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Acyclic domains of linear orders: a survey

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Abstract.
Among the many significant contributions of Fishburn to social choice theory some have borne on what he has called «acyclic sets», i.e. these sets of linear orders where majority rule applies without "Condorcet effect" (majority relation never has cycles). Search for large such domains is a fascinating topic. I review the works in this field and in particular a recent one allowing showing the connections between some of them unrelated up to now.

Key words: acyclic set, alternating scheme, distributive lattice, effet Condorcet, maximal chain, permutoedre lattice, weak Bruhat order, value restriction.
JEL classification number: D71 AMS MSC classification number 05, 06D, 91B

1 Notations and preliminaries

A = {1,2...i,j,k…n} is a finite set of n elements that I will generally call *alternatives* (but which could also be called issues, decisions, outcomes, candidates, objects, etc). The elements of A will be also denoted by letters like x,y,z etc. A subset of cardinality p of A will be called a p-set.
A^2 (respectively, A^3) denotes the set of all ordered pairs (x,y) (respectively, ordered triples (x,y,z) written for convenience as xyz) of A. When the elements of A are denoted by the n first integers, $P^2(n)$ denotes the set of the n(n-1)/2 ordered pairs (i<j).

A binary relation on A is a subset R of A^2 and we write xRy or $(x,y) \in R$ when x is in the relation R with y. For ℓ integer ≥ 2, a *cycle of length* ℓ of R, called also a *ℓ-cycle*, is a subset $\{x_1,x_2,....x_\ell\}$ of A such that $x_1Rx_2Rx_3.......x_\ell Rx_1$. For $B \subseteq A$, the restriction of a relation R to B is denoted by $R_{/B}$.

A *strict linear order* on A is an irreflexive, transitive and complete ($x \neq y$ implies xRy or yRx) binary relation on A. Henceforth, we will omit the qualifier strict and sometimes, when there is no ambiguity, the qualifier linear. Linear orders on A are in a one-to-one correspondence with *permutations* of A. So if L is a linear order on A one can write it as a permutation $x_1...x_kx_{k+1}...x_n$. Then one says that x_k has *rank* k and is *covered* by x_{k+1} and that x_k and x_{k+1} are *consecutive* in L. I denote by τ_k the transposition which exchange x_k and x_{k+1} in L: $\tau_k(L) = x_1...x_{k+1}x_k...x_n$.

The set of all linear orders on A is denoted by $\mathcal{L}$ or $\mathcal{L}_n$ if $|A| = n$. $\mathcal{D}$ denotes any subset of $\mathcal{L}$.

In all this paper the preferences of what I will call a *voter* (but what could also be called agent, person, individual, criterion, etc) on a set A of alternatives is represented by a linear order $L = x_1...x_kx_{k+1}...x_n$ where x_1 is assumed to be the last preferred alternative, x_2 the next-to-last, etc. So, yLx or $(y,x) \in L$ means that alternative x is preferred to alternative y in the linear order L.

Remark

One could consider that the notation yLx should mean that y is preferred to x. But we are working in this paper with posets and, unfortunately, this choice would be not in accordance with the usual convention of poset theory. Indeed in this theory the symbol used for a (strict) order is generally $<$ what means that yLx is interpreted as $y<x$, and so as y is less than x. The reader must keep in mind a consequence of our choice: in a linear order of preference $L = x_1x_2...x_n$, the worst alternative x_1 (respectively, the best alternative x_n) has rank 1 (respectively, n).

The problem to get a collective preference from various voters' preferences was tackled by Borda and Condorcet at the end of 18th century. Condorcet criticized Borda's rank method and proposed to use the majority rule on the pairs of alternatives. Before to recall the definition of this rule, I introduce some notations. I consider v voters, which express their preferences on the alternatives by linear orders taken in a set $\mathcal{D}$ of linear orders ($\mathcal{D} \subseteq \mathcal{L}$). The state of their preferences is given by a *v-profile* π =

$(L_1,..L_q,..,L_v)$ where L_q is the linear order of $\mathcal{D}$ representing the preference of voter q. $\mathcal{D}^v$ denotes the set of all these v-profiles. For a subset B of alternatives, $\pi_{/B} = (L_{1/B},..L_{q/B},..,L_{v/B})$ denotes the profile of voters' preferences restricted to B.
For a v-profile $\pi = (L_1,..L_q,..,L_v)$ and two alternatives x and y, one denotes by $v_\pi(y,x)$ the number of voters q preferring x to y (i.e. such that yL_qx).
In his «*Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*» (1785) Condorcet recommended the rule now called Condorcet's majority rule[1]. This rule associates with a profile π the collective preference defined as the *strict* (*simple*) *majority relation*[2] $R_{SmAJ}(\pi)$:

$$y\ R_{SMAJ}(\pi)\ x \text{ if } v_\pi(y,x) > v/2$$

i.e. alternative x is collectively preferred to alternative y if it is preferred by a (strict) majority of voters. It is clear that this majority relation is asymmetric i. e., has no 2-cycles. But Condorcet discovered that majority relations can have cycles of length $l \geq 3$: $x_1R_{MAJ}x_2R_{MAJ}x_3.......x_lR_{MAJ}x_1$. This fact rediscovered for instance by Dodgson, Black or Arrow has been called the «Condorcet effect» by Guilbaud (1952) and is also known as the «voting paradox»[3]. I prefer the first appellation, which emphasizes the fact that this occurrence of cycles is not really a paradox (see Guilbaud 1952 or Monjardet 2006).
The simplest cases of Condorcet effect occur when A = {i,j,k} and v = 3, with the profiles (ijk, jki, kij) and (jik, ikj, kji)) since then majority relations are the 3-cycles $iR_{SMAJ}jR_{SMAJ}k\ R_{SMAJ}i$ and $jR_{SMAJ}iR_{SMAJ}kR_{SMAJ}j$. I say that such profiles are 3-*cyclic profile*s. More generally, for an integer $\ell \geq 3$, I say that a profile like $\pi = (x_1x_2x_3....x_\ell;\ x_2x_3....x_\ell x_1;\ .....;\ x_1x_1x_2....x_{\ell-1})$ is a ℓ-*cyclic profile*. The strict majority relation associated with such a profile is a ℓ-cycle. Observe that arbitrary profiles can contain the same linear order several times, but that ℓ-cyclic profiles are subsets of $\mathcal{L}$.

1 Condorcet uses other terms like «plurality».
2 The *(simple) majority relation* is the relation defined by $yR_{MAJ}(\pi)x$ if $v_\pi(y,x) \geq v/2$. Observe that since π is a profile of linear orders one has for $x \neq y$, $(y,x) \in R_{MAJ}(\pi)$ if and only if $(x,y) \notin R_{SMAJ}(\pi)$.
3 Condorcet speaks of the «contradictory case». Dodgson and Black speak of «cyclical majorities» and I dont'know who has used the term paradox for the first time (it appears in Arrow's 1951 book).

A subset $\mathcal{D}$ of the set $\mathcal{L}$ of all linear orders on A is an *acyclic domain* (of linear orders) if for every integer v and every profile $\pi = (L_1,L_2,..,L_v) \in \mathcal{D}^V$, $R_{SMAJ}(\pi)$ has no cycles[4].

