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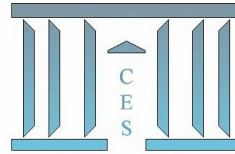


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# Borsuk's Antipodal and Fixed-Point Theorems for Correspondences Without Convex Values

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## Abstract

We present an extension of Borsuk's antipodal theorem (existence of a zero) for antipodally approachable correspondences without convex values. This result is a generalization of Borsuk-Ulam Theorem and has a fixed-point equivalent formulation.

**Key words and phrases:** Borsuk's antipodal Theorem, balanced set, approachable selection, fixed points.

**Classification-JEL:** C02, C65, C69.

The aim of this note is to extend Borsuk's Theorem to the antipodally approachable correspondences without convex values. Under suitable assumption, a correspondence with convex values is antipodally approachable. This concept is stable by composition which is not the case for correspondence with convex values. Our result generalizes those that use a correspondence with convex values.

In what follows  $X$  (resp  $Y$ ) is a nonempty subset of  $\mathbb{R}^n$  (resp  $\mathbb{R}^p$ ), capital letters  $F : X \rightarrow Y$  denote correspondences while non capital letters  $f : X \rightarrow Y$  will denote single-valued functions. We denote by  $\partial X$  the boundary of the subset  $X$  and  $\text{conv}X$  its convex hull. In the whole paper, we will assume that correspondences have nonempty values. Let  $\text{Gr}F = \{(x, y) \mid y \in F(x), x \in X\}$  be the graph of  $F$ ,  $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$  the unit  $n$ -sphere and  $B_X(A, r)$  the open ball of  $X$  with center  $A$  and radius  $r$ . Let  $\mathcal{N}^m$  be a fundamental basis of open symmetric neighborhood of the

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origin in  $\mathbb{R}^m$ . A set  $M \subset Y$  is said to be balanced if  $\lambda M \subset M$  for every real number  $\lambda$  with  $|\lambda| \leq 1$ . Suppose that  $X$  is symmetric, a correspondence  $F : X \rightarrow Y$  is said antipodal-preserving (resp. antipodal<sup>4</sup>) if for all  $x \in X$ ,  $F(x) = -F(-x)$  (resp.  $F(x) \cap -F(-x) \neq \emptyset$ ). It is easy to see that if  $F$  is antipodal-preserving then  $F$  is antipodal. Note that if the correspondence  $F$  is antipodal and single-valued then it is antipodal-preserving.

We recall that a correspondence  $F : X \rightarrow Y$ ,  $X$  and  $Y$  topological spaces, is upper semi-continuous (u.s.c) on  $X$  if and only if for any open subset  $V$  of  $Y$ , the set  $\{x \in X : F(x) \subset V\}$  is open in  $X$ .

**Definition 1** *Let  $X$  be a symmetric nonempty subset of  $\mathbb{R}^n$ ,  $Y$  a nonempty subset of  $\mathbb{R}^p$  and  $F$  a correspondence from  $X$  to  $Y$ .*

- (1) *For any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , a function  $s : X \rightarrow Y$  is said to be a  $(U, V)$ -approximative selection of  $F$  if for any  $x \in X$ ,  $s(x) \in (F[(x + U) \cap X] + V) \cap Y$  or equivalently  $\text{Grs} \subset \text{Gr}F + (U \times V)$ .*
- (2) *A correspondence  $F : X \rightarrow Y$  is said to be approachable if for any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , there exists a continuous  $(U, V)$ -approximative selection for  $F$ . We denote by  $\mathcal{A}(X, Y)$  the class of such correspondences and we write  $\mathcal{A}(X) = \mathcal{A}(X, X)$ .*

We will use the notion of approachable correspondences (see [B]).

