C} = \mathcal{N}\mathcal{I} \cap \mathcal{N}\mathcal{S}$.

Given a game \mathbf{g} , we introduce for each player i , the linear operator $\Pi_*^i : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, defined for all $\mathbf{s} \in S$ by,

$$\Pi_*^i g^i(\mathbf{s}) = g^i(\mathbf{s}) - \frac{1}{\prod_{i \in N} h_i} \sum_{\mathbf{s} \in S} g^i(\mathbf{s}). \quad (5.3.7)$$

We also have $\Pi_* : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, such that $\Pi_* \mathbf{g} = (\Pi_*^1 g^1, \dots, \Pi_*^n g^n)$.

We define $\Gamma^i : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, as follows:

$$\Gamma^i g^i(\mathbf{s}) = \frac{1}{\prod_{i \in N} h_i} \sum_{\mathbf{s} \in S} g^i(\mathbf{s}). \quad (5.3.8)$$

In like manner, we have $\Gamma : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, such that $\Gamma \mathbf{g} = (\Gamma^1 g^1, \dots, \Gamma^n g^n)$ and it clearly holds $I = \Pi_* + \Gamma$.

Proposition 5.3.13. *The space of games is the direct orthogonal sum with respect to the inner product in (5.2.2), of the total-sum zero games and constant games, i.e.,*

$$\mathcal{G} = \mathcal{G}_* \oplus \mathcal{C},$$

where $\mathcal{G}_* = \text{Im}(\Pi_*) = \text{Ker}(\Gamma)$ and $\mathcal{C} = \text{Ker}(\Pi_*) = \text{Im}(\Gamma)$.

Proof. It is easy to check that $\Pi_*^i \circ \Pi_*^i = \Pi_*^i$ and $\Gamma^i \circ \Gamma^i = \Gamma^i$. Therefore, Π_*^i and Γ^i are projection operators. It follows that $\mathcal{G}_* = \text{Im}(\Pi_*^i) \oplus \text{Im}(\Gamma^i)$ due to Γ^i is the complementary projection operator of Π_*^i . From (5.3.7), for any given game $\mathbf{g} \in \mathcal{G}_*$, for each player i , $\Pi_*^i g^i = g^i$. Conversely, let us consider $\mathbf{g} \in \text{Im}(\Pi_*)$, which using (5.3.7) implies $\sum_{\mathbf{s} \in S} g^i(\mathbf{s}) = 0$.

Now, let us assume that $\mathbf{g} \in \text{Im}(\Gamma)$. By (5.3.8) equivalently, for all $i \in N$, the payoff of player i at the strategy profile $\mathbf{s} \in S$, is defined as the average of the i -th player's payoffs over the strategy profiles. By Definition 5.3.12, we get that $\mathbf{g} \in \mathcal{C}$. To prove the inverse inclusion, consider game \mathbf{g} , such that for all i , $g^i \in \mathcal{C}^i$. Definition 5.3.12 implies that for all i and all $\mathbf{s} \in S$, we have $g^i(\mathbf{s}) = (1/\prod_{i \in N} h_i) \sum_{\mathbf{s} \in S} g^i(\mathbf{s})$ and we therefore get, $\Gamma^i g^i = g^i$. The result follows from the definitions of Π_* and Γ .

To prove orthogonality between the components, let us consider games $\mathbf{g}_{\mathcal{G}_*} \in \mathcal{G}_*$ and $\mathbf{g}_{\mathcal{C}} \in \mathcal{C}$. Using the inner product in (5.2.2), we have:

$$\langle \mathbf{g}_{\mathcal{G}_*}, \mathbf{g}_{\mathcal{C}} \rangle = \sum_{\mathbf{s} \in S} g_{\mathcal{G}_*}^i(\mathbf{s}) g_{\mathcal{C}}^i(\mathbf{s}) = c^i \prod_{i \in N} h_i \sum_{\mathbf{s} \in S} g_{\mathcal{G}_*}^i(\mathbf{s}) = 0,$$

where the median equality holds due to Definition 5.3.12 and the latter one by Definition 5.3.11. \square

Remark 5.3.14. The decomposition result of Proposition 5.3.13 holds true in the subspaces of uniform and non interactive games by projection. Namely,

$$\mathcal{U} = \mathcal{U}_* \oplus \mathcal{C} \quad , \quad \mathcal{NI} = \mathcal{NI}_* \oplus \mathcal{C}.$$

5.3.3 The offspring decomposition

In this section, we provide our main result. Next lemma will be useful to that end.

Lemma 5.3.15. *The following assertions hold:*

- $\mathcal{U}_0 = \mathcal{U}_*$
- $\mathcal{NI} \cap \mathcal{NO} = \mathcal{NI}_*$

Proof. Clearly $\mathcal{U}_0 \subset \mathcal{U}_*$. In order to show the inverse inclusion let us consider $\mathbf{g} \in \mathcal{U}_*$. Since \mathbf{g} is uniform total sum zero game, for each player i , it holds $h_i c^i = 0$ that implies $c^i = 0$ and thus, $\mathbf{g} \in \mathcal{U}_0$.

In order to prove the second assertion, let us consider $\mathbf{g} \in \mathcal{NI}_*$. Then, due to \mathbf{g} is supposed to be non interactive game, we equivalently get,

$$\sum_{s^i \in S^i} \prod_{j \neq i} h_j \lambda^i(s^i) = \sum_{s^i \in S^i} \lambda^i(s^i) \prod_{j \neq i} h_j = 0,$$

where the latter equality follows since \mathbf{g} is a total zero sum game by assumption. We equivalently have $\sum_{s^i \in S^i} \lambda(s^i) = 0$ and thus, $\mathbf{g} \in \mathcal{NI} \cap \mathcal{NO}$. □

Corollary 5.3.16. *The space of games is the direct orthogonal sum with respect to the inner product in (5.2.2), of the uniform with zero-constant, the non-interactive normalized and the constant games, i.e.,*

$$\mathcal{G} = \mathcal{U}_0 \oplus \mathcal{C} \oplus (\mathcal{NI} \cap \mathcal{NO})$$

Proof. The result clearly follows from Proposition 5.3.10, Proposition 5.3.13, Remark 5.3.14 and Lemma 5.3.15. □

Lemma 5.3.17. *The projection operators Θ, Π_* commute. Namely,*

$$\Theta \Pi_* = \Pi_* \Theta$$

Furthermore, we have $\Theta \Gamma = \Gamma \Theta = \Gamma$.

Proof. Let us consider a game \mathbf{g} and a player $i \in N$. Due to (5.3.6) and (5.3.7), for all $\mathbf{s} \in S$,

$$\Theta^i \Pi_*^i g^i(\mathbf{s}) = \frac{\sum_{s^{-i} \in S^{-i}} g^i(\mathbf{s})}{\prod_{j \neq i} h_j} - \frac{\sum_{\mathbf{s} \in S} g^i(\mathbf{s})}{\prod_{i \in N} h_i} = \Pi_*^i \Theta^i g^i(\mathbf{s}).$$

Since constant games are a subspace of non interactive games, it is immediate that $\Theta^i \Gamma^i = \Gamma^i \Theta^i = \Gamma^i$. We conclude the result on the operators Θ, Π_* and Γ using their definitions. □

Lemma 5.3.18. *The linear operator defined by $\Theta\Pi_* : \mathbb{R}^{n|S|} \rightarrow \mathbb{R}^{n|S|}$ is the projection onto $\text{Im}(\Theta) \cap \text{Im}(\Pi_*)$ and the linear operator defined by $\Theta\Gamma : \mathbb{R}^{n|S|} \rightarrow \mathbb{R}^{n|S|}$ is the projection onto $\text{Im}(\Gamma)$.*

Proof. We present the proof only for the former operator since for the latter one, the result follows from Lemma 5.3.17, due to $\text{im}(\Gamma) \subset \text{im}(\Theta)$. In doing so, again Lemma 5.3.17 implies,

$$(\Theta\Pi_*)(\Theta\Pi_*) = (\Pi_*\Theta)(\Theta\Pi_*) = \Pi_*\Theta^2\Pi_* = (\Pi_*\Theta)\Pi_* = \Theta\Pi_*^2 = \Theta\Pi_*$$

and thus, $\Theta\Pi_*$ is a projection.

Let us consider $\mathbf{u} \in \mathbb{R}^{n|S|}$, such that $\Theta\Pi_*\mathbf{g} = \mathbf{g}$. Then, $\Theta\mathbf{g} = \Theta^2\Pi_*\mathbf{g} = \Theta\Pi_*\mathbf{g} = \mathbf{g}$ and thus, $\mathbf{g} \in \text{Im}(\Theta)$. Moreover, $\Pi_*\Theta\mathbf{g} = \Theta\Pi_*\mathbf{g} = \mathbf{g}$ and it immediately follows that $\mathbf{g} \in \text{Im}(\Pi_*)$. Hence, $\mathbf{g} \in \text{Im}(\Theta) \cap \text{Im}(\Pi_*)$.

Conversely, assume $\mathbf{g} \in \text{Im}(\Theta) \cap \text{Im}(\Pi_*)$. Then, $\Theta\mathbf{g} = \mathbf{g}$ and $\Pi_*\mathbf{g} = \mathbf{g}$. Hence, $\Theta\Pi_*\mathbf{g} = \mathbf{g}$, which concludes the proof. \square

Corollary 5.3.19. *Let \mathbf{g} be a game. With respect to the induced distance in the space of games, the following hold:*

- *the closest non-interactive game is $\mathbf{g}_{\mathcal{N}\mathcal{I}} = (\Theta\Pi_* + \Gamma)\mathbf{g}$,*
- *the closest uniform game is given by $\mathbf{g}_{\mathcal{U}} = (\Sigma + \Gamma)\mathbf{g}$,*
- *the closest constant game is given by $\mathbf{g}_{\mathcal{C}} = \Gamma\mathbf{g}$,*

Furthermore, every dominant strategy equilibrium of its closest non-interactive game is an ϵ -equilibrium of \mathbf{g} , for some $\epsilon \leq \max_{i \in N} (2/\sqrt{h_i}) \|\mathbf{g}_{\mathcal{U}_0}\|_{\mathcal{G}}$ and the uniformly mixed strategy profile is an $\tilde{\epsilon}$ -equilibrium of \mathbf{g} , for some $\tilde{\epsilon} \leq \max_{i,s^i} \{\lambda^i(s^i)\}$, where $\lambda^i(s^i)$ is the payoff of player i for his strategy s^i , in $\mathbf{g}_{\mathcal{N}\mathcal{I}^}$.*

Proof. Using the decomposition result of Corollary 5.3.16, due to (5.3.8), it is immediate that the closest constant game to an arbitrary game \mathbf{g} , is given by $\mathbf{g}_{\mathcal{C}} = \Gamma\mathbf{g}$. From Lemma 5.3.15 and Lemma 5.3.18, the operator $\Theta\Pi_*$ is the projection onto $\mathcal{N}\mathcal{I} \cap \mathcal{N}\mathcal{O}$ and since constant games is a subspace of non-interactive games, it clearly follows that its closest non-interactive game is given by $\mathbf{g}_{\mathcal{N}\mathcal{I}} = (\Theta\Pi_* + \Gamma)\mathbf{g}$. Since constant games are also a subspace of uniform games, Corollary 5.3.17 implies that the closest uniform game to \mathbf{g} , is given by $\mathbf{g}_{\mathcal{U}} = (I - \Theta\Pi_*)\mathbf{g}$. By Lemma 5.3.18, it clearly follows that $\Sigma + \Theta\Pi_* + \Gamma = \Sigma + \Theta\Pi_* + \Theta\Gamma = \Sigma + \Theta(\Pi_* + \Gamma) = \Sigma + \Theta = I$. Hence, $I - \Theta\Pi_* = \Sigma + \Gamma$, which concludes this part of the proof.

From Corollary 5.3.16 and definition of the norm in (5.2.3), for all $i \in N$, $\mathbf{s} \in S$, it holds:

$$|g^i(\mathbf{s}) - g_{\mathcal{N}\mathcal{I}}^i(\mathbf{s})| = |g_{\mathcal{U}_0}^i(\mathbf{s})| \leq (1/\sqrt{h_i}) \|\mathbf{g}_{\mathcal{U}_0}\|_{\mathcal{G}} \leq \max_{i \in N} (1/\sqrt{h_i}) \|\mathbf{g}_{\mathcal{U}_0}\|_{\mathcal{G}}, \quad (5.3.9)$$

Let $\mathbf{s} = (s^1, \dots, s^i, \dots, s^n) \in S$ be a dominant strategy equilibrium in $\mathbf{g}_{\mathcal{N}\mathcal{I}}$ and thus,

a pure equilibrium in the closest non-interactive game to \mathbf{g} . For all $\mathbf{t} \in S$, such that $t^i \neq s^i$ and $t^{-i} = s^{-i}$, it holds:

$$\begin{aligned} g^i(\mathbf{t}) - g^i(\mathbf{s}) &\leq g^i(\mathbf{t}) - g^i(\mathbf{s}) - (g_{\mathcal{N}\mathcal{I}}^i(\mathbf{t}) - g_{\mathcal{N}\mathcal{I}}^i(\mathbf{s})) \\ &= (g_{\mathcal{N}\mathcal{I}}^i(\mathbf{s}) - g^i(\mathbf{s})) + (g^i(\mathbf{t}) - g_{\mathcal{N}\mathcal{I}}^i(\mathbf{t})) \\ &= g_{\mathcal{U}_0}^i(\mathbf{t}) - g_{\mathcal{U}_0}^i(\mathbf{s}), \end{aligned}$$

where the inequality follows since \mathbf{s} is a pure equilibrium in $\mathbf{g}_{\mathcal{N}\mathcal{I}}$. Thus, using (5.3.9) we get

$$\begin{aligned} g^i(\mathbf{t}) - g^i(\mathbf{s}) &\leq g_{\mathcal{U}_0}^i(\mathbf{t}) - g_{\mathcal{U}_0}^i(\mathbf{s}) \\ &\leq \max_{i \in N} (2/\sqrt{h_i}) \|\mathbf{g}_{\mathcal{U}_0}\|_{\mathcal{G}}. \end{aligned}$$

Therefore, the dominant strategy equilibrium \mathbf{s} in $\mathbf{g}_{\mathcal{N}\mathcal{I}}$, is an ϵ -equilibrium in \mathbf{g} , for some $\epsilon \leq \max_{i \in N} (2/\sqrt{h_i}) \|\mathbf{g}_{\mathcal{U}_0}\|_{\mathcal{G}}$.