Several classical characterizations of acyclic domains are given in the below theorem. I need some definitions. I say that a set $\mathcal{D}$ of linear orders contains a ℓ-cyclic profile if there exists a subset $B = \{x_1,x_2,....x_\ell\}$ of A and a subset $\{L_1,\ldots L_q,\ldots,L_\ell\}$ of ℓ linear orders in $\mathcal{D}$ such that the profile $\pi_{/B} = (L_{1/B},\ldots L_{q/B},..,L_{\ell/B})$ is a ℓ-cyclic profile. When a set of three alternatives is linearly ordered as $i<j<k$, then[5] i has rank 1, j has rank 2 and k has rank 3. I say that a set $\mathcal{D}$ of linear orders is *value-restricted* if for every subset $\{i,j,k\}$ of A, there exists an alternative which either never has rank 1 or never has rank 2 or never has rank 3 in the set $\mathcal{D}_{/\{i,j,k\}}$. Finally in condition 7) of the theorem I use the majority relation defined in footnote 2.

Theorem

Let $\mathcal{D}$ be a subset of the set $\mathcal{L}$ of all linear orders on a set A. The following conditions are equivalent:

1) $\mathcal{D}$ is acyclic (i.e., for every integer v and every profile $\pi \in \mathcal{D}^V$, $R_{SMAJ}(\pi)$ has no cycles),

2) For every integer v and every profile $\pi \in \mathcal{D}^V$, $R_{SMAJ}(\pi)$ is a (strict) partial order,

3) For every odd integer v and every profile $\pi \in \mathcal{D}^V$, $R_{SMAJ}(\pi)$ is a linear order,

4) $\mathcal{D}$ does not contain ℓ-cyclic profiles,

5) $\mathcal{D}$ does not contain 3-cyclic profiles,

6) $\mathcal{D}$ is value-restricted,

7) For every integer v, every profile $\pi \in \mathcal{D}^V$ and every $B \subseteq A$, $\{a \in B :$ for every $b \in B\backslash\{a\}, bR_{MAJ}(\pi)a\} \neq \emptyset\}$.

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[4] Acyclic domains have been also called consistent profiles (Ward 1965), valued-restricted domains (Kim and Roush 1980), transitive simple majority domains or consistent sets (Abello and Johnson 1984), «états d'opinion fortement Condorcéen» (Chameni-Nembua 1989), acyclic sets (Fishburn 1992,1997), majority-consistent sets (Craven 1996) or Condorcet domains (Monjardet 2006).

[5] See the Remark on the ranks of linearly ordered alternatives in the previous page.

Condition 2) means that when voters' preferences belong to an acyclic domain the collective preference given by majority rule is transitive (and asymmetric) what in particular implies that it can be extended into a linear order. For a given profile I say that an alternative is a *Condorcet winner* if it is preferred to all other alternatives in the majority relation (see footnote 2) associated with this profile. Condition 7) means that for every profile and every subset of candidates there exists at least a Condorcet winner. Condition 5) means that $\mathcal{D}$ is acyclic if and only if for every subset $C = \{L_1,L_2,L_3\}$ of three different linear orders of $\mathcal{D}$ and every subset {i,j,k} of three different alternatives, C is not a 3-cyclic profile on {i,j,k}. It was introduced by Ward (1965) which proved the equivalence of conditions 1), 4) and 5) of the above theorem. He called it the condition of *Latin-square-lessness* since a 3-cyclic profile forms a Latin square when it is disposed in a 3×3 array. Condition 6 of value-restriction was introduced by Sen (1966)[6].

In what follows I will use a form of condition 6) of value restriction. One assumes that the n alternatives of A are ranked in an arbitrary linear order, which in fact will be the natural order 1<2<...i<j<k<…..n. There are two 3-cyclic profiles on a 3-element set {i,j,k}, namely {ijk, jki, kij} and {jik, ikj, kji}. In each of these 3-cyclic profiles each element h of {i,j,k} appears at rank 1, 2 and 3 in one of the three linear orders of the profile. In order to avoid a 3-cyclic profile on {i,j,k}, it suffices to assume that one of the linear orders in {ijk, jki, kij} and one in {jik, ikj, kji} never occurs. There are 3.3 = 9 different ways to do that. But each of these ways comes back to assume that an element h of {i,j,k} never appears at rank 1, 2 or 3 in a linear order on {i,j,k}. For instance, to exclude ijk and jik comes back to assume that k never has rank 3 in the restrictions to {i,j,k} of the linear orders of $\mathcal{D}$. I will write this condition $kN_{\{i,j,k\}}3$. More generally for h in {i,j,k} and r in {1,2,3}, the *Never Condition* $hN_{\{i,j,k\}}r$ means that h never has rank r in the restrictions to {i,j,k} of the linear orders of $\mathcal{D}$. With these definitions a set of linear orders is an acyclic domain if and only if for every ordered triple i<j<k there exists $h \in \{i,j,k\}$ and $r \in \{1,2,3\}$ such that $hN_{\{i,j,k\}}r$. Since 1<2...<n contains n(n-1)(n-2)/6 ordered triples and that for each or-

[6] In fact Sen's value-restriction condition is more general since it bears on the case where voters' preferences are represented by weak orders (transitive and complete binary relations). But Sen has immediately pointed out that when voters' preferences are represented by linear orders his condition is equivalent to Ward's Latin-square-lessness condition. In this case Ward's result and Arrow's theorem are «dual» (see Monjardet 1978).

dered triple $i<j<k$, one can choose one of the nine possible Never Condition $hN_{\{i,j,k\}}r$, one sees that there are many ways to get acyclic domains[7]. I will say that an acyclic domain satisfies the Never Condition hNr if for every ordered triple $i<j<k$, the same Never Condition $hN_{\{i,j,k\}}r$ is satisfied. For instance $\mathcal{D}$ satisfies jN1 if for every ordered triple $i<j<k$, j never has rank 1 (i.e. is never last) in the restrictions to {i,j,k} of the orders of $\mathcal{D}$. I will say that an acyclic domain satisfies the Never Condition ijkNr if for every ordered triple $i<j<k$, one has either iNr or jNr or kNr (one of the three alternatives never has rank r).
An obvious but useful observation is that the Never Conditions are «hereditary». Firstly if a set $\mathcal{D}$ of linear orders satisfies a set of Never Conditions any subset of $\mathcal{D}$ satisfies the same set of Never Conditions. Secondly if a set $\mathcal{D}$ of linear orders defined on A satisfies a set of Never Conditions then for every $B \subseteq A$, $\mathcal{D}_{/B}$ (the set of linear orders restrictions to B of the linear orders of $\mathcal{D}$) satisfies the same set of Never Conditions. It is also interesting to mention the following fact on these conditions. Let us denote by L^d the *dual linear order* of the linear order L: xL^dy if and only if yLx, and for $\mathcal{D} \subseteq \mathcal{L}$, call $\mathcal{D}^d = \{L^d, L \in \mathcal{D}\}$ the *dual domain* of $\mathcal{D}$. Then a domain satisfies the Never Condition hNr if and only if its dual satisfies the Never Condition hN(4-r).
Now the interesting problem is: how large can be domains of linear orders where Condorcet's majority rule works well? or more concisely, how large can be acyclic domains ? Observe that the problem becomes a pure combinatorial problem: to construct large sets of linear orders satisfying the above restriction conditions. I introduce some definitions and notations. An acyclic domain $\mathcal{D}$ is *maximal* if for any linear order L not in $\mathcal{D}$, $\mathcal{D} \cup \{L\}$ is no more an acyclic domain. Moreover a (maximal) acyclic domain contained in $\mathcal{L}_n$ is *maximum* if it has the maximum size, denoted by **f(n)**, among all acyclic domains in $\mathcal{L}_n$. An acyclic domain $\mathcal{D} \subset \mathcal{L}_n$ is *con-*

[7] But the set of Never Conditions chosen must be satisfied by at least a linear order. For instance Raynaud 1981 has shown that for $n \geq 5$ there does not exist a linear order satisfying jN2 for every ordered triple $i<j<k$ (and that this condition is satisfied by only four orders for $n = 4$).

nected if there always exists a *path*[8] of $\mathcal{L}_n$ included in $\mathcal{D}$ between any two linear orders in $\mathcal{D}$; such a connected domain is of *diameter* d if the maximum length of a shortest path between two linear orders of $\mathcal{D}$ is d. One can observe that the diameter of $\mathcal{L}_n$ is n(n-1)/2. I denote by **g(n)** the maximum size of a connected acyclic domain of diameter n(n-1)/2 contained in $\mathcal{L}_n$. One will show that g(n) = f(n) for n ≤ 6, but seems to be less than f(n) at least for n ≥ 16.