**Definition 2** (1) *A correspondence  $F : X \rightarrow Y$  is said to be antipodally approachable if for any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , there exists a continuous antipodal  $(U, V)$ -approximative selection for  $F$ . We denote by  $\mathcal{A}_a(X, Y)$  the class of such correspondences and we write  $\mathcal{A}_a(X) = \mathcal{A}_a(X, X)$ .*

- (2) *A correspondence  $F : X \rightarrow Y$  is said to be antipodally approximable if its restriction to any symmetric compact subset  $K$  of  $X$ ,  $F|_K$ , is antipodally approachable.*

**Remark 1** *Let  $Z$  be a symmetric subset of  $\mathbb{R}^n$ . Let  $F : Z \rightarrow \mathbb{R}^p$  be an antipodal preserving correspondence with convex value. If there exists a continuous selection of  $F$  then there exists an antipodal-preserving selection of it. Indeed, it suffices to consider  $\tilde{h}(x) = \frac{h(x) - h(-x)}{2}$  where  $h$  is the continuous selection.*

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<sup>4</sup>It is a generalization of the original single-valued antipodal function to set-valued maps.

- Proposition 1** (1) *If a correspondence  $F : X \rightarrow Y$  is antipodally approachable then for any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , the correspondence  $F^{U,V} : X \rightarrow Y$  defined by  $(F((x + U) \cap X) + V) \cap Y$  is antipodal.*
- (2) *If a correspondence  $F : X \rightarrow Y$  is u.s.c. with compact values and for any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , the correspondence  $F^{U,V}$  is antipodal then  $F$  is antipodal.*
- (3) *If a correspondence  $F : X \rightarrow Y$  is antipodally approachable and u.s.c. with compact values then  $F$  is antipodal.*
- (4) *Let  $Z$  be a symmetric compact subset of  $\mathbb{R}^n$ . If a correspondence  $F : Z \rightarrow \mathbb{R}^p$  is antipodal and u.s.c. with convex value then  $F$  is antipodally approachable.*

Proof:

- (1) The correspondence  $F$  is antipodally approachable then for any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , there exists an antipodal-preserving function  $s : X \rightarrow Y$  such that  $s(x) \in (F((x + U) \cap X) + V) \cap Y$  for all  $x \in X$ . By the fact that  $s(x) = -s(-x)$  for all  $x \in X$  and the symmetry of  $U$  and  $V$ , it follows that  $s(x) = -s(-x) \in \{(F((x + U) \cap X) + V) \cap Y\} \cap \{-F((-x + U) \cap X) + V\} \cap Y$ , then  $F^{U,V}$  is antipodal.
- (2) Let  $x_0 \in X$ , for all  $n \geq 1$ , there exists  $y_n \in \{(F((x_0 + B_X(O, \frac{1}{n})) \cap X) + B_Y(O, \frac{1}{n})) \cap Y\} \cap \{-F((-x_0 + B_X(O, \frac{1}{n})) \cap X) + B_Y(O, \frac{1}{n})) \cap Y\}$ . Therefore, there exists  $t_{\frac{1}{n}} \in x_0 + B_X(O, \frac{1}{n})$  (resp.  $\tilde{t}_{\frac{1}{n}} \in -x_0 + B_X(O, \frac{1}{n})$ ) and  $h_n \in B_Y(O, \frac{1}{n})$  (resp.  $\tilde{h}_n \in B_Y(O, \frac{1}{n})$ ) such that  $y_n \in F(t_{\frac{1}{n}}) - h_n$  (resp.  $y_n \in -F(\tilde{t}_{\frac{1}{n}}) - \tilde{h}_n$ ) then  $(t_{\frac{1}{n}}, y_n + h_n) \in GrF$  (resp.  $(\tilde{t}_{\frac{1}{n}}, -y_n + \tilde{h}_n) \in GrF$ ). By a compactness argument, we can extract a subsequence  $(t_{\varphi(n)}, y_{\varphi(n)} + h_{\varphi(n)})$  which converges to  $(x_0, \bar{y}) \in GrF$  when  $n \rightarrow +\infty$ . Remark that  $(\tilde{t}_{\varphi(n)}, -y_{\varphi(n)} + \tilde{h}_{\varphi(n)})$  is a subsequence of  $(\tilde{t}_n, -y_n + \tilde{h}_n)$  which converges to  $(-x_0, -\bar{y})$  when  $n \rightarrow +\infty$ . Consequently,  $(-x_0, -\bar{y}) \in GrF$  and then  $\bar{y} \in F(x_0) \cap -F(-x_0)$ .
- (3) If a correspondence  $F : X \rightarrow Y$  is antipodally approachable then by (2), for any  $U \subset \mathcal{N}^n$ ,  $V \subset \mathcal{N}^p$ , the correspondence  $F^{U,V}$  is antipodal. Since, the correspondence  $F$  is u.s.c. with compact values then by (2), the correspondence  $F$  is antipodal.
- (4) For any  $V$  in  $\mathcal{N}^n$ , we define as in [CH]  $F^V : Z + V \rightarrow \mathbb{R}^p$  by  $F^V(x) = \text{conv}(\bigcup_{z \in (x+V) \cap Z} F(z))$ . Let  $W_1 \in \mathcal{N}^n$  and  $W_2 \in \mathcal{N}^p$ , with no loss of

