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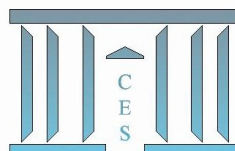
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# Least Square Approximations and Conic Values of Cooperative Games

Ulrich FAIGLE\*      Michel GRABISCH†

May 29, 2015

## Abstract

The problem of least square approximation for set functions by set functions satisfying specified linear equality or inequality constraints is considered. The problem has important applications in the field of pseudo-Boolean functions, decision making and in cooperative game theory, where approximation by additive set functions yields so-called least square values. In fact, it is seen that every linear value for cooperative games arises from least square approximation. We provide a general approach and problem overview. In particular, we derive explicit formulas for solutions under mild constraints, which include and extend previous results in the literature.

**Keywords:** least square approximation, cooperative game, pseudo-Boolean function, least square value, Shapley value, probabilistic value

**JEL Classification:** C71

## 1 Introduction

Approximation of high-dimensional quantities or complicated functions by simpler functions with linear properties from low-dimensional spaces has countless

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applications in physics, economics, operations research *etc.* In these applications, the quality of the approximation is usually measured by the Gaussian principle of the least squared error, which is also the guiding optimality criterion in the present investigation. Our study addresses a particular case of such an approximation context with many applications in different fields related to operations research, namely decision theory, game theory and the theory of pseudo-Boolean functions.

Where  $N$  is a finite set with  $n = |N|$  elements and collection  $2^N$  of subsets, a set function  $v : 2^N \rightarrow \mathbb{R}$  assigns to every subset of  $N$  a real number, and is by definition of exponential complexity (in  $n$ ). Identifying subsets of  $N$  with their characteristic (incidence) vectors (and thus  $2^N$  with  $\{0, 1\}^n$ ), a set function can be viewed as a so-called pseudo-Boolean function  $f : \{0, 1\}^n \rightarrow \mathbb{R}$  (*cf.* Hammer and Rudeanu [11]). Of particular interest are those set functions which vanish on the empty set, since they represent cooperative TU games with  $N$  being the set of players and the quantities  $v(S)$  expressing the benefit created by the cooperation of the members of  $S \subseteq N$  (see, *e.g.*, Peleg and Sudhölter [13]). Under the additional stipulation of monotonicity, *i.e.*, the property that  $v(S) \leq v(T)$  holds whenever  $S \subseteq T$ , one arrives at so-called capacities, which are a fundamental tool in the analysis of decision making under uncertainty (*cf.* Schmeidler [16]) or relative to several criteria (Grabisch and Labreuche [7]).

Being of exponential complexity, a natural question is to try to approximate general set functions by simpler functions, the simplest being the additive set functions, which are completely determined by the value they take on the empty set  $\emptyset$  and on the  $n$  singleton sets  $\{i\}$  and are thus of linear complexity (in  $n$ ). In the field of pseudo-Boolean functions, the question has been addressed by Hammer and Holzman [9] with respect to linear and quadratic approximations, while approximation of degree  $k$  was studied by Grabisch *et al.* [8]. In decision theory, linear approximation amounts to the approximation of a capacity  $\mu$  by a probability measure  $P$  (an additive capacity satisfying the additional constraint that  $P(N) = 1$ ).

In game theory, the approximation of a game  $v$  by an additive game (equivalently by a (payoff) vector in  $\mathbb{R}^N$ ) is related to the concept of value or solution of a game: given  $v$ , find  $x \in \mathbb{R}^N$  such that  $\sum_{i \in N} x_i = v(N)$  and the  $x_i$  represent as faithfully as possible the contribution of the individual players  $i$  in the total benefit  $v(N)$ . A very natural approach for a value is to define it as the best least square approximation of  $v$ , under the constraint  $\sum_{i \in N} x_i = v(N)$ , the approximation being possibly weighted. Such values are called least square values. An early and

important contribution to this cooperative solution concept is due to Charnes *et al.* [2], who gave the general solution for the weighted approximation with nonnegative weights, and exhibited the well-known Shapley value [17] as a least square value. Ruiz *et al.* [14], for example, generalized this approach and derived further values from least square approximation.

