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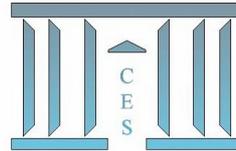
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# Detecting induced subgraphs

Benjamin Lévêque\*, David Y. Lin†, Frédéric Maffray\*, Nicolas Trotignon‡

October 9, 2007

## Abstract

An *s-graph* is a graph with two kinds of edges: *subdivisible* edges and *real* edges. A *realisation* of an *s-graph*  $B$  is any graph obtained by subdividing subdivisible edges of  $B$  into paths of arbitrary length (at least one). Given an *s-graph*  $B$ , we study the decision problem  $\Pi_B$  whose instance is a graph  $G$  and question is “Does  $G$  contain a realisation of  $B$  as an induced subgraph?”. For several  $B$ ’s, the complexity of  $\Pi_B$  is known and here we give the complexity for several more.

Our NP-completeness proofs for  $\Pi_B$ ’s rely on the NP-completeness proof of the following problem. Let  $\mathcal{S}$  be a set of graphs and  $d$  be an integer. Let  $\Gamma_{\mathcal{S}}^d$  be the problem whose instance is  $(G, x, y)$  where  $G$  is a graph whose maximum degree is at most  $d$ , with no induced subgraph in  $\mathcal{S}$  and  $x, y \in V(G)$  are two non-adjacent vertices of degree 2. The question is “Does  $G$  contain an induced cycle passing through  $x, y$ ?”. Among several results, we prove that  $\Gamma_{\emptyset}^3$  is NP-complete. We give a simple criterion on a connected graph  $H$  to decide whether  $\Gamma_{\{H\}}^{+\infty}$  is polynomial or NP-complete. On the other hand, the polynomial cases rely on the algorithm three-in-a-tree, due to Chudnovsky and Seymour.

**AMS Mathematics Subject Classification:** 05C85, 68R10, 68W05, 90C35

**Key words:** detecting, induced, subgraphs

## 1 Introduction

In this paper graphs are simple and finite. A *subdivisible graph* (*s-graph* for short) is a triple  $B = (V, D, F)$  such that  $(V, D \cup F)$  is a graph and  $D \cap F = \emptyset$ . The edges in  $D$  are said to be *real edges* of  $B$  while the edges in  $F$  are said to be *subdivisible edges* of  $B$ . A *realisation* of  $B$  is a graph obtained from  $B$  by subdividing edges of  $F$  into paths of arbitrary length (at least one). The problem

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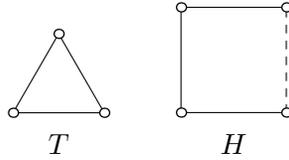


Figure 1: s-graphs yielding trivially polynomial problems

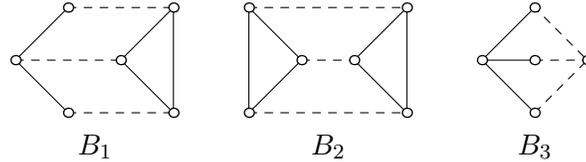


Figure 2: Pyramids, prisms and thetas

$\Pi_B$  is the decision problem whose input is a graph  $G$  and whose question is "Does  $G$  contain a realisation of  $B$  as an induced subgraph?". On figures, we depict real edges of an s-graph with straight lines, and subdivisible edges with dashed lines.

Several interesting instance of  $\Pi_B$  are studied in the litterature. For some of them, the existence of a polynomial time algorithm is trivial, but efforts are devoted toward optimized algorithms. For example, Alon, Yuster and Zwick [2] solve  $\Pi_T$  in time  $O(m^{1.41})$  (instead of the obvious  $O(n^3)$  algorithm), where  $T$  is the s-graph depicted on Figure 1. This problem is known as *triangle detection*. Tarjan and Yannakakis [10] solve  $\Pi_H$  in time  $O(n + m)$  where  $H$  is the s-graph depicted on Figure 1.

For some  $\Pi_B$ 's, the existence of a polynomial time algorithm is non-trivial. A *pyramid* (resp. *prism*, *theta*) is any realisation of the s-graph  $B_1$  (resp.  $B_2$ ,  $B_3$ ) depicted on figure 2. Chudnovsky and Seymour [5] gave an  $O(n^9)$ -time algorithm for  $\Pi_{B_1}$  (or equivalently, for detecting a pyramid). As far as we know, that is the first example of a solution to a  $\Pi_B$  whose complexity is non-trivial to settle. In contrast, Maffray and Trotignon [8] proved that  $\Pi_{B_2}$  (or detecting a prism) is NP-complete. Chudnovsky and Seymour [4] gave an  $O(n^{11})$ -time algorithm for  $P_{B_3}$  (or detecting a theta). Their algorithm relies on the solution of a problem called "three-in-a-tree", that we will define precisely and use in Section 2. The three-in-tree algorithm is quite general since it can be used to solve a lot of  $\Pi_B$  problems, including the detection of pyramids.

These facts are a motivation for a systematic study of  $\Pi_B$ . A further motivation is that very similar s-graphs can lead to a drastically different complexity. The following example may be more striking than pyramid/prism/theta :  $\Pi_{B_4}, \Pi_{B_6}$  are polynomial and  $\Pi_{B_5}, \Pi_{B_7}$  are NP-complete, where  $B_4, \dots, B_7$  are the s-graphs depicted on figure 3. This will be proved in section 3.1.

## Notation and remarks

By  $C_k$  ( $k \geq 3$ ) we denote the cycle on  $k$  vertices, by  $K_l$  ( $l \geq 1$ ) the clique on  $l$  vertices. We denote by  $I_l$  ( $l \geq 1$ ) the tree on  $l + 5$  vertices obtained by

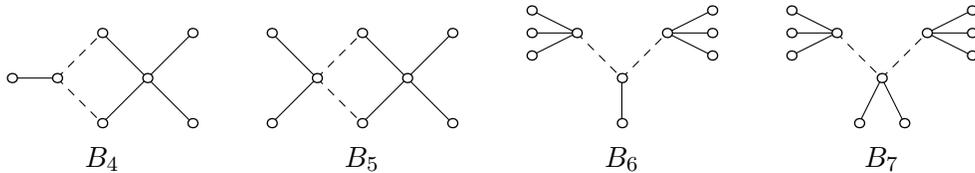


Figure 3: Some s-graphs with pending edges

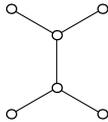


Figure 4:  $I_1$

taking a path of length  $l$  with ends  $a, b$ , and adding four vertices, two of them adjacent to  $a$ , the other two to  $b$ ; see Figure 4. When a graph  $G$  contains a graph isomorphic to  $H$  as an induced subgraph, we will often say “ $G$  contain an  $H$ ”.

Let  $(V, D, F)$  be an s-graph. Suppose that  $(V, D \cup F)$  has a vertex of degree one incident to an edge  $e$ . Then  $\Pi_{(V, D \cup \{e\}, F \setminus \{e\})}$  and  $\Pi_{(V, D \setminus \{e\}, F \cup \{e\})}$  have the same complexity, because a graph  $G$  contains a realisation of  $(V, D \cup \{e\}, F \setminus \{e\})$  if and only if it contains a realisation of  $(V, D \setminus \{e\}, F \cup \{e\})$ . For the same reason, if  $(V, D \cup F)$  has a vertex of degree two incident to the edges  $e \neq f$  then  $\Pi_{(V, D \setminus \{e\} \cup \{f\}, F \setminus \{f\} \cup \{e\})}$ ,  $\Pi_{(V, D \setminus \{f\} \cup \{e\}, F \setminus \{e\} \cup \{f\})}$  and  $\Pi_{(V, D \setminus \{e, f\}, F \cup \{e, f\})}$  have the same complexity. If  $|F| \leq 1$  then  $\Pi_{(V, D, F)}$  is clearly polynomial. Thus in the rest of the paper, we will consider only s-graphs  $(V, D, F)$  such that:

- $|F| \geq 2$ ;
- no vertex of degree one is incident to an edge of  $F$ ;
- every induced path of  $(V, D \cup F)$  with all interior vertices of degree 2 and whose ends have degree  $\neq 2$  has at most one edge in  $F$ . Moreover, this edge is incident to an end of the path;
- every induced cycle with at most one vertex  $v$  of degree at least 3 in  $(V, D \cup F)$  has at most one edge in  $F$  and this edge is incident to  $v$  if  $v$  exists (if it does not then the cycle is a component of  $(V, D \cup F)$ ).

## 2 Detection of holes with prescribed vertices

Let  $\Delta(G)$  be the maximum degree of  $G$ . Let  $\mathcal{S}$  be a set of graphs and  $d$  be an integer. Let  $\Gamma_{\mathcal{S}}^d$  be the problem whose instance is  $(G, x, y)$  where  $G$  is a graph such that  $\Delta(G) \leq d$ , with no induced subgraph in  $\mathcal{S}$  and  $x, y \in V(G)$  are two non-adjacent vertices of degree 2. The question is “Does  $G$  contain a hole passing through  $x, y$ ?”. For simplicity, we write  $\Gamma_{\mathcal{S}}$  instead of  $\Gamma_{\mathcal{S}}^{+\infty}$  (so, the graph in the instance of  $\Gamma_{\mathcal{S}}$  has unbounded degree). Also we write  $\Gamma^d$  instead

of  $\Gamma_\emptyset^d$  (so the graph in the instance of  $\Gamma^d$  has no restriction on its induced subgraphs). Bienstock [3] proved that  $\Gamma = \Gamma_\emptyset$  is NP-complete. For  $\mathcal{S} = \{K_3\}$  and  $\mathcal{S} = \{K_{1,4}\}$ ,  $\Gamma_{\mathcal{S}}$  can be shown to be NP-complete, and a consequence is the NP-completeness of several problems of interest: see [8] and [9].