generality, we may assume that  $W_2$  is nonempty convex. By lemma 1 in [CH], there exists  $V \in \mathcal{N}^n$  such that  $Gr F^V \subset Gr F + W_1 \times W_2$  and  $F^V$  is antipodal on  $Z+V$  with open lower sections. Let us now consider  $F|_Z^V$  the restriction of  $F^V$  to  $Z$ , then  $F|_Z^V$  is antipodal with convex values and open lower sections. Let us now consider the correspondence  $G : Z \rightarrow \mathbb{R}^p$  defined by  $G(x) = F|_Z^V(x) \cap -F|_Z^V(-x)$ . This correspondence has nonempty convex values and open lower sections, hence with the Theorem of Michael [M], it has a continuous selection. Since the correspondence  $G$  is antipodal preserving then, in view of Remark 1,  $G$  (hence  $F|_Z^V$ ) has a continuous antipodal selection  $f : Z \rightarrow \mathbb{R}^p$ . Consequently,  $Gr f \subset Gr F + W_1 \times W_2$ . ■

**Remark 2** (1) *Under the assumptions of Proposition 2.5 in [B], the composition of two antipodally approachable correspondences is antipodally approachable.*

(2) *Remark that the class of correspondences with convex values is not stable by composition.*

We will give two examples in order to show that the u.s.c. assumption (respectively compactness) can't be dropped in the assertion (2) of Proposition 1. Let  $\mathcal{C}(O, r)$  be the circle in  $\mathbb{R}^2$  with center  $O = (0, 0)$  and radius  $r$ .

**Example 1** *Define  $F : \mathcal{C}(O, 1) \rightarrow \mathbb{R}^2$  by  $F(x) = x$  if  $x \neq (1, 0)$  and  $F(1, 0) = (-1, 0)$ . The correspondence  $F$  which can be viewed as a function has compact values but is not u.s.c. It is clear that for any  $U \in \mathcal{N}^2$ ,  $V \in \mathcal{N}^2$ , the correspondence  $F^{U, V}$  is antipodal (not antipodal-preserving) but  $F$  is not antipodal in  $(1, 0)$ .*

**Example 2** *Define  $F : \mathcal{C}(O, 1) \rightarrow \mathbb{R}^2$  by  $F(x) = x$  if  $x \neq (1, 0)$  and  $F(1, 0) = \mathcal{C}(O, 1) \setminus (1, 0)$ . The correspondence  $F$  is u.s.c with no compact values. In  $(1, 0)$ , it is clear that  $F$  is not antipodal but for any  $U \in \mathcal{N}^2$ ,  $V \in \mathcal{N}^2$ , the correspondence  $F^{U, V}$  is antipodal (not antipodal-preserving).*

We recall Borsuk's antipodal theorem:

**Theorem 1 Borsuk's antipodal theorem** *A single-valued antipodal continuous map  $f : S^n \rightarrow \mathbb{R}^n$  has a zero value.*

The following remark will be used to extend the domain in Borsuk's antipodal theorem from  $S^n$  to the boundary of any open bounded symmetric balanced subset of  $\mathbb{R}^{n+1}$ .

**Remark 3** Let  $K$  be a symmetric compact subset of  $\mathbb{R}^{n+1}$  and  $T : S^n \rightarrow K$  a u.s.c. antipodal correspondence with convex values. If we consider  $T$  as a correspondence from  $S^n$  to  $\mathbb{R}^{n+1}$  then, by Proposition 1 assertion (4), the correspondence  $T$  is antipodally approachable that is for all  $\delta > 0$ ,  $\eta > 0$ , there exists a continuous  $B_{\mathbb{R}^{n+1}}(O, \delta) \times B_{\mathbb{R}^{n+1}}(O, \eta)$ -antipodal-preserving selection  $\tilde{\varphi} : S^n \rightarrow K + B_{\mathbb{R}^{n+1}}(O, \eta)$  of  $T$ .