The aim of this paper is to provide a clear, rigorous and general view of the approximation problem for set functions by placing it in the context of quadratic optimization and bringing the tools of convex analysis to bear on the problem. This approach not only generalizes existing results but also points to interesting connections and facts. Our formulation will remain general, although we will adopt most of the time the notation and ideas from cooperative game theory, due to the great interest in this field towards values and how to obtain them.

For example, it is seen that approximation under linear inequality constraints generalizes the idea of the core of a cooperative game in a natural way and gives rise to values that are conic – but not necessarily linear (see the *core values* in Section 2.4). Under linear equality constraints, we find that *every* least square problem yields a linear value and that *every* linear value arises as a least square value (Section 2.3).

The paper is organized as follows. Section 2 settles the general problem of least square approximation and gives the most fundamental results which will be useful in the sequel. In particular, it is seen how linear (or more generally, conic) values arise from least square approximation. Section 3 concentrates on least square values, and establishes explicit solution formulas under mild constraints on the weights used in the approximation with application to the Shapley value and to an optimization problem given in Ruiz *et al.* [14]. (Interestingly, the weights do not necessarily have to be all positive under our constraints.) Finally, Section 4 shows how Weber’s [19] so-called probabilistic values arise naturally in our model.

## 2 Least square approximations

We begin by reviewing some basic facts from convex optimization<sup>1</sup>. For integers  $k, m \geq 1$ , we denote by  $\mathbb{R}^k$  the vector space of all  $k$ -dimensional (column) vectors and by  $\mathbb{R}^{m \times k}$  the vector space of all  $(m \times k)$ -matrices. The standard (euclidian)

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<sup>1</sup>see, *e.g.*, Faigle *et al.* [4] or any other textbook for more details

inner product on  $\mathbb{R}^k$  is given as

$$\langle x|y \rangle = x^T y \quad (x, y \in \mathbb{R}^k).$$

Let  $B \in \mathbb{R}^{k \times k}$  be a matrix of full rank  $k$  so that the columns of  $B$  form a basis of  $\mathbb{R}^k$ . Then the basis (coordinate) change  $x = B\bar{x}$  yields the inner product

$$\langle \bar{x}|\bar{y} \rangle_Q = \langle B\bar{x}|B\bar{y} \rangle = \bar{x}^T Q \bar{y} \quad \text{with} \quad Q = B^T B.$$

Recall from linear algebra that  $Q$  is a *positive definite* matrix and that every positive definite matrix  $P$  is of the form  $P = C^T C$  for some  $C \in \mathbb{R}^{k \times k}$  of full rank. The associated *norm* is

$$\|\bar{x}\|_Q = \sqrt{\langle \bar{x}|\bar{x} \rangle_Q} = \sqrt{\bar{x}^T Q \bar{x}} = \|B\bar{x}\|.$$

The standard norm  $\|x\| = \|x\|_I$  corresponds to the choice  $Q = I$  as the identity matrix.

## 2.1 Approximation and quadratic optimization

Recall that a *polyhedron*  $P$  in  $\mathbb{R}^k$  is a subset of the form

$$P = P(A, b) = \{x \in \mathbb{R}^k \mid Ax \geq b\},$$

where  $A \in \mathbb{R}^{m \times k}$  and  $b \in \mathbb{R}^m$  for some  $m \geq 1$ . The (least square) *approximation problem* of  $c \in \mathbb{R}^k$  relative to  $P$  is

$$\min_{x \in P} \|c - x\|^2 = \langle c|c \rangle - 2\langle c|x \rangle + \langle x|x \rangle. \quad (1)$$

Assuming  $P = P(A, b) \neq \emptyset$ , it is well-known that (1) has a unique optimal solution  $\hat{c}$ , which can be computed by solving the associated *Karush-Kuhn-Tucker* (KKT) system

$$\begin{array}{rcl} x & - & A^T y = c \\ Ax & & \geq b \\ & & y \geq 0. \end{array} \quad (2)$$