In this section, we try to settle  $\Gamma_{\mathcal{S}}^d$  for as many  $\mathcal{S}$ 's and  $d$ 's as we can. In particular, we give the complexity of  $\Gamma_{\mathcal{S}}$  when  $\mathcal{S}$  contains only one connected graph and of  $\Gamma^d$  for all  $d$ . We also settle  $\Gamma_{\mathcal{S}}^d$  for some cases when  $\mathcal{S}$  is a set of cycles. The polynomial cases are either trivial, or are a direct consequence of an algorithm of Chudnovsky and Seymour. The NP-complete cases follow from several extensions of Bienstock's construction.

## Polynomial cases

Chudnovsky and Seymour [4] proved that the problem whose instance is a graph and three vertices  $a, b, c$ , whose question is "Does the graph contains a tree passing through  $a, b, c$  as an induced subgraph?" can be solved in  $O(n^4)$ . We call this algorithm "three-in-a-tree". Three-in-a-tree can be used directly to solve  $\Gamma_{\mathcal{S}}$  for several  $\mathcal{S}$ 's. Let us call *subdivided claw* any tree with one vertex  $u$  of degree 3, three vertices  $v_1, v_2, v_3$  of degree 1 and all the other vertices of degree 2.

**Theorem 2.1** *Let  $H$  be a graph on  $k$  vertices that is either a path or a subdivided claw. There is an  $O(n^k)$ -time algorithm for  $\Gamma_{\{H\}}$ .*

PROOF — Here is an algorithm for  $\Gamma_{\{H\}}$ . Let  $(G, x, y)$  be an instance of  $\Gamma_H$ . If  $H$  is a path on  $k$  vertices then every hole in  $G$  is on at most  $k$  vertices. Hence, by a brute-force search on every  $k$ -tuple, we will find a hole through  $x, y$  if there is any. Now we suppose that  $H$  is a subdivided claw. So  $k \geq 4$ . For convenience, we put  $x_1 = x, y_1 = y$ . Let  $x_0, x_2$  (resp.  $y_0, y_2$ ) be the two neighbors of  $x_1$  (resp.  $y_1$ ).

First check whether there is in  $G$  a hole  $C$  through  $x_1, y_1$  such that the distance between  $x_1$  and  $y_1$  in  $C$  is at most  $k - 2$ . If  $k = 4$  or  $k = 5$  then the vertex-set of any such hole must be included in  $\{x_0, x_1, x_2, y_0, y_1, y_2\}$ , so it can be found in constant time. Now suppose  $k \geq 6$ . For every  $l$ -tuple  $(x_3, \dots, x_{l+2})$  of vertices of  $G$ , with  $l \leq k - 5$ , test whether  $P = x_0 - x_1 - \dots - x_{l+2} - y_2 - y_1 - y_0$  is an induced path, and if so delete the interior vertices of  $P$  and their neighbors except  $x_0, y_0$ , and look for a shortest path from  $x_0$  to  $y_0$ . This will find the desired hole if there is one, after possibly swapping  $x_0, x_2$  and doing the work again. This takes time  $O(n^{k-3})$ .

Now we assume that in every hole through  $x_1, y_1$ , the distance between  $x_1, y_1$  is at least  $k - 1$ .

Let  $k_i$  be the length of the unique path of  $H$  from  $u$  to  $v_i$ ,  $i = 1, 2, 3$ . Note that  $k = k_1 + k_2 + k_3 + 1$ . Let us check every  $(k - 4)$ -tuple  $z = (x_3, \dots, x_{k_1+1}, y_3, \dots, y_{k_2+k_3})$  of vertices of  $G$ . For such a  $(k - 4)$ -tuple, test whether  $x_0 - x_1 - \dots - x_{k_1+1}$  and  $P = y_0 - y_1 - \dots - y_{k_2+k_3}$  are induced paths of  $G$  with no edge between them except possibly  $x_{k_1+1}y_{k_2+k_3}$ . If not, go to the next  $(k - 4)$ -tuple, but if yes, delete the interior vertices of  $P$  and their neighbors except  $y_0, y_{k_2+k_3}$ . Also delete the neighbors of  $x_2, \dots, x_{k_1}$ , except  $x_1, x_2, \dots, x_{k_1}, x_{k_1+1}$ . Call  $G_z$  the resulting graph and run three-in-a-tree in

$G_z$  for the vertices  $x_1, y_{k_2+k_3}, y_0$ . We claim that the answer to three-in-a-tree is YES for some  $(k-4)$ -tuple if and only if  $G$  contains a hole through  $x_1, y_1$  (after possibly swapping  $x_0, x_2$  and doing the work again).

If  $G$  contains a hole  $C$  through  $x_1, y_1$  then up to a symmetry this hole visits  $x_0, x_1, x_2, y_2, y_1, y_0$  in this order. Let us name  $x_3, \dots, x_{k_1+1}$  the vertices of  $C$  that follow after  $x_1, x_2$ , and let us name  $y_3, \dots, y_{k_2+k_3}$  those that follow after  $y_1, y_2$ . Note that all these vertices exist and are pairwise distinct since in every hole through  $x_1, y_1$  the distance between  $x_1, y_1$  is at least  $k-1$ . So the path from  $y_0$  to  $y_{k_2+k_3}$  in  $C \setminus y_1$  is a tree of  $G_z$  passing through  $x_1, y_{k_2+k_3}, y_0$ , where  $z$  is the  $(k-4)$ -tuple  $(x_3, \dots, x_{k_1+1}, y_3, \dots, y_{k_2+k_3})$ .

Conversely, suppose that  $G_z$  contains a tree  $T$  passing through  $x_1, y_{k_2+k_3}, y_0$ , for some  $(k-4)$ -tuple  $z$ . We suppose that  $T$  is vertex-inclusion-wise minimal. If  $T$  is a path visiting  $y_0, x_1, y_{k_2+k_3}$  in this order, then we obtain the desired hole of  $G$  by adding  $y_1, y_2, \dots, y_{k_2+k_3-1}$  to  $T$ . If  $T$  is a path visiting  $x_1, y_0, y_{k_2+k_3}$  in this order, then we denote by  $y_{k_2+k_3+1}$  the neighbor of  $y_{k_2+k_3}$  along  $T$ . Note that  $T$  contains either  $x_0$  or  $x_2$ . If  $T$  contains  $x_0$ , then there are three paths in  $G$ :  $y_0 - T - x_0 - x_1 - \dots - x_{k_1}$ ,  $y_0 - T - y_{k_2+k_3+1} - \dots - y_{k_3+2}$  and  $y_0 - y_1 - \dots - y_{k_3}$ . These three paths form a subdivided claw centered at  $y_0$  that is long enough to contain an induced subgraph isomorphic to  $H$ , a contradiction. If  $T$  contains  $x_2$  then the proof works similarly with  $y_0 - T - x_{k_1+1} - x_{k_1} - \dots - x_1$  instead of  $y_0 - T - x_0 - x_1 - \dots - x_{k_1}$ . If  $T$  is a path visiting  $x_1, y_{k_2+k_3}, y_0$  in this order, the proof is similar, except that we find a subdivided claw centered at  $y_{k_2+k_3}$ . If  $T$  is not a path, then it is a subdivided claw centered at a vertex  $u$  of  $G$ . We obtain again an induced subgraph of  $G$  isomorphic to  $H$  by adding to  $T$  sufficiently many vertices of  $\{x_0, \dots, x_{k_1+1}, y_0, \dots, y_{k_2+k_3}\}$ .  $\square$

## NP-complete cases (unbounded degree)

Many NP-completeness results can be proved by adapting Bienstock's construction. We give here several polynomial reductions from the problem 3-SATISFIABILITY of Boolean functions. These results are given in a framework that involves a few parameters, so that our result can possibly be used for other problems of the same type. Recall that a Boolean function with  $n$  variables is a mapping  $f$  from  $\{0, 1\}^n$  to  $\{0, 1\}$ . A Boolean vector  $\xi \in \{0, 1\}^n$  is a *truth assignment* for  $f$  if  $f(\xi) = 1$ . For any Boolean variable  $z$  on  $\{0, 1\}$ , we write  $\bar{z} := 1 - z$ , and each of  $z, \bar{z}$  is called a *literal*. An instance of 3-SATISFIABILITY is a Boolean function  $f$  given as a product of clauses, each clause being the Boolean sum  $\vee$  of three literals; the question is whether  $f$  admits a truth assignment. The NP-completeness of 3-SATISFIABILITY is a fundamental result in complexity theory, see [6].

Let  $f$  be an instance of 3-SATISFIABILITY, consisting of  $m$  clauses  $C_1, \dots, C_m$  on  $n$  variables  $z_1, \dots, z_n$ . For every integer  $k \geq 3$  and parameters  $\alpha \in \{1, 2\}$ ,  $\beta \in \{0, 1\}$ ,  $\gamma \in \{0, 1\}$ ,  $\delta \in \{0, 1, 2, 3\}$ ,  $\varepsilon \in \{0, 1\}$ ,  $\zeta \in \{0, 1\}$  such that if  $\alpha = 2$  then  $\varepsilon = \beta = \gamma$ , let us build a graph  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  with two specialized vertices  $x, y$  of degree 2. There will be a hole containing  $x$  and  $y$  in  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  if and only if there exists a truth assignment for  $f$ . In  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  (we will sometimes write  $G_f$  for short), there will be two

kinds of edges: blue and red. The reason for this distinction will appear later.