**Theorem 2** Let  $U$  be an open bounded symmetric balanced subset of  $\mathbb{R}^{n+1}$ , then any antipodal single valued continuous function  $s : \partial U \rightarrow \mathbb{R}^n$  has a zero value.

Proof: By contradiction, suppose that for all  $x \in \partial U$ ,  $s(x) \neq 0$ . Let us consider the correspondence  $S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  defined by  $S(x) = s(x)$  if  $x \in \partial U$  and  $S(x) = \mathbb{R}^n$  if not. By the Dugundji extension Theorem<sup>5</sup> (see [DG] p. 163), there exists a continuous function that extends  $s$  over  $\mathbb{R}^{n+1}$ , then there exists a continuous selection of  $S$ . Since the correspondence  $S$  is antipodal preserving with convex values then by Remark 1, there exists an antipodal selection  $\tilde{s}$  of  $S$  such that  $\tilde{s}(x) = S(x) = s(x)$  for all  $x \in \partial U$ . Let  $Z = \{x \in \mathbb{R}^{n+1}, \tilde{s}(x) \neq 0\}$ , then  $Z$  is an open set containing  $\partial U$ . Since  $\partial U$  is compact, then there exists  $\eta > 0$  such that  $\partial U + B_{\mathbb{R}^{n+1}}(0, \eta) \subset Z$ .

Let  $\varphi : S^n \rightarrow \partial U$  defined by  $\varphi(x) = \partial U \cap \mathbb{R}_+x$ . It is easy to show that the correspondence  $\varphi$  is antipodal-preserving and u.s.c. with nonempty (compact) values. We will prove that  $\varphi$  has convex values. Let  $a$  and  $b$  in  $\partial U \cap \mathbb{R}_+x$  with  $a \neq b$ . Without loss of generality, we may assume that there exists  $\alpha$  in  $]0, 1[$  such that  $a = \alpha b$ . For any  $c$  in  $\text{conv}(a, b)$ , there exists  $\beta \in [\alpha, 1]$  such that  $c = \beta b$ , which leads to the existence of  $\lambda$  in  $]0, 1]$  such that  $a = \lambda c$ . Let us first remark that  $c \in \overline{U}$  and  $c \in \mathbb{R}_+x$ , since  $\overline{U}$  is a balanced set and  $b \in \overline{U}$ . Moreover if  $c \in U$  then  $a \in U$  which is absurd. Consequently  $c \in \partial U$  and then  $\varphi$  has convex values. By Remark 3, there exists a continuous antipodal selection  $\tilde{\varphi} : S^n \rightarrow \partial U + B_{\mathbb{R}^{n+1}}(0, \eta)$  of  $\varphi$ . Finally, for all  $x \in S^n$ ,  $\tilde{s}(\tilde{\varphi}(x))$  is a continuous antipodal function without zero. This is a contradiction to Borsuk's antipodal Theorem. ■

The main result of this paper is an extension of Theorem 2 to a correspondence. This extension generalizes Borsuk antipodal, Borsuk-Ulam Theorem<sup>6</sup> and Theorem 4 in [CH].

**Theorem 3** Let  $U$  be an open bounded symmetric balanced subset of  $\mathbb{R}^{n+1}$  and let  $F : \partial U \rightarrow \mathbb{R}^n$  be u.s.c antipodally approachable correspondence with nonempty closed values. Then  $F$  has a zero on  $\partial U$ .

<sup>5</sup>Let  $X$  be any metrizable space and  $A \subset X$  a closed subset. Then any continuous function  $f : A \rightarrow \mathbb{R}$  has an extension  $F : X \rightarrow \mathbb{R}$ .

<sup>6</sup>see [DG] for the statement of Borsuk-Ulam Theorem and the equivalence with Borsuk antipodal Theorem.