Let $\mathbf{x} = (x^1, \dots, x^i, \dots, x^n) \in \prod_{i \in N} \mathbf{P}(S^i)$ be the uniformly mixed strategy profile. By Definition 5.3.2, \mathbf{x} is an equilibrium in $\mathbf{g}_{\mathcal{U}}$. For all $\mathbf{y} \in \prod_{i \in N} \mathbf{P}(S^i)$ such that $y^i \neq x^i$ and $y^{-i} = x^{-i}$ for some $i \in N$, we have:

$$\begin{aligned} g^i(\mathbf{y}) - g^i(\mathbf{x}) &= \sum_{\mathbf{s} \in S} \mathbf{y}(\mathbf{s}) g^i(\mathbf{s}) - \sum_{\mathbf{s} \in S} \mathbf{x}(\mathbf{s}) g^i(\mathbf{s}) \\ &= \sum_{\mathbf{s} \in S} (\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})) g^i(\mathbf{s}) \\ &\leq \sum_{\mathbf{s} \in S} (\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})) g^i(\mathbf{s}) - \sum_{\mathbf{s} \in S} (\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})) g_{\mathcal{U}}^i(\mathbf{s}), \end{aligned}$$

where the inequality follows since \mathbf{x} is an equilibrium in $\mathbf{g}_{\mathcal{U}}$. Hence, from Corollary 5.3.16, it follows

$$g^i(\mathbf{y}) - g^i(\mathbf{x}) \leq \sum_{\mathbf{s} \in S} (\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})) g_{\mathcal{N}\mathcal{I}^*}^i(\mathbf{s}).$$

Observe that $\mathbf{y} = (x^1, \dots, x^{i-1}, y^i, x^{i+1}, \dots, x^n)$ and $\mathbf{x} = (x^i, x^{-i})$ is the uniformly mixed strategy profile (see Definition 5.3.2). If y_{s^i} stands for the probability whereby player i chooses his strategy s^i , then we have:

$$\begin{aligned} g^i(\mathbf{y}) - g^i(\mathbf{x}) &\leq \sum_{\mathbf{s} \in S} \left[\left(\frac{y_{s^i}}{\prod_{j \neq i} h_j} - \frac{1}{\prod_{i \in N} h_i} \right) g_{\mathcal{N}\mathcal{I}^*}^i(\mathbf{s}) \right] \\ &\leq \sum_{\mathbf{s} \in S} \left(\frac{y_{s^i}}{\prod_{j \neq i} h_j} g_{\mathcal{N}\mathcal{I}^*}^i(\mathbf{s}) \right), \end{aligned}$$

where the last inequality follows since the game $\mathbf{g}_{\mathcal{N}\mathcal{I}^*}$ is a total sum zero game,

which means that $(1/\prod_{i \in N} h_i) \sum_{\mathbf{s} \in S} g^i_{\mathcal{N}\mathcal{I}^*}(\mathbf{s}) = 0$. Equivalently,

$$\begin{aligned} g^i(\mathbf{y}) - g^i(\mathbf{x}) &\leq \frac{1}{\prod_{j \neq i} h_j} \sum_{s^i \in S^i} \sum_{s^{-i} \in S^{-i}} y_{s^i} g^i_{\mathcal{N}\mathcal{I}^*}(s^i, s^{-i}) \\ &\leq \sum_{s^i \in S^i} y_{s^i} \lambda^i(s^i) \\ &\leq \max_{i, s^i} \{\lambda^i(s^i)\}, \end{aligned}$$

where the second inequality is obtained by Definition 5.3.7. Hence, the uniformly mixed strategy equilibrium \mathbf{x} is an $\tilde{\epsilon}$ -equilibrium in \mathbf{g} , for some $\tilde{\epsilon} \leq \max_{i, s^i} \{\lambda^i(s^i)\}$. \square

The dimensions of the uniform, non-interactive and constant games' subspaces are presented below.

Corollary 5.3.20. *The dimensions of the uniform, non-interactive and constant games subspaces are given as follows:*

- $\dim(\mathcal{U}) = \sum_{i \in N} \left(\prod_{i \in N} h_i - h_i \right) + n$,
- $\dim(\mathcal{N}\mathcal{I}) = \sum_{i \in N} h_i$,
- $\dim(\mathcal{C}) = n$.

Proof. The result on the dimension of constant games is trivial. Definition 5.3.7 implies that $\dim(\mathcal{N}\mathcal{I}) = \sum_{i \in N} \dim(\mathcal{N}\mathcal{I}^i) = \sum_{i \in N} h_i$. Thus, Remark 5.3.14 ensures that $\dim(\mathcal{N}\mathcal{I}^*) = \dim(\mathcal{N}\mathcal{I}) - \dim(\mathcal{C}) = \sum_{i \in N} h_i - n$. Since $\dim(\mathcal{G}) = n \prod_{i \in N} h_i$, using the decomposition result of Corollary 5.3.16, it follows the dimension of the uniform games subspace,

$$\dim(\mathcal{U}) = n \prod_{i \in N} h_i - \left(\sum_{i \in N} h_i - n \right)$$

that concludes the proof. \square

5.4 Examples

In this section two examples are provided. The first one is a celebrated game in which our decomposition result coincides with this one proposed in Candogan et al. [2011], the decomposition presented in Kalai and Kalai [2010], and the one in Jessie and Saari [2013]. Let us further notice that in the context of two-players with equal number of actions uniform games, one may obtain coincidence between the three mentioned above decompositions results. The second example is a decomposition of

a game that can be considered as a perturbation in the payoffs of a uniform with zero constant game.

Example 5.4.1. *Rock-Paper-Scissors game.*

The payoff matrix of the Rock-Paper-Scissors game is given by:

$$\mathbf{g} = \begin{pmatrix} 0, 0 & -w, w & w, -w \\ w, -w & 0, 0 & -w, w \\ -w, w & w, -w & 0, 0 \end{pmatrix}$$

where $w \in \mathbb{R}$. The uniformly mixed strategy profile is an equilibrium of \mathbf{g} and thus, \mathbf{g} is a uniform game. It is also a harmonic game with respect to the definition of this class of games by [Candogan et al. \[2011\]](#).

Example 5.4.2. *A perturbed uniform with zero constant game.*

Let us consider the game \mathbf{g} whose payoffs are given by the following matrix:

$$\mathbf{g} = \begin{pmatrix} 100, 100 & 100, 105 & -190, 1 \\ 105, 100 & 95, 95 & -195, 200 \\ 1, -190 & 200, -195 & -190, -195 \end{pmatrix}$$

The uniform with zero constant component-game is given by:

$$\mathbf{g}_{\mathcal{U}_0} = \begin{pmatrix} 290/3, 290/3 & 290/3, 310/3 & -580/3, -1 \\ 310/3, 290/3 & 280/3, 280/3 & -590/3, 594/3 \\ -8/3, -580/3 & 589/3, -590/3 & -581/3, -591/3 \end{pmatrix}$$

and the non interactive one by:

$$\mathbf{g}_{\mathcal{N}\mathcal{I}} = \begin{pmatrix} 10/3, 10/3 & 10/3, 5/3 & 10/3, 2 \\ 5/3, 10/3 & 5/3, 5/3 & 5/3, 2 \\ 11/3, 10/3 & 11/3, 5/3 & 11/3, 2 \end{pmatrix}$$

Moreover, the non interactive normalized component-game is given by:

$$\mathbf{g}_{\mathcal{N}\mathcal{I} \cap \mathcal{N}\mathcal{O}} = \begin{pmatrix} 4/9, 1 & 4/9, -6/9 & 4/9, -3/9 \\ -11/9, 1 & -11/9, -6/9 & -11/9, -3/9 \\ 7/9, 1 & 7/9, -6/9 & 7/9, -3/9 \end{pmatrix} = \mathbf{g}_{\mathcal{N}\mathcal{I}^*}$$

and the uniform one is given by:

$$\mathbf{g}_{\mathcal{U}} = \begin{pmatrix} 896/9, 99 & 896/9, 951/9 & -1714/9, 12/9 \\ 956/9, 99 & 866/9, 861/9 & -1744/9, 1803/9 \\ 2/9, -191 & 1793/9, -1749/9 & -1717/9, -1752/9 \end{pmatrix}$$

Lastly, the constant-component is given as follows:

$$\mathbf{g}_{\mathcal{C}} = \begin{pmatrix} 26/9, 7/3 & 26/9, 7/3 & 26/9, 7/3 \\ 26/9, 7/3 & 26/9, 7/3 & 26/9, 7/3 \\ 26/9, 7/3 & 26/9, 7/3 & 26/9, 7/3 \end{pmatrix}$$

In $\mathbf{g}_{\mathcal{U}}$, the uniformly mixed strategy profile $\mathbf{x} = (x^1, x^2)$, where $x^1 = x^2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, is an equilibrium. Observe that in $\mathbf{g}_{\mathcal{N}\mathcal{I}^*}$, $\max_{i,s^i} \{\lambda^i(s^i)\} = 1$. We therefore get that \mathbf{x} is an ϵ -equilibrium in \mathbf{g} , for some $\epsilon \leq 1$.

5.5 Conclusion and perspectives

We have introduced the classes of uniform and non-interactive games. Using an appropriate projection operator, we classify games in terms of uniform games with zero-constant. More precisely, games that share equal payoff differences, in deviations realized by the rest of the players, are represented by the same uniform with zero-constant game. Under the natural inner product, the structure of the proposed decomposition enables the appearance of non interactive games as the orthogonal complement of the uniform with zero-constant component, in the space of games. The main feature of this debut class of games, is that each player has a best choice, regardless of the other players' choices and thus, players do not need to interact each other in order to take their decisions. The uniform with zero constant component-game forms the interactive part of an arbitrary game.

A similar projection operator helps us to define a strategical equivalence between games in terms of their Nash equilibria. Games that share the same Nash equilibrium set are represented by the same normalized game. It follows a direct sum decomposition where the orthogonal complement of normalized games, with respect to the standard inner product, consists of games in which unilateral deviations are identically zero. This decomposition distinguishes the strategic and non strategic parts of a given game. Non strategic games belong to the uniform class and thus, they can be decomposed in non strategic with zero constant and constant games. Consequently, the *strategic-interactive* part of a given game consists of the intersection of uniform and normalized components in the corresponding decompositions.

We introduce total-sum zero games and using a suitable projection operator, a decomposition result follows, in which games are uniquely decomposed in two components; namely, the total-sum zero and constant components. Hence, we establish a classification of games in which a total-sum zero game represents games that share the same Nash equilibrium set and equal payoff differences for all players, in any coalitional deviation comprised of all the players but the i -th one. Constant games are in the intersection of non-interactive and uniform games, leading us to define an offspring canonical decomposition of games in three components, where the former is the uniform with zero-constant games, the second is the non interactive normalized games and the latter one consists of constant games. In this context, explicit expressions of the closest uniform and non interactive games to an arbitrary game, are provided. Furthermore, it is given a characterization of the approximate equilibria of a given game in terms of the uniformly mixed strategy profile that is an equilibrium in its closest uniform game. An additional characterization of its approximate equilibrium set can be given through the equilibria in dominant strategies of its closest non interactive game. If the non-interactive component-game has more than one equilibria, then all of them are ϵ -equilibria in the original game with the same ϵ , since in non interactive games, equilibria in dominant strategies have a unique payoff.

Chapter 6

Metrics and harmonicity in games

Abstract¹. In this work we generalize the orthogonal decomposition of games presented in Candogan et al. [2011] to several metrics induced in the space of games. We introduce a family of inner products in the space of finite games, which then leads to the definition of μ -harmonic games. Using the Helmholtz-Hodge decomposition theorem we are able to establish an orthogonal μ -decomposition of finite games in the *potential*, *μ -harmonic* and *non-strategic* components. The decomposition components admit distinct and tractable equilibria; we prove that the selected inner product yields a completely mixed strategy profile that appears as an equilibrium in the μ -harmonic component and that μ -harmonic games do not generically have pure equilibria. We further generalize well-known results concerning bimatrix harmonic games by showing that the sets of correlated and mixed equilibria coincide and if players have equal number of actions, the equilibrium must be unique.

6.1 Introduction

Games including players' preferences that are aligned with a global function are called *potential games* and they have been introduced by Monderer and Shapley [1996]. Such games always admit a pure Nash equilibrium. Games in which all players obtain identical payoffs at any strategy profile, are called *common-interest games* and constitute a subset of the potential class. Several notions of strategic equivalence have been developed in the literature in order to describe static and dynamic properties of games. For instance, Moulin and Vial [1978] define strategically equivalent games to be games that share the same better-response correspondence and then study the equilibrium properties of games which are equivalent to zero-sum games. Morris and Ui [2004] propose to use a different notion of strategical equivalence according to which games are defined as strategically equivalent if best responses of each player are identical. Yet another, more general definitions appear in Candogan et al. [2011] and Hwanga and Rey-Belletb [2014], who call two games *strategically equivalent* if the payoff differences of each player, among two actions, fixing other players' actions, are identical. Note that the latter notion of strategical equivalence implies the former two. In this context, it follows that potential games are always strategically equivalent to common interest games. In order to determine

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a representative of each equivalence class, the notion of *normalized* game has been proposed in the literature. In a such game, each player's payoffs, given the other players' actions, add up to zero. Since the space of finite games can be viewed as a vector space, the difference between a given game and its associated normalized one, is defined as a *non-strategic* game. In such a game each player's payoffs depend only on the actions selected by his adversaries. On the other hand, conflicting strategic interactions are reflected in *harmonic games* that first appeared in Candogan et al. [2011]. In this context, the famous *rock-paper-scissors* and *matching pennies* games are examples of harmonic games. The *uniformly mixed strategy* consists in selecting with equal probability each action. Harmonic games do not have pure Nash equilibria and the *uniformly mixed strategy profile* appears as an equilibrium in this class of games. In addition, if we restrict ourselves to games where players have equal number of actions, harmonic games are zero-sum games². Clearly potential and harmonic games admit distinct and tractable equilibrium properties.

