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On the emergence of scale-free production networks

Stanislao Gualdi[‡] and Antoine Mandel[§]

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Abstract

Building upon the standard model of monopolistic competition on the market for intermediary goods, we propose a simple dynamical model of the formation of production networks. The model subsumes the standard general equilibrium approach and robustly reproduces key stylized facts of firms' demographics. Firms' growth rates are negatively correlated with size and follow a core double-exponential distribution followed by fat tails. Firms' size and production network are power-law distributed. These properties emerge because continuous inflow of new firms shifts away the model from a steady state to a disequilibrium regime in which firms get scaled according to their resistance to competitive forces.

1 Introduction

The scale-free nature of a wide range of socio-economic networks has been extensively documented in the recent literature [see e.g Barabási et al., 2009, Gabaix, 2009, Schweitzer et al., 2009]. An example of central concern for macroeconomics are production networks whose scale-free nature has recently been put forward by Acemoglu et al. [2012] as a potentially major driver of macroeconomic fluctuations [see also Battiston et al., 2007]. Relatedly, the scale-free distribution of firms' size [see Axtell, 2001] has also been identified as a key micro-economic source of aggregate volatility [see Gabaix, 2011].

It therefore seems problematic that the central tenet of economic theory with respect to the formation of structures, namely general equilibrium theory, has essentially nothing to say about the scale-free nature, or the nature in general, of the distribution of firms' size or this of production networks. Indeed, in a general equilibrium framework firms' size are either indeterminate (when there are constant returns to scale) or completely determinate by the primitives of

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the model (when there are decreasing returns to scale, the equilibrium size of the firm is completely determinate by its production technique.) In particular when firms have the same production technique, they have the same size at equilibrium.

The present paper addresses this wide gap in the theory through a dynamic extension of the general equilibrium model that accounts for three key stylized facts about the structure of the productive sector: firms' growth rates follow a Laplace distribution [see e.g Bottazzi and Secchi, 2006], firms' sizes are Zipf distributed and the degree distribution of production networks are scale-free.

The backbone of our approach is a model of monopolistic competition on the markets for intermediate goods, akin to the one introduced by Ethier [1982] (on the basis of Dixit and Stiglitz [1977]) and popularized by the endogenous growth literature [see e.g Romer, 1990]. In this framework, we represent supply relationships as the weighted edges of a network and consider out-of-equilibrium dynamics in which (i) demands are made in nominal terms and sellers adjust their prices to balance real supply and nominal demand (ii) firms progressively adjust their production technologies (i.e the network weights) to prevailing market prices. When the set of relationships is fixed (i.e only the weights of the network can evolve), the identification with the underlying general equilibrium model is perfect in the sense that (i) the adjacency matrix of the network is in a one to one correspondence with the underlying ge economy (ii) the model does converge to the underlying general equilibrium. However, the context of interest for us is this where the technological structure is not fixed a priori and where, the different production goods being assumed substitutable, firms can, in the long-run, adjust their production technologies/ supply relationships (i.e the adjacency matrix) as a function of market prices. Then, we show that the model does not in general admit a steady-state but rather settles in a non-equilibrium regime where the distribution of firms 'size and the structure of the production network are scale-free [see Bak et al., 1987, for related results].

Our approach offers a much more systemic perspective on firms' demographics that this existing in the literature. Indeed, from Kalecki [1945] and Simon et al. [1977] to more recent contributions such as Bottazzi and Secchi [2006], the problem of the distribution of firms' size has been approached almost solely through "island-models" in which the growth of each firm is studied in isolation and driven by exogenous shocks. On the contrary, in our model, growth opportunities are endogenous and we account for general equilibrium linkages.

More broadly, our paper contributes to the literature on the formation of socio-economic networks [see e.g Jackson et al., 2008] by providing micro-foundations for the emergence of scale-free networks which have been largely lacking in this literature, but for the notable exception of Jackson and Rogers [2007]. The paper also has close relationships with the infra marginal analysis pioneered by Xiaokai Yang [see Yang and Borland, 1991, Cheng and Yang, 2004] and Bak and co-authors's approach to the importance of self-organized criticality in economic networks [see Bak et al., 1993, Scheinkman and Woodford, 1994].