For each variable  $z_i$  ( $i = 1, \dots, n$ ), prepare a graph  $G(z_i)$  with  $4k$  vertices  $a_{i,r}, b_{i,r}, a'_{i,r}, b'_{i,r}$ ,  $r \in \{1, \dots, k\}$  and  $4(m+2)2k$  vertices  $t_{i,2pk+r}, f_{i,2pk+r}, t'_{i,2pk+r}, f'_{i,2pk+r}$ ,  $p \in \{0, \dots, m+1\}$ ,  $r \in \{0, \dots, 2k-1\}$ . Add blue edges so that the four sets  $\{a_{i,1}, \dots, a_{i,k}, t_{i,0}, \dots, t_{i,2k(m+2)-1}, b_{i,1}, \dots, b_{i,k}\}$ ,  $\{a_{i,1}, \dots, a_{i,k}, f_{i,0}, \dots, f_{i,2k(m+2)-1}, b_{i,1}, \dots, b_{i,k}\}$ ,  $\{a'_{i,1}, \dots, a'_{i,k}, t'_{i,0}, \dots, t'_{i,2k(m+2)-1}, b'_{i,1}, \dots, b'_{i,k}\}$ ,  $\{a'_{i,1}, \dots, a'_{i,k}, f'_{i,0}, \dots, f'_{i,2k(m+2)-1}, b'_{i,1}, \dots, b'_{i,k}\}$  all induce paths (and the vertices appear in this order along these paths). See Figure 5.

Add red edges according to the value of  $\alpha, \beta, \gamma$ , as follows.

If  $\alpha = 1$  then, for every  $p = 1, \dots, m+1$ , add all edges between  $\{t_{i,2kp}, t_{i,2kp+\beta}\}$  and  $\{f_{i,2kp}, f_{i,2kp+\gamma}\}$ , between  $\{f_{i,2kp}, f_{i,2kp+\gamma}\}$  and  $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$ , between  $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$  and  $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$ , between  $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$  and  $\{t_{i,2kp}, t_{i,2kp+\beta}\}$ .

If  $\alpha = 2$  then, for every  $p = 1, \dots, m$ , add all edges between  $\{t_{i,2kp+k-1}, t_{i,2kp+k-1+\beta}\}$  and  $\{f_{i,2kp+k-1}, f_{i,2kp+k-1+\gamma}\}$ ; for every  $p = 1, \dots, m+1$ , add all edges between  $\{f_{i,2kp+k-1}, f_{i,2kp+k-1+\gamma}\}$  and  $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$ , between  $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$  and  $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$ , between  $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$  and  $\{t_{i,2k(p-1)+k-1}, t_{i,2k(p-1)+k-1+\beta}\}$ . See Figures 6, 7.

For each clause  $C_j$  ( $j = 1, \dots, m$ ), with  $C_j = y_j^1 \vee y_j^2 \vee y_j^3$ , where each  $y_j^q$  ( $q = 1, 2, 3$ ) is a literal from  $\{z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n\}$ , prepare a graph  $G(C_j)$  with  $2k$  vertices  $c_{j,p}, d_{j,p}$ ,  $p \in \{1, \dots, k\}$  and  $6k$  vertices  $u_{j,p}^q$ ,  $q \in \{1, 2, 3\}$ ,  $p \in \{1, \dots, 2k\}$ . Add blue edges so that the three sets  $\{c_{j,1}, \dots, c_{j,k}, u_{j,1}^q, \dots, u_{j,2k}^q, d_{j,1}, \dots, d_{j,k}\}$ ,  $q \in \{1, 2, 3\}$  all induce paths (and the vertices appear in this order along these paths). Add red edges according to the value of  $\delta$ . If  $\delta = 0$ , add no edge. If  $\delta = 1$ , add  $u_{j,1}^1 u_{j,1}^2, u_{j,2k}^1 u_{j,2k}^2$ . If  $\delta = 2$ , add  $u_{j,1}^1 u_{j,1}^2, u_{j,1}^1 u_{j,1}^3, u_{j,2k}^1 u_{j,2k}^2, u_{j,2k}^1 u_{j,2k}^3$ . If  $\delta = 3$ , add  $u_{j,1}^1 u_{j,1}^2, u_{j,1}^1 u_{j,1}^3, u_{j,2k}^1 u_{j,2k}^2, u_{j,2k}^1 u_{j,2k}^3$ . See Figure 8.

The graph  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  is obtained from the disjoint union of the  $G(z_i)$ 's and the  $G(C_j)$ 's as follows. For  $i = 1, \dots, n-1$ , add blue edges  $b_{i,k} a_{i+1,1}$  and  $b'_{i,k} a'_{i+1,1}$ . Add a blue edge  $b'_{n,k} c_{1,1}$ . For  $j = 1, \dots, m-1$ , add a blue edge  $d_{j,k} c_{j+1,1}$ . Introduce the two special vertices  $x, y$  and add blue edges  $xa_{1,1}, xa'_{1,1}$  and  $yd_{m,k}, yb_{n,k}$ . See Figure 9.

Add red edges according to  $f, \varepsilon, \zeta$ . For  $q = 1, 2, 3$ , if  $y_j^q = z_i$ , then add all possible edges between  $\{f_{i,2kj+k-1}, f_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$  and between  $\{f'_{i,2kj+k-1}, f'_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$ ; while if  $y_j^q = \bar{z}_i$  then add all possible edges between  $\{t_{i,2kj+k-1}, t_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$  and between  $\{t'_{i,2kj+k-1}, t'_{i,2kj+k-1+\varepsilon}\}$  and  $\{u_{j,k}^q, u_{j,k+\zeta}^q\}$ . See Figure 10.

Clearly the size of  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  is polynomial (actually quadratic) in the size  $n+m$  of  $f$ , and  $x, y$  are non-adjacent and both have degree two.

**Lemma 2.2**  *$f$  admits a truth assignment if and only if  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  contains a hole passing through  $x, y$ .*

PROOF — Recall that if  $\alpha = 2$  then  $\varepsilon = \beta = \gamma$ . We will prove the lemma for  $\beta = 0, \gamma = 0, \varepsilon = 0, \zeta = 0$  because the proof is essentially the same for the other possible values.

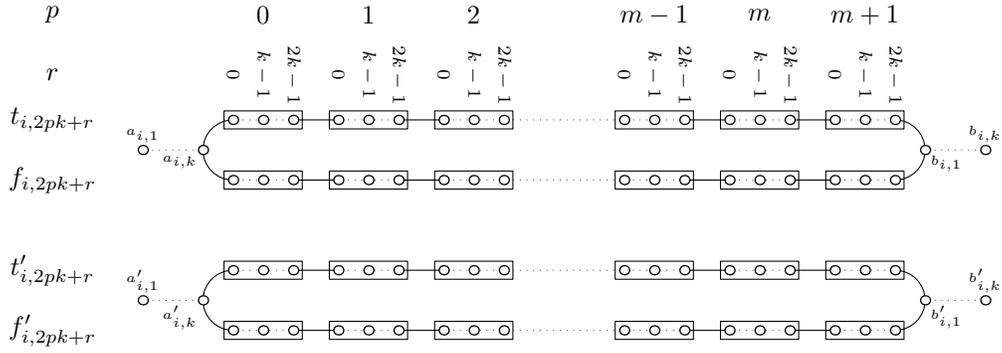


Figure 5: The graph  $G(z_i)$  (only blue edges are depicted)