Reformulating the approximation problem (1) with respect to the  $Q$ -norm, we arrive at

$$\min_{\bar{x}: B\bar{x} \geq b} \|\bar{c} - \bar{x}\|_Q^2 = \langle \bar{c}|\bar{c} \rangle_Q - 2\langle \bar{c}|\bar{x} \rangle_Q + \langle \bar{x}|\bar{x} \rangle_Q. \quad (3)$$

with  $\bar{A} = AB$  and the KKT system

$$\begin{aligned} \frac{Q\bar{x}}{\bar{A}\bar{x}} - \bar{A}^T y &= Q\bar{c} \\ &\geq b \\ y &\geq 0. \end{aligned} \quad (4)$$

Finally, setting  $\tilde{c} = Q\bar{c}$  and observing that  $\langle \bar{c} | \bar{c} \rangle_Q$  is a constant, we note that the approximation problem (1) is equivalent with the *quadratic optimization problem* under linear constraints

$$\min_{\bar{x}: \bar{A}\bar{x} \geq b} \bar{x}^T Q\bar{x} - 2\tilde{c}^T \bar{x}. \quad (5)$$

The optimal solutions of (5) are obtained from the KKT system

$$\begin{aligned} \frac{Q\bar{x}}{\bar{A}\bar{x}} - \bar{A}^T y &= \tilde{c} \\ &\geq b \\ y &\geq 0. \end{aligned} \quad (6)$$

## 2.2 Conic maps from quadratic optimization

Let  $N$  be a set of  $n$  players and denote by  $\mathcal{N} = 2^N$  the family of all *coalitions*  $S \subseteq N$ . Recall that a *cooperative TU game* on  $N$  is described by a *characteristic function*  $v : \mathcal{N} \rightarrow \mathbb{R}$  that is *zero-normalized* in the sense  $v(\emptyset) = 0^2$ .

Clearly, linear combinations of characteristic functions yield characteristic functions. So the collection  $\mathcal{G} = \mathcal{G}(N)$  of all characteristic functions forms a  $(2^n - 1)$ -dimensional vector space relative to the field  $\mathbb{R}$  of scalars.

We call a map  $f : \mathcal{G} \rightarrow \mathbb{R}^k$  *conic* if  $f$  is additive and positively homogeneous, *i.e.*, if for all  $v, w \in \mathcal{G}$  and scalars  $\lambda \geq 0$ ,

$$f(v + w) = f(v) + f(w) \quad \text{and} \quad f(\lambda v) = \lambda f(v).$$

Consider a fixed constraint matrix  $A \in \mathbb{R}^{m \times k}$  and a positive definite matrix  $Q \in \mathbb{R}^{k \times k}$ . Let furthermore  $c : \mathcal{G} \rightarrow \mathbb{R}^k$  and  $b : \mathcal{G} \rightarrow \mathbb{R}^m$  be linear operators such that  $Ax \geq b(v)$  has a solution for every  $v$ . Denote by  $\hat{v}$  the optimal solution of the quadratic (approximation) problem

$$\min_{x: Ax \geq b(v)} x^T Qx - 2c(v)^T x. \quad (7)$$

Our key observation is now the following:

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<sup>2</sup>The condition  $v(\emptyset) = 0$  is in fact not a restriction in our optimization problem, as this term does not play any role. Therefore, our results concern any type of set function.

**Lemma 2.1 (Conicity lemma)**  $v \mapsto \hat{v}$  is a conic map from  $\mathcal{G}$  into  $\mathbb{R}^k$ .

*Proof.* Let  $v, w \in \mathcal{G}$  and  $\lambda \geq 0$  be arbitrary. So  $\hat{v}$  and  $\hat{w}$  satisfy the associated KKT system

$$\begin{array}{rcl} Qx & - & A^T y = c(v) \\ Ax & & \geq b(v) \\ & & y \geq 0. \end{array}$$

It is straightforward to verify that  $\hat{v} + \hat{w}$  and  $\lambda \hat{v}$  satisfy the KKT system relative to  $v + w$  and  $\lambda v$ .