The problem of determining f(n) or g(n) for all n is daunting. Up to now these numbers are known only for n ≤ 6 (where they are equal). Then one rather searches good lower or upper bounds for them. Lower bounds are obtained by producing (maximal) acyclic domains. The first maximal connected acyclic domain obtained by Black contains only 2^{n-1} linear orders (compare to the n! possible linear orders). For a long time the other maximal acyclic domains found were also connected and contained no more orders. I will present some of them in section 2. This perhaps raised up the conjecture f(n) = 2^{n-1}; but it was too bad since it can be disproved for n = 4 (see footnote 13 and Figure 4). Breakthroughs come first in the eighties with Abello and Chameni-Nembua's works which I will present in sections 3 and 4. They use the order on the «permutoèdre» and don't explicitly use Never Conditions. For instance for n = 6, maximal connected acyclic domains with 44 or 45 linear orders were obtained (instead of 32 = 2^5). A clever use of the Never Conditions by Fishburn and Craven allowed to find larger maximal connected acyclic domains for n > 6 (all of diameter n(n-1)/2). They will be presented in section 5 along with Fishburn's construction allowing to get still larger -but not connected- maximal acyclic domains. Finally in section 6, I will state Galambos and Reiner's work which allows to get a unified version of almost all the known results on maximal connected acyclic domains of diameter n(n-1)/2. In the conclusion I will point out two conjectures. The Appendix contains a Table giving numerical results on lower or upper bounds of f(n) and g(n).

8 A path in $\mathcal{L}_n$ is a sequence of different linear orders $L_1 \ldots L_k L_{k+1} \ldots L_s$ such that for k = 1,2…s-1, L_k and L_{k+1} differ only by a transposition (of two consecutive elements). In fact it is a path in the «permutoèdre graph» defined in section 3.

2 The beginnings : small maximal acyclic domains

As already said the first maximal (connected) acyclic domain was produced by Black (1948, 1958) who called it the domain of the single peaked preferences. Assume that the set of alternatives is linearly ordered as $1 < 2 < \ldots < p \ldots < n$ by a «reference» order. Let L be a linear order of preference on A for which p is the preferred alternative. L is said *single-peaked* (with respect to the reference order <) if $i<j<p$ implies $iLj(Lp)$ and $p<q<r$ implies $rLq(Lp)$. This condition means that given p the preferred alternative of the voter, he prefers alternative x to alternative y if x is «closer» from p than y in the reference order (for instance such a condition can be satisfied for political preferences, when the political parties can be ranked from extreme left to extreme right). Now it is not difficult to see that a linear order L is single-peaked (w.r.t. <) if and only if for every ordered triple $i<j<k$, jLi implies kLj and jLk implies iLj, which is true if and only if L satisfies the condition jN1, i.e. for every ordered triple $i<j<k$, $jN_{\{i,j,k\}}1$ (in other words, the middle alternative of the triple is never the least preferred). Then the domain of single-peaked (w.r.t. <) linear orders is the domain of all linear orders satisfying jN1. It is also easy to see that for n alternatives its size is 2^{n-1} (see for instance Kreweras (1962) who used the fact, already observed by Ward that no more than two alternatives can have rank 1 in these single-peaked linear orders). The set of the eight single-peaked linear orders on {1,2,3,4} w.r.t the linear order 1<2<3<4 is shown on Figure 4 : each of these orders has a black square.

Black's single-peakedness condition is a subcase of *Arrow-Black's single-peakedness condition*[9] (1951), which is the condition ijkN1 (for every 3-subset {i,j,k}, there exists h in {i,j,k} such that $hN_{\{i,j,k\}}1$). An acyclic domain satisfying Arrow-Black's single-peakedness condition does not necessarily satisfies Black's single-peakedness condition. But such an acyclic domain contains also at most 2^{n-1} linear orders. This results immediately from the point already mentioned that a Never Condition is hereditary and from another easy observation: the set of elements ranked 1 in the linear

[9] The terminology of these conditions depends on authors. For instance what I call Black's single-peakedness condition (respectively, Arrow-Black's single-peakedness condition) has been called unimodality condition by Romero 1978 (respectively, pseudo-unimodality condition by Romero and single-peakedness on the triples by Kelly 1978). In fact as it was observed by Inada 1964, Arrow-Black's single-peakedness appears only implicitly in the proof of Theorem 4 in Arrow's book. This condition appears also in Dumett and Farquharson (1961). What is somewhat confusing is that the term single–peakedness condition is sometimes used without making clear in which of the two above senses.

orders belonging to a domain satisfying Arrow-Black's single-peakedness condition has size at most 2.

Some other interesting domains satisfying Arrow-Black's condition have been investigated. For instance let be L and L' two linear orders ranking the alternatives of A according two different criteria. A decision maker can rank the alternatives from the last by using alternatively the two criteria: he gives rank 1 to an alternative ranked 1 by one of the criteria (i.e. to the worst alternative according to this criteria) ; then he deletes this alternative from the two linear orders and he uses the same procedure on the restrictions obtained to determine his next to last alternative, and so on. Romero (1978) said that a set of linear orders obtained by this procedure satisfies the *quasi-unimodality condition* and he proved that this set satisfies Arrow-Black's single-peakedness condition. When the two linear orders L and L' are dual (xLy if and only if yL'x) one gets again the set of all single-peaked linear orders (w.r.t. L).

It is obvious that the dual of an acyclic domain is an acyclic domain. For instance the dual of Black's (respectively, Arrow-Black's) single–peaked linear orders, i.e., the set of linear orders satisfying jN3 (respectively, ijkN3) was called by Vickrey (1961) the domain of *single-troughed* (respectively, by Inada (1964) the domain of *single-caved*) linear orders. One can find a systematic study of the domains of linear orders satisfying one of the Never Conditions in Arrow and Raynaud's book (1986, see also Kohler 1978, Romero 1978 and Raynaud 1981-1982).

Another type of acyclic domains was discovered by Blin (1973) under the name of *multidimensional consistency*: the chains of the «permutoèdre lattice». It will be described in the following section but one can already say that the size of such a domain is at most n(n-1)/2 +1 and so less than 2^{n-1} (for n > 3).