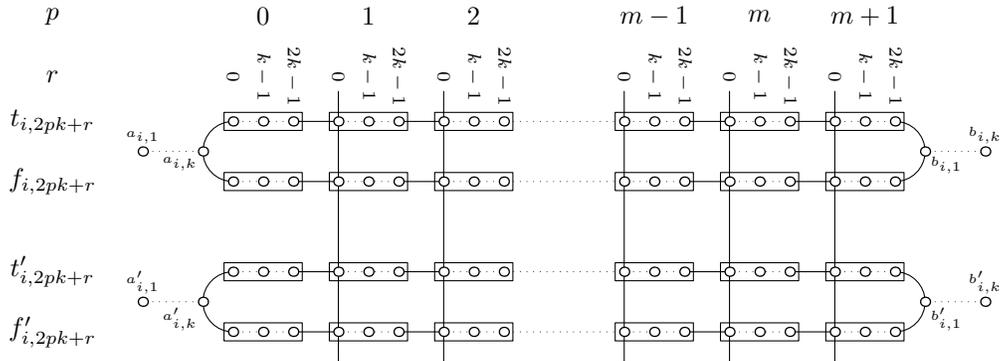


Figure 6: The graph  $G(z_i)$  when  $\alpha = 1, \beta = 0, \gamma = 0$

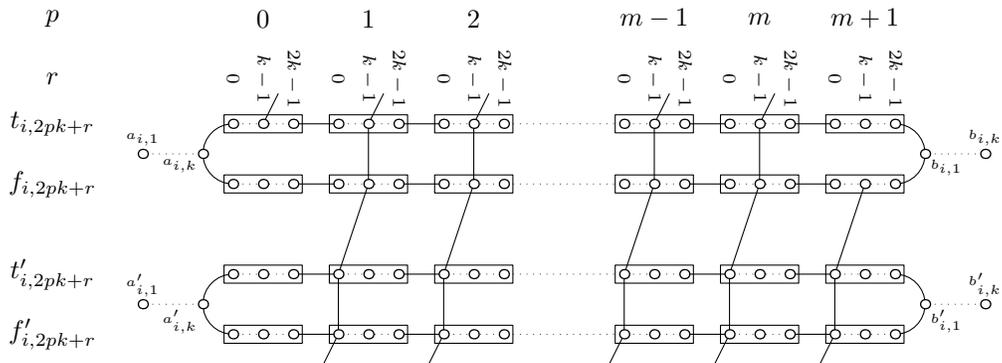


Figure 7: The graph  $G(z_i)$  when  $\alpha = 2, \beta = 0, \gamma = 0$

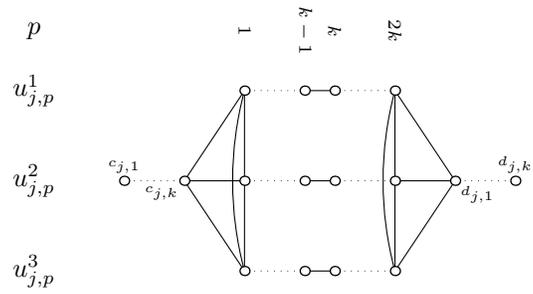


Figure 8: The graph  $G(c_j)$  when  $\delta = 3$

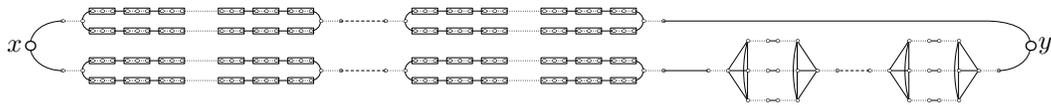


Figure 9: The whole graph  $G_f$

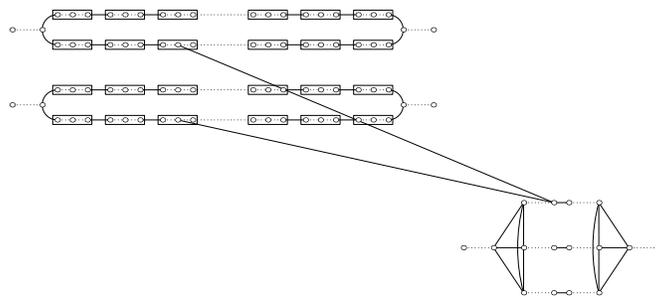


Figure 10: Red edges between  $G(z_i)$  and  $G(c_j)$  when  $\varepsilon = \zeta = 0$

Suppose that  $f$  admits a truth assignment  $\xi \in \{0, 1\}^n$ . We can build a hole in  $G$  by selecting vertices as follows. Select  $x, y$ . For  $i = 1, \dots, n$ , select  $a_{i,p}, b_{i,p}, a'_{i,p}, b'_{i,p}$  for all  $p \in \{1, \dots, k\}$ . For  $j = 1, \dots, n$ , select  $c_{j,p}, d_{j,p}$  for all  $p \in \{1, \dots, k\}$ . If  $\xi_i = 1$  select  $t_{i,p}, t'_{i,p}$  for all  $p \in \{0, \dots, 2k(m+2) - 1\}$ . If  $\xi_i = 0$  select  $f_{i,p}, f'_{i,p}$  for all  $p \in \{0, \dots, 2k(m+2) - 1\}$ . For  $j = 1, \dots, m$ , since  $\xi$  is a truth assignment for  $f$ , at least one of the three literals of  $C_j$  is equal to 1, say  $y_j^q = 1$  for some  $q \in \{1, 2, 3\}$ . Then select  $u_{j,p}^q$  for all  $p \in \{1, \dots, 2k\}$ . Now it is a routine matter to check that the selected vertices induce a cycle  $Z$  that contains  $x, y$ , and that  $Z$  is chordless, so it is a hole. The main point is that there is no chord in  $Z$  between some subgraph  $G(C_j)$  and some subgraph  $G(z_i)$ , for that would be either an edge  $t_{i,p}u_{j,r}^q$  with  $y_j^q = z_i$  and  $\xi_i = 1$ , or, symmetrically, an edge  $f_{i,p}u_{j,r}^q$  with  $y_j^q = \bar{z}_i$  and  $\xi_i = 0$ , and in either case this would contradict the way the vertices of  $Z$  were selected.

Conversely, suppose that  $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$  admits a hole  $Z$  that contains  $x, y$ .

(1) For  $i = 1, \dots, n$ ,  $Z$  contains at least  $4k + 4k(m+2)$  vertices of  $G(z_i)$ :  $4k$  of these are  $a_{i,p}, a'_{i,p}, b_{i,p}, b'_{i,p}$  where  $p \in \{1, \dots, k\}$ , and the others are either the  $t_{i,p}, t'_{i,p}$ 's or the  $f_{i,p}, f'_{i,p}$ 's where  $p \in \{0, \dots, 2k(m+2) - 1\}$ .

Let us first deal with the case  $i = 1$ . Since  $x \in Z$  has degree 2,  $Z$  contains  $a_{1,1}, \dots, a_{1,k}$  and  $a'_{1,1}, \dots, a'_{1,k}$ . Hence exactly one of  $t_{1,0}, f_{1,0}$  is in  $Z$ . Likewise exactly one of  $t'_{1,0}, f'_{1,0}$  is in  $Z$ . If  $t_{1,0}, f'_{1,0}$  are both in  $Z$  then there is a contradiction: indeed, if  $\alpha = 1$  then,  $t_{1,0}, \dots, t_{1,2k}$  and  $f'_{1,0}, \dots, f'_{1,2k}$  must all be in  $Z$ , and since  $t_{1,2k}$  sees  $f'_{1,2k}$ ,  $Z$  cannot go through  $y$ ; and if  $\alpha = 2$  the proof is similar. Similarly,  $t_{1,0}, f'_{1,0}$  cannot both be in  $Z$ . So, there exists a largest integer  $p \leq 2k(m+2) - 1$  such that either  $t_{1,0}, \dots, t_{1,p}$  and  $t'_{1,0}, \dots, t'_{1,p}$  are all in  $Z$  or  $f_{1,0}, \dots, f_{1,p}$  and  $f'_{1,0}, \dots, f'_{1,p}$  are all in  $Z$ .

We claim that  $p = 2k(m+2) - 1$ . For otherwise, some vertex  $w$  in  $\{t_{1,p}, t'_{1,p}, f_{1,p}, f'_{1,p}\}$  is incident to a red edge  $e$  of  $Z$ . If  $\alpha = 1$  then, up to a symmetry, we assume that  $t_{1,0}, \dots, t_{1,p}$  and  $t'_{1,0}, \dots, t'_{1,p}$  are all in  $Z$ . Let  $w'$  be the vertex of  $e$  that is not  $w$ . Then  $w'$  (which is either an  $f_{1,\cdot}$ , an  $f'_{1,\cdot}$  or a  $u_{j,\cdot}$ ) is a neighbor of both  $t_{1,p}, t'_{1,p}$ . Hence,  $Z$  cannot go through  $y$ , a contradiction. This proves our claim when  $\alpha = 1$ . If  $\alpha = 2$ , we distinguish between the following six cases.

*Case 1:*  $p = k - 1$ . Then  $e = t_{1,k-1}f'_{1,2k}$ . Clearly  $t_{1,0}, \dots, t_{1,k-1}$  must all be in  $Z$ . If  $t'_{1,0}, \dots, t'_{1,2k}$  are in  $Z$ , there is a contradiction because of  $t'_{1,2k}f'_{1,2k}$ , and if  $f'_{1,0}, \dots, f'_{1,2k}$  are in  $Z$ , there is a contradiction because of  $e$ .

*Case 2:*  $p = 2kl$  where  $1 \leq l \leq m+1$  and  $w = t'_{1,2kl}$ . Then  $e$  is  $t'_{1,2kl}f_{1,2kl+k-1}$  or  $t'_{1,2kl}f'_{1,2kl}$ . In either case,  $t_{1,2kl}, \dots, t_{1,2kl+k-1}$  are all in  $Z$ , and there is a contradiction because of the red edge  $f_{1,2kl+k-1}t_{1,2kl+k-1}$  or  $t_{1,2(l-1)k+k-1}f'_{1,2kl}$ , or when  $l = m+1$  because of  $b_{1,1}$ .

*Case 3:*  $p = 2kl$  where  $1 \leq l \leq m+1$  and  $w = f'_{1,2kl}$ . Then  $e$  is  $f'_{1,2kl}t_{1,2(l-1)k+k-1}$  or  $t'_{1,2kl}f'_{1,2kl}$ . In either case,  $f_{1,2kl}, \dots, f_{1,2kl+k-1}$  are all in  $Z$ , and there is a contradiction because of the red edge  $t_{1,2(l-1)k+k-1}f_{1,2(l-1)k+k-1}$  or  $t'_{1,2kl}f_{1,2kl+k-1}$ , or when  $l = 1$  because of  $a_{1,k}$ .

*Case 4:*  $p = 2kl + k - 1$  where  $1 \leq l \leq m$  and  $w = t_{1,2kl+k-1}$ . Then  $e$  is  $t_{1,2kl+k-1}f_{1,2kl+k-1}$ ,  $t_{1,2kl+k-1}f'_{1,2(l+1)k}$ , or  $t_{1,2kl+k-1}u_{j,k}^q$  for some  $j, q$ . In the last case, there is a contradiction since  $t'_{1,2kl+k-1} \in Z$  also sees  $u_{j,k}^q$ . For the same reason,  $t'_{1,2kl+k-1}u_{j,k}^q$  is not an edge of  $Z$  and  $t'_{1,2kl+k-1}, \dots, t'_{1,2(l+1)k}$  are

all in  $Z$ . So there is a contradiction because of the red edge  $t'_{1,2kl}f_{1,2kl+k-1}$  or  $t'_{1,2(l+1)k}f'_{1,2(l+1)k}$ .