◇

REMARK. For Lemma 2.1 to remain valid, one can relax the linearity hypothesis. For  $c$ , it is enough to require conicity. For  $b$ , subadditivity and positive semi-homogeneity is sufficient:

$$b(v) + b(w) \geq b(v + w) \quad \text{and} \quad \lambda b(v) \geq b(\lambda v)$$

for all  $v, w \in \mathcal{G}$  and  $\lambda \geq 0$ .

### 2.3 Linear equality constraints

In the case of linear constraints of type  $Ax = b$ , the solution space is an affine subspace of  $\mathbb{R}^k$  and the KKT system of the optimization problem

$$\min_x x^T Qx - 2c^T x \quad \text{s.t.} \quad Ax = b$$

becomes the linear equality system

$$\begin{array}{rcl} Qx & - & A^T y = c \\ Ax & & = b. \end{array} \tag{8}$$

Hence we conclude

**Proposition 2.1** Assume that  $c : \mathcal{G} \rightarrow \mathbb{R}^k$  and  $b : \mathcal{G} \rightarrow \mathbb{R}^m$  are linear operators such that  $Ax = b$  is always solvable. Then  $v \mapsto \hat{v}$  is a linear map.

◇

Notice that the linear maps arising in the manner of Proposition 2.1 are not special. In fact, every linear map  $f : \mathcal{G} \rightarrow \mathbb{R}^k$  can be obtained in this way. To see

this, observe that  $f(v)$  is (trivially) the unique optimal solution of the approximation problem

$$\min_{x \in \mathbb{R}^k} \|f(v) - x\|^2 = \|f(v)\|^2 - 2f(v)^T x + \|x\|^2.$$

So  $f(v)$  is the unique optimal solution of the quadratic problem

$$\min_x x^T Q x - 2c^T x \quad \text{with } Q = I \text{ and } c = f(v).$$

## 2.4 Examples

In the case  $k = n = |N|$ , a map  $\Phi : \mathcal{G} \rightarrow \mathbb{R}^n$  is called a *value* in cooperative game theory. So the discussion around Proposition 2.1 shows that linear values arise from least square approximation (or quadratic minimization) problems relative to linear constraints. For example, the value  $\Phi$  is said to be *efficient* if it satisfies the restriction

$$\sum_{i \in N} \Phi_i(v) = v(N),$$

which is linear in  $v$ .

The concept of values can be refined by considering the space  $\mathbb{R}^{\mathcal{N}}$  of all maps  $\mathcal{N} \rightarrow \mathbb{R}$ . The *additive (cooperative) games* correspond to exactly those  $x \in \mathbb{R}^{\mathcal{N}}$  that satisfy the homogeneous system of linear equations

$$x(S) - \sum_{i \in S} x_i = 0 \quad (S \in \mathcal{N})$$

and one may be interested in the approximation of a game function  $v$  by an additive game with certain properties. More general approximations might be of interest. For example, the linear constraints

$$\begin{aligned} \sum_{i \in N} x_i &= v(N) \\ \sum_{S \in \mathcal{N}} x_S &= \sum_{S \in \mathcal{N}} v(S) \end{aligned}$$

would stipulate an approximation of  $v$  by a game that induces an efficient value (the first equality) and, furthermore, preserves the total sum of the  $v(S)$  (second equality).

Inequality constraints  $Ax \geq b$  connect the idea of values naturally with the notion of the core of a cooperative game. Consider, for example, the system of core inequalities

$$\sum_{i \in S} x_i \geq v(S) \quad (S \in \mathcal{N}),$$

which is clearly always solvable. So Lemma 2.1 guarantees the existence of a conic (but not necessarily linear) map  $v \mapsto \hat{v}$ , where  $\hat{v}$  is the solution of the least square approximation problem

$$\min_x \|v - x\|^2 \quad \text{s.t.} \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N,$$

inducing the *core value*  $\Gamma : \mathcal{G} \rightarrow \mathbb{R}^N$ , with components

$$\Gamma_i(v) = \hat{v}_i \quad (i \in N),$$

for the class  $\mathcal{G}$  of cooperative TU games.