3 Abello's work

I begin by Abello's contributions contained in his doctoral dissertation (1985) and several papers (1981, 1984 with Johnson, 1985, 1987, 1988, 1991, 2004). In all these papers Abello works with S_n the set of all permutations on a set of cardinality n. I will describe some of his results but I will continue to rather speak of linear orders belonging to $\mathcal{L}_n$. These results use the partial order known as the *weak Bruhat order* (on S_n)[10]. Let L be an arbitrary linear order of $\mathcal{L}_n$; it will be convenient to take L = 1<2<….n. For L' $\in \mathcal{L}_n$, one sets InvL' = {{i,j} $\subseteq$ A such that iLj and jL'i} (i.e. the set of pairs {i,j} on which L and L' «disagree»). For L', L'' $\in \mathcal{L}_n$, one sets L'' $\leq$ L' if InvL' $\subseteq$ InvL''. It has been shown by Guilbaud and Rosenstiehl (1963) that the poset ($\mathcal{L}_n$, $\leq$) denoted henceforth simply by $\mathcal{L}_n$ is a lattice[11] called the "*permutoèdre*" *lattice* in French tradition (see for instance Barbut et Monjardet 1970). Its maximum element is 1<2<…n denoted by ϖ, and its minimum element is the dual linear order n<…2<1 denoted by α.

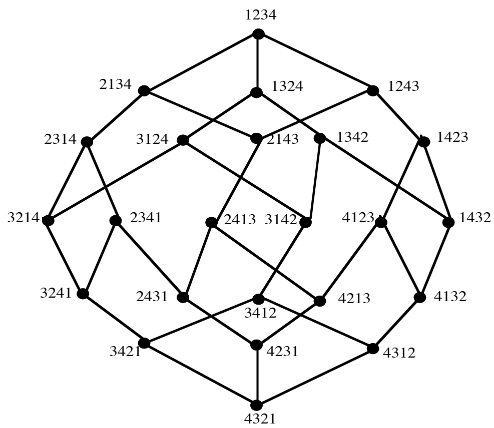

Fig. 1 The permutoèdre lattice $\mathcal{L}_4$

[10] S_n the symmetric group of all permutations on {1,2….n} is an example of a finite Coxeter group. All Coxeter groups can be partially ordered by the so-called weak Bruhat order (and also by the strong Bruhat order).

[11] That is two linear orders have a least upper bound and a greatest lower bound in this partial order. Some authors attribute this result to Yanagimoto and Okamoto (1969). One can admit than a paper published in French will be less known that a paper written in English. But Guilbaud and Rosenstiehl's paper which precedes Yanagimoto and Okamoto's paper has been quoted in many English-written papers ; moreover its proof that S_n is a lattice is reproduced in *Principles of combinatorics* (Berge 1971) and above all Yanagimoto and Okamoto's paper does not contain a real proof of their assertion (read it !). One can add that properties of the permutoèdre lattice are studied in Barbut and Monjardet (1970), Le Conte de Poly-Barbut (1990), Duquenne and Cherfouh (1994), Markowsky (1994) and Caspard (2000) and that more generally Björner (1984) proved that all finite Coxeter groups partially ordered by the weak Bruhat order are lattices.

The lattice $\mathcal{L}_4$ is represented on Figure 1 by a (Hasse) diagram giving its covering relation. The undirected covering relation of this lattice is the *adjacency relation* between linear orders where a linear order is adjacent to another one if they differ on a unique pair of elements. The set of all linear orders endowed with this adjacency relation is called the *permutoèdre graph*.
Come back to acyclic domains. The first easy observation is that the set of ordered triples ijk contained in the linear orders of an acyclic domain $\mathcal{D}$ of $\mathcal{L}_n$ has size at most 4n(n-1)(n-2)/6 (if not $\mathcal{D}$ contains a 3-cyclic profile). So when one adds to an acyclic domain $\mathcal{D}$ all the linear orders which don't increase the set of ordered triples already present in $\mathcal{D}$ one gets a maximal acyclic domain. More generally the map, which adds to an arbitrary set of linear orders all the linear orders that don't increase the set of ordered triples, is a closure operator on the subsets of $\mathcal{L}_n$[12]. The second –also easy but significant– observation is that a maximal chain of $\mathcal{L}_n$ is an acyclic domain (a fact already observed by Blin (1973) as said above) which contains exactly 4n(n-1)(n-2)/6 ordered triples. So by applying the above closure operator to a maximal chain one obtains a maximal acyclic set. Now Abello has proved several significant results and in particular the following ones:
1) a maximal acyclic domain $\mathcal{D}$ obtained by the closure operator applied to a maximal chain of $\mathcal{L}_n$ is a connected subset of $\mathcal{L}_n$ of diameter n(n-1)/2 and an upper semimodular sublattice of the permutoèdre lattice ;
2) for any maximal connected acyclic domain of $\mathcal{L}_n$ of diameter n(n-1)/2, there exists a maximal acyclic domain with the same size obtained by the closure of a maximal chain ;
3) let us say that two maximal chains of the permutoèdre lattice $\mathcal{L}_n$ are equivalent if they have the same closure (and so are two maximal chains of the associated lattice). One goes from one of these chains to the other by «quadrangular transformations» of linear orders : let $L = x_1\ldots x_k x_{k+1}\ldots x_i x_{i+1}\ldots x_n$ be a linear order such that $x_k x_{k+1}$ and $x_i x_{i+1}$ are four different alternatives ; then L is transformed into $L' = x_1\ldots x_{k+1} x_k\ldots x_{i+1} x_i\ldots x_n$ ($= \tau_i\tau_k(L) = \tau_k\tau_i(L)$).
Property 2 means that to search maximal connected acyclic domains of diameter n(n-1)/2 with large size it suffices to consider those obtained by the closure of a maximal chain. Abello gives an algorithm to get the maximal

[12] This closure operator appears already in Kim and Roush's 1980 book (see Definition 5.12)

connected acyclic domain obtained from a maximal chain $L_0 \prec L_1 \ldots \prec L_{n(n-1)/2}$ of $\mathcal{L}_n$. The algorithm constructs a sequence $\mathcal{D}_0 = \{L_0\}, \mathcal{D}_1, \ldots \mathcal{D}_{n(n-1)/2}$ of acyclic domains. One goes from $\mathcal{D}_s$ to $\mathcal{D}_{s+1}$ by adding to $\mathcal{D}_s$ the linear order L_{k+1} and the set of linear orders obtained by applying to all the linear orders of a subset $\mathcal{E}_s$ of $\mathcal{D}_s$ the transposition τ_i (of x_i and x_{i+1}) used to obtain L_{k+1} from L_k ; a linear order M is in $\mathcal{E}_s$ if there exists in $\mathcal{D}_s \cup \{L_{k+1}\}$ a maximal chain from M to L_{k+1}, for which none of the transpositions along this chain act on x_i or x_{i+1}.

A similar algorithm can be used with other acyclic domains to get maximal connected acyclic domains. With this algorithm Abello and Johnson 1981 show that $f(n) \geq 3(2^{n-2})-4$ (for $n \geq 4$). Except for n = 4, where one gets a lower bound of 8 and where a maximal acyclic domain of size 9 has been already found[13], the acyclic domains so found were the first of size $> 2^{n-1}$. One will see in the following sections that there exist maximal connected acyclic domains with a much greater size.