Historical review on the games decomposition

In the context of non-cooperative game theory, decomposition methods have been proven to be useful tools for the characterization of potential games and their tractable equilibrium analysis. They also provide an extended analysis of the dynamics, such as the gradient evolutionary dynamics based on potential games and near-potential games as in Candogan et al. [2011]. Furthermore, endowing the space of games with an inner product, if the achieved decomposition is orthogonal, then given a game, one may provide characterizations of its approximate equilibria set through the equilibria of its components using the distance between games with respect to the norm induced by the inner product.

In Sandholm [2010], for each subset of players, the authors define a component game and obtain a decomposition of normal form games. They suggest alternative methods to verify whether a game is potential or not. In Kalai and Kalai [2010], the proposed decomposition relies on identifying zero-sum and common interest components for a given game. The authors deal with cooperation-related issues that emerge in games with strategic players. In general, the common interest component is a potential game, and it can be used to approximate a given game with a potential game. However, in Candogan et al. [2011], the authors provide an example, in which despite the original game is a potential game, the zero-sum and common interest game decomposition may lead to a potential game which is much farther away than the closest potential game. Recently, Jessie and Saari [2013] provide a *strategic* and *behavioral* decomposition of games in which all players have two actions, highlight that certain solution concepts are determined by a game's strategic part, or influenced by the behavioral portion. More recently, Hwanga and Rey-Belletb [2014] view the space of games as a Hilbert space and prove several decomposition theorems for arbitrary games identifying components such as potential games and games that are strategically equivalent to zero-sum games. They further extend their results to games with continuous strategy sets. Using this decomposition, they also derive a new characterization of zero-sum equivalent games and provide an alternative proof

²In multi-player zero-sum games, at each action profile the sum of the players' payoffs are equal to zero.

for the well-known characterization of potential games by [Monderer and Shapley \[1996\]](#).

In the leading paper of [Candogan et al. \[2011\]](#), the key point is a novel flow representation of the preference structure in finite games, which enables to outline the fundamental characteristics in preferences that lead to potential games. In doing so, they associate an undirected graph to each finite game. The idea of representing finite games as graphs appeared in [Goemans et al. \[2005\]](#) and [Christodoulou et al. \[2012\]](#), to study different solution concepts in games. Here, we follow [Candogan et al. \[2011\]](#). Accordingly, nodes represent the set of action profiles and edges are defined between nodes that differ in the action of a single player. Their main result relies on the *Helmholtz-Hodge decomposition theorem* applied to graphs. The authors provide a canonical orthogonal decomposition of an arbitrary game into a *potential*, a *harmonic* and a *non-strategic* components, each with its distinct equilibrium properties. Endowing the space of real-valued functions defined on the nodes of the graph with the standard inner product, they then end up providing the space of finite games with an inner product naturally induced by the standard one. Therefore, distance between games can be defined so that for each finite game, the authors furnish a characterization of its approximate equilibria in terms of the pure Nash equilibria of its closest potential game.

The Helmholtz-Hodge decomposition and its applications

Hodge theory is a synthesis of topology, geometry and analysis that has been developed in the context of Riemannian manifolds. The *Helmholtz-Hodge decomposition* allows separating any vector field into the sum of three uniquely defined components: curl-free, divergence-free and harmonic. Using this powerful tool, several interesting results on the domains of physics, economic and social sciences have been established. For instance, [Bartholdi et al. \[2012\]](#) develop Hodge theory on metric spaces endowed with a probability measure. In particular, in graph theory, the Helmholtz-Hodge decomposition theorem enables the decomposition of a flow on a graph into globally consistent, locally consistent (but globally inconsistent), and locally inconsistent components. The globally consistent component reflects a gradient flow which is always curl-free³, while the locally consistent flow represents flows around global cycles, that is harmonic flows. Lastly, the locally inconsistent component corresponds to flows around 3-cliques of the graph, the so-called divergence-free flow. Using this interpretation, [Jiang et al. \[2011\]](#) achieve an implementation of Hodge theory in Statistical Ranking. From raw ranking data, the authors define pairwise rankings, represented as edge flows on an appropriate graph, and they show that every edge flow can be decomposed into three orthogonal components, where each one reflects distinct ranking properties. In the orthogonal decomposition result of the finite games space, presented in [Candogan et al. \[2011\]](#), the payoff differences for the deviating players along the edges define a flow on that associated graph. The authors remove the non-strategic component and they then apply the Helmholtz-Hodge flow-decomposition of graphs to the remainder. Flows resulted from games are always locally consistent since the only triangles of the graph are

³It corresponds to the well-known identity in vector-calculus, $\text{curl} \circ \text{grad} = 0$.

due to unilateral deviations of the same player. Therefore, the Helmholtz-Hodge decomposition yields the two remaining components of a game: the potential component that corresponds to a gradient flow and the harmonic one that reflects global cycles.

Main results

In the leading paper of Candogan et al. [2011], the decomposition result relies on the standard inner product. Here, we introduce a family of inner products on the set of real-valued functions defined on the nodes of a graph. Similarly, we define the inner products on the space of flows and triangular flows. Depending on the selected inner product we further define the adjoints of the gradient and curl operators. Under these definitions, we use the Helmholtz-Hodge decomposition theorem of the space of flows defined on an undirected graph. We provide a normalization of finite games with respect to the chosen inner product and as a consequence, it is possible to identify Nash equilibria of an arbitrary game through its representative normalized game. We then provide a generalized definition of harmonic games depending on the selected inner product and prove that the space of finite games is the direct orthogonal sum of potential, harmonic and non-strategic games. We secondly show that a completely mixed strategy profile induced by the chosen inner product appears as a Nash equilibrium in the harmonic component. If all the coefficients of the selected inner product are equal to one, our decomposition reduces to the one of Candogan et al. [2011]. We then study the class of bimatrix games and generalize the results of Candogan et al. [2011] by showing that in harmonic bimatrix games correlated equilibria are Nash equilibria and that in the particular case of games with equal number of actions, the completely mixed strategy profile induced by the chosen inner product is the unique correlated and Nash equilibrium.

Fixing an inner product in the space of games induces a norm that allows us to quantify the distance between games. Hence, given an arbitrary finite game, one may find closed-form expressions for the *closest* potential and harmonic games with respect to the selected inner product. Consequently, it is possible to characterize the approximate equilibria set of a given game in terms of the completely mixed equilibrium and the pure equilibria of the closest harmonic and potential games respectively. With respect to our decomposition result, given a finite game, one may choose an appropriate inner product in order to establish an advantaged decomposition that allows us to achieve *good* approximations of its equilibria set through the distinct equilibria of its components.

Structure of the paper

In Section 6.2, we introduce a family of inner products on the space of real-valued bounded functions defined over the nodes of an undirected graph and we then study the Helmholtz decomposition of the graph flows. Section 6.3 deals with a first decomposition result of the space of finite games. Section 6.4 presents our main results and Section 6.5 concludes and outlines future research.

6.2 Helmholtz-Hodge decomposition on graphs

This section is largely based on Candogan et al. [2011]. We first provide a brief overview of the notation and tools required for the analysis of flows on graphs using operators related to the Helmholtz-Hodge decomposition. We endow the space of real-valued functions defined on the nodes of an undirected graph and the sets of flows and triangular flows with a family of inner products. We then provide, for each fixed inner product, the corresponding Helmholtz-Hodge decomposition of its graph flow.

6.2.1 Graphs and their chain complex

Given an undirected graph $Gr = (V, E)$, where V stands for the set of nodes and E for the set of edges, one can define the set of real-valued functions on the nodes of the graph, i.e., $C_0 = \{f | f : V \rightarrow \mathbb{R}\}$ and the set of flows on this graph as the set of skew-symmetric matrices denoted by C_1 ; i.e., a flow on the edges of the graph is a mapping $X : V \times V \rightarrow \mathbb{R}$ such that for all $u, v \in V$ we have $X(u, v) = -X(v, u)$ and $X(u, v) = 0$ if $(u, v) \notin E$. We denote the set of 3-cliques of the graph by $T = \{(u, v, w) : (u, v), (v, w), (w, u) \in E\}$. Let us consider a permutation of three elements, denoted by $\theta : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$. The *triangular flows* $\Psi : V \times V \times V \rightarrow \mathbb{R}$ are functions defined for all $(u, v, w) \in T$ as $\Psi(u_\theta, v_\theta, w_\theta) = \text{sign}(\theta)\Psi(u, v, w)$ and $\Psi(u, v, w) = 0$, if $(u, v, w) \notin T$. C_2 stands for the set of all possible triangular flows. Notice that C_0, C_1 and C_2 have a natural structure of vector spaces, with the obvious operations of addition and scalar multiplication. Let $W : V \times V \rightarrow \{0, 1\}$ be the indicator function for the edges of the graph, i.e.,

$$W(u, v) = \begin{cases} 1, & \text{if } (u, v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

The *gradient operator* $\delta_0 : C_0 \rightarrow C_1$ is defined for all $f \in C_0$, such that

$$(\delta_0 f)(u, v) = W(u, v)(f(v) - f(u)), \quad u, v \in V. \quad (6.2.1)$$

We next present the *curl operator* $\delta_1 : C_1 \rightarrow C_2$, which is defined for all $X \in C_1$ by:

$$(\delta_1 X)(u, v, w) = \begin{cases} X(u, v) + X(v, w) + X(w, u), & \text{if } (u, v, w) \in T \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6.2.1. *The vector spaces C_0, C_1, C_2 connected by the operators δ_0 and δ_1 are a cochain complex, i.e.,*

$$0 \rightarrow C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2 \rightarrow 0.$$

Proof. It suffices to prove that $\delta_1 \circ \delta_0 = 0$. Indeed, for all $(u, v, w) \in T$,

$$\begin{aligned} (\delta_1(\delta_0 f))(u, v, w) &= W(u, v)(f(v) - f(u)) + W(v, w)(f(w) - f(v)) + W(w, u)(f(u) - f(w)) \\ &= f(u)(W(w, u) - W(u, v)) + f(v)(W(u, v) - W(v, w)) + f(w)(W(v, w) - W(w, u)) \\ &= 0, \end{aligned}$$

since $W(u, v) = W(v, w) = W(w, u) = 1$. □

6.2.2 A family of inner products and related results

Let us fix $\mu : V \rightarrow \mathbb{R}_+^*$. We introduce the following inner products in C_0, C_1 and C_2 :

$$\langle f_1, f_2 \rangle_0 = \sum_{v \in V} \mu(v) f_1(v) f_2(v), \quad (6.2.2)$$

$$\langle X, Y \rangle_1 = \frac{1}{2} \sum_{(u,v) \in E} \mu(u) \mu(v) X(u, v) Y(u, v), \quad (6.2.3)$$

$$\langle \Psi, Z \rangle_2 = \frac{1}{6} \sum_{(u,v,w) \in T} \mu(u) \mu(v) \mu(w) \Psi(u, v, w) Z(u, v, w). \quad (6.2.4)$$

Recall that for given inner products $\langle \cdot, \cdot \rangle_k$ on C_k with $k \in \{0, 1, 2\}$, the adjoint of δ_k , denoted by $\delta_k^* : C_{k+1} \rightarrow C_k$, is the unique linear operator satisfying

$$\langle \delta_k f_k, F_{k+1} \rangle_{k+1} = \langle f_k, \delta_k^* F_{k+1} \rangle_k, \quad (6.2.5)$$

for all $f_k \in C_k, F_{k+1} \in C_{k+1}$.

Proposition 6.2.2. *For any $u \in V$, the adjoint of the gradient operator, $\delta_0^* : C_1 \rightarrow C_0$ is given by*

$$\delta_0^* X(u) = - \sum_{v:(u,v) \in E} \mu(v) X(u, v) = - \sum_{v \in V} \mu(v) W(u, v) X(u, v).$$

Proof. Let us consider the orthonormal basis of C_0 , $(\varepsilon_w)_{w \in V}$, defined with respect to the inner product introduced in (6.2.2) as follows:

$$\varepsilon_w(u) = \begin{cases} \frac{1}{\sqrt{\mu(w)}}, & \text{if } w = u, \\ 0, & \text{otherwise.} \end{cases}$$

Given $X \in C_1$, by definition for all $u_0 \in V$, we have:

$$\delta_0^* X(u_0) = \sum_{w \in V} \langle \varepsilon_w, \delta_0^* X \rangle_0 \varepsilon_w(u_0)$$

and thus, $\delta_0^* X(u_0) = \frac{1}{\sqrt{\mu(u_0)}} \langle \varepsilon_{u_0}, \delta_0^* X \rangle_0$. Using (6.2.5), we get:

$$\begin{aligned} \delta_0^* X(u_0) &= \frac{1}{\sqrt{\mu(u_0)}} \langle \delta_0 \varepsilon_{u_0}, X \rangle_1 \\ &= \frac{1}{2\sqrt{\mu(u_0)}} \sum_{u,v \in V} \mu(u) \mu(v) (\delta_0 \varepsilon_{u_0})(u, v) X(u, v) \\ &= \frac{1}{2\sqrt{\mu(u_0)}} \left(\sum_{u,v \in V} \mu(u) \mu(v) W(u, v) \varepsilon_{u_0}(v) X(u, v) - \sum_{u,v \in V} \mu(u) \mu(v) W(u, v) \varepsilon_{u_0}(u) X(u, v) \right) \end{aligned}$$

Since $\varepsilon_{u_0}(v) = \varepsilon_{u_0}(u) = \frac{1}{\sqrt{\mu(u_0)}}$ if and only if $u_0 = v = u$, it clearly follows:

$$\begin{aligned} \delta_0^* X(u_0) &= \frac{1}{2} \left(\sum_{u \in V} \mu(u) W(u, u_0) X(u, u_0) - \sum_{v \in V} \mu(v) W(u_0, v) X(u_0, v) \right) \\ &= -\frac{1}{2} \sum_{u \in V} \mu(u) W(u_0, u) X(u_0, u) - \frac{1}{2} \sum_{v \in V} \mu(v) W(u_0, v) X(u_0, v) \\ &= -\frac{1}{2} \sum_{v: (u_0, v) \in E} \mu(v) X(u_0, v) - \frac{1}{2} \sum_{v: (u_0, v) \in E} \mu(v) X(u_0, v) \\ &= - \sum_{v: (u_0, v) \in E} \mu(v) X(u_0, v), \end{aligned}$$

where the first equality follows by summing over v and u respectively and the second one is due to the skew-symmetric structure of X . \square

Note that given a node $u \in V$, the value of $-(\delta_0^* X)(u)$ represents the total flow leaving u . The operator $-\delta_0^*$ is said to be *divergence operator*.

