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Shahid Rahman

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## Tableaux for Basic Modal Logic

Shahid Rahman

[shahid.rahman@univ-lille3.fr](mailto:shahid.rahman@univ-lille3.fr)

Univ. Lille; Philosophie, UMR 8163: STL

### DEFINITION

- A *label* is finite sequence of positive integers. A *labelled formula* is an expression of the form  $i\varphi$ , where  $i$  is the label of the formula  $\varphi$ .
- If the label  $i$  is a sequence of length  $>1$  the positive integers of the sequence will be separated by periods. Thus, if  $i$  is a label and an  $n$  is positive integer, then  $i.n$  is a new label, called *an extension of  $i$* . The label  $i$  is then an *initial segment* of  $i.n$ .

<i>Restricted Labelling</i>	Unrestricted Labelling
$(\mathbf{V})\Box A_i$ ----- $(\mathbf{V})A_j$  $(\mathbf{F})\Diamond A_i$ ----- $(\mathbf{F})A_j$	$(\mathbf{F})\Box A_i$ ----- $(\mathbf{F})A_j$  $(\mathbf{T})\Diamond A_i$ ----- $(\mathbf{T})A_j$
<p><b>K:</b> <math>j</math> must occur already in the branch</p> <p><b>Reflexivity:</b> <math>j=i</math></p> <p><b>Seriality:</b> <math>j</math> can be new</p> <p><b>Symmetry:</b> Assume that <math>i = k.n</math>, then <math>j= k</math> ( <math>k.n</math>. is an immediate extension of <math>k</math>, and <math>k</math> occurs already in the branch).</p> <p><b>Transitivity:</b> <math>i</math> is an initial segment of <math>j</math> (such as <math>j = i.m</math> or <math>j = i.m.n\dots</math>).</p> <p><b>Euclideanity:</b> If in the branch occur, <math>i</math>, <math>i.n</math> and it is the case that <math>(\mathbf{V})\Box A_i.m</math>(or <math>(\mathbf{F})\Diamond A_i.m</math>), then <math>j = i.n</math>:</p>	<p><math>j</math>: is new</p>
$[i]$ $[i.n]$ $(\mathbf{F})\Diamond A_i.m$ ----- $(\mathbf{F})A_i.n$	
$[i]$ $[i.n]$ $(\mathbf{V})\Box A_i.m$	

$$\text{-----}$$

$$(\mathbf{V})A_{i.n}$$

Tableaux and as generated by winning strategies

- If  $\Sigma$  is a set of dialogically signed formulae and  $X$  is a single dialogically signed formula, we will write  $\Sigma, X$  for  $\Sigma \cup \{X\}$ .
- Observe that the formulae below the line always represent pairs of attack and defence moves. In other words, they represent rounds.
- The vertical bar '|' indicates alternative choices for  $\mathbf{O}$ ,  $\mathbf{P}$ 's strategy must have a defence for both possibilities (dialogical games defining two possible plays).
- The rules containing yielding two lines indicate that it is  $\mathbf{P}$  who has the choice – an he thus might need only one of both possible choices.
- Note that the expressions between the symbols "<" and ">", such as  $\langle(\mathbf{P})?\rangle$  or  $\langle(\mathbf{O})?\rangle$  are moves – more precisely they are attacks – but not formulae (assertions) which could be attacked. These expressions are not actually part of the tableau. They are formulae which are included in the set of formulae. **These expressions are rather part of the algorithmic device to reconstruct the correspondent dialogues.**

### Intuitionistic Tableaux

(O)-Cases	(P)-Cases
$\Sigma, (\mathbf{O})A \vee B$ $\text{-----}$ $\Sigma, \langle(\mathbf{P})?\text{-}\vee\rangle(\mathbf{O})A \mid \Sigma, \langle(\mathbf{P})?\text{-}\vee\rangle(\mathbf{O})B$	$\Sigma, (\mathbf{P})A \vee B$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, \langle(\mathbf{O})?\text{-}\vee\rangle(\mathbf{P})A$ $\Sigma_{[\mathbf{O}]}, \langle(\mathbf{O})?\text{-}\vee\rangle(\mathbf{P})B$
$\Sigma, (\mathbf{O})A \wedge B$ $\text{-----}$ $\Sigma, \langle(\mathbf{P})?\text{-}L\rangle(\mathbf{O})A$ $\Sigma, \langle(\mathbf{P})?\text{-}R\rangle(\mathbf{O})B$	$\Sigma, (\mathbf{P})A \wedge B$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, \langle(\mathbf{O})?\text{-}L\rangle(\mathbf{P})A \mid \Sigma_{[\mathbf{O}]}, \langle(\mathbf{O})?\text{-}R\rangle(\mathbf{P})B$
$\Sigma, (\mathbf{O})A \rightarrow B$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, (\mathbf{P})A \dots \mid \langle(\mathbf{P})A\rangle(\mathbf{O})B$	$\Sigma, (\mathbf{P})A \rightarrow B$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, (\mathbf{O})A; (\mathbf{P})B$
$\Sigma, (\mathbf{O})\neg A$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, (\mathbf{P})A; \text{---}$	$\Sigma, (\mathbf{P})\neg A$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, (\mathbf{O})A; \text{---}$
$\Sigma, (\mathbf{O})\forall xA$ $\text{-----}$ $\Sigma, \langle(\mathbf{P})?\text{-}\forall x/k_i\rangle(\mathbf{O})A_{[x/k_i]}$	$\Sigma, (\mathbf{P})\forall xA$ $\text{-----}$ $\Sigma_{[\mathbf{O}]}, \langle(\mathbf{O})?\text{-}\forall x/k_i\rangle(\mathbf{P})A_{[x/k_i]}$

$$\begin{array}{c}
\frac{\Sigma, (\mathbf{O})\exists xA}{\Sigma, \langle (\mathbf{P})?-\exists \rangle (\mathbf{O})A_{[x/ki]} \\
k_i \text{ is new}}
\end{array}
\quad
\begin{array}{c}
\frac{k_i \text{ is new}}{\Sigma, (\mathbf{P})\exists xA} \\
\frac{\Sigma_{[\mathbf{O}]}, \langle (\mathbf{O})?-\exists \rangle (\mathbf{P})A_{[x/ki]}
\end{array}$$