4 Chameni-Nembua's work

Chameni-Nembua's work on acyclic domains is contained in his 1970 «thèse de 3ème cycle» and in a paper appeared the same year. I was his thesis' director and his work has answered some questions that I had asked him to investigate. The origin of these questions comes back to Guilbaud's paper (1962). In this paper one finds an analysis of Black's domain showing that the set of single-peaked linear orders has a distributive lattice structure and that the majority relation of a profile taken in this domain is the *median* of the elements of the profile in this lattice[14]. In particular one finds (page 289 of the English translation) a figure showing the distributive lattice of the sixteen single-peaked linear orders on a set of five alternatives. This figure is reproduced below at Figure 2a**.** One can observe that

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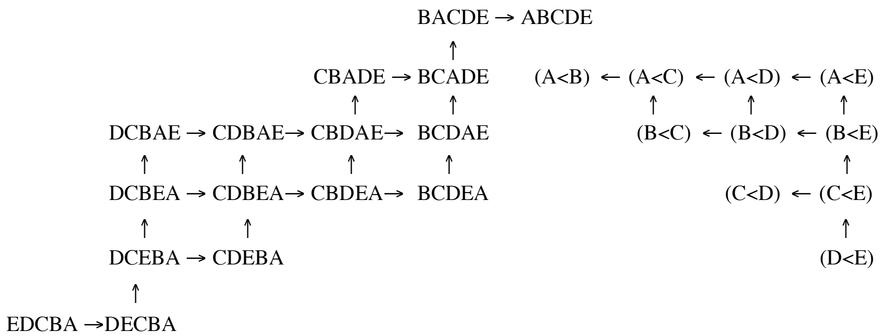
[13] An acyclic domain of size 9 in $\mathcal{L}_4$ is given in Kim and Roush's book (1980) or in Raynaud (1982). Such an acyclic domain is represented Figure 4 as $\mathcal{AS}(4)$ (see section 5).

[14] The fact that in this case majority relation is both a *metric* and an a*lgebraic* median is a special case of median's theory in distributive lattices (or more generally in median semilattices). One will find elements of this theory and references in Barthélemy and Monjardet (1981), Monjardet (2006a) and in Day and McMorris's 2005 book.

this lattice is a *covering sublattice* of the permutoèdre lattice $\mathcal{L}_5$ that means that the covering relation in this sublattice is the same as the covering relation in $\mathcal{L}_5$. Indeed a single-peaked linear order is covered by another single-peaked linear order if and only if they differ on a unique pair of elements.



2a **2b**

Fig.2. The distributive lattice of the 16 single-peaked linear orders on a 5-set and the associated poset of the ordered pairs

Several other acyclic domains that are covering distributive sublattices of the permutoèdre lattice were given in Frey (1971) and in Frey and Barbut's 1971 book. For instance the so-called «fuseaux bipolaires d'insertions» which are in fact the sets of all linear orders containing a partial order formed by the (cardinal) sum of two unrelated chains. Figure 3 here reproduces the Figure page 121 of Frey and Barbut's book showing the case where the two unrelated chains are $1<2<3$ and $4<5<6$ (I have replaced letters by integers) ; one obtains a (not maximal) covering distributive sublattice of $\mathcal{L}_6$. Other examples given in this book are the so-called «faisceaux d'indifférence» which are the set of linear orders which differ from a given linear order L only on consecutive elements of L[15] and the set of «co-blackiens» (= single-troughed) linear orders.

[15] Like the «fuseaux bipolaires», the «faisceaux d'indifférence» are also the set of linear extensions of some posets P of *width* (the maximum number of incomparable elements of P) 2. More generally the set of linear extensions of any poset of width 2 is a covering distributive sublattice of $\mathcal{L}_n$ (Chameni-Nembua 1989).

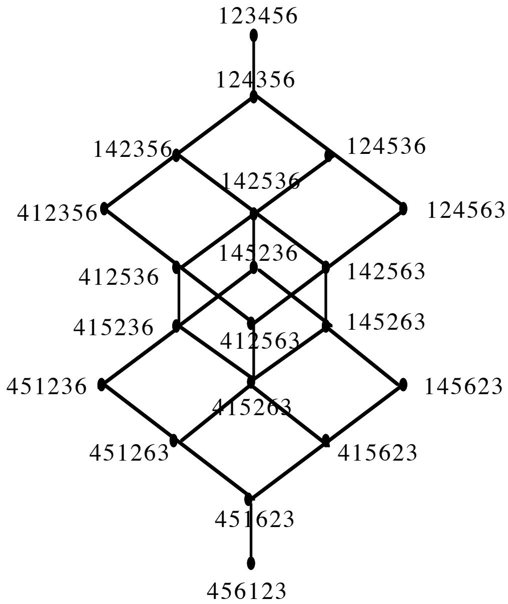


Fig. 3. The distributive lattice of the linear extensions of a poset sum of two chains

So I asked Chameni-Nembua to answer the following question: is any covering distributive sublattice of the permutoèdre lattice an acyclic domain ? His answer based on properties of the meet and join in this lattice and the fact that a distributive lattice must not contain some sublattices (see any book on lattice theory and Monjardet 1971 for the case of $\mathcal{L}_n$) was positive. Moreover he showed that maximal covering distributive sublattices are maximal acyclic domains which contain the minimum and the maximum elements of $\mathcal{L}_n$ (i.e. n<…2<1 and 1<2<….n) and so a maximal chain of $\mathcal{L}_n$. These results led us to search such large maximal covering distributive sublattices of $\mathcal{L}_n$. For n = 4, one founds the sublattice $\mathcal{AS}(4)$ of size 9 represented Figure 4.

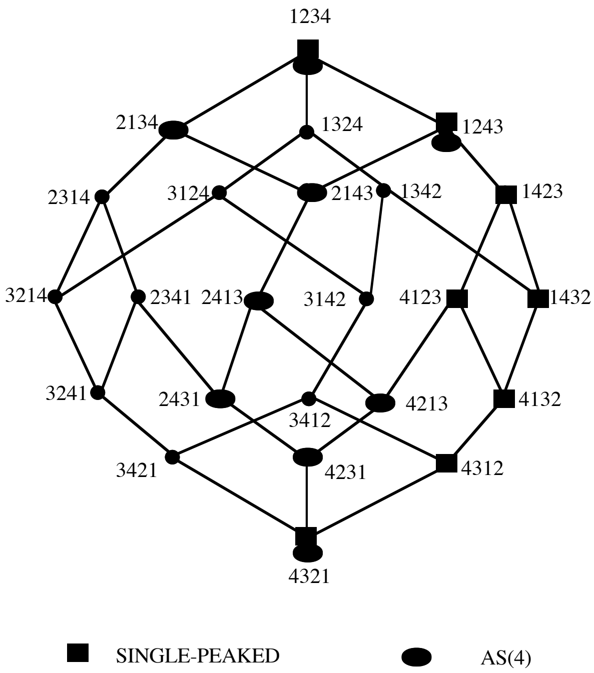


Fig. 4. Two distributive lattices acyclic domains on a 4-set

For n = 5, we found such a sublattice of size 20, and for n = 6, I found a sublattice of size 45 which is the last Figure in Chameni-Nembua's paper and which is reproduced here at Figure 5 (with integers instead of letters for the elements of A). This last sublattice had the interest to show that it was possible to surpass the best Abello and Johnson's lower bound known at this date ($f(6) \geq 45 > 44 = 3(2^6-2)-4$). I was pretty sure that there was a general construction to get such large acyclic domains but since I didn't find it I sent these examples to Peter who was already working on the topic and (obviously) found the construction described in the next section[16].

[16] I should be ashamed to have not found this construction since as it will seen in section 5 it was sufficient to look the triples and in fact it was also found by Dridi (1994 private letter). But Fishburn achieved a much more difficult task: to compute up to n = 25 the size of the corresponding acyclic domains (Dridi computed this size up to n = 8 with the exact values for n ≤ 7 but he found 220 instead of 222 for n = 8). By the way, it is worthwhile to mention here Fishburn's practice, what one should like more wide-spread in our scientific world. In his works on acyclic domains he always quoted the example that I have sent to him. He always did the same when in some other circumstances and/or for some other authors I indicated to him a result preceding one of his works.