Proposition 6.2.3. *For any $(u, v) \in E$, the adjoint of the curl operator, $\delta_1^* : C_2 \rightarrow C_1$ is given by*

$$\delta_1^* \Psi(u, v) = \sum_{w \in V: (u, v, w) \in T} \mu(w) \Psi(u, v, w).$$

Proof. Let us consider the orthonormal basis of C_1 , $(\varepsilon_{(w_1, w_2)})_{(w_1, w_2) \in E}$ defined with respect to the inner product introduced in (6.2.3):

$$\varepsilon_{(w_1, w_2)}(u, v) = \begin{cases} \frac{1}{\sqrt{\mu(w_1)\mu(w_2)}}, & \text{if } (w_1, w_2) = (u, v) \\ 0, & \text{otherwise.} \end{cases}$$

Given $\Psi \in C_2$, by definition for all $(u_0, v_0) \in E$ we have:

$$\delta_1^* \Psi(u_0, v_0) = \sum_{(w_1, w_2) \in E} \langle \varepsilon_{(w_1, w_2)}, \delta_1^* \Psi \rangle_1 \varepsilon_{(w_1, w_2)}(u_0, v_0)$$

and thus, $\delta_1^* \Psi(u_0, v_0) = \frac{1}{\sqrt{\mu(u_0)\mu(v_0)}} \langle \varepsilon_{(u_0, v_0)}, \delta_1^* \Psi \rangle_1$. Using (6.2.5) and adapting the argument of the previous proof, one gets:

$$\begin{aligned} \delta_1^* \Psi(u_0, v_0) &= \frac{1}{\sqrt{\mu(u_0)\mu(v_0)}} \langle \delta_1 \varepsilon_{(u_0, v_0)}, \Psi \rangle_2 \\ &= \frac{1}{6\sqrt{\mu(u_0)\mu(v_0)}} \sum_{(u, v, r) \in T} \mu(u)\mu(v)\mu(r) (\delta_1 \varepsilon_{(u_0, v_0)})(u, v, r) \Psi(u, v, r). \end{aligned}$$

By definition of the curl operator,

$$\begin{aligned} \sum_{(u,v,r) \in T} \mu(u)\mu(v)\mu(r)(\delta_1 \varepsilon_{(u_0,v_0)})(u,v,r) \Psi(u,v,r) &= \sum_{(u,v,r) \in T} \mu(u)\mu(v)\mu(r) \varepsilon_{(u_0,v_0)}(u,v) \Psi(u,v,r) \\ &+ \sum_{(u,v,r) \in T} \mu(u)\mu(v)\mu(r) \varepsilon_{(u_0,v_0)}(v,r) \Psi(u,v,r) \\ &+ \sum_{(u,v,r) \in T} \mu(u)\mu(v)\mu(r) \varepsilon_{(u_0,v_0)}(r,u) \Psi(u,v,r). \end{aligned}$$

By definition of $(\varepsilon_{(w_1,w_2)})_{(w_1,w_2) \in E}$, the right hand side of the above equation can be re-written as follows:

$$2\sqrt{\mu(u_0)\mu(v_0)} \left(\sum_{r:(u_0,v_0,r) \in T} \mu(r) \Psi(u_0,v_0,r) + \sum_{u:(u,u_0,v_0) \in T} \mu(u) \Psi(u,u_0,v_0) + \sum_{v:(v_0,v,u_0) \in T} \mu(v) \Psi(v_0,v,u_0) \right)$$

Therefore,

$$\begin{aligned} \delta_1^* \Psi(u_0,v_0) &= \frac{1}{3} \left(\sum_{r:(u_0,v_0,r) \in T} \mu(r) \Psi(u_0,v_0,r) + \sum_{u:(u,u_0,v_0) \in T} \mu(u) \Psi(u,u_0,v_0) + \sum_{v:(u_0,v,v_0) \in T} \mu(v) \Psi(u_0,v,v_0) \right) \\ &= \sum_{w:(u_0,v_0,w) \in T} \mu(w) \Psi(u_0,v_0,w). \end{aligned}$$

□

By taking the adjoints we clearly have $\delta_0^* \circ \delta_1^* = 0$, so the vector spaces C_0, C_1, C_2 connected by the operators δ_0^* and δ_1^* are a chain complex, i.e.,

$$0 \leftarrow C_0 \xleftarrow{\delta_0^*} C_1 \xleftarrow{\delta_1^*} C_2 \leftarrow 0.$$

We next present the *Laplace* operator $\Delta_k : C_k \rightarrow C_k$, which is defined by:

$$\Delta_k = \delta_k^* \circ \delta_k + \delta_{k-1} \circ \delta_{k-1}^*. \quad (6.2.6)$$

The value of $\Delta_0 f$ corresponding to the node u is given by

$$\begin{aligned} \Delta_0 f(u) &= \delta_0^* \circ \delta_0 f(u,v) = \delta_0^* \left(W(u,v)(f(v) - f(u)) \right) \\ &= - \sum_{v \in V} \mu(v) W^2(u,v)(f(v) - f(u)) \\ &= \sum_{v \in V} \mu(v) W(u,v)(f(u) - f(v)). \end{aligned}$$

Δ_0 gives a measure of the aggregate value of a node over all its neighbors. The Laplace equation refers to $\Delta_0 f = 0$ and then, f is said to be a *harmonic function*. The equation $\Delta_1 X = 0$ is known in the literature as the *Helmholtz-Hodge equation* and then, X is said to be a *harmonic flow*. Given $X \in C_1$, if there exists $f \in C_0$ such that, $X = \delta_0 f$, then f is a *potential function* for X and X is said to be a *potential*

flow. When $\delta_1 X = 0$, X is said to be a *locally consistent flow*. Since $\delta_1 \circ \delta_0 = 0$, it follows that every potential flow X satisfies: $\delta_1 X = 0$. We further mention that if the graph $Gr = (V, E)$ has a nontrivial topology, there exist flows that are locally consistent, but not potential. Consequently, $\ker(\delta_1)$ consists of the locally consistent flows that may be or may not be potential.

6.2.3 The Helmholtz-Hodge decomposition of graph flows

Notation. In the sequel, for any $X_1, X_2 \subseteq C_1$, the notation $X_1 \oplus X_2$ stands for the direct orthogonal sum of X_1 and X_2 .

A basic flow-decomposition theorem known as the Helmholtz-Hodge decomposition, is presented below. The theorem states that any graph flow can be decomposed into three orthogonal flows. For completeness, we provide a proof of the result.

Theorem 6.2.4. *The vector space of flows C_1 admits an orthogonal decomposition with respect to the inner product introduced in (6.2.3), i.e.,*

$$C_1 = \text{Im}(\delta_0) \oplus \ker(\Delta_1) \oplus \text{Im}(\delta_1^*), \quad (6.2.7)$$

where $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$.

Proof. We first show that $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$. By (6.2.6),

$$\ker(\Delta_1) = \ker(\delta_1^* \circ \delta_1 + \delta_0 \circ \delta_0^*).$$

Therefore, it is immediate that $\ker(\delta_1) \cap \ker(\delta_0^*) \subseteq \ker(\Delta_1)$. In order to show the converse inclusion, let us consider $X \in \ker(\Delta_1)$. Then,

$$\delta_1^* \circ \delta_1(X) + \delta_0 \circ \delta_0^*(X) = 0.$$

It clearly follows that

$$\begin{aligned} 0 &= \langle \delta_1^* \circ \delta_1 X + \delta_0 \circ \delta_0^* X, X \rangle_1 = \langle \delta_1^* \circ \delta_1 X, X \rangle_1 + \langle \delta_0 \circ \delta_0^* X, X \rangle_1, \\ &= \langle \delta_1 X, \delta_1 X \rangle_1 + \langle \delta_0^* X, \delta_0^* X \rangle_1, \\ &= \|\delta_1 X\|^2 + \|\delta_0^* X\|^2, \end{aligned}$$

which necessarily implies $\delta_1 X = 0$ and $\delta_0^* X = 0$ and thus, this part of the proof is completed.

For the rest of the proof, since $C_1 = \text{Ker}(\delta_0^*) \oplus \text{Im}(\delta_0)$, it suffices to show that $\text{Ker}(\delta_0^*) = (\text{Ker}(\delta_0^*) \cap \text{Ker}(\delta_1)) \oplus \text{Im}(\delta_1^*)$. In doing so, recall that $C_1 = \text{Ker}(\delta_1) \oplus \text{Im}(\delta_1^*)$ and let $X \in \text{Ker}(\delta_0^*)$. Then, there exist unique $U \in \text{Ker}(\delta_1)$ and $V \in \text{Im}(\delta_1^*)$, such that $X = U + V$. Since $V \in \text{Im}(\delta_1^*)$, there exists $W \in C_2$ such that $\delta_1^* W = V$. Thus, $X = U + \delta_1^* W$, which implies $\delta_0^* X = \delta_0^* U + \delta_0^* \delta_1^* W$. Hence, $U \in \text{Ker}(\delta_0^*) \cap \text{Ker}(\delta_1)$ and the proof is completed. \square

Remark 6.2.5. Using the introduced vocabulary, let us summarize the interpretation of each component of the Helmholtz-Hodge decomposition of C_1 :

- $\text{Im}(\delta_0)$: potential flows.
- $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$: harmonic flows, which are non potential but locally consistent. Observe that $\ker(\delta_1)$ consists of locally consistent flows that may or may not be potential, while $\ker(\delta_0^*)$ consists of non potential flows that may or may not be locally consistent.
- $\text{Im}(\delta_1^*)$: locally inconsistent flows (since $\text{Im}(\delta_1^*) = \ker(\delta_1)^\perp$).

Given a graph $Gr = (V, E)$, if for any node, the total flow leaving the node is zero, i.e., $\delta_0^* X = 0$, then we say that X satisfies the *flow conservation condition*. The Helmholtz-Hodge decomposition theorem implies that X satisfies the flow conservation condition only when

$$X \in \ker(\delta_0^*) = \text{Im}(\delta_0)^\perp = \ker(\Delta_1) \oplus \text{Im}(\delta_1^*).$$

Thus, the flow conservation condition is satisfied for harmonic flows and locally inconsistent flows but not for potential flows.

6.3 The space of finite games

6.3.1 Basic definitions and notations

A *finite game*, consists of:

- A finite set of players, denoted by $N = \{1, \dots, n\}$.
- Strategy spaces: A finite set of actions S^i , for every $i \in N$. The joint strategy space is denoted by $S = \prod_{i \in N} S^i$.
- Payoff functions $g^i : S \rightarrow \mathbb{R}$, $i \in N$.

The notation $s^i \in S^i$ stands for an action of player i . A collection of players' actions is given by $\mathbf{s} = (s^i)_{i \in N}$ and is referred to as an action profile. A collection of actions for all players but the i -th one, is denoted by $s^{-i} \in S^{-i}$. Let $\mathbf{P}(S^i)$ be the simplex of probabilities on the set of actions of player i and denote by $\mathbf{P}_0(S^i)$ the set of completely mixed strategies of player i . A completely mixed strategy profile is denoted by $\eta = (\eta^i)_{i \in N} \in \prod_i \mathbf{P}_0(S^i)$. Notice that for any $\mathbf{s} \in S$, $\eta(\mathbf{s}) = \prod_{i \in N} \eta^i(s^i)$ and for all $i \in N$, $\eta^{-i}(s^{-i}) = \prod_{j \neq i} \eta^j(s^j)$. Given a set of players N and a strategy set S^i for each $i \in N$, \mathbf{g} stands for the corresponding game, where each player i has the payoff function g^i . For given N and S , every game is uniquely defined by its set of payoff functions. Hence, the space of games with set of players N and strategy profiles S can be identified as $\mathcal{G} \cong \mathbb{R}^{n|S|}$. Consequently, $\dim(\mathcal{G}) = n \prod_{i \in N} h_i$, where h_i stands for the number of actions of player i .

6.3.2 First decomposition

For all $i \in N$, let $\mu^i : S^i \rightarrow \mathbb{R}_+^*$ and for all $\mathbf{s} \in S$, we put $\mu(\mathbf{s}) = \prod_{i \in N} \mu^i(s^i)$. In this paragraph, we provide a first decomposition result of the space of games, which depends on the coefficients μ^i . The first component of the decomposition admits μ -centered payoffs and it will be said *μ -normalized games* and the second one consists of the *non-strategic games*.

In doing so, let C_0 be the vector space of all real-valued functions $f : S \rightarrow \mathbb{R}$. The payoff functions of each player can be viewed as elements of C_0 , i.e., $g^i \in C_0$ for all $i \in N$ and so a game \mathbf{g} can be seen as an element of C_0^n . Hence, the space of games with set of players N and action profiles S can be identified as $\mathcal{G} \cong C_0^n$.

The class of games in which each player's payoff depends only on the actions selected by the rest of players, is called *non-strategic games* and it is formally defined as follows:

Definition 6.3.1. A *non-strategic game* \mathbf{g} is a finite game in which for all $i \in N$, there exists a function $\ell^i : S^{-i} \rightarrow \mathbb{R}$, such that for all $s^{-i} \in S^{-i}$, $\ell^i(s^{-i}) = g^i(s^i, s^{-i})$. The space of non-strategic games will be denoted by \mathcal{NS} .

Definition 6.3.2. The games \mathbf{g}_1 and \mathbf{g}_2 are *strategically equivalent*, if for all $i \in N$ and all $s^{-i} \in S^{-i}$

$$g_1^i(s^i, s^{-i}) - g_1^i(t^i, s^{-i}) = g_2^i(s^i, s^{-i}) - g_2^i(t^i, s^{-i}),$$

for any $s^i, t^i \in S^i$. If \mathbf{g}_1 is equivalent to \mathbf{g}_2 then we denote $\mathbf{g}_1 \sim \mathbf{g}_2$.

Proposition 6.3.3. If $\mathbf{g}_1 \sim \mathbf{g}_2$ then the game defined as $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$ is a *non-strategic game*.