*Case 5:*  $p = 2kl + k - 1$  where  $2 \leq l \leq m$  and  $w = f_{1,2kl+k-1}$ . Then  $e$  is  $f_{1,2kl+k-1}t_{1,2kl+k-1}$ ,  $f_{1,2kl+k-1}t'_{1,2kl}$ , or  $f_{1,2kl+k-1}u_{j,k}^q$  for some  $j, q$ . In the last case, there is a contradiction since  $f'_{1,2kl+k-1} \in Z$  also sees  $u_{j,k}^q$ . For the same reason,  $f'_{1,2kl+k-1}u_{j,k}^q$  is not an edge of  $Z$  and  $f'_{1,2kl+k-1}, \dots, f'_{1,2(l+1)k}$  are all in  $Z$ . So there is a contradiction because of the red edge  $t'_{1,2kl}f'_{1,2kl}$  or  $t_{1,2kl+k-1}f'_{1,2(l+1)k}$ .

*Case 6:*  $p = 2k(m+1) + k - 1$  and  $w = f_{1,2k(m+1)+k-1}$ . Then there is a contradiction because of the red edge  $t'_{1,2k(m+1)}f'_{1,2k(m+1)}$ . This proves our claim.

Since  $p = 2k(m+2) - 1$ ,  $b_{1,1}$  is in  $Z$ . We claim that  $b_{1,2}$  is in  $Z$ . For otherwise, the two neighbors of  $b_{1,1}$  in  $Z$  are  $t_{1,2k(m+2)-1}$  and  $f_{1,2k(m+2)-1}$ . This is a contradiction because of the red edges  $t_{1,2km+k-1}f'_{1,2k(m+1)}$ ,  $t'_{1,2k(m+1)}f_{1,2k(m+1)+k-1}$  (if  $\alpha = 2$ ) or  $t_{1,2k(m+1)}f'_{1,2k(m+1)}$ ,  $t'_{1,2k(m+1)}f_{1,2k(m+1)}$  (if  $\alpha = 1$ ). Similarly,  $b'_{1,1}, b'_{1,2}$  are in  $Z$ . So  $b_{1,1}, \dots, b_{1,k}$  and  $b'_{1,1}, \dots, b'_{1,k}$  are all in  $Z$ .

This proves (1) for  $i = 1$ . The proof for  $i = 2$  is essentially the same as for  $i = 1$ , and by induction the claim holds up to  $i = n$ . This proves (1).

(2) For  $j = 1, \dots, m$ ,  $Z$  contains  $c_{j,1}, \dots, c_{j,k}$ ,  $d_{j,1}, \dots, d_{j,k}$  and exactly one of  $\{u_{j,1}^1, \dots, u_{j,2k}^1\}$ ,  $\{u_{j,1}^2, \dots, u_{j,2k}^2\}$ ,  $\{u_{j,1}^3, \dots, u_{j,2k}^3\}$ .

Let us first deal with the case  $j = 1$ . By (1),  $b'_{n,k}$  is in  $Z$  and so  $c_{1,1}, \dots, c_{1,k}$  are all in  $Z$ . Consequently exactly one of  $u_{1,1}^1, u_{1,1}^2, u_{1,1}^3$  is in  $Z$ , say  $u_{1,1}^1$  up to a symmetry. Note that the neighbour of  $u_{1,1}^1$  in  $Z \setminus c_1$  cannot be a vertex among  $u_{1,1}^2, u_{1,1}^3$  for this would imply that  $Z$  contains a triangle. Hence  $u_{1,2}^1, \dots, u_{1,k}^1$  are all in  $Z$ . The neighbour of  $u_{1,k}^1$  in  $Z \setminus u_{1,k-1}^1$  cannot be in some  $G(z_i)$  ( $1 \leq i \leq n$ ). Else, up to a symmetry we assume that this neighbor is  $t_{1,p}$ ,  $p \in \{0, \dots, 2k(m+2) - 1\}$ . If  $t_{1,p} \in Z$ , there is a contradiction because then  $t'_{1,p}$  is also in  $Z$  by (1) and  $t'_{1,p}$  would be a third neighbour of  $u_{1,k}^1$  in  $Z$ . If  $t_{1,p} \notin Z$ , there is a contradiction because then the neighbor of  $t_{1,p}$  in  $Z \setminus u_{1,k}^1$  must be  $t_{1,p+1}$  (or symmetrically  $t_{1,p-1}$ ) for otherwise  $Z$  contains a triangle. So,  $t_{1,p+1}, t_{1,p+2}, \dots$  must be in  $Z$ , till reaching a vertex having a neighbor  $f_{1,p'}$  or  $f'_{1,p'}$  in  $Z$  (whatever  $\alpha$ ). Thus the neighbour of  $u_{1,k}^1$  in  $Z \setminus u_{1,k-1}^1$  is  $u_{1,k+1}^1$ . Similarly, we prove that  $u_{1,k+2}, \dots, u_{1,2k}$  are in  $Z$ , that  $d_{1,1}, \dots, d_{1,k}$  are in  $Z$ , and so the claim holds for  $j = 1$ . The proof of the claim for  $j = 2$  is essentially the same as for  $j = 1$ , and by induction the claim holds up to  $j = m$ . This proves (2).

Together with  $x, y$ , the vertices of  $Z$  found in (1) and (2) actually induce a cycle. So, since  $Z$  is a hole, they are the members of  $Z$  and we can replace “at least” by “exactly” in (1). We can now make a Boolean vector  $\xi$  as follows. For  $i = 1, \dots, n$ , if  $Z$  contains  $t_{i,0}, t'_{i,0}$  set  $\xi_i = 1$ ; if  $Z$  contains  $f_{i,0}, f'_{i,0}$  set  $\xi_i = 0$ . By (1) this is consistent. Consider any clause  $C_j$  ( $1 \leq j \leq m$ ). By (2) and up to symmetry we may assume that  $u_{j,k}^1$  is in  $Z$ . If  $y_j^1 = z_i$  for some  $i \in \{1, \dots, n\}$ , then the construction of  $G$  implies that  $f_{i,2kj+k-1}, f'_{i,2kj+k-1}$  are not in  $Z$ , so  $t_{i,2kj+k-1}, t'_{i,2kj+k-1}$  are in  $Z$ , so  $\xi_i = 1$ , so clause  $C_j$  is satisfied by  $x_i$ . If  $y_j^1 = \bar{z}_i$  for some  $i \in \{1, \dots, n\}$ , then the construction of  $G_f$  implies that

$t_{i,2kj+k-1}, t'_{i,2kj+k-1}$  are not in  $Z$ , so  $f_{i,2kj+k-1}, f'_{i,2kj+k-1}$  are in  $Z$ , so  $\xi_i = 0$ , so clause  $C_j$  is satisfied by  $\bar{z}_i$ . Thus  $\xi$  is a truth assignment for  $f$ .  $\square$

**Theorem 2.3** *Let  $k \geq 5$  be an integer.*

*Then  $\Gamma_{\{C_3, \dots, C_k, K_{1,6}\}}$  and  $\Gamma_{\{I_1, \dots, I_k, C_5, \dots, C_k, K_{1,4}\}}$  are NP-complete.*

PROOF — It is a routine matter to check that the graph  $G_f(k, 2, 0, 0, 0, 0, 0)$  contains no  $C_l$  ( $3 \leq l \leq k$ ) and no  $K_{1,6}$  (in fact it has no vertex of degree at least 6). So Lemma 2.2 implies that  $\Gamma_{\{C_3, \dots, C_k, K_{1,6}\}}$  is NP-complete.

It is a routine matter to check that the graph  $G_f(k, 1, 1, 1, 3, 1, 1)$  contains no  $K_{1,4}$ , no  $C_l$  ( $5 \leq l \leq k$ ) and no  $I_{l'}$  ( $3 \leq l' \leq k$ ). So Lemma 2.2 implies that  $\Gamma_{\{K_{1,4}, C_5, \dots, C_k, I_3, \dots, I_k\}}$  is NP-complete.  $\square$

## Complexity of $\Gamma_{\{H\}}$ when $H$ is a connected graph

**Theorem 2.4** *Let  $H$  be a connected graph. Then either :*

- *$H$  is a path or a subdivided claw and  $\Gamma_{\{H\}}$  is polynomial.*
- *$H$  contains one of  $K_{1,4}, I_k$  for some  $k \geq 1$ , or  $C_l$  for some  $l \geq 3$  as an induced subgraph and  $\Gamma_{\{H\}}$  is NP-complete.*

PROOF — If  $H$  contains one of  $K_{1,4}, I_k$  for some  $k \geq 1$ , or  $C_l$  for some  $l \geq 3$  as an induced subgraph then  $\Gamma_{\{H\}}$  is NP-complete by Theorem 2.3. Else,  $H$  is a tree since it contains no  $C_l, l \geq 3$ . If  $H$  has no vertex of degree at least 3, then  $H$  is a path and  $\Gamma_{\{H\}}$  is polynomial by Theorem 2.1. If  $H$  has a single vertex of degree at least 3, then this vertex has degree 3 because  $H$  contains no  $K_{1,4}$ . So,  $H$  is a subdivided claw and  $\Gamma_{\{H\}}$  is polynomial by Theorem 2.1. If  $H$  has at least two vertices of degree at least 3 then  $H$  contains an  $I_l$ , where  $l$  is the length of the unique path of  $H$  joining these two vertices. This is a contradiction.  $\square$

Interestingly, a similar theorem was proved by Alekseev:

**Theorem 2.5 (Alekseev, [1])** *Let  $H$  be a connected graph that is not a path nor a subdivided claw. Then the problem of finding a maximum stable set in  $H$ -free graphs is NP-hard.*

But the complexity of the maximum stable set problem is not known in general for  $H$ -free graphs when  $H$  is a path or a subdivided claw. See [7] for a survey.