REMARK. The core value is studied with respect to decomposition properties of convex games in Fujishige [6].

### 3 Least square values

We have seen that every linear value  $\Phi : \mathcal{G} \rightarrow \mathbb{R}^N$  can be interpreted as arising from a least square approximation problem. Special cases of seemingly more general least square problems have received considerable attention in the literature and led to the concept of *least square values* and *semivalues*. Take, for example, the weighted least square problem

$$\min_{x \in \mathbb{R}^N} \sum_{S \in \mathcal{N}} \alpha_S (v(S) - x(S))^2 \quad \text{s.t.} \quad \sum_{i \in N} x_i = v(N), \quad (9)$$

where we set  $x(S) = \sum_{i \in S} x_i$ . So (9) asks for the best ( $\alpha$ -weighted) least square approximation of a game  $v$  by an additive game  $x$  under the additional efficiency constraint  $x(N) = v(N)$ .

This problem has a long history. Hammer and Holzman ([9])<sup>3</sup> studied both the above version and the unconstrained version with equal weights ( $\alpha_S = 1 \quad \forall S$ ), and

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<sup>3</sup>later published in [10]

proved that the optimal solutions of the unconstrained version yield the Banzhaf value [1] (see also Section 4 below). More general versions of the unconstrained problem were solved by Grabisch *et al.* [8] with the approximation being relative to the space of  $k$ -additive games (*i.e.*, games whose Möbius transform vanishes for subsets of size greater than  $k$ )<sup>4</sup>.

In 1988, Charnes *et al.* [2] gave a solution for the case with the coefficients  $\alpha_S$  being *uniform* (*i.e.*,  $\alpha_S = \alpha_T$  whenever  $|S| = |T|$ ) and strictly positive. As a particular case, the Shapley value was shown to result from the coefficient choice

$$\alpha_S = \alpha_s = \binom{n-2}{s-1} = \frac{(n-2)!}{(s-1)!(n-1-s)!} \quad (s = |S|). \quad (10)$$

REMARK. Ruiz *et al.* [14] state that problem (9) has a unique optimal solution for any choice of weights (see Theorem 3 there). In this generality, however, the statement is not correct as neither the existence nor the uniqueness can be guaranteed. So additional assumptions on the weights must be made.

We will first present a general framework for dealing with such situations and then illustrate it with the example of regular weight approximations and probabilistic values.

### 3.1 Weighted approximation

For the sake of generality, consider a general linear subspace  $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{N}}$  of dimension  $k = \dim \mathcal{F}$ , relative to which the approximation will be made.

Let  $W = [w_{ST}] \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$  be a given matrix of weights  $w_{ST}$ . Let  $c : \mathbb{R}^{\mathcal{N}} \rightarrow \mathbb{R}^{\mathcal{N}}$  be a linear function and consider, for any game  $v$ , the optimization problem

$$\min_{u \in \mathcal{F}} (v - u)W(v - u)^T + c(v - u)^T \quad \text{with } c = c(v), \quad (11)$$

which is equivalent with

$$\min_{u \in \mathcal{F}} uWu^T - \tilde{c}u^T, \quad (12)$$

where  $\tilde{c} \in \mathbb{R}^{\mathcal{N}}$  has the components  $\tilde{c}_S = c_S + 2 \sum_T w_{ST}v_T$ . A further simplification is possible by choosing a basis  $B = \{b_1, \dots, b_k\}$  for  $\mathcal{F}$ . With the identification

$$x = (x_1, \dots, x_k) \in \mathbb{R}^k \quad \longleftrightarrow \quad u = \sum_{i=1}^k x_i b_i \in \mathcal{F},$$

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<sup>4</sup>see also Ding [3], and Marichal and Mathonet [12]

problem (12) becomes

$$\min_{x \in \mathbb{R}^k} \sum_{i=1}^k \sum_{j=1}^k q_{ij} x_i x_j - \sum_{i=1}^k \bar{c}_i x_i \quad (13)$$

with the coefficients

$$q_{ij} = \sum_S \sum_T w_{ST} b_i(S) b_j(T) \quad \text{and} \quad \bar{c}_i = \sum_S \tilde{c}_S b_i(S).$$

Note that  $\bar{c} : \mathbb{R}^{\mathcal{N}} \rightarrow \mathbb{R}^k$  is a linear function.