Proof. By Definition 6.3.2, for all $i \in N$, all $s^{-i} \in S^{-i}$ and all $s^i, t^i \in S^i$

$$g_1^i(s^i, s^{-i}) - g_2^i(s^i, s^{-i}) = g_1^i(t^i, s^{-i}) - g_2^i(t^i, s^{-i}),$$

proving the result. □

Clearly, equivalent games share the same equilibrium sets.

In order to fix a representative for strategically equivalent games with respect to μ , we present the notion of *μ -normalized games*.

Definition 6.3.4. A finite game \mathbf{g} is *μ -normalized*, if for all $i \in N$ and all $s^{-i} \in S^{-i}$,

$$\sum_{s^i \in S^i} \mu^i(s^i) g^i(s^i, s^{-i}) = 0.$$

The space of μ -normalized games will be denoted by \mathcal{NO} .

Notation. In the sequel, for all $i \in N$ and all $s^i \in S^i$, we set $\bar{\mu}^i(s^i) = \frac{\mu^i(s^i)}{\sum_{s^i \in S^i} \mu^i(s^i)}$. Clearly $\sum_{s^i \in S^i} \bar{\mu}^i(s^i) = 1$. We also use the notation I_C for the identity operator defined on a vector space C .

We define the linear operator $\Lambda^i : C_0 \rightarrow C_0$ such that, for all $\mathbf{s} = (s^i, s^{-i}) \in S$,

$$\Lambda^i g^i(s^i, s^{-i}) = \sum_{t^i \in S^i} \bar{\mu}^i(t^i) g^i(t^i, s^{-i}). \quad (6.3.1)$$

We further define $\Lambda : C_0^n \rightarrow C_0^n$, such that $\Lambda \mathbf{g} = (\Lambda^1 g^1, \dots, \Lambda^n g^n)$, where $\mathbf{g} = (g^i)_{i \in N}$. Moreover, for each player i , the linear operator $\Pi^i : C_0 \rightarrow C_0$ is defined, for all $\mathbf{s} \in S$, as $\Pi^i = I_{C_0} - \Lambda^i$, i.e.,

$$\Pi^i g^i(s^i, s^{-i}) = g^i(s^i, s^{-i}) - \sum_{t^i \in S^i} \bar{\mu}^i(t^i) g^i(t^i, s^{-i}) \quad (6.3.2)$$

and we also define $\Pi : C_0^n \rightarrow C_0^n$, such that $\Pi \mathbf{g} = (\Pi^1 g^1, \dots, \Pi^n g^n)$.

Remark 6.3.5. By Definition 6.3.4, using (6.3.2) and definition of Π , $\mathcal{NO} = \text{Im}(\Pi)$.

Remark 6.3.6. By Definition 6.3.1, using (6.3.2) and definition of Π , $\mathcal{NS} = \text{Ker}(\Pi)$.

Lemma 6.3.7. Π^i and Λ^i are orthogonal projectors of the vector space C_0 .

Proof. By definitions, $\text{Ker}(\Pi^i) = \text{Im}(\Lambda^i)$ and $\text{Ker}(\Lambda^i) = \text{Im}(\Pi^i)$. Thus, it suffices to prove that $\Lambda^i \circ \Lambda^i = \Lambda^i$ and $\text{Ker}(\Lambda^i) \perp \text{Im}(\Lambda^i)$. Indeed,

$$\Lambda^i \circ \Lambda^i(g^i)(s^i, s^{-i}) = \sum_{t^i \in S^i} \bar{\mu}^i(t^i) (\Lambda^i(g^i)(t^i, s^{-i})) = \sum_{t^i \in S^i} \bar{\mu}^i(t^i) \sum_{t^i \in S^i} \bar{\mu}^i(t^i) g^i(t^i, s^{-i}) = \Lambda^i(g^i)(s^i, s^{-i}).$$

Now, let us consider $f_K \in \text{Ker}(\Lambda^i)$ and $f_I \in \text{Im}(\Lambda^i)$. From (6.3.1), for all $\mathbf{s} \in S$, there exists $\ell : S^{-i} \rightarrow \mathbb{R}$ such that $\ell(s^{-i}) = f_I(s^i, s^{-i})$. Therefore,

$$\langle f_K, f_I \rangle_0 = \sum_{\mathbf{s} \in S} \mu(\mathbf{s}) f_K(\mathbf{s}) f_I(\mathbf{s}) = \sum_{s^{-i} \in S^{-i}} \mu^{-i}(s^{-i}) \ell(s^{-i}) \sum_{s^i \in S^i} \mu^i(s^i) f_K(s^i, s^{-i}) = 0,$$

since $f_K \in \text{Ker}(\Lambda^i)$. □

Theorem 6.3.8. *The space of games is the direct sum of the μ -normalized and non-strategic games, i.e.,*

$$\mathcal{G} = \mathcal{NO} \oplus \mathcal{NS}.$$

Proof. The result clearly follows from Remark 6.3.5, Remark 6.3.6 and Lemma 6.3.7 using the definitions of Π and Λ . □

6.4 The Helmholtz-Hodge μ -decomposition on games

In this section we provide our main results. We first show that a completely mixed strategy profile induced by the selected inner product appears as an equilibrium in harmonic games and we prove coincidence of mixed and correlated equilibria in bimatrix harmonic games. Secondly, we prove that any finite game can be decomposed into a potential, a μ -harmonic and a non-strategic game and with respect to an appropriately chosen inner product on the space of games, we further establish orthogonality between the components.

6.4.1 Graphs generated by games and related results

In this paragraph, we first provide some preliminary results, which will be helpful in the sequel of the paper.

Graph of the game

Given a pair of action profiles $(\mathbf{s}, \mathbf{t}) \in S \times S$, if there exists a unique $i \in N$, such that $s^i \neq t^i$ then (\mathbf{s}, \mathbf{t}) will be referred to as a *comparable profile couple*. We denote the set of comparable profile couples by $A \subset S \times S$. A pair of action profiles that differ only in the strategy of player i is called a pair of *i -comparable profile couple*. The set of pairs of i -comparable profile couples is denoted by $A^i \subset S \times S$. Clearly, $\cup_i A_i = A$ and $A^i \cap A^j = \emptyset$ for any two different players i and j . Following [Candogan et al. \[2011\]](#), we associate to any given game \mathbf{g} , an undirected graph defined as follows:

$$\Gamma := Gr(N, S^1, \dots, S^n) = (S, A),$$

which can be equivalently seen as the disjoint union of the graphs Γ^i defined as:

$$\Gamma^i := Gr^i(N, S^1, \dots, S^n) = (S, A^i).$$

Short Chain Complex of a game

Let $W^i : S \times S \rightarrow \{0, 1\}$ be the indicator function for *i -comparable profiles couple*, i.e.,

$$W^i(\mathbf{s}, \mathbf{t}) = \begin{cases} 1, & \text{if } (\mathbf{s}, \mathbf{t}) \in A^i \\ 0, & \text{otherwise.} \end{cases}$$

Recalling that any pair of action profiles cannot be comparable by more than a single player, we have

$$W^i(\mathbf{s}, \mathbf{t})W^j(\mathbf{s}, \mathbf{t}) = 0, \quad \text{for all } j \neq i \quad \text{and } \mathbf{s}, \mathbf{t} \in S.$$

Consequently, the indicator function of comparable profile couples is $W = \sum_{i \in N} W^i$. The *partial gradient operator* associated to player i is denoted by $\delta_0^i : C_0 \rightarrow C_1$ and it is defined for all $\mathbf{s}, \mathbf{t} \in S$ as

$$\delta_0^i(g^i)(\mathbf{s}, \mathbf{t}) = W^i(\mathbf{s}, \mathbf{t})(g^i(\mathbf{t}) - g^i(\mathbf{s})). \quad (6.4.1)$$

It is easy to see that the gradient operator δ_0 presented in (6.2.1) satisfies $\delta_0 = \sum_{i \in N} \delta_0^i$. We associate to game \mathbf{g} the *joint gradient operator* $D : C_0^n \rightarrow C_1$, defined as

$$D(\mathbf{g}) = \sum_{i \in N} \delta_0^i g^i. \quad (6.4.2)$$

It follows easily by Definition 6.3.2 that $\mathbf{g}_1 \sim \mathbf{g}_2$ if and only if $D(\mathbf{g}_1) = D(\mathbf{g}_2)$.

Lemma 6.4.1. *For all $\mathbf{g} \in \mathcal{G}$, it holds true that $D(\mathbf{g}) \in \text{Ker}(\delta_1)$.*

Proof. Let $\mathbf{g} \in \mathcal{G}$. Then,

$$\delta_1 \circ D(\mathbf{g}) = \delta_1 \left(\sum_{i \in N} \delta_0^i g^i \right) = \delta_1 (\delta_0^1 g^1 + \dots + \delta_0^n g^n) = \delta_1 (\delta_0^1 g^1) + \dots + \delta_1 (\delta_0^n g^n) = 0,$$

since for all $i \in N$ and for all $\mathbf{s}, \mathbf{t}, \mathbf{r}$, such that $s^{-i} = t^{-i} = r^{-i}$ and $s^i \neq t^i \neq r^i$, one has:

$$\delta_1 (\delta_0^i (g^i)) = (g^i(\mathbf{t}) - g^i(\mathbf{s})) + (g^i(\mathbf{r}) - g^i(\mathbf{t})) + (g^i(\mathbf{s}) - g^i(\mathbf{r})) = 0.$$

□

Lemma 6.4.1 implies that the vector spaces C_0^n, C_1 and C_2 connected by the operator D and δ_1 are a cochain complex, i.e.,

$$0 \rightarrow C_0^n \xrightarrow{D} C_1 \xrightarrow{\delta_1} C_2 \rightarrow 0.$$

Lemma 6.4.2. *Let \mathbf{g} be a finite game. Then, for all $\mathbf{s} \in S$, it holds true that*

$$\delta_0^* (D(\mathbf{g})) (\mathbf{s}) = - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}).$$

Proof. By Definition 6.2.2,

$$\begin{aligned} \delta_0^* (D(\mathbf{g})) (\mathbf{s}) &= - \sum_{\mathbf{t}: (\mathbf{s}, \mathbf{t}) \in A} \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}) \\ &= - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) W(\mathbf{s}, \mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}) \\ &= - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) \sum_{i \in N} (W^i(\mathbf{s}, \mathbf{t}))^2 (g^i(\mathbf{t}) - g^i(\mathbf{s})) \\ &= - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) \sum_{i \in N} W^i(\mathbf{s}, \mathbf{t}) (g^i(\mathbf{t}) - g^i(\mathbf{s})) \\ &= - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}), \end{aligned}$$

where the fifth and latter equalities are due to (6.4.2), (6.4.1) and definition of the indicator function. □

The adjoint of δ_0^i , denoted by $\delta_0^{i*} : C_1 \rightarrow C_0$ is defined for all $\mathbf{s} \in S$ as

$$\delta_0^{i*}(X)(\mathbf{s}) = - \sum_{\mathbf{t}:(\mathbf{s},\mathbf{t}) \in A^i} \mu(\mathbf{t})X(\mathbf{s}, \mathbf{t}). \quad (6.4.3)$$

The divergence operator defined in Proposition 6.2.2 satisfies $\delta_0^* = \sum_{i \in N} \delta_0^{i*}$.

Notation. In the sequel, we set for all $i \in N$ and all $s^{-i} \in S^{-i}$, $\lambda^i(s^{-i}) = \sqrt{\mu^{-i}(s^{-i})\mu^i(S^i)}$, where $\mu^i(S^i) = \sum_{s^i \in S^i} \mu^i(s^i)$. Moreover,

$$\begin{aligned} \mathcal{N}\mathcal{S}^i &= \left\{ f \in C_0 : \exists \ell : S^{-i} \rightarrow \mathbb{R}, f(\mathbf{s}) = \ell(s^{-i}), \forall s^{-i} \in S^{-i} \right\} \\ \mathcal{N}\mathcal{O}^i &= \left\{ f \in C_0 : \forall \mathbf{s} \in S, \sum_{s^i \in S^i} \bar{\mu}^i(s^i) f(s^i, s^{-i}) = 0 \right\}. \end{aligned}$$

and we immediately have: $\mathcal{N}\mathcal{O}^i = \text{Im}(\Pi^i)$ and $\mathcal{N}\mathcal{S}^i = \text{Im}(\Lambda^i)$. By Theorem 6.3.8, we recall $\mathcal{N}\mathcal{O}^i \oplus \mathcal{N}\mathcal{S}^i = C_0$. One also has $\mathcal{N}\mathcal{S}^i = \text{Ker}(\delta^i) = \text{Ker}(\Pi^i) = \text{Im}(\Lambda^i)$ and $\mathcal{N}\mathcal{O}^i = \text{Im}(\Pi^i) = \text{Ker}(\Lambda^i)$.

Let us define the operator $M^i : C_0 \rightarrow C_0$ by

$$(M^i f)(\mathbf{s}) = \lambda^i(s^{-i})f(\mathbf{s}). \quad (6.4.4)$$

This is clearly an isomorphism on C_0 . We also define $M : C_0^n \rightarrow C_0^n$ as $M\mathbf{g} = (M^1 g^1, \dots, M^n g^n)$. We will say that a space is *stable* by M^i if the image of any vector of that space through M^i belongs to the space. The following properties of M^i clearly follow from its definition:

- $\mathcal{N}\mathcal{O}^i$ and $\mathcal{N}\mathcal{S}^i$ are stable by M^i ,
- $\Pi^i \circ M^i = M^i \circ \Pi^i$, $\Lambda^i \circ M^i = M^i \circ \Lambda^i$ and $M^i \circ \delta^i = \delta^i \circ M^i$,
- $M^{i*} = M^i$.