Proof: Let  $\mathbf{0}$  denote the zero function from  $\partial U$  into  $\{0\}$ . Suppose that  $F$  does not have zero, then  $d(Gr(\mathbf{0}), Gr(F)) > 0$ . Let  $\varepsilon = \frac{1}{3}d(Gr(\mathbf{0}), Gr(F))$  then

$$Gr(\mathbf{0}) \cap [Gr(F) + B_{\mathbb{R}^{n+1}}(O, \varepsilon) \times B_{\mathbb{R}^n}(O, \varepsilon)] = \emptyset.$$

The correspondence  $F$  is antipodally approachable then for  $V = B_{\mathbb{R}^{n+1}}(O, \varepsilon)$  and  $W = B_{\mathbb{R}^n}(O, \varepsilon)$ , there exists  $f^{V,W}$  such that  $Gr(f^{V,W}) \subset [Gr(F) + V \times W]$  which imply that  $Gr(f^{V,W}) \cap Gr(\mathbf{0}) = \emptyset$ . Consequently,  $f^{V,W}$  is a continuous antipodal function with zero free, this is a contradiction to Theorem 2.  $\blacksquare$

We will give two examples of u.s.c. antipodal-preserving correspondence  $F : \partial B_{\mathbb{R}^3} \rightarrow \mathbb{R}^2$  with nonempty closed values. We assume that  $F(x, y, z)$  depend only on  $z$ . The following two examples coincide when  $z \neq 0$  but in the second example, the correspondence is antipodally approachable which is not the case in the first example.

Let us consider the following spiral  $\zeta$ :

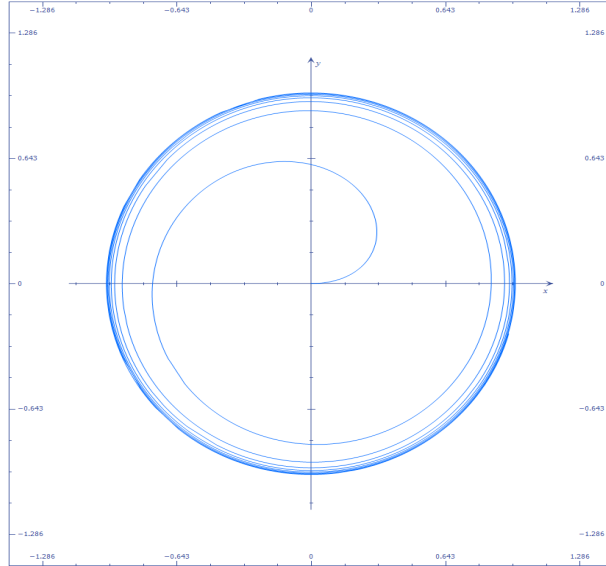
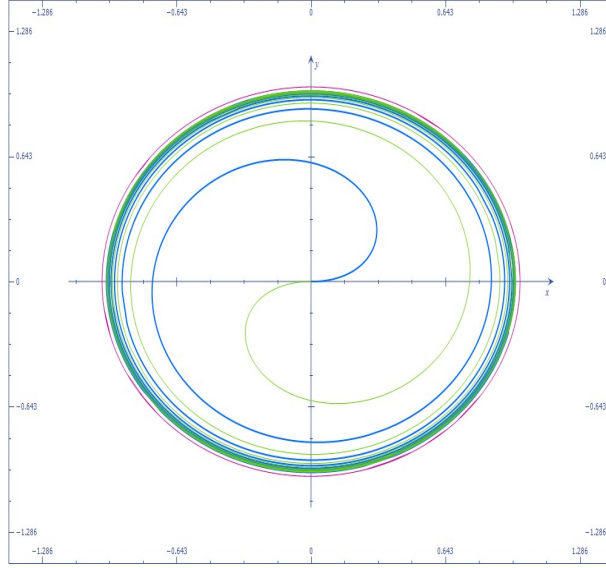


Figure 1:  $\zeta$  spiral with polar equation  $r = \frac{\theta}{1+\theta}$ ,  $\theta \geq 0$ .