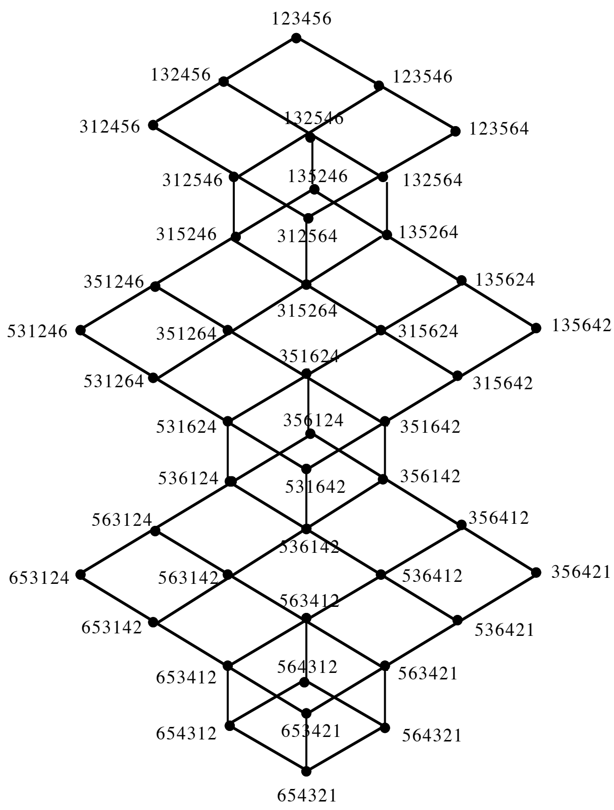


Fig.5. $\mathcal{AS}(6)$, a distributive lattice acyclic domain of size 45 on a 6-set

5 Fishburn's and Craven's works

It seems that Peter's interest for acyclic domains was motivated by Craven's conjecture reported in Kelly's 1991 paper. In his 1992 book Craven conjectures that $f(n) = 2^{n-1}$ and he gives an example of an acyclic domain of size 8 for n = 4 (but see [17]). Kelly exhibits for $n \geq 4$ a maximal acyclic domain of size 2^{n-1} generalizing Craven' example (in fact, this domain is a

[17] This is another example of the bad circulation of some scientific results since this conjecture has been already made by Johnson (1978) and disproved at least since 1980 (see footnote 13 and Figure 4).

maximal Arrow-Black's single-peaked domain). In his 1992 note Fishburn mentions that the above conjecture is false for $n \geq 4$ (see footnote 13) and that in fact $f(n)/2^{n-1} \to \infty$. This is proved by using in particular an iterative construction of acyclic domains where the first one is the domain of size 9 on a 4-element set and one goes from an acyclic domain of size p on a n-set to an acyclic domain of size $2p^2$ on a 2n-set. Fishburn's paper contains also a replacement construction which for n = 2m and $m \geq 4$ gives a much better lower bound than Abello and Johnson's lower bound : $f(16) \geq 68049 > 3(2^{14}-2)-4 = 49148$. In fact when Peter wrote his Notes on Craven's conjecture he didn't remember that Abello had worked on the topic. He remembered only after he read Kim, Roush and Intriligator's 1992 *Overview of Mathematical Social Sciences* where the problem to find f(n) was mentioned. Therefore, when (in January 1993) I sent him Chameni-Nembua's paper with my example of Figure 5 they were welcome. A week later he sent me a 7 pages memo containing the first elements of what will become his 1996 and 1997's papers (for which I was referee or editor) and the personal details mentioned above. These papers contain many significant results.

Firstly Peter defines the alternating scheme which is the construction allowing to generalize my example. Let 1 <2…<p…<n be a linear order on A. An acyclic domain $\mathcal{D}$ of $\mathcal{L}_n$ satisfies the *alternating scheme*, if for all i<j<k

either (1) jN1 if j is even and jN3 if j is odd. or (2) jN3 if j is even and jN1 if j is odd

(observe that these two domains are dual). So to define such a domain, denoted by $\mathcal{AS}(n)$, one combines the Never Conditions used for the single-peaked and single-troughed domains. The size of $\mathcal{AS}(n)$ is computed by recursion up to n = 25. Concerning these sizes Peter writes that he was unable to find a closed formula for them. Such a formula has been since obtained by Galambos and Reiner (2006 see next section). The number of linear orders satisfying the alternating scheme is:

$$2^{n-3}(n+3) - C(n-2, n/2-1)\ (n-3/2), \text{ for even } n > 2$$

$$2^{n-3}(n+3) - C(n-1, (n-1)/2)\ (n-1)/2), \text{ for odd } n > 1$$

where C(p,q) = p!/(p-q)!q! is the binomial coefficient.

Secondly Fishburn proves that f(4) = 9, f(5) = 20 and that for $n \leq 5$, an acyclic domain is maximum if and only if satisfies the alternating scheme. He conjectured the same for n = 6 and 7 the first conjecture having been proved in his 2002 paper (a not easy task!).

Thirdly it is shown that at least for $n \geq 16$, the alternating scheme is not optimal since the *replacement scheme* is better. This scheme uses two acyclic domains $\mathcal{D}$ defined on {0,1,2…m} and $\mathcal{D}$' defined on {m+1,

…m+p}. For every order in $\mathcal{D}$ one replaces 0 by each of the orders in $\mathcal{D}$'. It is easy to check that the domain of linear orders obtained on {0,1,2…m, m+1,…m+p} is acyclic. Hence one gets f(m+p) ≥ f(p)f(m+1)[18] and in particular f(16) ≥ 108336 > 105884 the size of the acyclic domain given by the alternating scheme. Another result allows to show that f(n) > $(2.17)^n$ for all large n and that $|\mathcal{AS}(n)|/f(n) \to 0$ as $n \to \infty$ i.e., that the lower bound given by the alternating scheme becomes more and more bad.
Finally the paper attacks the «major challenge» to find good upper bounds for f(n). The only upper bound already known 2[(n-1)!] has been given in Arrow and Raynaud's book, but for instance it gives f(9) ≤ 103.698 whereas a clever Fishburn's Lemma allows to obtain f(9) ≤ 22.680. The paper raises two conjectures. The first one is f(n+m) ≤ f(n+1)f(m+1) for all n, m ≥1 and in Fishburn 2002's paper it is shown that it would imply f(n) < $(2.591)^{n-2}$. The second conjecture is f(n) ≤ $(c)^n$ for some constant c and it was proved later by Raz (2000).
I come back now to Craven's works. In his 1994 note he gives a *partition scheme* which generalizes a construction given in Fishburn's 1992 note and which in a particular case is equivalent to Fishburn's replacement scheme. So he obtains the same formula f(m+p) ≥ f(p)f(m+1) allowing him to improve some lower bounds of Fishburn's note. In his 1996 paper, after reproving the fact that there are 2^{n-1} single-peaked linear orders on a n-set (see section 2), he studies the acyclic domains generated by Fishburn's alternating scheme. In particular he makes more precise the linear orders generated by this scheme and he gives some recurrence relations allowing him to obtain the sizes of the corresponding acyclic domains up to n = 15.