The remaining of the paper is organized as follows. In section 2, we propose a model of production networks as monopolistically competitive markets for intermediate goods. In section 3, we propose a numerical exploration of the dynamics of the model. Section 4 gives an analytical proof of the main results and section 5 concludes.

2 The Model

2.1 A general equilibrium primer

We consider an economy consisting in a finite set of (monopolistically competitive) firms producing differentiated goods and of a representative household. We denote the set of firms by $M = \{1, \cdots, m\}$, the representative household by the index 0 and the set of agents by $N = \{0, \cdots, m\}$.

Our central concern is the endogenous formation of supply relationships between firms. Therefore, to assume away any exogenous determinism, we place ourselves in a setting where there is no a priori distinction between potential intermediary goods. More precisely, we consider that the production possibilities of firm i are given by a C.E.S production function of the form:

$$f_i(x_0, (x_j)_{j=1,\cdots,n_i}) = x_0^\alpha (\sum_{j=1}^{n_i} x_j^\sigma)^{(1-\alpha)/\sigma} \tag{1}$$

where x_0 is the quantity of labor used, n_i the number of intermediary goods/ components combined and x_j the quantity of input j used in the production process.

This representation assumes that each good can be used interchangeably in the production process (as the production function depends only on the number of inputs). It is standard in models of monopolistic competition on the intermediate goods markets [see Ethier, 1982, Romer, 1990]. One of its key implications is that productivity grows with the number of components/suppliers[1].

As for the representative household, we consider that he supplies a constant quantity of labor (normalized to 1) and has preferences represented by a Cobb-Douglas utility function of the form $u(x_1, \cdots, x_m) = \prod_{i=1}^m x_i^{\alpha_{0,i}}$. He hence spends his income on each good $i \in M$ proportionally to $\alpha_{0,i}$ (we assume that for all $i \in M, \alpha_{0,i} > 0$, so that the household consumes a positive quantity of each and every good).

As such, this model is incomplete. The micro-economic choices of the agents in terms of production or consumption can not be determined without further assumptions on the structure of interactions. In our firm-focused setting, these interactions are mainly characterized by the production network, which specifies the flows of goods between firms. The formation of this production network is the key focus of the reminder of this paper.

[1] This feature is also at the core of the infra-marginal approach to economic growth [see Yang and Borland, 1991] and of Adam Smith's original description of the effects of the division of labor.

A general equilibrium approach to the issue would consist, in our setting, in defining the production network through an adjacency matrix $A = (a_{i,j})_{i,j \in M}$ such that $a_{i,j} = 1$ if j is a supplier of i and $a_{i,j} = 0$ otherwise. Consistency with equation (1) would then require that for all $i \in M$, $\sum_{j=1}^{m} a_{i,j} = n_i$ and, denoting by $S_i(A) := \{j \in M \mid a_{i,j} = 1\}$ the set of suppliers of firm i, the production function of firm i would be further specialized into:

$$f_i(x_0, (x_j)_{j \in S_i}) = x_0^{\alpha} (\sum_{j \in S_i(A)} x_j^{\sigma})^{(1-\alpha)/\sigma} \tag{2}$$

One could then define a general equilibrium of the economy $\mathcal{E}(A)$ associated to the production network A as follows.