**Example 3** We define the correspondence  $F : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$  by:

$$F(x, y, z) = \varphi(z) = \begin{cases} \mathcal{C}(O, \sqrt{1-z^2}) \cap \zeta & \text{if } z > 0 \\ \mathcal{C}(O, \sqrt{1-z^2}) \cap -\zeta & \text{if } z < 0 \\ \mathcal{C}(O, 1) & \text{if } z = 0 \end{cases}$$



We deduce that the correspondence  $F$  is antipodal-preserving, u.s.c. with closed nonempty non convex values ( $\varphi(0)$  is not convex). Note that it is easy to check that  $\varphi$  is a continuous antipodal-preserving function on  $[-1, 0[ \cup ]0, 1]$  and that  $\varphi(0)$  is the limit superior in the sense of Painlevé Kuratowski of the family  $(C_n)_{n>0}$  where  $C_n = \varphi(]0, \frac{1}{n}[)$ <sup>7</sup>. Let us now prove that  $F$  is not antipodally approachable. Suppose by contradiction that the correspondence  $F$  is antipodally approachable then for all  $\varepsilon > 0$ , there exists a continuous antipodal function  $f^\varepsilon : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$  such that  $Gr f^\varepsilon \subset Gr F + B_{\mathbb{R}^2}(O, \varepsilon) \times B_{\mathbb{R}^2}(O, \varepsilon)$ . Let us now fix  $\varepsilon \in ]0, \frac{1}{2}[$ , for each fixed  $z \in [-1, 1]$ , we define the closed path  $\gamma_z : [0, 2\pi] \rightarrow \mathbb{R}^2$  by

$$\gamma_z(t) = f^\varepsilon(\cos(t)\sqrt{1-z^2}, \sin(t)\sqrt{1-z^2}, z),$$

<sup>7</sup>The limit superior of a sequence of set  $(C_n)_{n>0}$  in the sense of Painlevé Kuratowski [AF] is defined by:

$$\limsup_{n \rightarrow \infty} C_n := \{x \in \mathbb{R}^2 \mid \liminf_{n \rightarrow \infty} d(x, C_n) = 0\} \text{ where } d(x, X) = \inf_{y \in X} d(x, y).$$

we denote by  $\tilde{\gamma}_z = \{\gamma_z(t) \mid t \in [0, 2\pi]\}$ . For all  $0 \leq z < 1 - 2\varepsilon$ , the origin  $O \notin \tilde{\gamma}_z$  then let  $I(\gamma_z, O) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{dz}{z}$  be the index of  $\gamma_z$  with respect to  $O$ . For a fixed  $z \in [0, 1 - 2\varepsilon[$ ,  $\tilde{\gamma}_z$  is a subset of  $F(z) + B_{\mathbb{R}^2}(O, 2\varepsilon)$  which is an open simply connected set <sup>8</sup> then  $\gamma_z$  is homotopic to one point  $a_z$ . Recall that in Theorem 2 p. 60 [CA], if a path  $\bar{\gamma}_z$  is homotopic to a path  $\gamma_z$  then,  $I(\gamma_z, O) = I(\bar{\gamma}_z, O)$ , in particular  $I(\gamma_z, O) = I(a_z, O) = 0$ . By the continuity of  $f^\varepsilon$  and the fact that the index  $I(\gamma_z, O)$  is a constant when  $\gamma_z$  is continuously deformed, we deduce that  $I(\gamma_0, O) = 0$ . Since  $\gamma_0(t + \pi) = -\gamma_0(t)$ , a simple computation of index proves that  $I(\gamma_0, O)$  is odd then it can't be equal to 0.

**Example 4** Let the correspondence  $F : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$  defined by:

$$F(x, y, z) = \psi(z) = \begin{cases} \mathcal{C}(O, \sqrt{1 - z^2}) \cap \zeta & \text{if } z > 0 \\ \mathcal{C}(O, \sqrt{1 - z^2}) \cap -\zeta & \text{if } z < 0 \\ \mathcal{C}(O, 1) \cup ([-1, 1] \times \{0\}) & \text{if } z = 0 \end{cases}$$