6 Galambos and Reiner's work

In this section I consider the problem to compute g(n) or rather good lower bounds to this number, i.e., to provide large connected acyclic domains. We have seen that Abello had constructed such domains by applying a closure operator to some maximal chains of the permutoèdre lattice. I have given an example showing that it was possible to find larger such domains that are covering distributive sublattices of the permutoèdre lattice (shown to be acyclic domains by Chameni-Nembua). Generalizing this exemple by means of his alternating scheme using the two Never Conditions jN3 and

[18] If instead to use this replacement procedure one concatenates each order in $\mathcal{D}$ with each order in $\mathcal{D}$' (a procedure proposed in Abello 1985) one gets only f(m+p) ≥ f(p)f(m).

jN1, Fishburn obtained the up to now best lower bound known for g(n). I present now the link between these various results as it is established in recent Galambos and Reiner's 2006 work (and anticipated in Guilbaud's 1952 paper; see Remark later).
Abello constructs maximal connected acyclic domains which are (upper) semimodular sublattices of the permutoèdre lattice by using the fact that the maximal chains of these lattices have an invariant, namely the set of the ordered triples of elements appearing in the orders of the chain. Galambos and Reiner show that these lattices are the same as Chameni-Nembua's lattices, i.e., that they are (maximal) covering distributive sublattices of the permutoèdre lattice and that their maximal chains have another invariant, namely a poset defined on $P^2(n)$ (the set of n(n-1)/2 ordered pairs (i<j)). The fact that Abello's maximal connected acyclic domains are distributive lattices is significant since it allows to use the well-known Birkhoff's duality between posets and distributive lattices.
We need some notions of lattice theory. A join-irreducible element of a lattice is an element covering a unique element and an ideal (respectively, a filter) of a poset (X,<) is a subset I of X such that $x \in I$ and $y < x$ implies $y \in I$ (respectively, a subset F of X such that $x \in F$ and $x < y$ implies $y \in F$). Now by Birkhoff's duality between posets and distributive lattices, a distributive lattice D is isomorphic to the set ordered by inclusion of all the ideals of the poset J_D of its join-irreducible elements (or to the set ordered by $\supseteq$ of all the filters of J_D). It is well-known that in this duality the maximal chains of a distributive lattice are in a one-to-one correspondence with the linear extensions of the poset J_D (i.e. with the linear orders containing the partial order between the join-irreducible elements) ; indeed when x_k is covered by x_{k+1} in a maximal chain of a distributive lattice then there exists a unique join-irreducible element j_k such that $x_{k+1} = x_k \vee j_k$; so the covering relation $x_k \prec x_{k+1}$, can be labeled by j_k and the linear order $j_1 j_2 \ldots j_{|J_D|}$ obtained on J_D is a linear extension of the poset J_D.
What are the join-irreducible elements of a covering distributive sublattice of the permutoèdre lattice? I consider a covering distributive sublattice $\mathcal{D}$ containing a maximal chain of $\mathcal{L}_n$ (then containing the maximum element $\varpi = 1<2<\ldots n$ and the minimum element $\alpha = n<\ldots 2<1$ of the permutoèdre lattice). A linear order L is a join-irreducible element of $\mathcal{D}$ if it covers a unique other element L' of $\mathcal{D}$. Then one has $L = x_1 \ldots x_k x_{k+1} \ldots x_n = \tau_k(L' = x_1 \ldots x_{k+1} x_k \ldots x_n)$ with $x_k < x_{k+1}$ (in the order $1<2<\ldots.n$). Yet, since on a maximal chain between α and ϖ any of the n(n-1)/2 ordered pairs j>i of α has to be transposed exactly once to get ϖ, the transposition of the elements x_k

and x_{k+1} appears for the first time in any maximal chain between α and $x_1 \ldots x_k x_{k+1} \ldots x_n$. So we can identify the join-irreducible $L = x_1 \ldots x_k x_{k+1} \ldots x_n$ with the ordered pair (x_k, x_{k+1}), and finally the poset of join-irreducible elements of $\mathcal{D}$ is isomorphic to a poset $P_{\mathcal{D}} = [P^2(n), <_{\mathcal{D}}]$ defined on the set $P^2(n)$ of all the ordered pairs $i<j$. Now any linear order L in $\mathcal{D}$ corresponds to an ideal of $P_{\mathcal{D}}$: L is obtained from $\alpha = n<\ldots2<1$ by applying all the transpositions of the ordered pairs belonging to this ideal. And any maximal chain of $\mathcal{D}$ corresponds to a linear order on $P^2(n)$, which is a linear extension of the poset $P_{\mathcal{D}}$.

Using more general results on Bruhat orders (Ziegler, 1993) Galambos and Reiner characterize the linear orders on $P^2(n)$ which are *admissible* i.e., which correspond to the sequence of transpositions of a maximal chain C of $\mathcal{L}_n$: a linear order λ on $P^2(n)$ is admissible if and only if it contain only triples (of ordered pairs) ordered in the lexicographic order or in its dual, i.e triples of the form $ij<ik<jk$ or $jk<ik<ij$. Moreover, these two sets of ordered triples are the same for the linear orders corresponding to any maximal chain of the distributive lattice closure of the chain C. For instance a maximal chain of the domain of single peaked-linear orders of $\mathcal{L}_4$ is $4321\prec4312\prec4132\prec1432\prec1423\prec1243\prec1234$, the associated linear order on $P^2(4)$ is $12\prec13\prec14\prec23\prec24\prec34$ and the set of ordered triples corresponding to any of the maximal chains in this domain is {(12,13,23), (12,14,24), {(13,14,34), {(23,24,34)} (so it does not contain triples dually lexicographically ordered). The domain $\mathcal{AS}(4)$ contains the maximal chain $4321\prec4231\prec4213\prec2413\prec2143\prec2134\prec1234$ that gives on $P^2(4)$ the linear order $23\prec13\prec24\prec14\prec34\prec12$, the set {(13,14,34), {(23,24,34)} of lexicographically ordered triples and the set {(23,13,12), (24,14,12)} of dually lexicographically ordered triples.

When one takes an arbitrary maximal chain $C = \alpha \prec L_1 \prec L_2 \ldots \prec \varpi$ of $\mathcal{L}_n$ it is a maximal chain in a maximal covering distributive sublattice $\mathcal{D}$ of the permutoèdre lattice. In order to determine $\mathcal{D}$ it suffices to determine the poset $P_{\mathcal{D}}$ associated to this maximal chain. Galambos and Reiner constructs $P_{\mathcal{D}}$ by using a notion of «arrangement of pseudolines» allowing to represent $P_{\mathcal{D}}$ and its ideals and so to recover the linear orders in $\mathcal{D}$. Another algorithm to get $P_{\mathcal{D}}$ is proposed in Monjardet 2006b.

When $P_{\mathcal{D}}$ is known, to computing the size of $\mathcal{D}$ comes back to computing the numbers of ideals of this poset, a not easy task in general, since this computation is known to be #$\mathcal{P}$-complete (Provan and Ball 1983). In the case when $\mathcal{D}$ is given by the alternating scheme, the corresponding poset has a very regular structure (its covering relation is given in Monjardet 2006b). Galambos and Reiner describe it by means of a certain arrangement of pseudolines and show that computing the ideals of this poset comes back computing some lattice paths. Using cleverly path enumeration techniques they get the formula for $|\mathcal{AS}(n)|$ given in the previous section.