## NP-complete cases (bounded degree)

Here, we will show that  $\Gamma^d$  is NP-complete when  $d \geq 3$  and polynomial when  $d = 2$ . If  $\mathcal{S}$  is any finite list of cycles  $C_{k_1}, C_{k_2}, \dots, C_{k_m}$ , then we will also show that  $\Gamma_{\mathcal{S}}^3$  is NP-complete as long as  $C_6 \notin \mathcal{S}$ .

Let  $f$  be an instance of 3-SATISFIABILITY, consisting of  $m$  clauses  $C_1, \dots, C_m$  on  $n$  variables  $z_1, \dots, z_n$ . For each clause  $C_j$  ( $j = 1, \dots, m$ ), with  $C_j = y_{3i-2} \vee y_{3i-1} \vee y_{3i}$ , then  $y_i$  ( $i = 1, \dots, 3m$ ) is a literal from  $\{z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n\}$ .

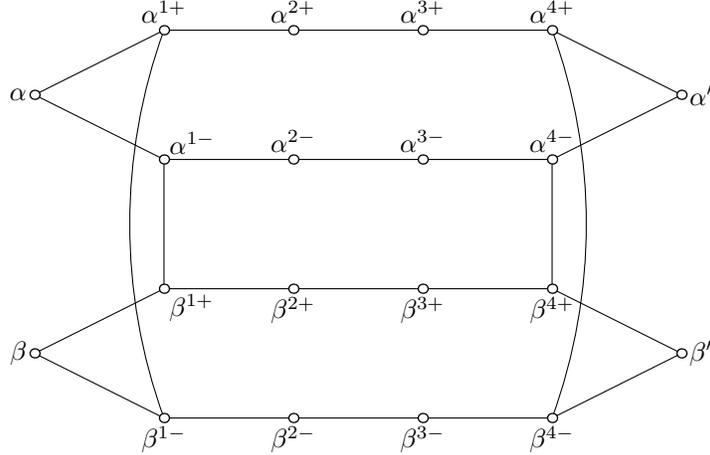


Figure 11: The graph  $G(y_j)$

Let us build a graph  $G_f$  with two specialized vertices  $x$  and  $y$  of degree 2 such that  $\Delta(G_f) = 3$ . There will be a hole containing  $x$  and  $y$  in  $G_f$  if and only if there exists a truth assignment for  $f$ .

For each literal  $y_j$  ( $j = 1, \dots, 3m$ ), prepare a graph  $G(y_j)$  on 20 vertices  $\alpha, \alpha', \alpha^{1+}, \dots, \alpha^{4+}, \alpha^{1-}, \dots, \alpha^{4-}, \beta, \beta', \beta^{1+}, \dots, \beta^{4+}, \beta^{1-}, \dots, \beta^{4-}$ . (We drop the subscript  $j$  in the labels of the vertices for clarity).

For  $i = 1, 2, 3$  add the edges  $\alpha^{i+}\alpha^{(i+1)+}, \beta^{i+}\beta^{(i+1)+}, \alpha^{i-}\alpha^{(i+1)-}, \beta^{i-}\beta^{(i+1)-}$ . Also add the edges  $\alpha^{1+}\beta^{1-}, \alpha^{1-}\beta^{1+}, \alpha^{4+}\beta^{4-}, \alpha^{4-}\beta^{4+}, \alpha\alpha^{1+}, \alpha\alpha^{1-}, \alpha^{4+}\alpha', \alpha^{4-}\alpha', \beta\beta^{1+}, \beta\beta^{1-}, \beta^{4+}\beta', \beta^{4-}\beta'$ .

For each clause  $C_j$  ( $j = 1, \dots, m$ ), prepare a graph  $G(C_j)$  with 10 vertices  $c^{1+}, c^{2+}, c^{3+}, c^{1-}, c^{2-}, c^{3-}, c^{0+}, c^{12+}, c^{0-}, c^{12-}$ . (We drop the subscript  $j$  in the labels of the vertices for clarity).

Add the edges  $c^{12+}c^{1+}, c^{12+}c^{2+}, c^{12-}c^{1-}, c^{12-}c^{2-}, c^{0+}c^{12+}, c^{0+}c^{3+}, c^{0-}c^{12-}, c^{0-}c^{3-}$ .

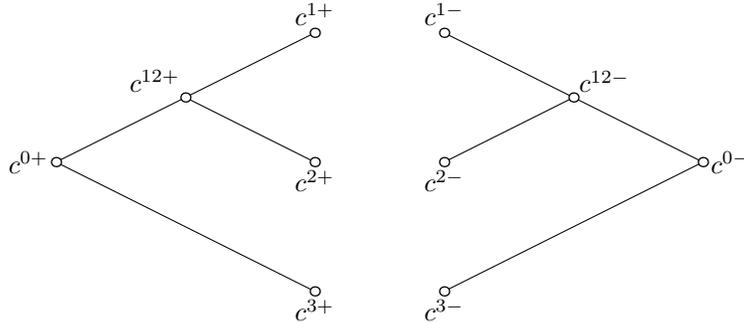


Figure 12: The graph  $G(C_j)$

For each variable  $z_i$  ( $i = 1, \dots, n$ ), prepare a graph  $G(z_i)$  with  $2z_i^- + 2z_i^+$  vertices, where  $z_i^-$  is the number of times  $\bar{z}_i$  appears in clauses  $C_1, \dots, C_m$  and  $z_i^+$  is the number of times  $z_i$  appears in clauses  $C_1, \dots, C_m$ .

Let  $G(z_i)$  consist of two internally disjoint paths  $P_i^+$  and  $P_i^-$  with common endpoints  $d_i^+$  and  $d_i^-$  and lengths  $1 + 2z_i^-$  and  $1 + 2z_i^+$  respectively. Label the vertices of  $P_i^+$  as  $d_i^+, p_{i,1}^+, \dots, p_{i,2f_i}^+, d_i^-$  and label the vertices of  $P_i^-$  as  $d_i^+, p_{i,1}^-, \dots, p_{i,2g_i}^-, d_i^-$ .

The final graph  $G_f$  (see figure 2) will be constructed from the disjoint union of all the graphs  $G(y_j)$ ,  $G(C_i)$ , and  $G(x_i)$  with the following modifications:

- For  $j = 1, \dots, 3m - 1$ , add the edges  $\alpha'_j \alpha_{j+1}$  and  $\beta'_j \beta_{j+1}$ . This creates one connected chain of the graphs  $G(y_j)$ .
- For  $j = 1, \dots, m - 1$ , add the edge  $c_j^{0-} c_{j+1}^{0+}$ . This creates one connected chain of the graphs  $G(C_j)$ .
- For  $i = 1, \dots, n - 1$ , add the edge  $d_i^- d_{i+1}^+$ . This creates one connected chain of the graphs  $G(x_i)$ .
- For  $i = 1, \dots, n$ , let  $y_{n_1}, \dots, y_{n_{z_i^-}}$  be the occurrences of  $\bar{x}_i$  over all literals. For  $j = 1, \dots, z_i^-$ , delete the edge  $p_{i,2j-1}^+ p_{i,2j}^+$  and add the four edges  $p_{i,2j-1}^+ \alpha_{n_j}^{2+}, p_{i,2j-1}^+ \beta_{n_j}^{2+}, p_{i,2j}^+ \alpha_{n_j}^{3+}, p_{i,2j}^+ \beta_{n_j}^{3+}$ .
- For  $i = 1, \dots, n$ , let  $y_{n_1}, \dots, y_{n_{z_i^+}}$  be the occurrences of  $x_i$  over all literals. For  $j = 1, 2, \dots, z_i^+$ , delete the edge  $p_{i,2j-1}^- p_{i,2j}^-$  and add the four edges  $p_{i,2j-1}^- \alpha_{n_j}^{2+}, p_{i,2j-1}^- \beta_{n_j}^{2+}, p_{i,2j}^- \alpha_{n_j}^{3+}, p_{i,2j}^- \beta_{n_j}^{3+}$ .
- For  $i = 1, \dots, m$  and  $j = 1, 2, 3$ , add the edges  $\alpha_{3(i-1)+j}^{2-} c_i^{j+}, \alpha_{3(i-1)+j}^{3-} c_i^{j-}, \beta_{3(i-1)+j}^{2-} c_i^{j+}, \beta_{3(i-1)+j}^{3-} c_i^{j-}$ .
- Add the edges  $\alpha'_{3m} d_1^+$  and  $\beta'_{3m} c_1^{0+}$
- Add the vertex  $x$  and add the edges  $x\alpha_1$  and  $x\beta_1$ .
- Add the vertex  $y$  and add the edges  $yc_m^{0-}$  and  $yd_n^-$ .

It is easy to verify that  $\Delta(G_f) = 3$ , that  $G_f$  is polynomial (actually linear) in the size  $n + m$  of  $f$ , and that  $x, y$  are non-adjacent and both have degree two.

**Lemma 2.6**  *$f$  admits a truth assignment if and only if  $G_f$  contains a hole passing through  $x$  and  $y$ .*

PROOF — First assume that  $f$  admits a truth assignment  $\xi \in \{0, 1\}^n$ . We will pick a set of vertices that induce a hole containing  $x$  and  $y$ .