Note that  $\psi(0)$  is not convex. It is clear that the correspondence  $F$  is u.s.c. with nonempty closed values. Let us prove that the correspondence  $F$  is antipodally approachable: For every  $\varepsilon > 0$ ,  $\delta > 0$ , let us consider  $\eta = \min(\varepsilon, \delta)$  and

$$Q = \{(x, y) \in \overline{B}_{\mathbb{R}^2}(O, 1) \text{ such that } d((x, y), \psi(0)) < 1 - |\psi(2\eta)|\},$$

then  $\psi(\eta) \in Q$ . It is easy to see that  $Q$  is a path-connected subset of  $\mathbb{R}^2$  and  $O \in Q$ . Then there exists a path  $\gamma : [0, 1] \rightarrow Q$  such that  $\gamma(0) = O$  and  $\gamma(1) = \psi(\eta)$ . We define the function  $f^\eta : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$  by:

$$f^\eta(x, y, z) = h^\eta(z) = \begin{cases} \psi(z) & \text{if } |z| \geq \eta \\ \gamma\left(\frac{z}{\eta}\right) & \text{if } 0 \leq z \leq \eta \\ -\gamma\left(\frac{-z}{\eta}\right) & \text{if } -\eta \leq z \leq 0 \end{cases}$$

Then  $f^\eta$  is a continuous, antipodal-preserving function with  $\text{Gr } f^\eta \subset \text{Gr } F + B_{\mathbb{R}^2}(O, \varepsilon) \times B_{\mathbb{R}^2}(O, \delta)$ . Consequently the correspondence  $F$  is antipodally approachable.

We will now focus on the correspondence from  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^{n+1}$ . The following theorem is a generalization of Theorem 6 in [CH]. Indeed, the correspondence are assumed to be approachable which is more general than a correspondence with convex values (see [CE], Remark 1, Proposition 1 assumption (4) and Remark 2).

<sup>8</sup>An open simply connected set  $D$  is a connected set in which every closed path is homotopic to one point in  $D$ . (see [CA] p. 61)

**Theorem 4** *Let  $U$  be an open bounded symmetric balanced subset of  $\mathbb{R}^{n+1}$ . Let  $F : \bar{U} \rightarrow \mathbb{R}^{n+1}$  be an u.s.c. correspondence with nonempty closed values and approachable on  $\bar{U}$  by a function which is antipodal on  $\partial U$ . Then  $F$  has a zero value and a fixed point on  $\bar{U}$ .*

Proof: Let  $\mathbf{0}$  denote the zero map from  $\partial U$  into  $\{0\}$ . Suppose that  $F$  does not have zero, then  $d(\text{Gr}(\mathbf{0}), \text{Gr}(F)) > 0$ . Let  $\varepsilon = \frac{1}{3}d(\text{Gr}(\mathbf{0}), \text{Gr}(F))$ , then  $\text{Gr}(\mathbf{0}) \cap \{\text{Gr}(F) + B_{\mathbb{R}^{n+1}}(O, \varepsilon) \times B_{\mathbb{R}^{n+1}}(O, \varepsilon)\} = \emptyset$ . The correspondence  $F$  is antipodally approachable then for  $V = W = B_{\mathbb{R}^{n+1}}(O, \varepsilon)$ , there exists a continuous  $(V, W)$ -approximative selection  $f^{V,W}$  such that  $\text{Gr}(f^{V,W}) \subset \text{Gr}(F) + V \times W$  and  $f^{V,W}(x) = -f^{V,W}(-x)$  for all  $x \in \partial U$ . Consequently  $\text{Gr}(f^{V,W}) \cap \text{Gr}(\mathbf{0}) = \emptyset$  then  $f^{V,W}$  is a continuous antipodal function on  $\partial U$  with zero free. This is a contradiction to Theorem 6 [CH].