Another significant Galambos and Reiner's result is the characterization of the maximal covering distributive sublattice $\mathcal{D}$ of $\mathcal{L}_n$ by a set of Never Conditions. Let C be a maximal chain of $\mathcal{D}$ and λ be the corresponding linear order admissible on $P^2(n)$, i.e. the linear order corresponding to the sequence of transpositions of this maximal chain. It has been said above that the restrictions of λ to any subset {(ij), (ik),(jk)} of three ordered pairs are ordered either lexicographically (ij<ik<jk) or dually lexicographically (jk<ik<ij). Let us denote by $LEX_3\lambda$ (respectively, $ALEX_3\lambda$) the set of triples ijk for which the set {(ij),(ik),(jk)} is lexicographically ordered (respectively, dually lexicographically ordered) in λ. As also already said, $LEX_3\lambda$ and $ALEX_3\lambda$ are the same for any other maximal chain of $\mathcal{D}$. Then, $\mathcal{D}$ is the set of all linear orders satisfying the following Never Conditions:

$$jN1, \forall\ i<j<k \text{ with } ijk \in LEX_3\lambda$$
$$jN3, \forall\ i<j<k \text{ with } ijk \in ALEX_3\lambda$$

For instance, for any maximal chain λ of $\mathcal{AS}(4)$, $LEX_3\lambda$ = {134,234} and $ALEX_3\lambda$ = {123,124} and one gets again the Never Conditions 3N1 and 2N3 of formula (2) in section 5.

Remark

As said before Guilbaud's paper contains an anticipation of a Galambos and Reiner's result in a particular case. Indeed Guilbaud not only pointed out the distributive lattice structure of the domain of single-peaked linear orders but he also gave an explanation for it. He writes (page 286, English translation): «These remarks focus attention on a sort of hierarchy of the judgments[19] ; one judgment dominates several others…This subordination is easy to designate in the form of an ordered network» (he adds in note: «This is a partially ordered structure, called a lattice»[20]). He represents this partial order by a triangular tableau for the domain of single-peaked linear orders on a 6-element set (this tableau is reproduced here Figure 2b) and he adds below it: «Note that the affirmation of any one of these judgments implies the affirmation of all the «consequents» ; that is, the affirmation of those located either in the same row and to the left, or in the same column and thus of all the judgments located to the left and above». He concludes that single-peaked orders corresponds to frontiers separating judgments + (i.e., $x>y$) and judgments – (i.e., $x\leq y$) in the triangular tableau. In other terms he shows that single-peaked orders correspond to filters in the partial order defined between the ordered pairs.

7 Conclusion

The search for large acyclic domains appears as a fascinating quest all the more that I have not said all. For instance maximal chains of the permutoèdre lattice are in one-to-one correspondence with other significant combinatorial objects the standard Young tableaux and the balanced tableaux (see Edelman and Greene 1987, Abello 2004) and this allows other interpretations of the raised problems[21].

There are also interesting algorithmic problems to answer the question of recognizing acyclic domains. Some answers have been given –especially

[19] In Guilbaud's paper a (simple) judgment is an ordered pair of alternatives expressing a preference between them; for example, $x >y$ (see page 285 of the translation)

[20] Indeed in the case of the covering distributive sublattice corresponding to single-peaked orders, it is not difficult to prove that the associated poset on $P^2(n)$ is the lattice where $(i,j)\vee(k,l) = (\max(i,k), \max(j,l))$ and $(i,j)\wedge(k,l) = (\min(i,k), \min(j,l))$. See also Monjardet 2006b.

[21] A *balanced tableau* of format (n-1,…2,1) is a tableau of $n(n-1)/2$ cases – corresponding to the ordered pairs ($i<j$)– containing the integers from 1 to $n(n-1)/2$. Such a tableau codes a maximal chain of $\mathcal{L}_n$ by coding the linear order λ on $P^2(n)$ associated to this chain: the integer in the case corresponding to (i,j) is the rank of (i,j) in λ. Conversely a balanced tableau induces the maximal chain obtained by effecting the sequence of transpositions of the ordered pairs in the order of the cases of the tableau. The –much more sophisticated– bijection between maximal chains of $\mathcal{L}_n$ and standard Young tableaux allows to Edelman and Greene to give a formula for computing the number of these chains.

for Black's single peaked domains– by Romero (1978, see also Arrow and Raynaud 1986)), Bartholdi and Trick (1986) and Doignon and Falmagne (1994).
I end this paper by mentioning a last result and two conjectures. Instead to search maximal covering distributive sublattices of the permutoèdre lattice which have a maximum size, one can ask what are those that have a minimum size. Since such a sublattice is the closure of a maximal chain one gets the answer if there exist maximal chains that are closed. It's actually the case as it is shown in Monjardet 2006b. This paper contains also some results on the distributive lattices given by Fishburn's alternating scheme and by Black's single-peakedness condition.

Conjecture 1 (Fishburn 1996, 1997)

$$f(n+m) \leq f(n+1)f(m+1) \text{ for all } n, m \geq 1$$

The proof of this conjecture would imply $(2.17)^n < f(n) < (2.591)^{n-2}$ for all large n since Fishburn (1997) proved the lower bound and the implication for the upper bound (2002). Then if true it would give a much better upper bound that the bound 4^{n-1} conjectured by Abello (1991). In the same paper Abello conjectures $g(n) \leq 3^{n-1}$ for which the conjectured upper bound $(2.591)^{n-2}$ would still be much better.

Let $|\mathcal{AS}(n)|$ be the size of the acyclic domain given by the alternating scheme.
Conjecture 2 (Galambos and Reiner 2006)

$$g(n) = |\mathcal{AS}(n)|$$

This conjecture is true for $n \leq 6$ since in this case $f(n) = |\mathcal{AS}(n)|$ and Galambos and Reiner checked it for n = 7.

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APPENDIX

Table . Exact values and bounds for **g(n)** (maximum size of a maximal connected acyclic domain) and **f(n)** (maximum size of a maximal acyclic domain)

	A	B	C	D	E	F	G	H
n	2^{n-1}	$2^{n-1}+2^{n-3}-1$	$3.2^{n-2}-4$	$\mathcal{AS}(n)$	**g(n)**	C(n)	RS(n)	**f(n)**
3	4	4	2	4	**4**	5	4	**4**
4	8	9	8	9	**9**	14	8	**9**
5	16	19	20	20	**20**	42	16	**20**
6	32	39	44	45	**45**	132	36	**45**
7	64	79	92	100	**100 ?**	429	81	?
8	128	159	188	222	?	1430	180	?
9	256	319	380	488	?	4862	400	?
10	512	639	764	1069	?	16796	900	?
11	1024	1279	1532	2324	?	58786	2025	?
12	2048	2559	3068	5034	?	208012	4500	?
13	4096	5119	6140	10840	?	742900	10000	?
14	8192	10239	12284	23266	?	2674440	22200	?
15	16384	20479	24572	49704	?	9694845	49284	?
16	32768	40959	49148	105884	?	35357670	108336	?
17	65536	81919	98300	224720	?		238144	?
18	131072	163840	196604	475773	?		521672	?
19	262144	826680	393216	1004212	?		1142761	?
20	524288	671359	805628	2115186	?		2484356	?

EXACT VALUES
E: $n \le 4$ folklore, n = 5,6 Fishburn 1997, 2002
H: $n \le 4$ folklore, n = 5,6 Fishburn 1997, 2002

LOWER BOUNDS
A: Craven's conjecture, 1992 (!)
B: Kim and Roush,1980
C: Abello and Johnson 1984 (N.B. $3.2^{n-2}-4 = 2^{n-1}+2^{n-2}-4$)
D: Fishburn 1997 (Alternating scheme, n = 6 BM 1989)
G: Fishburn 1997 (Replacement scheme $f(n+m) \ge f(n).f(m+1)$)
For all large n, $(2.17)^n < f(n)$ (Fishburn 1997)

UPPER BOUNDS
F: $g(n) < C(n)$ = Catalan number 2n!/n!(n+1)! (Abello 1991)
For all n, $f(n) < c^n$ for some c > 0 (Raz 2000)