Lemma 6.4.3. *For all $i \in N$, it holds true that*

$$\delta^{i*} \circ \delta^i = (M^i)^2 \circ \Pi^i \quad (6.4.5)$$

Proof. Using (6.4.3), for all $f \in C_0$ and any $\mathbf{s} \in S$, we have:

$$\begin{aligned} (\delta^{i*}(\delta^i f))(\mathbf{s}) &= \delta^{i*}(W^i(\mathbf{s}, \mathbf{t})(f(\mathbf{t}) - f(\mathbf{s}))) \\ &= \sum_{t^i \in S^i} \mu^i(t^i) \mu^{-i}(s^{-i}) (f(s^i, s^{-i}) - f(t^i, s^{-i})) \\ &= \mu^{-i}(s^{-i}) \sum_{t^i \in S^i} \mu^i(t^i) (f(s^i, s^{-i}) - f(t^i, s^{-i})) \\ &= \mu^{-i}(s^{-i}) \sum_{t^i \in S^i} \mu^i(t^i) \sum_{t^i \in S^i} \bar{\mu}^i(t^i) (f(s^i, s^{-i}) - f(t^i, s^{-i})) \\ &= \mu^{-i}(s^{-i}) \mu^i(S^i) (\Pi^i f)(\mathbf{s}) \\ &= ((M^i)^2 \Pi^i f)(\mathbf{s}), \end{aligned}$$

which ends the proof. \square

Let \mathbf{g}_1 and \mathbf{g}_2 in \mathcal{G} . We define the inner product on \mathcal{G} as follows:

$$\langle \mathbf{g}_1, \mathbf{g}_2 \rangle_{\mathcal{G}} = \sum_{i \in N} \langle M^i g_1^i, M^i g_2^i \rangle_0, \quad (6.4.6)$$

where the inner product in the right-hand side is the inner product on C_0 , defined in (6.2.1).

Lemma 6.4.4. *The adjoint of the operator D , denoted by $D^* : C_1 \rightarrow C_0^n$ is given by*

$$D^* = \left(((M^1)^2)^{-1} \delta_0^{1*}, \dots, ((M^n)^2)^{-1} \delta_0^{n*} \right).$$

Proof. Using (6.4.2), for all $X \in C_1$ and all $\mathbf{h} \in C_0^n$,

$$\langle D^* X, \mathbf{h} \rangle_{\mathcal{G}} = \langle X, D\mathbf{h} \rangle_1 = \langle X, \sum_{i \in N} \delta_0^i h^i \rangle_1 = \sum_{i \in N} \langle X, \delta_0^i h^i \rangle_1 = \sum_{i \in N} \langle \delta_0^{i*} X, h^i \rangle_0$$

and using (6.4.6), it follows:

$$\langle D^* X, \mathbf{h} \rangle_{\mathcal{G}} = \sum_{i \in N} \langle M^i D_i^* X, M^i h^i \rangle_0 = \sum_{i \in N} \langle (M^i)^2 D_i^* X, h^i \rangle_0,$$

and the result follows immediately. \square

By taking the adjoints, $D^* \delta_1^* = 0$ and so the vector spaces C_0^n, C_1 and C_2 , connected by the operators D^* and δ_1^* , are a chain complex, i.e.,

$$0 \leftarrow C_0^n \xleftarrow{D^*} C_1 \xleftarrow{\delta_1^*} C_2 \leftarrow 0.$$

6.4.2 Classes of games with distinguished equilibria

In this section, we describe the main features of potential and harmonic games and we provide some useful characterizations of their equilibrium sets.

Potential games

Here, we briefly provide a description of the space of potential games. In these games, the preferences are aligned with a global potential function.

Definition 6.4.5. A *potential game* \mathbf{g} is a finite game such that, there exists a function $\varphi \in C_0$ satisfying for all $\mathbf{s}, \mathbf{t} \in S$,

$$D(\mathbf{g})(\mathbf{s}, \mathbf{t}) = (\delta_0 \varphi)(\mathbf{s}, \mathbf{t}).$$

The function φ is referred to as a *potential function* of the game and the space of potential games will be denoted by \mathcal{P} .

Next result is due to [Monderer and Shapley \[1996\]](#):

Theorem 6.4.6. *A potential game \mathbf{g} admits a pure Nash equilibrium.*

To see the above statement, we remark that since in finite games the potential function φ necessarily has a maximum, that maximum is a Nash equilibrium of \mathbf{g} .

Next, we present a particular class of potential games.

Definition 6.4.7. A *common interest game* \mathbf{g} is a finite game, such that for all $i, j \in N$ with $i \neq j$, one has $g^i(\mathbf{s}) = g^j(\mathbf{s})$ for all $\mathbf{s} \in S$. This space of games will be denoted by $\mathcal{C}\mathcal{I}$.

Harmonic games

In this section, we define the class of harmonic games and we then prove that a completely mixed strategy profile induced by the selected inner product is a Nash equilibrium in such a game.

Definition 6.4.8. A finite game \mathbf{g} is said to be a *harmonic game* if

$$D(\mathbf{g}) \in \text{Ker}(\delta_0^*).$$

Since $\delta_0^*(D(\mathbf{g})) = \sum_{i \in N} \delta_0^{i*} \delta_0^i g^i$, a finite game \mathbf{g} is harmonic if and only if $\sum_{i \in N} \delta_0^{i*} \delta_0^i g^i = 0$.

Proposition 6.4.9. *A finite game \mathbf{g} is a μ -normalized harmonic game if and only if*

$$\sum_{i \in N} (M^i)^2 g^i = 0 \quad \text{and} \quad \Pi \mathbf{g} = \mathbf{g}$$

Proof. Let $\mathbf{g} \in \mathcal{H} \cap \mathcal{N}\mathcal{O}$. Since $\mathbf{g} \in \mathcal{H}$, $\delta_0^*(D\mathbf{g})(\mathbf{s}) = 0$, for any $\mathbf{s} \in S$, i.e.,

$$\sum_{i \in N} \sum_{t^i \in S^i} \mu(t^i, s^{-i}) (g^i(s^i, s^{-i}) - g^i(t^i, s^{-i})) = 0.$$

Due to $\mathbf{g} \in \mathcal{N}\mathcal{O}$, by Definition 6.3.4 and Remark 6.3.5, $\Pi^i g^i = g^i$ for all $i \in N$. Moreover,

$$\sum_{i \in N} \sum_{t^i \in S^i} \mu(t^i, s^{-i}) g^i(s^i, s^{-i}) = \sum_{i \in N} \mu^i(S^i) \mu^{-i}(s^{-i}) g^i(s^i, s^{-i}) = \sum_{i \in N} (\lambda^i(s^{-i}))^2 g^i(s^i, s^{-i}),$$

and the result follows by definition of M^i in (6.4.4).

Conversely, let $\mathbf{g} \in \mathcal{G}$, such that for all $i \in N$, $\Pi^i g^i = g^i$. By Remark 6.3.5, $\mathbf{g} \in \mathcal{N}\mathcal{O}$ and by Definition 6.3.4, for all $s^{-i} \in S^{-i}$, $\sum_{t^i \in S^i} \mu^i(t^i) g^i(t^i, s^{-i}) = 0$. Since it is also assumed that for all $\mathbf{s} \in S$, $\sum_{i \in N} \sum_{t^i \in S^i} \mu(t^i, s^{-i}) g^i(s^i, s^{-i}) = 0$, it clearly follows that for all $\mathbf{s} \in S$, $\delta_0^*(D\mathbf{g})(\mathbf{s}) = 0$, which implies $\mathbf{g} \in \mathcal{H}$. \square

The following lemma will help us proving the main theorem of this section.

Lemma 6.4.10. *Let $X \in C_1$ and T be a subset of S . Then, it holds true that*

$$\sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in S} \mu(\mathbf{s})\mu(\mathbf{t})X(\mathbf{s}, \mathbf{t}) = \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s})\mu(\mathbf{t})X(\mathbf{s}, \mathbf{t})$$

Proof. We have:

$$\sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in S} \mu(\mathbf{s})\mu(\mathbf{t})X(\mathbf{s}, \mathbf{t}) = \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T} \mu(\mathbf{s})\mu(\mathbf{t})X(\mathbf{s}, \mathbf{t}) + \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s})\mu(\mathbf{t})X(\mathbf{s}, \mathbf{t})$$

and due to the skew-symmetric structure of X for any $\mathbf{s}, \mathbf{t} \in S$, $X(\mathbf{s}, \mathbf{t}) + X(\mathbf{t}, \mathbf{s}) = 0$ and thus the first term of the right hand side is equal to 0. \square

In the sequel and for the rest of this section, let us fix the completely mixed strategy profile $\eta = \{\eta^i\}_{i \in N}$ defined for all $i \in N$ and any $s^i \in S^i$ as

$$\eta^i(s^i) = \frac{(\mu^i(s^i))^2}{\sum_{s^i \in S^i} (\mu^i(s^i))^2} \in \mathbf{P}_0(S^i). \quad (6.4.7)$$

In terms of notations, it follows that for all $s^{-i} \in S^{-i}$,

$$\eta^{-i}(s^{-i}) = \frac{(\mu^{-i}(s^{-i}))^2}{\prod_{j \neq i} \sum_{s^j \in S^j} (\mu^j(s^j))^2} \in \prod_{j \neq i} \mathbf{P}_0(S^j).$$

Theorem 6.4.11. *Let \mathbf{g} be a harmonic game. Then, the completely mixed strategy profile $\eta = \{\eta^i\}_{i \in N}$ is a Nash equilibrium, i.e., for all $i \in N$ and any $r^i, t^i \in S^i$,*

$$\sum_{s^{-i} \in S^{-i}} \prod_{j \neq i} \eta^j(s^j) g^i(r^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} \prod_{j \neq i} \eta^j(s^j) g^i(t^i, s^{-i}).$$

Proof. Let \mathbf{g} be a harmonic game. Then, $\delta_0^* D(\mathbf{g})(\mathbf{s}) = 0$ for all $\mathbf{s} \in S$. Let $i \in N$ and let $r^i \in S^i$ and denote by T the subset of action profiles $\{(r^i, s^{-i}) | s^{-i} \in S^{-i}\}$ and by T^c its complement in the set S . Then, multiplying by $\mu(r^i, s^{-i})$ and adding on $s^{-i} \in S^{-i}$ implies:

$$\begin{aligned} 0 &= \sum_{s^{-i} \in S^{-i}} \mu(r^i, s^{-i}) \delta_0^* D(\mathbf{g})(r^i, s^{-i}) = \sum_{\mathbf{s} \in T} \mu(\mathbf{s}) \delta_0^* D(\mathbf{g})(\mathbf{s}) = - \sum_{\mathbf{s} \in T} \mu(\mathbf{s}) \left(\sum_{\mathbf{t} \in S} \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}) \right) \\ &= - \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in S} \mu(\mathbf{s}) \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}), \end{aligned}$$

where we used Lemma 6.4.2. Thus, in view of Lemma 6.4.10,

$$0 = \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s}) \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t})$$

Hence,

$$\begin{aligned}
0 &= - \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s}) \mu(\mathbf{t}) D(\mathbf{g})(\mathbf{s}, \mathbf{t}) \\
&= \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s}) \mu(\mathbf{t}) \sum_{j \in N} \delta_0^j(g^j)(\mathbf{s}, \mathbf{t}) \\
&= \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s}) \mu(\mathbf{t}) (\delta_0^i(g^i)(\mathbf{s}, \mathbf{t}) + \sum_{j \neq i} \delta_0^j(g^j)(\mathbf{s}, \mathbf{t})) \\
&= \sum_{\mathbf{s} \in T} \sum_{\mathbf{t} \in T^c} \mu(\mathbf{s}) \mu(\mathbf{t}) \delta_0^i(g^i)(\mathbf{s}, \mathbf{t}) + 0 \quad (\mathbf{s} \in T \text{ and } \mathbf{t} \in T^c \text{ are not } j\text{-comparable}) \\
&= \sum_{s^{-i} \in S^{-i}} \sum_{s^i \in S^i} \mu(r^i, s^{-i}) \mu(s^i, s^{-i}) (g^i(r^i, s^{-i}) - g^i(s^i, s^{-i})) \\
&= \mu^i(r^i) \sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 \sum_{s^i \in S^i} \mu^i(s^i) (g^i(r^i, s^{-i}) - g^i(s^i, s^{-i}))
\end{aligned}$$

Therefore,

$$0 = \sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 \sum_{s^i \in S^i} \mu^i(s^i) (g^i(r^i, s^{-i}) - g^i(s^i, s^{-i})).$$

It follows that for all $r^i \in S^i$,

$$\sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 \sum_{s^i \in S^i} \mu^i(s^i) g^i(r^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 \sum_{s^i \in S^i} \mu^i(s^i) g^i(s^i, s^{-i}),$$

and thus we equivalently have:

$$\sum_{s^i \in S^i} \mu^i(s^i) \sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 g^i(r^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 \sum_{s^i \in S^i} \mu^i(s^i) g^i(s^i, s^{-i}),$$

If we divide both parts of the equation by $\sum_{s^i \in S^i} \mu^i(s^i)$, we then get that for all $r^i \in S^i$,

$$\sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 g^i(r^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} (\mu^{-i}(s^{-i}))^2 \sum_{s^i \in S^i} \bar{\mu}^i(s^i) g^i(s^i, s^{-i}).$$

Dividing by $\prod_{j \neq i} (\sum_{s^j \in S^j} \mu(s^j)^2)$ we obtain:

$$\sum_{s^{-i} \in S^{-i}} \eta^{-i}(s^{-i}) g^i(r^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} \eta^{-i}(s^{-i}) \sum_{s^i \in S^i} \bar{\mu}^i(s^i) g^i(s^i, s^{-i}),$$

Notice that the right hand side is independent of r^i . This concludes the proof since r^i is arbitrary. \square

Proposition 6.4.12. *Let \mathbf{g} be a harmonic game with a pure Nash equilibrium. Then, \mathbf{g} is a non-strategic game.*

Proof. Let \mathbf{g} be a harmonic game. By Definition 6.4.8, since $\delta_0^* = \sum_{i \in N} \delta_0^{i*}$, using (6.4.3), for all $\mathbf{s} \in S$,

$$\delta_0^* D(\mathbf{g})(\mathbf{s}) = \sum_{i \in N} \sum_{\mathbf{t} | (\mathbf{s}, \mathbf{t}) \in A^i} \mu(\mathbf{t})(g^i(\mathbf{s}) - g^i(\mathbf{t})) = 0. \quad (6.4.8)$$

Let us assume that $\mathbf{s} \in S$ is a pure Nash equilibrium in the game \mathbf{g} . It follows that for all $i \in N$ and all $\mathbf{t} \in S$, such that $(\mathbf{s}, \mathbf{t}) \in A^i$, we have $g^i(\mathbf{s}) - g^i(\mathbf{t}) \geq 0$. Combining this with (6.4.8), for all $i \in N$ and all $\mathbf{t} \in S$, such that $(\mathbf{s}, \mathbf{t}) \in A^i$, we conclude that $g^i(\mathbf{s}) = g^i(\mathbf{t})$, which ends the proof. \square

Corollary 6.4.13. *Let \mathbf{g} a μ -normalized harmonic game, such that $\mathbf{g} \neq \mathbf{0}$. Then, \mathbf{g} does not admit pure equilibria.*

Proof. It clearly follows by Proposition 6.4.12. \square

6.4.3 Equilibria in two-player harmonic games

In this paragraph, we investigate useful properties of bimatrix harmonic games that allow us to highlight interesting links between their mixed Nash and correlated equilibria sets.