Definition 1 *A general equilibrium of the economy $\mathcal{E}(A)$ is a collection of prices $(p_1^*, \cdots, p_m^*) \in \mathbb{R}_+^N$, production levels $(q_1^*, \cdots, q_m^*) \in \mathbb{R}_+^M$ and commodity flows $(x_{i,j}^*)_{i,j=0\cdots n} \in \mathbb{R}_+^{M \times M}$ such that:*

1. *Markets clear. That is one has for all $j \in N$, $q_j^* = \sum_{i=0}^{n} x_{i,j}^*$ (with $q_0^* = 1$ by normalization).*

2. *The representative consumer maximizes his utility. That is $(q_0^*, (x_{0,j}^*)_{j=1,\cdots,n})$ is a solution to*
$$\begin{cases} \max\ u_i((x_{0,j})_{j=1,\cdots,n}) \\ \\ s.t\ \sum_{j=1}^{n} p_j^* x_{0,j}^* \leq 1 \end{cases}$$
(with the price of labor normalized to 1)

3. *Firms maximize profits. That is for all $i \in M$, $(q_i^*, (x_{i,j}^*)_{j \in S_i(A)})$ is a solution to*
$$\begin{cases} \max\ p_i^* q_i - \sum_{j \in S_i(A)} p_j^* x_{i,j} \\ \\ s.t\ f_i((x_{i,j})_{j \in S_i(A)}) \geq q_i \end{cases}$$

Hence, in a general equilibrium setting, the adjacency structure of the production network is fixed and the magnitude of the physical flows between firms is determined at equilibrium. A particular case that has received widespread attention in the literature [see Acemoglu et al., 2012, Long and Plosser, 1983] is the Cobb-Douglas case (i.e when $\sigma \to 0$) in which the value of flows between firms at equilibrium is given by the corresponding exponents in the production function (uniformly equal to one in our framework).

Our aim in the following is to subsume this general equilibrium approach within an endogenous model of the formation of production networks.

2.2 An endogenous model of network formation

We consider a coupled model of network formation and out of equilibrium dynamics in which firms adaptively search for profit maximizing/cost minimizing input combinations. More precisely, we consider that time is discrete and indexed by $t \in \mathbb{N}$. Each agent $i \in N$ is initially endowed with a wealth $w_i^0 \in \mathbb{R}_+$ and a quantity of output $q_i^0 \in \mathbb{R}_+$ (normalized to 1 throughout in the case of the representative household). As for the production network, we assume its initial structure is given by the matrix of weights $\mathbb{A}^0 = (\alpha_{i,j}^0)_{i,j \in N}$, where $\alpha_{i,j}$ represents the share of agent i's expenses directed towards agent j.

We are concerned with the time evolution of the wealths $(w_i^t)_{i \in N}^{t \in \mathbb{N}}$, the quantities produced $(q_i^t)_{i \in N}^{t \in \mathbb{N}}$, the production network $\mathbb{A}^t = (\alpha_{i,j}^0)_{i,j \in N}^{t \in \mathbb{N}}$, as well as this of prices $(p_i^t)_{i \in N}^{t \in \mathbb{N}}$. This evolution is driven by the interplay between the workings of the market out of equilibrium and the evolution of the production network. More precisely, during each period $t \in \mathbb{N}$, the following sequence of events takes place:

1. Each agent i receives the nominal demand $\sum_{j \in N} \alpha_{i,j} w_j^t$.

2. Given the nominal demand $\sum_{j \in N} \alpha_{i,j} w_j^t$ and the output stock q_i^t, the market clearing price for firm i would be

$$\overline{p}_i^t = \frac{\sum_{j \in N} \alpha_{i,j} w_j^t}{q_i^t}. \tag{3}$$

   Now, we shall assume that prices adjust frictionally to their market-clearing values and hence consider that firm actually set their prices according to

$$p_i^t = \tau_p \overline{p}_i^t + (1 - \tau_p) p_i^{t-1} \tag{4}$$

   where $\tau_p \in [0, 1]$ is a parameter measuring the speed of price adjustment (the case $\tau_p = 1$ corresponding to instantaneous price adjustment).