Let  $A \in \mathbb{R}^{m \times k}$  be a constraint matrix and  $b : \mathbb{R}^{\mathcal{N}} \rightarrow \mathbb{R}^m$  a linear function such that  $Ax = b(v)$  has a solution for every  $v \in \mathbb{R}^{\mathcal{N}}$ . If  $Q = [q_{ij}] \in \mathbb{R}^{k \times k}$  is positive definite, the problem

$$\min_{x \in \mathbb{R}^k} \sum_{i=1}^k \sum_{j=1}^k q_{ij} x_i x_j - \sum_{i=1}^k \bar{c}_i x_i \quad \text{s.t.} \quad Ax = b \quad (14)$$

has a unique optimal solution  $x^*$  which is linear in  $v$  (Proposition 2.1). So we obtain the a linear value  $v \mapsto \hat{v}$  with components

$$\hat{v}_j = u_{\{j\}}^* \quad (j \in N), \quad \text{where} \quad u^* = \sum_{i=1}^k x_i^* b_i \in \mathcal{F}.$$

In the model (9), for example,  $\mathcal{F}$  is the space  $\mathcal{C}$  of all additive games and has dimension  $n$ . The matrix  $W$  is diagonal with the diagonal elements  $w_{SS} = \alpha_S$ . If  $\alpha_S > 0$  holds for all  $S$ , then  $W$  is positive definite and the linearity of the implied value  $v \mapsto \hat{v}$  follows directly from Proposition 2.1.

Otherwise, let us choose for  $B$  the basis of unanimity games  $\zeta_i, i \in N$ , for  $\mathcal{C}$ , where

$$\zeta_i(S) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

The associated matrix  $Q = [q_{ij}]$  in model (9) has the coefficients

$$q_{ij} = \sum_{S \in \mathcal{N}} \alpha_S \zeta_i(S) \zeta_j(S) = \sum_{S \ni \{i,j\}} \alpha_S. \quad (15)$$

For establishing a linear value, it suffices that  $Q$  be positive definite, which is possible even when some of the  $\alpha_S$  are negative (see Examples 3.1 and 3.2 below).

## 3.2 Regular weights

While Proposition 2.1 guarantees the existence of linear values resulting from approximation, explicit formulas can be given under additional assumptions on the weights. Restricting ourselves to objectives of type

$$\sum_{S \in \mathcal{N}} \alpha_S (v_S - u_S)^2 + \sum_{S \in \mathcal{N}} c_S u_S,$$

we propose a simple framework that nevertheless includes all the cases treated in the literature so far. We say that the weights  $\alpha_S$  are *regular* if the resulting matrix  $Q$  has just two types of coefficients  $q_{ij}$ , *i.e.*, if there are real numbers  $p, q$  such that

$$q_{ij} = \begin{cases} q & \text{if } i = j \\ p & \text{if } i \neq j. \end{cases}$$

**Example 3.1** Assume that the weights  $\alpha_S$  are uniform and set  $\alpha(|S|) = \alpha_S$ . Then formula (15) yields

$$q_{ij} = \sum_{s=2}^n \binom{n-2}{s-2} \alpha(s) \quad \text{and} \quad q_{ii} = \sum_{s=1}^n \binom{n-1}{s-1} \alpha(s)$$

holds for all  $i \neq j$ . So  $Q = [q_{ij}]$  is regular.

**Lemma 3.1** Let  $Q = [q_{ii}] \in \mathbb{R}^{k \times k}$  be regular with  $q = q_{ii}$  and  $p = q_{ij}$  for  $i \neq j$ . Then  $Q$  is positive definite if and only if  $q > p \geq 0$ .