It is clear that the correspondence  $G$  defined by  $G(x) = F(x) - x$  for all  $x \in \bar{U}$  have a zero value then the correspondence  $F$  have a fixed point. ■

In a topological vector space, it is classical to extend the usual notion of bounded subset of a normed space using the following definition (see for example [K]):

**Definition 3** *A subset  $Q$  of a topological vector space  $E$  is said to be bounded if for each neighbourhood  $U$  of 0 there is a  $\rho > 0$  with  $Q \subset \rho U$ .*

Theorem 4 is easily extends to any finite dimensional vector space and this is an intermediate step towards topological vector spaces:

**Proposition 2** *Let  $U$  be an open bounded symmetric balanced subset in a finite dimensional vector space  $E$ . Let  $F : \bar{U} \rightarrow E$  be an u.s.c. correspondence with nonempty closed values and approachable on  $\bar{U}$  by a selection which is antipodal on  $\partial U$ . Then  $F$  has a zero value and a fixed point on  $\bar{U}$ .*

Proof: Let  $\mathcal{B} = \{x_1, \dots, x_{n+1}\}$  be a basis of  $E$ , which allows to consider  $\Phi$  the usual linear homeomorphism between  $\mathbb{R}^{n+1}$  and  $E$ . If we define  $V = \Phi^{-1}(U)$ , it is easy to show that it is bounded in the usual sense. Moreover, letting  $G = \Phi^{-1} \circ F \circ \Phi$ , it is routine to check that  $(V, G)$  satisfies the assumptions of Theorem 4, which leads to the conclusion. ■

We now extend our result to infinite dimensional space. Note that in view of Proposition 1 (4), this allows to generalize Theorem 7 of [CH].

**Theorem 5** *Let  $M$  be a closed bounded symmetric balanced set in a Hausdorff locally convex topological vector space  $E$ . Let  $F : M \rightarrow E$  be u.s.c. correspondence with nonempty closed values such that the closure of  $F(M)$  is compact. Assume that  $F$  is approximable on  $M$  by a selection which is antipodal on  $\partial M$ , then  $F$  has at least one fixed point.*

Proof: We will construct this fixed point as a limit of “approximated fixed point”. Let  $\mathcal{B}$  denote a closed bounded symmetric convex neighborhood base at 0 in  $E$ . Since the closure of  $F(M)$  is compact, then for each  $V$  in  $\mathcal{B}$ , there exists a finite subset  $S_V$  of  $F(M)$  such that  $(y + V) \cap S_V \neq \emptyset$  for each  $y \in F(M)$ . Let  $H_{S_V}$  the vector space spanned by  $S_V$ . In the following of this proof we will refer to the topology of  $H_{S_V}$ . Define  $F_V : M \cap H_{S_V} \rightarrow H_{S_V}$  by  $F_V(x) = (F(x) + V) \cap H_{S_V}$ , then the correspondence  $F_V$  is u.s.c. with nonempty compact values. Note that  $M \cap H_{S_V}$  (resp  $\partial_{H_{S_V}}(M \cap H_{S_V})$ , the boundary of  $M \cap H_{S_V}$  with respect to the topology of  $H_{S_V}$ ) is a compact subset of  $M$  (resp  $\partial M$ ). Since  $F$  is approximable on  $M$  by a selection which is antipodal on  $\partial M$  then  $F_V$  is approachable on  $M \cap H_{S_V}$  by a selection which is antipodal on  $\partial_{H_{S_V}}(M \cap H_{S_V})$ . let us remark that  $0 \in M \cap H_{S_V}$ , consequently either  $0 \in \text{int}_{H_{S_V}}(M \cap H_{S_V})$  or  $0 \in \partial_{H_{S_V}}(M \cap H_{S_V})$ . In the first case, we can apply Proposition 2 with  $U = \text{int}_{H_{S_V}}(M \cap H_{S_V})$  and there exists  $x_V \in F_V(x_V)$ . In the second case,  $F_V$  is antipodally approachable on  $\partial_{H_{S_V}}(M \cap H_{S_V})$ , then there exists an antipodal approximative selection  $s_V$  of  $F_V$  such that  $s_V(x) \in F_V(x+V)+V$ , in particular  $s_V(0) = 0 \in F_V(0+V)+V$ .

In both case, for each  $V \in \mathcal{B}$ , there exists  $x_V \in M$  such that  $x_V \in F_V(x_V + V) + V$ . Since  $V$  is symmetric, there exists  $(v, w) \in V^2$  such that  $y_V \in F(z_V)$  where  $y_V = x_V + v$  and  $z_V = x_V + w$ . A standard argument based on the compactness of  $\overline{F(M)}$ , the upper semicontinuity of  $F$  and the closedness of its values ends the proof (see for example [BI]). ■

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