3. Whenever $\tau_p < 1$ markets do not clear (except if the system is at a stationary equilibrium). In case of excess demand, we assume that clients are rationed proportionally to their demand. In case of excess supply, we assume that the amount $\overline{q}_i^t := \sum_{j \in N} \alpha_{i,j} w_j^t / p_i^t$ is actually sold and that the rest of the output is stored as inventory. Together with production occurring on the basis of purchased inputs, this yields the following evolution of the product stock:

$$q_i^{t+1} = q_i^t - \overline{q}_i^t + f_i\left(\frac{\alpha_{0,i} w_i^t}{p_0^t}, \left(\frac{\alpha_{j,i} w_i^t}{p_j^t}\right)_{j \in S_i(A)}\right) \tag{5}$$

   Note that in the case where $\tau_p = 1$, one necessarily has $\overline{q}_i^t = q_i^t$ and equation (5) reduces to

$$q_i^{t+1} = f_i\left(\frac{\alpha_{0,i} w_i^t}{p_0^t}, \left(\frac{\alpha_{j,i} w_i^t}{p_j^t}\right)_{j \in S_i(A[t])}\right) \tag{6}$$

4. As for the evolution of agents' wealth, it is determined on the one hand by their purchases of inputs and their sales of output. On the other hand, we assume that the firm sets its expenses for next period at $(1-\lambda)$ times its current revenues and distributes the rest as dividends to the representative household. That is one has:

$$\forall i \in M,\ w_i^{t+1} = (1-\lambda)\overline{q}_i^t p_i^t \tag{7}$$

$$w_0^{t+1} = q_0^t p_0^t + \lambda \sum_{i \in M} \overline{q}_i^t p_i^t \tag{8}$$

   Note that equation (8) can be interpreted as assuming that firms have myopic expectations about their nominal demand (i.e they assume they will face the same nominal demand next period) and target a fixed profit/dividend share $\lambda \in (0,1)$.

This first sequence of operations defines out of equilibrium dynamics for a given production network. As for the evolution of the network, it takes place at the end of the period according to two process: one governs the evolution of weights, the other the evolution of the adjacency structure.

5. As for the evolution of weights, given prevailing prices the optimal input weights for a firm i are those that minimize production costs. These are defined as the solution to the following optimization problem:

$$\begin{cases} \max & f_i(\frac{\alpha_0, i}{p_0^t}, (\frac{\alpha_j, i}{p_j^t})_{j \in S_i(A)}) \\ \text{s.t} & \sum_{j \in S_i(A)} \alpha_{j,i} = 1 \end{cases} \tag{9}$$

   Now, as in the case of prices, we shall consider that the process of technological adjustment can be subject to frictions and that input weights are actually updated according to the following rule:

$$\alpha_i^{t+1} = \tau_w \overline{\alpha}_i^t + (1-\tau_w)\alpha_i^t \tag{10}$$

   where $\overline{\alpha}_i^t \in \mathbb{R}^M$ denotes the solution of 9 and $\tau_w \in [0,1]$ measures the speed of technological adjustment of the production network.

6. As for the evolution of the adjacency structure, each firm independently receives the opportunity to change one of its suppliers with probability $\rho_{chg} \in [0,1]$. If the opportunity actually arises for firm i in period t, it selects randomly one of its suppliers $\overline{j}_i$ and another random firm j among those to which it is not already connected. It then shifts its connection from firm $\overline{j}_i$ to firm j if and only if the price of j is cheaper than this of $\overline{j}_i$. In other words, the adjacency matrix A^t evolves according to:

$$\begin{gathered} a_{i,\overline{j}_i}^{t+1} = \begin{cases} 1 & \text{if } p_{i,\overline{j}_i} \leq p_{i,j} \\ 0 & \text{otherwise} \end{cases} \\ a_{i,j}^{t+1} = 1 - a_{i,\overline{j}_i}^{t+1} \end{gathered} \tag{11}$$

The actual weight of the new connection is then determined according to an average of these of other suppliers.