*Proof.* For any  $x \in \mathbb{R}^k$ , we have after some algebra

$$x^T Q x = (q - p) \sum_{i=1}^k x_i^2 + p \bar{x}^2$$

where  $\bar{x} = \sum_{i=1}^n x_i$ , which makes the claim of the Lemma obvious. ◇

Note that our model allows for possibly negative uniform coefficients, as shown in the following example.

**Example 3.2** Let  $n = 3$ . We get  $p = \alpha_2 + \alpha_3$  and  $q = \alpha_1 + 2\alpha_2 + \alpha_3$ . Letting  $\alpha > 0$ , the following vectors  $(\alpha_1, \alpha_2, \alpha_3)$  lead to a positive definite matrix  $Q$ :

$$(0, \alpha, 0), \quad (\alpha, 0, \alpha), \quad (0, \alpha, -\alpha), \quad \text{etc.}$$

For the remainder of this section, let  $Q \in \mathbb{R}^{N \times N}$  be a regular matrix with parameters  $q > p \geq 0$ ,  $c \in \mathbb{R}^N$  a vector and  $g \in \mathbb{R}$  a scalar. Setting  $\mathbf{1}^T = (1, 1, \dots, 1)$ , the optimization problem

$$\min_{x \in \mathbb{R}^N} x^T Q x - c^T x \quad \text{s.t.} \quad \mathbf{1}^T x = x(N) = g \quad (16)$$

has a unique optimal solution  $x^* \in \mathbb{R}^N$ . Moreover, there is a unique scalar  $z^* \in \mathbb{R}$  such that  $(x^*, z^*)$  is the unique solution of the associated KKT-system

$$\begin{aligned} Qx &- z\mathbf{1} = c/2 \\ \mathbf{1}^T x &= g. \end{aligned} \quad (17)$$

Verifying this KKT-system, the proof of the following explicit solution formulas is straightforward.

**Theorem 3.1** If  $Q$  is regular, the solution  $(x^*, z^*)$  of the KKT-system (17) is:

$$\begin{aligned} z^* &= (2(q + (n-1)p)g - C)/n \quad (\text{with } C = c\mathbf{1}^T = \sum_{i \in N} c_i) \\ x_i^* &= (c_i + z^* - 2pg)/(2q - 2p) \quad (i \in N). \end{aligned}$$

If  $Q$  is furthermore positive definite, then  $x^*$  is an optimal solution for (16). ◇

In the case of uniform weights  $\alpha(s)$ , the formulas in Theorem 3.1 yield the formulas derived by Charnes *et al.*[2] for problem (9). To demonstrate the scope of Theorem 3.1, let us look at the extremal problem<sup>5</sup> studied by Ruiz *et al.* [15]

$$\min_{x \in \mathbb{R}^N} \sum_{S \subseteq N} m_S d(x, S)^2 \quad \text{s.t.} \quad x(N) = v(N), \quad (18)$$

where  $m_S > 0$  and

$$d(x, S) = \frac{v(S) - x(S)}{|S|} - \frac{v(N \setminus S) - x(N \setminus S)}{n - |S|}.$$

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<sup>5</sup>see also Sun *et al.* [18] for similar problems

Letting  $v^*(S) = v(N) - v(N \setminus S)$  and

$$\bar{v}(S) = \frac{(n - |S|)v(S) + |S|v^*(S)}{n}$$

(and thus  $n\bar{v}(N) = v(N)$ ), we find that problem (18) becomes

$$\min_{x \in \mathbb{R}^N} \sum_{S \subseteq N} \alpha_S (\bar{v}(S) - x(S))^2 \text{ s.t. } x(N) = n\bar{v}(N).$$

with  $\alpha_S = n^2 m_S (|S|^2 (n - |S|)^2)^{-1}$ . Because  $v \mapsto \bar{v}$  and  $v \mapsto g(v) = n\bar{v}(N)$  are linear mappings, the optimal solutions of (18) yield an efficient linear value for any choice of parameters  $m_S$  such that the associated matrix  $Q$  is positive definite.