Notation. In this paragraph, we adopt the following notation: $I = \{1, \dots, m\}$ and $J = \{1, \dots, n\}$ are the action sets of player 1 and 2 respectively.

Coincidence of correlated and mixed equilibria sets

According to Aumann [1987], given a finite bimatrix game \mathbf{g} , a probability distribution $\mathbf{z} \in \mathbf{P}(I \times J)$ is a correlated equilibrium if,

$$\sum_{j \in J} (g^1(i, j) - g^1(i_0, j)) z(i, j) \geq 0, \quad \forall i, i_0 \in I$$

and,

$$\sum_{i \in I} (g^2(i, j) - g^2(i, j_0)) z(i, j_0) \geq 0, \quad \forall j, j_0 \in J.$$

This definition implies that every mixed Nash equilibrium is a correlated equilibrium where the corresponding distribution $(\mathbf{x}, \mathbf{y}) \in \mathbf{P}(I) \times \mathbf{P}(J) \subset \mathbf{P}(I \times J)$ is a product distribution, i.e., for all $(i, j) \in I \times J$, one has $z(i, j) = x(i)y(j)$. It further follows that the set of correlated equilibria is independent of the non-strategic component.

Lemma 6.4.14. *Let \mathbf{g} be a bimatrix harmonic game and let $\mathbf{z} \in \mathbf{P}(I \times J)$. Then, \mathbf{z} is a correlated equilibrium if and only if for all $i, i_0, i_1 \in I$*

$$\sum_{j \in J} z(i, j) g^1(i_0, j) = \sum_{j \in J} z(i, j) g^1(i_1, j),$$

and for all $j, j_0, j_1 \in J$,

$$\sum_{i \in I} z(i, j) g^2(i, j_0) = \sum_{i \in I} z(i, j) g^2(i, j_1).$$

Proof. Let \mathbf{g} be a bimatrix harmonic game and let $\mathbf{z} \in \mathbf{P}(I \times J)$ be a correlated equilibrium in \mathbf{g} . Then, $\delta^*(D(\mathbf{g}))(i, j) = 0$ for all $(i, j) \in I \times J$, i.e.,

$$\sum_{i_0 \in I} \mu(i_0, j) (g^1(i, j) - g^1(i_0, j)) + \sum_{j_0 \in J} \mu(i, j_0) (g^2(i, j) - g^2(i, j_0)) = 0.$$

It follows that:

$$\sum_{i, j} \frac{z(i, j)}{\mu(i, j)} \left(\sum_{i_0 \in I} \mu(i_0, j) (g^1(i, j) - g^1(i_0, j)) + \sum_{j_0 \in J} \mu(i, j_0) (g^2(i, j) - g^2(i, j_0)) \right) = 0.$$

Hence,

$$\sum_{i_0 \in I} \mu^1(i_0) \sum_{i \in I} \frac{1}{\mu^1(i)} \sum_{j \in J} z(i, j) (g^1(i, j) - g^1(i_0, j)) + \sum_{j_0 \in J} \mu^2(j_0) \sum_{j \in J} \frac{1}{\mu^2(j)} \sum_{i \in I} z(i, j) (g^2(i, j) - g^2(i, j_0)) = 0.$$

Since $\mathbf{z} \in \mathbf{P}(I \times J)$ is a correlated equilibrium in \mathbf{g} , the added quantities are positive and by the latter equation, they further add up to 0. It follows that for all $i, i_0 \in I$,

$$\sum_{j \in J} z(i, j) (g^1(i, j) - g^1(i_0, j)) = 0,$$

and for all $j, j_0 \in J$,

$$\sum_{i \in I} z(i, j) (g^2(i, j) - g^2(i, j_0)) = 0.$$

Writing the former equation for two actions $i_0, i_1 \in I$ and the latter one for $j_0, j_1 \in J$ and subtracting the resulting equations from each other, it then follows that for all $i, i_0, i_1 \in I$

$$\sum_{j \in J} z(i, j) (g^1(i_0, j) - g^1(i_1, j)) = 0,$$

and for all $j, j_0, j_1 \in J$

$$\sum_{i \in I} z(i, j) (g^2(i, j_0) - g^2(i, j_1)) = 0.$$

Conversely, if the two conditions hold true, then it is immediate that $\mathbf{z} \in \mathbf{P}(I \times J)$ is a correlated equilibrium. \square

Notation. In the sequel we will often use the notation (A, B) for a bimatrix game \mathbf{g} , where $A, B \in \mathbb{R}^{m \times n}$. Moreover, for any $i_0 \in I$, let us set: $Z_{i_0} = \sum_{j \in J} z(i_0, j)$.

Theorem 6.4.15. *Let (A, B) be a bimatrix harmonic game and $\mathbf{z} \in \mathbf{P}(I \times J)$ be a correlated equilibrium in this game. Generically, the family of mixed strategies $(x_{i_0})_{i_0 \in I}$ and $(y_{i_0})_{i_0 \in I}$, defined respectively for all $i \in I$ by $x_{i_0}(i) = \frac{z_{i_0}}{\alpha_i}$ for some $\alpha_i \in \mathbb{R}$ and for all $j \in J$ by $y_{i_0}(j) = \frac{z(i_0, j)}{Z_{i_0}}$ are mixed Nash equilibria of \mathbf{g} .*

Proof. Let $\mathbf{g} = (A, B)$ be a bimatrix harmonic game. Without loss of generality, we assume that $n \leq m$. Generically $\text{Rank}(A) = n$. Let us consider the linear map $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$, defined for all $\mathbf{y} \in \mathbb{R}^n$, as

$$\Phi(\mathbf{y}) = A\mathbf{y}.$$

It is clear that Φ is an injective map. Let $\mathbf{z} \in \mathbf{P}(I \times J)$ be a correlated equilibrium in \mathbf{g} and let us denote the diagonal in \mathbb{R}^m by \mathbf{d} , i.e., \mathbf{d} is generated by the vector $(1, \dots, 1) \in \mathbb{R}^m$. Then,

$$\Phi^{-1}(\mathbf{d}) = \{\mathbf{y} \in \mathbb{R}^n : \Phi(\mathbf{y}) \in \mathbf{d}\}.$$

Since Φ is an injective map, it follows that $\dim(\Phi^{-1}(\mathbf{d})) = 1$. Let us put $w(i) := z(i, \cdot) \in \mathbb{R}^n$. Then, from Lemma 6.4.9, for all $i \in I$, $\Phi(w(i)) \in \mathbf{d}$. It follows that for all i, i_0 , there exist $\alpha_i, \alpha_{i_0} \in \mathbb{R}$ respectively, such that $\alpha_i w(i) = \alpha_{i_0} w(i_0)$. To simplify, we assume that $\alpha_{i_0} = 1$ and let us set $x_i := \frac{Z_{i_0}}{\alpha_i}$ and $y_j := \frac{z(i_0, j)}{Z_{i_0}}$. Clearly $\sum_{j \in J} y_j = 1$ and since for all $(i, j) \in I \times J$,

$$x_i y_j = \frac{z(i_0, j)}{\alpha_i} = z(i, j), \quad \mathbf{z} \in \mathbf{P}(I \times J) \quad (6.4.9)$$

it also follows that $\sum_{i \in I} x_i = 1$. Consequently, $\mathbf{x} \in \mathbf{P}(I)$ and $\mathbf{y} \in \mathbf{P}(J)$, where $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_j)_{j \in J}$. Then,

$$A\mathbf{y} = \Phi(\mathbf{y}) = \Phi\left(\frac{w(i_0)}{Z_{i_0}}\right) = \frac{1}{Z_{i_0}} \Phi(w(i_0)) \in \mathbf{d},$$

and thus, \mathbf{y} is an equalizing strategy for player 2. Concerning player 1, from Lemma 6.4.9, for all $j, j_0, j_1 \in J$,

$$\sum_{i \in I} z(i, j) g^2(i, j_0) = \sum_{i \in I} z(i, j) g^2(i, j_1).$$

Using (6.4.9), we get:

$$\sum_{i \in I} x_i g^2(i, j_0) = \sum_{i \in I} x_i g^2(i, j_1),$$

which proves that \mathbf{x} is an equalizing strategy for player 1. Therefore, the family of mixed strategy profiles $(\mathbf{x}_{i_0}, \mathbf{y}_{i_0})_{i_0 \in I}$ constitutes the set of mixed Nash equilibria in the game \mathbf{g} . \square

Corollary 6.4.16. *In bimatrix harmonic games with equal number of actions, there exists a unique Nash equilibrium given by the mixed strategies η^1 and η^2 defined as follows:*

$$\eta^1(i) = \frac{(\mu^1(i))^2}{\sum_{i \in I} (\mu^1(i))^2} \quad \text{and} \quad \eta^2(j) = \frac{(\mu^2(j))^2}{\sum_{j \in J} (\mu^2(j))^2}.$$

Proof. Let \mathbf{g} be bimatrix harmonic game, such that $n = m$. From Theorem 6.4.11, the strategy profile (η^1, η^2) defined by $\eta^1(i) = \frac{(\mu^1(i))^2}{\sum_i (\mu^1(i))^2}$ and $\eta^2(j) = \frac{(\mu^2(j))^2}{\sum_j (\mu^2(j))^2}$ is a pair of equalizing strategies and thus, a mixed Nash equilibrium. It is immediate that $\mathbf{x} \in \mathbb{P}(I \times J)$ defined for all $(i, j) \in I \times J$ by

$$x(i, j) = \frac{(\mu(i, j))^2}{\sum_i (\mu^1(i))^2 \sum_j (\mu^2(j))^2}.$$

is a correlated equilibrium in \mathbf{g} . From Theorem 6.4.15, under the assumption $m \leq n$, and using (6.4.9), player 2 chooses η^2 at the equilibrium. Following the same argument, if $n \leq m$, player 1 chooses η^1 at the equilibrium and by using (6.4.9) we get that $\alpha_i = 1$ for all $i \in I$, which proves uniqueness. \square

6.4.4 The μ -decomposition of finite games.

Notation. We use the notation δ^i , δ and Δ instead of δ_0^i , δ_0 and Δ_0 respectively.

For all $i \in N$, let $\delta^{i\dagger} : C_1 \rightarrow C_0$ be the Moore-Penrose pseudoinverse defined as follows:

- on $\text{Im}(\delta^i)$, as the inverse of the restriction of δ^i on $\mathcal{N}\mathcal{O}^i$, i.e.,

$$\delta^{i\dagger} \upharpoonright_{\text{Im}(\delta^i)} = [\delta^i \upharpoonright_{\mathcal{N}\mathcal{O}^i}]^{-1},$$

- on $\text{Im}(\delta^i)^\perp$, $\delta^{i\dagger} = 0$,
- on C_1 , $\delta^{i\dagger}$ is defined linearly, i.e., $\delta^{i\dagger}(X) = \delta^{i\dagger}(X_I + X_I^\perp) = \delta^{i\dagger}(X_I)$, where $X_I \in \text{Im}(\delta^i)$.

The following properties characterize the operator $\delta^{i\dagger}$:

- $\delta^{i\dagger} \circ \delta^i \circ \delta^{i\dagger} = \delta^{i\dagger}$
- $\delta^i \circ \delta^{i\dagger} \circ \delta^i = \delta^i$
- $(\delta^i \circ \delta^{i\dagger})^* = \delta^i \circ \delta^{i\dagger}$
- $(\delta^{i\dagger} \circ \delta^i)^* = \delta^{i\dagger} \circ \delta^i$

Next, we provide the main result of this work. In doing so, we first prove two lemmas that will be useful for the proof of the direct sum decomposition result.

Lemma 6.4.17. *Let $\varphi \in \mathcal{N}\mathcal{O}^i$ and $\tilde{\varphi} \in C_0$ and assume that $\delta^i\varphi = \delta^i\tilde{\varphi}$. Then, for all $\psi \in \mathcal{N}\mathcal{O}^i$:*

$$\langle M^i\varphi, M^i\psi \rangle_0 = \langle M^i\tilde{\varphi}, M^i\psi \rangle_0.$$

Proof. Since $M^i\varphi \in \mathcal{N}\mathcal{O}^i$ and $\delta^{i\dagger}\delta^i$ is the identity operator on $\mathcal{N}\mathcal{O}^i$, one may write:

$$\langle M^i\varphi, M^i\psi \rangle_0 = \langle \delta^{i\dagger}\delta^i M^i\varphi, M^i\psi \rangle_0.$$

Using the property of $M^i\delta^i = \delta^i M^i$ and our initial assumption, we get:

$$\langle M^i\varphi, M^i\psi \rangle_0 = \langle \delta^{i\dagger} M^i \delta^i \varphi, M^i\psi \rangle_0 = \langle \delta^{i\dagger} M^i \delta^i \tilde{\varphi}, M^i\psi \rangle_0 = \langle \delta^{i\dagger} \delta^i M^i \tilde{\varphi}, M^i\psi \rangle_0.$$

Since $(\delta^{i\dagger}\delta^i)^* = \delta^{i\dagger}\delta^i$, it clearly follows that

$$\langle M^i\varphi, M^i\psi \rangle_0 = \langle M^i\tilde{\varphi}, \delta^{i\dagger}\delta^i M^i\psi \rangle_0 = \langle M^i\tilde{\varphi}, M^i\psi \rangle_0,$$

since $\delta^{i\dagger}\delta^i$ is the identity on $\mathcal{N}\mathcal{O}^i$ and $M^i\psi \in \mathcal{N}\mathcal{O}^i$. \square

Recall that $\Delta = \delta^* \circ \delta = \sum_{i \in N} \delta^{i*} \delta^i$. We define the space of common interest with harmonic payoff games as

$$\mathcal{H}_{\mathcal{C}\mathcal{I}} = \{(\varphi, \dots, \varphi) \in C_0^n \mid \Delta\varphi = 0\}.$$

Next lemma will be useful in order to prove our main decomposition result:

Lemma 6.4.18. *The following assertions hold true:*

1. $\mathcal{G} = \mathcal{H} + \mathcal{C}\mathcal{I}$,
2. $\mathcal{H} \cap \mathcal{C}\mathcal{I} = \mathcal{H}_{\mathcal{C}\mathcal{I}}$.
3. $\mathcal{G} = (\mathcal{N}\mathcal{O} \cap \mathcal{P}) + \mathcal{H}$

Proof. Let $\mathbf{g} = (g^1, \dots, g^n) \in \mathcal{G}$. Since $C_1 = \text{Im}(\delta) \oplus \text{Ker}(\delta^*)$, there exists $X_p \in \text{Im}(\delta)$ and $X_h \in \text{Ker}(\delta^*)$, such that $D\mathbf{g} = X_p + X_h$, where $X_p = \delta\varphi$ for some $\varphi \in C_0$. For all $i \in N$, we set $g_p^i = \varphi$ and $g_h^i = g^i - \varphi$. Then, we have:

$$D\mathbf{g}_p = \sum_{i \in N} \delta^i g_p^i = \sum_{i \in N} \delta^i \varphi = \delta\varphi = X_p.$$

and

$$D\mathbf{g}_h = D\mathbf{g} - D\varphi = D\mathbf{g} - X_p = X_h.$$

Therefore, $\mathbf{g}_h \in \mathcal{H}$, $\mathbf{g}_p \in \mathcal{C}\mathcal{I}$ and $\mathbf{g} = \mathbf{g}_h + \mathbf{g}_p$, which concludes the proof of assertion (1).