Intuitionistic tableaux are generated with the addition of the set  $\Sigma_{[\mathbf{O}]}$ . which only contains O-signed formulae): the totality of the previous **P**-formulae on the same branch of the tree are eliminated.

### DEFINITION

Let us look at two examples, namely one for classical logic and one for intuitionistic logic. We use the tree shape of the tableau made popular by Smullyan ([18]):

If  $\Theta$  is a given set of (**P**-, or **O**-) signed formulae, we say one of the rules **R** of above rules of the tableaux-system *applies* to  $\Theta$  if by appropriate choice of  $\Theta$ , the collection of signed formulae above the line in rule **R** becomes  $\Theta$ .

By an *application of **R** to the set  $\Theta$* , we mean the replacement of  $\Theta$  by  $\Theta_1$  (or by  $\Theta_1$  and  $\Theta_2$ , if **R** is (**P**) $\wedge$ , (**O**) $\vee$ , or (**O**) $\rightarrow$ ) where  $\Theta$  is the set of formulae above the line in rule **R** (after suitable substitution for  $\Sigma$ , and for the formulae **A** (and **B**)) and  $\Theta_1$  (or  $\Theta_1$  and  $\Theta_2$ ) is the set of formulae below. This assumes **R** to apply to  $\Theta$ . Otherwise the result is again  $\Theta$ . For example, by applying rule (**P**) $\rightarrow$  to the set  $\Theta$ :  $\{(\mathbf{O})\mathbf{A}, (\mathbf{P})\mathbf{B}, (\mathbf{P})(\mathbf{C}\rightarrow\mathbf{D})\}$  we may get  $\Sigma_{[\mathbf{O}]}4\Theta_1$ :  $\{(\mathbf{O})\mathbf{A}, (\mathbf{O})\mathbf{C}, (\mathbf{P})\mathbf{D}\}$  – notice that (**P**)**B** disappeared because we have  $\Sigma_{[\mathbf{O}]}$  and not  $\Sigma$ .

By *configuration* we mean a finite collection  $\{\Sigma_1, \Sigma_2, \dots, \Sigma_n\}$  of sets of signed formulae, where  $\Sigma$  might stand for  $\Sigma$  and/or  $\Sigma_{[\mathbf{O}]}$ .

By an *application of **R** to the configuration  $\{\Sigma_1, \Sigma_2, \dots, \Sigma_n\}$*  we mean the replacement of this configuration with a new one which is like the first except for containing instead of some  $\Sigma_i$  the result (or results) of applying rule **R** to  $\Sigma_i$ .

By a tableau we mean a finite sequence of configurations  $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_n$  in which each configuration except the first is the result of applying one of the above rules to the preceding configuration.

A set of signed formulae is closed if it contains both  $(\mathbf{O})a$  and  $(\mathbf{P})a$  (for  $a$  atomic). A configuration  $\{\Sigma_1, (\Sigma_2, \dots, \Sigma_n)\}$  is closed if each  $\Sigma_i$  is closed. A tableau  $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_n$  is closed if some  $\mathfrak{C}_i$  is closed.

By a tableau for a set  $\Sigma$  of signed formulae we mean a tableau  $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_n$  in which  $\mathfrak{C}_1$  is  $\{\Sigma\}$ .

## Modal Logic

<i>Restricted Labelling</i>	<i>Unrestricted Labelling</i>
$(\mathbf{O})\Box A_i$	$(\mathbf{P})\Box A_i$
-----	-----
$\langle (\mathbf{P}) ?j \rangle (\mathbf{O})A_j$	$\langle (\mathbf{O}) ?j \rangle (\mathbf{P})A_j$
$(\mathbf{P})\Diamond A_i$	$(\mathbf{O})\Diamond A_i$
-----	-----
$\langle (\mathbf{O}) ?\Diamond \rangle (\mathbf{P})A_j$	$\langle (\mathbf{P}) ?\Diamond \rangle (\mathbf{O})A_j$
<b>K:</b> $j$ must occur already in the branch	$j$ : is new
<b>Reflexivity:</b> $j=i$	
<b>Seriality:</b> $j$ can be new	
<b>Symmetry:</b> Assume that $i = k.n$ , then $j = k$ ( $k.n$ . is an immediate extension of $k$ , and $k$ occurs already in the branch).	
<b>Transitivity:</b> $i$ is an initial segment of $j$ (such as $j = i.m$ or $j = i.m.n\dots$ ).	
<b>Euclideanity:</b> If in the branch occur, $i, i.n$ and it is the case that $(\mathbf{O})\Box A_i.m$ (or $(\mathbf{P})\Diamond A_i.m$ ), then $j = i.n$ :	
$[i]$ $[i.n]$ $(\mathbf{P})\Diamond A_i.m$ ----- $(\mathbf{P})A_i.n$	
$[i]$ $[i.n]$ $(\mathbf{O})\Box A_i.m$ ----- $(\mathbf{O})A_i.n$	