If furthermore the weights  $m_S$  (and hence the  $\alpha_S$ ) are uniform,  $Q$  is regular and the optimal solution can be explicitly computed from the formulas of Theorem 3.1.

## 4 Probabilistic values

Weber [19] introduced the idea of a *probabilistic* value arising as the expected marginal contribution of players relative to a probability distribution on the coalitions. For example, a *semivalue* is a probabilistic value relative to probabilities that are equal on coalitions of equal cardinality.

For our purposes, it suffices to think of the *marginal contribution* of an element  $i \in N$  as a linear functional  $\partial_i : \mathcal{G} \rightarrow \mathbb{R}$ , where  $\partial_i^v(S)$  is interpreted as the marginal contribution of  $i \in N$  to the coalition  $S \subseteq N$  relative to the characteristic function  $v$ .

Probabilistic values can be studied quite naturally in the context of weighted approximations. Indeed, let  $p$  be an arbitrary probability distribution on  $\mathcal{N}$ . Then the expected marginal contribution of  $i \in N$  relative to the game  $v$  is

$$E(\partial_i^v) = \sum_{S \subseteq N} \partial_i^v(S) p_S.$$

Let  $\mu_i \in \mathbb{R}$  be an estimate value for the marginal contribution of  $i \in N$ . Then the expected observed deviation from  $\mu_i$  is

$$\sigma(\mu_i) = \sqrt{\sum_{S \subseteq N} p_S (\partial_i^v(S) - \mu_i)^2}.$$

A well-known fact in statistics says that the deviation function  $\mu_i \mapsto \sigma(\mu_i)$  has the unique minimizer  $\mu = E(\partial_i^v)$ , which can also be immediately deduced from the KKT conditions for the least square problem

$$\min_{\mu \in \mathbb{R}} \sum_{S \in \mathcal{N}} p_S (\partial_i^v(S) - \mu)^2.$$

**The values of Shapley and Banzhaf.** Shapley's [17] model assumes that player  $i$  contributes to a coalition  $S$  only if  $i \in S$  holds and that, in this case,  $i$ 's marginal contribution is evaluated as

$$\partial_i^v(S) = v(S) - v(S \setminus i).$$

So only coalitions in  $\mathcal{N}_i = \{S \subseteq N \mid i \in S\}$  need to be considered. In order to speak about the "average marginal contribution", the model furthermore assumes:

- (i) The cardinalities  $|X|$  of the coalitions  $X \in \mathcal{N}_i$  are distributed uniformly.
- (ii) The coalitions  $X \in \mathcal{N}_i$  of the same cardinality  $|X| = s$  are distributed uniformly.

Under these probabilistic assumptions, the coalition  $S \in \mathcal{N}_i$  of cardinality  $|S| = s$  occurs with probability

$$p_S = \frac{1}{n} \cdot \frac{1}{\binom{n-1}{s-1}} = \frac{(s-1)!(n-s)!}{n!}, \quad (19)$$

which exhibits the Shapley value as a probabilistic (and hence approximation) value:

$$\sum_{S \in \mathcal{N}_i} p_S [v(S) - v(S \setminus i)] = \sum_{S \in \mathcal{N}} p_S [v(S) - v(S \setminus i)] = \Phi_i^{\text{Sh}}(v).$$

REMARK. Among the probabilistic values, the Shapley value can also be characterized as the one with the largest entropy (Faigle and Voss [5]).

In contrast to the Shapley model, the assumption that all coalitions in  $\mathcal{N}_i$  are equally likely assigns to any coalition  $S \in \mathcal{N}_i$  the probability

$$\bar{p}_S = \frac{1}{2^{n-1}} \quad (20)$$

with the Banzhaf value [1] as the associated probabilistic value:

$$\sum_{S \in \mathcal{N}_i} \bar{p}_S [v(S) - v(S \setminus i)] = \sum_{S \in \mathcal{N}} \bar{p}_S [v(S) - v(S \setminus i)] = B_i^v.$$

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