Concerning the assertion (2), let $\mathbf{g} \in \mathcal{H} \cap \mathcal{C}\mathcal{I}$. Then, there exists $\varphi \in C_0$ such that $\mathbf{g} = (\varphi, \dots, \varphi)$ and $\delta^* D\mathbf{g} = 0$. Therefore,

$$0 = \sum_{i \in N} \delta^{i*} D\mathbf{g} = \sum_{i \in N} \delta^{i*} \delta^i g^i = \left(\sum_{i \in N} \delta^{i*} \delta^i \right) \varphi = \Delta\varphi = 0,$$

and since the converse inclusion is straightforward, the proof is concluded.

To prove assertion (3), let $\mathbf{g} \in \mathcal{G}$. Then, in view of assertion (1) for all $i \in N$ and for any $\varphi \in C_0$, $g^i = g_h^i + \varphi$. By Theorem 6.3.8, for all $i \in N$, there exists $\tilde{g}^i \in \mathcal{N}\mathcal{O}^i$ and $\ell^i \in \mathcal{N}\mathcal{S}^i$, such that $\varphi = \tilde{g}^i + \ell^i$. Let $\tilde{\mathbf{g}} = (\tilde{g}^1, \dots, \tilde{g}^n)$ and $\tilde{\mathbf{h}} = \mathbf{g} - \tilde{\mathbf{g}}$. Then, we have:

$$D\tilde{\mathbf{g}} = D\varphi = \delta\varphi = X_p \quad \text{and} \quad D\tilde{\mathbf{h}} = X_h.$$

Hence, $\tilde{\mathbf{g}} \in \mathcal{P} \cap \mathcal{N}\mathcal{O}$ and $\tilde{\mathbf{h}} \in \mathcal{H}$. \square

Theorem 6.4.19. *The space of games is a direct sum of the μ -normalized potential, μ -normalized harmonic and non strategic subspaces, i.e.,*

$$\mathcal{G} = (\mathcal{N}\mathcal{O} \cap \mathcal{P}) \oplus (\mathcal{N}\mathcal{O} \cap \mathcal{H}) \oplus \mathcal{N}\mathcal{S}. \quad (6.4.10)$$

Proof. In view of Theorem 6.3.8, it suffices to prove that $\mathcal{N}\mathcal{O} = (\mathcal{N}\mathcal{O} \cap \mathcal{P}) \oplus (\mathcal{N}\mathcal{O} \cap \mathcal{H})$. If $\mathbf{g} \in \mathcal{N}\mathcal{O}$ then, by statement 3 of Lemma 6.4.18, we also have that $\mathbf{h} \in \mathcal{N}\mathcal{O}$ and thus, $\mathcal{N}\mathcal{O} = (\mathcal{N}\mathcal{O} \cap \mathcal{P}) + (\mathcal{N}\mathcal{O} \cap \mathcal{H})$.

In order to prove that the sum is direct, let $\mathbf{g} \in \mathcal{N}\mathcal{O} \cap \mathcal{P} \cap \mathcal{H}$. Since $\delta^i g^i = \delta^i \varphi$, from Lemma 6.4.17 one has:

$$\langle M^i g^i, M^i g^i \rangle_0 = \langle M^i \varphi, M^i g^i \rangle_0.$$

Therefore by Lemma 6.4.3

$$\sum_{i \in N} \langle M^i g^i, M^i g^i \rangle_0 = \left\langle \varphi, \sum_{i \in N} (M^i)^2 g^i \right\rangle_0 = \left\langle \varphi, \sum_{i \in N} \delta^{i*} \delta^i g^i \right\rangle_0 = \langle \varphi, \delta^* D\mathbf{g} \rangle_0 = \langle \varphi, 0 \rangle_0 = 0,$$

since by assumption $\mathbf{g} \in \mathcal{H}$. Consequently, for all $i \in N$, $M^i g^i = 0$, and since M^i is an isomorphism on C_0 , we get $g^i = 0$. \square

Lemma 6.4.20. *For all $i, j \in N$, such that $i \neq j$ it holds true that $\delta^{i*} \delta^j = 0$.*

Proof. Let $\mathbf{g} \in \mathcal{G}$ and let $i, j \in N$, such that $i \neq j$. Then, by (6.4.1), (6.4.3) for all $\mathbf{s} \in S$,

$$\delta^{i*}(\delta^j g^j)(\mathbf{s}) = - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) W^i(\mathbf{s}, \mathbf{t}) (\delta^j g^j)(\mathbf{s}, \mathbf{t}) = - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) W^i(\mathbf{s}, \mathbf{t}) W^j(\mathbf{s}, \mathbf{t}) (g^j(\mathbf{t}) - g^j(\mathbf{s})) = 0,$$

since for $i \neq j$, $W^i(\mathbf{s}, \mathbf{t}) W^j(\mathbf{s}, \mathbf{t}) = 0$. \square

Characterization of non-strategic games

In the context of graphs generated by games, one may further define the Laplace operator $\Delta_0^N : C_0^n \rightarrow C_0^n$ as $\Delta_0^N = D^* \circ D$. Next lemma will be useful for a characterization of non-strategic games.

Lemma 6.4.21. *Let \mathbf{g} be a finite game. Then, $\mathbf{g} \in \text{Ker}(\Delta_0^N)$ if and only if $\delta_0^i g^i \in \text{Ker}(\delta_0^{i*})$ for all $i \in N$.*

Proof. Given a game \mathbf{g} , by definition of Δ_0^N it follows:

$$\begin{aligned} D^*(D(\mathbf{g})) &= D^*\left(\sum_{i \in N} \delta_0^i g^i\right) \\ &= \left(((M^1)^2)^{-1} \delta_0^{1*} \left(\sum_{i \in N} \delta_0^i g^i \right), \dots, ((M^n)^2)^{-1} \delta_0^{n*} \left(\sum_{i \in N} \delta_0^i g^i \right) \right) \\ &= \left(((M^1)^2)^{-1} \delta_0^{1*} (\delta_0^1(g^1)), \dots, ((M^n)^2)^{-1} \delta_0^{n*} (\delta_0^n(g^n)) \right), \end{aligned}$$

where the latter equality is obtained in view of Lemma 6.4.20 and since M^i is an isomorphism, the proof is concluded. \square

Corollary 6.4.22. *Let \mathbf{g} be a finite game. Then, $\mathbf{g} \in \text{Ker}(\Delta_0^N)$ if and only if $\mathbf{g} \in \mathcal{NS}$.*

Proof. Let us consider $\mathbf{g} \in \mathcal{G}$. From (6.4.3), for all $\mathbf{s} \in S$,

$$\delta_0^{i*}(\delta_0^i g^i)(\mathbf{s}) = - \sum_{\mathbf{t}: (\mathbf{s}, \mathbf{t}) \in A^i} \mu(\mathbf{t}) \delta_0^i(g^i)(\mathbf{s}, \mathbf{t}) = - \sum_{\mathbf{t} \in S} \mu(\mathbf{t}) W^i(\mathbf{s}, \mathbf{t}) (g^i(\mathbf{t}) - g^i(\mathbf{s})).$$

Thus,

$$\delta_0^{i*}(\delta_0^i g^i)(\mathbf{s}) = -\mu^{-i}(s^{-i}) \sum_{t^i \in S^i} \mu^i(t^i) (g^i(t^i, s^{-i}) - g^i(s^i, s^{-i})) \quad (6.4.11)$$

Hence, by Lemma 6.4.21, $\Delta_0^N(\mathbf{g}) = 0$ if and only if for all $\mathbf{s} \in S$,

$$\sum_{t^i \in S^i} \mu^i(t^i) g^i(\mathbf{s}) = \sum_{t^i \in S^i} \mu^i(t^i) g^i(t^i, s^{-i}),$$

which is equivalent to $g^i(\mathbf{s}) = \sum_{t^i \in S^i} \bar{\mu}^i(t^i) g^i(t^i, s^{-i})$. By Definition 6.3.1, $\mathbf{g} \in \mathcal{NS}$. \square

Orthogonality between the components

Notation. For any $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{G}$, the notation $\mathbf{g}_1 \oplus \mathbf{g}_2$ stands for the direct orthogonal sum of $\mathbf{g}_1, \mathbf{g}_2$.

Theorem 6.4.23. *The μ -normalized potential, μ -normalized harmonic and non strategic subspaces are orthogonal under the inner product introduced in (6.4.6), i.e.,*

$$(\mathcal{NO} \cap \mathcal{P}) \perp (\mathcal{NO} \cap \mathcal{H}) \perp \mathcal{NS}. \quad (6.4.12)$$

Proof. To prove that μ -normalized potential games are orthogonal to μ -normalized harmonic games let $\mathbf{g}_P \in \mathcal{NO} \cap \mathcal{P}$ and $\mathbf{g}_H \in \mathcal{NO} \cap \mathcal{H}$. Then, one has:

$$\langle \mathbf{g}_P, \mathbf{g}_H \rangle_{\mathcal{G}} = \sum_{i \in N} \langle M^i g_P^i, M^i g_H^i \rangle_0 = \sum_{i \in N} \langle g_P^i, (M^i)^2 g_H^i \rangle_0.$$

Since \mathbf{g}_H is a normalized game using (6.4.5) and thus,

$$\langle \mathbf{g}_P, \mathbf{g}_H \rangle_{\mathcal{G}} = \sum_{i \in N} \langle g_P^i, \delta^{i*} \delta^i g_H^i \rangle_0 = \left\langle g_P^i, \sum_{i \in N} \delta^{i*} \delta^i g_H^i \right\rangle_0 = \langle g_P^i, \delta^*(D(\mathbf{g}_H)) \rangle_0 = 0,$$

by Definition 6.4.8.

In order to prove that μ -normalized potential and μ -normalized harmonic games are both orthogonal to the non-strategic one, let $\mathbf{g}_{NO} \in \mathcal{NO}$ and $\mathbf{g}_{NS} \in \mathcal{NS}$. Then, for all $i \in N$,

$$\langle M^i g_{NO}^i, M^i g_{NS}^i \rangle_0 = \sum_{\mathbf{s} \in S} \mu(\mathbf{s}) M^i g_{NO}^i(\mathbf{s}) M^i g_{NS}^i(\mathbf{s})$$

From (6.3.1), for all $\mathbf{s} \in S$, there exists $\ell : S^{-i} \rightarrow \mathbb{R}$ such that $\ell(s^{-i}) = g_{NS}^i(s^i, s^{-i})$ and it thus using the definition of M^i , clearly follows that for all $i \in N$,

$$\begin{aligned} \langle M^i g_{NO}^i, M^i g_{NS}^i \rangle_0 &= \sum_{s^{-i} \in S^{-i}} \mu^{-i}(s^{-i}) \lambda^i(s^{-i}) \ell(s^{-i}) \sum_{s^i \in S^i} \mu^i(s^i) \lambda^i(s^{-i}) g_{NO}^i(s^i, s^{-i}) \\ &= \sum_{s^{-i} \in S^{-i}} \mu^{-i}(s^{-i}) (\lambda^i(s^{-i}))^2 \ell(s^{-i}) \sum_{s^i \in S^i} \mu^i(s^i) g_{NO}^i(s^i, s^{-i}) \\ &= 0, \end{aligned}$$

where the latter equality follows from Definition 6.3.4. Hence, the proof is completed. \square

6.5 Conclusion and perspectives

In this paper, we introduce a family of inner products in the finite games space. We then propose a generalized definition of μ -harmonic games and provide an orthogonal μ -decomposition of the finite games space into the potential, μ -harmonic and non-strategic component. We further prove that a completely mixed strategy profile induced by the selected inner product appears as an equilibrium in the μ -harmonic component. In the particular class of bimatrix harmonic games, we show that correlated and mixed equilibria sets coincide and if players have equal number of actions the equilibrium is unique.

A first consequence of the μ -decomposition result is that one may define appropriate projection operators and thereby identify, with respect to the induced distance between games, the closest potential and μ -harmonic games to any arbitrary finite game. Thus, a characterization of its approximate equilibrium set is

possible in terms of the pure equilibria and the completely mixed equilibrium of its potential and harmonic component respectively. This approach makes more sense for perturbed potential or μ -harmonic games since the distance between the given game and its closest component game should be small.

A further contribution of this work concerns the countable games, i.e., games with countable action sets. Since the decomposition result proposed by [Candogan et al. \[2011\]](#) is based on a natural extension of the standard inner product on the space of games, it induces the uniformly mixed strategy profile as an equilibrium in the harmonic component. But since the uniform measure is not σ -additive, it is impossible to extend their decomposition result to countable games. The results established in this paper indicate that working in the space of square-summable sequences ℓ^2 and following the present formalization, makes the generalization of the decomposition result to countable games attainable.

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