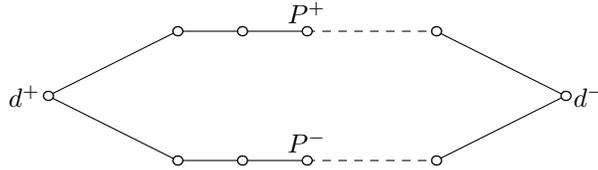


Figure 13: The graph  $G(z_i)$

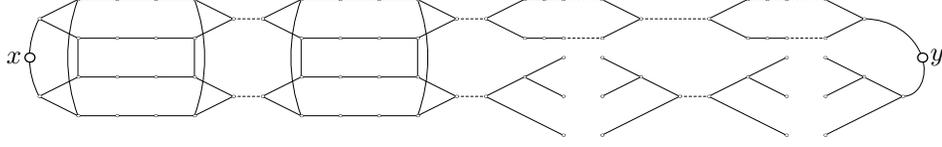


Figure 14: The final graph  $G_f$

1. Pick vertices  $x$  and  $y$ .
2. For  $i = 1, \dots, 3m$ , pick the vertices  $\alpha_i, \alpha'_i, \beta_i, \beta'_i$ .
3. For  $i = 1, \dots, 3m$ , if  $y_i$  is satisfied by  $\xi$ , then pick the vertices  $\alpha_i^{1+}, \alpha_i^{2+}, \alpha_i^{3+}, \alpha_i^{4+}, \beta_i^{1+}, \beta_i^{2+}, \beta_i^{3+}, \beta_i^{4+}$ . Otherwise, pick the vertices  $\alpha_i^{1-}, \alpha_i^{2-}, \alpha_i^{3-}, \alpha_i^{4-}, \beta_i^{1-}, \beta_i^{2-}, \beta_i^{3-}, \beta_i^{4-}$ .
4. For  $i = 1, \dots, n$ , if  $\xi_i = 1$ , then pick all the vertices of the path  $P_i^+$  and all the neighbors of the vertices in  $P_i^+$  of the form  $\alpha_k^{2+}$  or  $\alpha_k^{3+}$  for any  $k$ .
5. For  $i = 1, \dots, n$ , if  $\xi_i = 0$ , then pick all the vertices of the path  $P_i^-$  and all the neighbors of the vertices in  $P_i^-$  of the form  $\alpha_k^{2+}$  or  $\alpha_k^{3+}$  for any  $k$ .
6. For  $i = 1, \dots, m$ , pick the vertices  $c_i^{0+}$  and  $c_i^{0-}$ . Choose any  $j \in \{3i - 2, 3i - 1, 3i\}$  such that  $\xi$  satisfies  $y_j$ . Pick vertices  $\alpha_j^{2-}$ , and  $\alpha_j^{3-}$ . If  $j = 3i - 2$ , then pick the vertices  $c_i^{12+}, c_i^{1+}, c_i^{1-}, c_i^{12-}$ . If  $j = 3i - 1$ , then pick the vertices  $c_i^{12+}, c_i^{2+}, c_i^{2-}, c_i^{12-}$ . If  $j = 3i$ , then pick the vertices  $c_i^{3+}$  and  $c_i^{3-}$ .

It suffices to show that the chosen vertices induce a hole containing  $x$  and  $y$ . The only potential problem is that for some  $k$ , one of the vertices  $\alpha_k^{2+}, \alpha_k^{3+}, \alpha_k^{2-}$ , or  $\alpha_k^{3-}$  was chosen multiple times. If  $\alpha_k^{2+}$  and  $\alpha_k^{3+}$  were picked in Step 3, then  $y_k$  is satisfied by  $\xi$ . Therefore,  $\alpha_k^{2+}$  and  $\alpha_k^{3+}$  were not chosen in Step 4 or Step 5. Similarly, if  $\alpha_k^{2-}$  and  $\alpha_k^{3-}$  were picked in Step 6, then  $y_k$  is satisfied by  $\xi$  and  $\alpha_k^{2-}$  and  $\alpha_k^{3-}$  were not picked in Step 3. Thus, the chosen vertices induce a hole in  $G$  containing vertices  $x$  and  $y$ .

Now assume  $G_f$  contains a hole  $H$  passing through  $x$  and  $y$ . The hole  $H$  must contain  $\alpha_1$  and  $\beta_1$  since they are the only two neighbors of  $x$ . Next, either both  $\alpha_1^{1+}$  and  $\beta_1^{1+}$  are in  $H$ , or both  $\alpha_1^{1-}$  and  $\beta_1^{1-}$  are in  $H$ .

Without loss of generality, let  $\alpha_1^{1+}$  and  $\beta_1^{1+}$  be in  $H$  (the same reasoning that follows will hold true for the other case). Since  $\beta_1^{1-}$  and  $\alpha_1^{1-}$  are both neighbors of two members in  $H$ , they cannot be in  $H$ . Thus,  $\alpha_1^{2+}$  and  $\beta_1^{2+}$  must be in  $H$ . Since  $\alpha_1^{2+}$  and  $\beta_1^{2+}$  have the same neighbor outside  $G(y_1)$ , it follows that  $H$  must contain  $\alpha_1^{3+}$  and  $\beta_1^{3+}$ . Also,  $H$  must contain  $\alpha_1^{4+}$  and  $\beta_1^{4+}$ . Suppose that  $\alpha_1^{4-}$  and  $\beta_1^{4-}$  are in  $H$ . Because  $\alpha_1^{i-}$  has the same neighbor as  $\beta_1^{i-}$  outside  $G(y_1)$  for  $i = 2, 3$ , it follows that  $H$  must contain  $\alpha_1^{3-}, \alpha_1^{2-}$ , and  $\alpha_1^{1-}$ . But then  $H$  is not a hole containing  $b$ , a contradiction. Therefore,  $\alpha_1^{4-}$  and  $\beta_1^{4-}$  cannot both be in  $H$ , so  $H$  must contain  $\alpha'_1, \beta'_1, \alpha_2$ , and  $\beta_2$ .

By induction, we see for  $i = 1, 2, \dots, 3m$  that  $H$  must contain  $\alpha_i, \alpha'_i, \beta_i, \beta'_i$ . Also, for each  $i$ , either  $H$  contains  $\alpha_i^{1+}, \alpha_i^{2+}, \alpha_i^{3+}, \alpha_i^{4+}, \beta_i^{1+}, \beta_i^{2+}, \beta_i^{3+}, \beta_i^{4+}$  or  $H$  contains  $\alpha_i^{1-}, \alpha_i^{2-}, \alpha_i^{3-}, \alpha_i^{4-}, \beta_i^{1-}, \beta_i^{2-}, \beta_i^{3-}, \beta_i^{4-}$ .

As a result,  $H$  must also contain  $d_1^+$  and  $c_1^{0+}$ . By symmetry, we may assume  $H$  contains  $p_{1,1}^+$  and  $\alpha_k^{2+}$  for some  $k$ . Since  $\alpha_k^{1+}$  is adjacent to two vertices

in  $H$ ,  $H$  must contain  $\alpha_k^{3+}$ . Similarly,  $H$  cannot contain  $\alpha_k^{4+}$ , so  $H$  contains  $p_{1,2}^+$  and  $p_{1,3}^+$ . By induction, we see that  $H$  contains  $p_{1,i}^+$  for  $i = 1, 2, \dots, z_i^+$  and  $d_1^-$ . If  $H$  contains  $p_{1,z_i}^-$ , then  $H$  must contain  $p_{1,i}^-$  for  $i = z_i^-, \dots, 1$ , a contradiction. Thus,  $H$  must contain  $d_2^+$ . By induction, for  $i = 1, 2, \dots, n$ , we see that  $H$  contains all the vertices of the path  $P_i^+$  or  $P_i^-$  and by symmetry, we may assume  $H$  contains all the neighbors of the vertices in  $P_i^+$  or  $P_i^-$  of the form  $\alpha_k^{2+}$  or  $\alpha_k^{3+}$  for any  $k$ .

Similarly, for  $i = 1, 2, \dots, m$ , it follows that  $H$  must contain  $c_i^{0+}$  and  $c_i^{0-}$ . Also,  $H$  contains one of the following:

- $c_i^{12+}, c_i^{1+}, c_i^{1-}, c_i^{12-}$  and either  $\alpha_j^{2-}$  and  $\alpha_j^{3-}$  or  $\beta_j^{2-}$  and  $\beta_j^{3-}$  (where  $\alpha_j^{2-}$  is adjacent to  $c_i^{1+}$ ).
- $c_i^{12+}, c_i^{2+}, c_i^{2-}, c_i^{12-}$  and either  $\alpha_j^{2-}$  and  $\alpha_j^{3-}$  or  $\beta_j^{2-}$  and  $\beta_j^{3-}$  (where  $\alpha_j^{2-}$  is adjacent to  $c_i^{2+}$ ).
- $c_i^{3+}$  and  $c_i^{3-}$  and either  $\alpha_j^{2-}$  and  $\alpha_j^{3-}$  or  $\beta_j^{2-}$  and  $\beta_j^{3-}$  (where  $\alpha_j^{2-}$  is adjacent to  $c_i^{3+}$ ).

We can recover the satisfying assignment  $\xi$  as follows. For  $i = 1, 2, \dots, n$ , set  $\xi_i = 1$  if the vertices of  $P_i^+$  are in  $H$  and set  $\xi_i = 0$  if the vertices of  $P_i^-$  are in  $H$ . By construction, it is easy to verify that at least one literal in every clause is satisfied, so  $\xi$  is indeed a satisfying assignment.  $\square$

**Theorem 2.7** *The following statements hold:*

- For any  $d \in \mathbb{Z}$  with  $d \geq 2$ , the problem  $\Gamma^d$  is NP-complete when  $d \geq 3$  and polynomial when  $d = 2$ .
- If  $\mathcal{H}$  is any finite list of cycles  $C_{k_1}, C_{k_2}, \dots, C_{k_m}$  such that  $C_6 \notin \mathcal{H}$ , then  $\Gamma_{\mathcal{H}}^3$  is NP-complete.