7. Finally, the possibility for a firm to lose connections implies that it can eventually be driven out of the market. Indeed, we consider that a firm that has lost all its connections toward other firms exits the market. To sustain competition in the economy, we assume that those exits are compensated by entries of new firms according to the following process. Every period, each (potential) firm that is out of the market independently enters with probability p_{new}. When entering, the firm is endowed with the following characteristics:

   - The number of suppliers is drawn from a binomial distribution $B(p, n)$. The success probability p is adjusted in order to preserve the mean degree in the network in the long-run.
   - The price is initially set equal to the average price in the economy.
   - Each firm in the economy rewires to the newly created firm independently with probability $\bar{k}/n$, where $\bar{k}$ is the average number of clients at time 0.
   - The wealth of the firm is set equal to the average wealth of other firms[2] and its initial output stock is empty.

2.3 The linear case

In order to gain an understanding of the basic dynamics of the model, let us first consider the case where the network is fixed, both in terms of weights and adjacency structure. Note that this fixed weights assumption is equivalent to the assumption used in Acemoglu et al. [2012] that production functions are Cobb-Douglas (with the corresponding weights). In this respect, this simplified version of our model can be seen as an out of equilibrium extension of Acemoglu et al. [2012].

The dynamic properties of this model are relatively straightforward. First, it is clear that the evolution of wealths follows a linear dynamic, which can be written matricially as:

$$w^{t+1} = R(\lambda)\mathbb{A}w^t \tag{12}$$

where

$$R(\lambda) := \begin{pmatrix} \lambda & \cdots & \cdots & \lambda \\ 0 & & & \\ \vdots & & (1-\lambda)I & \\ 0 & & & \end{pmatrix}$$

accounts for the redistribution of firms' revenues. The matrix of weights $\mathbb{A}$ being moreover row-stochastic, it is straightforward to check using Perron-Froebenius

[2]To ensure conservation of money in the long term, this initial wealth of the firm is in practice considered as a loan that the firm has to reimburse before it can pay any dividend

theorem that the linear system in 12 is globally asymptotically stable. Accordingly, as illustrated in Fig. 1, we observe convergence in our simulations towards a stationary equilibrium determined by $\overline{w} \in \mathbb{R}^N$ such that:

$$\overline{w} = R(\lambda)A\overline{w} \tag{13}$$

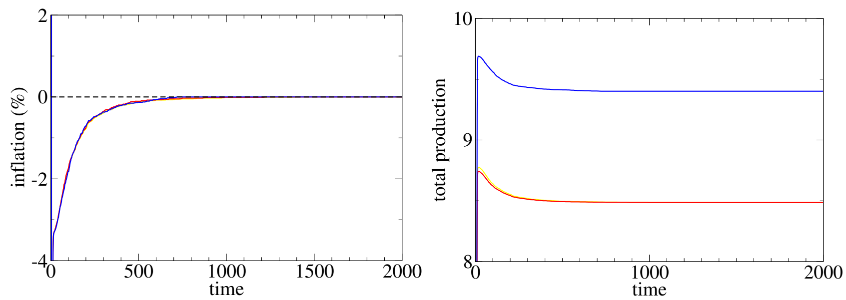

Figure 1: One time-step inflation rate and total production as a function of time for the basic model ($\rho_{chg} = 0$) and different values of τ_p, τ_w: $\tau_p = 1$ and $\tau_w = 0$ (yellow), $\tau_p = 0.9$ and $\tau_w = 0$ (red), $\tau_p = 0.9$ and $\tau_w = 0.9$ (blue). Other parameters are: $\sigma = 0.5$, $\lambda = 0.05$, $M = 1000$.