PROOF — In the above reduction,  $\Delta(G_f) = 3$  so  $\Gamma^d$  is NP-complete for  $d \geq 3$ . When  $d = 2$ , there is a simple  $O(n)$  algorithm. Any hole containing  $x$  and  $y$  must be a component of  $G$  so pick the vertex  $x$  and consider the component  $C$  of  $G$  that contains  $x$ . It takes  $O(n)$  time to verify whether  $C$  is a hole containing  $x$  and  $y$  or not.

To show the second statement, let  $K$  be the length of the longest cycle in  $\mathcal{H}$ . In the above reduction, do the following modifications.

- For  $i = 1, 2, 3$  and  $j = 1, 2, \dots, 3m$ , replace the edges  $\alpha_j^{i+} \alpha_j^{(i+1)+}$ ,  $\alpha_j^{i-} \alpha_j^{(i+1)-}$ ,  $\beta_j^{i+} \beta_j^{(i+1)+}$ , and  $\beta_j^{i-} \beta_j^{(i+1)-}$  by paths of length  $K$ .
- For  $j = 1, 2, \dots, 3m - 1$ , replace the edges  $\alpha'_j \alpha_{j+1}$  and  $\beta'_j \beta_{j+1}$  by paths of length  $K$ .
- Replace the edges  $x\alpha_1$  and  $x\beta_1$  by paths of length  $K$ .

This new reduction is polynomial in  $n$  and  $m$  and is  $\mathcal{H}$ -free. The proof of Lemma 2.6 still holds for this new reduction so therefore,  $\Gamma_{\mathcal{H}}^3$  is NP-complete.  $\square$

### 3 $\Pi_B$ for some special s-graphs

#### 3.1 Holes with pending edges and trees

Here, we study  $\Pi_{B_4}, \dots, \Pi_{B_7}$  where  $B_4, \dots, B_7$  are the s-graphs depicted on Figure 3. Our motivation is simply to give a striking example and to point out that surprisingly, pending edges of s-graphs matter and that even an s-graph with no cycle can lead to NP-complete problems.

**Theorem 3.1** *There is an  $O(n^{13})$ -time algorithm for  $\Pi_{B_4}$  but  $\Pi_{B_5}$  is NP-complete.*

PROOF — A realisation of  $B_4$  has exactly one vertex of degree 3 and one vertex of degree 4. Let us say that the realisation  $H$  is *short* if the distance between these two vertices in  $H$  is at most 3. Detecting short realisations of  $B_4$  can be done in time  $n^9$  as follows: for every 6-tuple  $F = (a, b, x_1, x_2, x_3, x_4)$  such that  $G[F]$  has edge-set  $\{x_1a, ax_2, x_2b, bx_3, bx_4\}$  and for every 7-tuple  $F = (a, b, x_1, x_2, x_3, x_4, x_5)$  such that  $G[F]$  has edge-set  $\{x_1a, ax_2, x_2x_3, x_3b, bx_4, bx_5\}$ , delete  $x_1, \dots, x_5$  and their neighbors except  $a, b$ . In the resulting graph, check whether  $a$  and  $b$  are in the same component. The answer is YES for at least one 7-or-6-tuple if and only if  $G$  contains at least one short realisation of  $B_4$ .

Here is an algorithm for  $\Pi_{B_4}$ , assuming that the entry graph  $G$  has no short realisation of  $B_4$ . For every 9-tuple  $F = (a, b, c, x_1, \dots, x_6)$  such that  $G[F]$  has edge-set  $\{x_1a, bx_2, x_2x_3, x_3x_4, cx_5, x_5x_3, x_3x_6\}$  delete  $x_1, \dots, x_6$  and their neighbors except  $a, b, c$ . In the resulting graph, run three-in-a-tree for  $a, b, c$ . It is easily checked that the answer is YES for some 7-tuple if and only if  $G$  contains a realisation of  $B_4$ .

Let us prove that  $\Pi_{B_5}$  is NP-complete by a reduction of  $\Gamma^3$  to  $\Pi_{B_5}$ . Since by Theorem 2.7,  $\Gamma^3$  is NP-complete, this will complete the proof. Let  $(G, x, y)$  an instance of  $\Gamma^3$ . Prepare a new graph  $G'$ : add four vertices  $x', x'', y', y''$  to  $G$  and add four edges  $xx', xx'', yy', yy''$ . Since  $\Delta(G) \leq 3$ , it is easily seen that  $G$  contain a hole passing through  $x, y$  if and only if  $G'$  contains a realisation of  $B_5$ .  $\square$

The proof of the theorem below is omitted since it is similar to the proof of Theorem 3.1.

**Theorem 3.2** *There is an  $O(n^{14})$ -time algorithm for  $\Pi_{B_6}$  but  $\Pi_{B_7}$  is NP-complete.*

#### 3.2 Induced subdivisions of $K_5$

Here, we study the problem of deciding whether a graph contain an induced subdivision of  $K_5$ . More precisely, we put :  $sK_5 = (\{a, b, c, d, e\}, \emptyset, (\{a, b, c, d, e\}))$ .

**Theorem 3.3**  *$\Pi_{sK_5}$  is NP-complete.*

PROOF — We consider an instance  $(G, x, y)$  of  $\Gamma^3$ . Let us denote by  $x', x''$  the two neighbors of  $x$  and by  $y', y''$  the two neighbors of  $y$ .

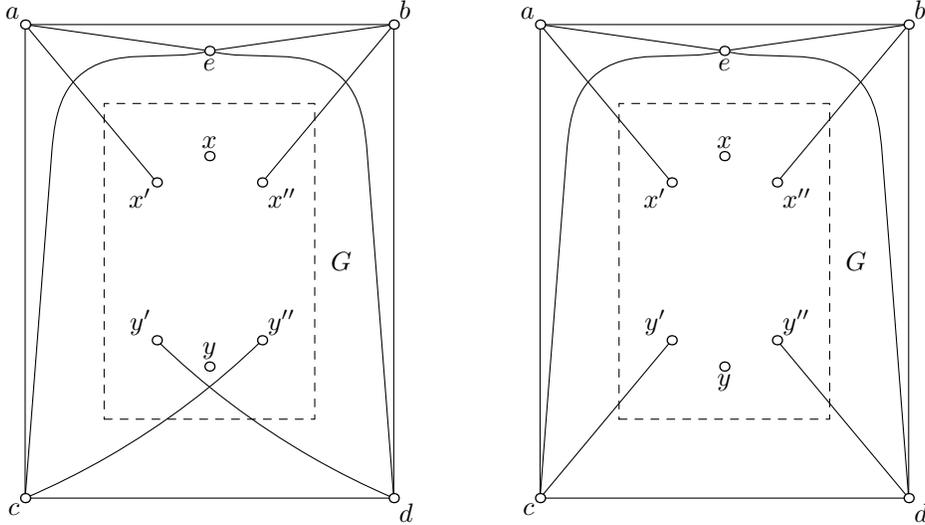


Figure 15: Graphs  $G'$  and  $G''$

Let us build a graph  $G'$  by adding five vertices  $a, b, c, d, e$ . We add the edges  $ab, bd, dc, ca, ea, eb, ec, ed, ax', bx'', cy'', dy'$ . We delete the edges  $xx', xx'', yy', yy''$ . We define a very similar graph  $G''$ , the only change being that we do not add edges  $cy'', dy'$  but edges  $cy', dy''$  instead. See figure 15.

Now in  $G'$  (and similarly  $G''$ ) every vertex has degree at most 3, except for  $a, b, c, d, e$ . We claim that  $G$  contains a hole going through  $x$  and  $y$  if and only if at least one of  $G', G''$  contains an induced subdivision of  $K_5$ . Indeed, if  $G$  contains a hole passing through  $x, x', y', y, y'', x''$  in that order then  $G'$  obviously contains an induced subdivision of  $K_5$ , and the hole passes in order through  $x, x', y'', y, y', x''$  then  $G''$  contains such a subgraph. Conversely, if  $G'$  (or symmetrically  $G''$ ) contains an induced subdivision of  $K_5$  then  $a, b, c, d, e$  must be the vertices of the underlying  $K_5$ , because they are the only vertices with degree at least 4. Hence there is a path from  $x'$  to  $y'$  in  $G \setminus \{x, y\}$  and a path from  $x''$  to  $y''$  in  $G \setminus \{x, y\}$ , and consequently a hole going through  $x, y$  in  $G$ .  $\square$

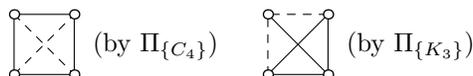
### 3.3 $\Pi_B$ for small $B$ 's

Here, we survey the complexity  $\Pi_B$  when  $B$  has at most four vertices. By the remarks in the introduction, if  $|V| \leq 3$  then  $\Pi_{(V,D,F)}$  is polynomial. Up to symmetries, we are left with twelve s-graphs on four vertices.

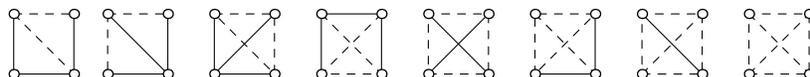
For the following two s-graphs, there is a polynomial algorithm using three-in-a-tree:



The next two s-graphs yield an NP-complete problem:



For the remaining eight ones, we do not know the answer:



As a conclusion, we would like to point out that every detection problem associated to an s-graph for which a polynomial time algorithm is known can be solved by using three-in-a-tree or by some easy brute-force enumeration.

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