2.4 Frictions and general equilibrium

Proceeding stepwise, we now consider the case where the adjacency structure of the network is fixed but the weights evolve according to equation (10). This setting is akin to the general equilibrium one introduced in section 2.1 but for the fact that firms aim at enforcing a mark-up proportional to λ on their production costs rather than at maximizing profits. More precisely, steady states of the dynamical system defined by equations (4) to (10) are mark-up equilibria in the following sense:

Definition 2 *A mark-up equilibrium of the economy $\mathcal{E}(A)$ is a collection of prices $(p_0^*, \cdots, p_n^*) \in \mathbb{R}_+^M$, production levels $(q_0^*, \cdots, q_n^*) \in \mathbb{R}_+^M$ and commodity flows $(x_{i,j}^*)_{i,j=0\cdots n} \in \mathbb{R}_+^{M\times M}$ such that:*

- *Markets clear. That is for all $i \in M$, one has*

$$q_i^* = \sum_{j=1}^{M} x_{i,j}^*.$$

- *The representative consumer maximizes his utility. That is $(q_0^*, (x_{0,j}^*)_{j=1,\cdots,n})$ is a solution to*

$$\begin{cases} max\ u_i((x_{0,j})_{j=1,\cdots,n}) \\ \\ s.t\ \sum_{j=1}^n p_j^* x_{0,j}^* \leq 1 \end{cases}$$

(with the price of labor normalized to 1)

- *Production costs are minimized. That is for all $i \in M$, $(x^*_{i,j})_{j=0\cdots n}$ is the solution to*

$$\begin{cases} min & \sum_{j \in S_i(A)} p^*_j x_j \\ s.t & f_i(x_j) \geq q^*_i \end{cases}$$

- *Prices are set as a mark-up over production costs at rate $\dfrac{\lambda}{1-\lambda}$. That is one has for all $i \in N$:*

$$p^*_i = (1 + \frac{\lambda}{1-\lambda}) \frac{\sum_{j \in S_i(A)} p^*_j x^*_{i,j}}{q^*_i}$$

Note that for $\lambda = 0$, mark-up equilibria coincide with general equilibria in the sense of Definition 1. Indeed in a setting with constant returns to scale, profits are zero at a general equilibrium[3].

In this sense our model can be seen as a (dynamic) extension of the conventional general equilibrium approach. Yet, this identification between economic equilibria and steady states of our dynamical system only makes sense if these steady states are stable. We investigate the issue numerically by performing, for different values of the elasticity of substitution σ, Monte-Carlo simulations in which we let vary the speeds of price and technology adjustment, i.e τ_p and τ_w.

The results of these simulations are reported in Figure 2 as phase diagrams in the (τ_p, τ_w) plane. As long as σ is in a neighborhood of 1, the system exhibits three distinct phases.

- There is first a stable phase in which the system converges to equilibrium: excess demand vanishes and prices converge to their equilibrium values (see Figure 3).
- As the speed of price and technological adjustment increase, the system reaches an "excess demand" phase with rationing. There is persistent mismatch between supply and demand, positive inflation and sustained volatility in the network (see Figure 3 as well). Note that disequilibrium appears to materialize via inflation and excess demand rather than via deflation and excess supply. This is however an aggregate view which averages excess demand and excess supply, increasing and decreasing prices. Yet, there is a bias towards inflation and excess demand because the price adjustment process (equation 4) induces an asymmetry in the magnitude of the price adjustments upwards and downwards.
- Last, for slow values of the price adjustment process, the system reaches a phase of "cyclical volatility". As illustrated in Figure 3, the system

[3]Yet in a dynamic setting like ours assuming $\lambda > 0$ seems necessary to prevent firms from remaining permanently at the brink of bankruptcy.

then oscillates between inflation and deflation, excess demand and excess supply, positive and negative profits. In this phase, as the prices evolve too slowly, output stocks carry the burden of adjustment. This leads to very strong feedback effects which entail a synchronized state of the economy (as in Gualdi et al. [2015b] and Gualdi et al. [2015a]). Figure 4 illustrates the processes at play during a cycle. During a first phase, production is larger than demand and firms build up stocks while the price adjusts downwards. During a second phase (after the inflexion of the supply-demand curve), demand is larger than supply, the stocks get built down but prices keep adjusting downwards (there is still excess supply in the stock that the stock of output is non-empty). The price keeps adjusting downwards until the stocks are completely depleted. At this stage, excess demand is at a maximum and profits at a minimum because prices are low and stocks are empty. Yet,in absence of buffer stocks, prices become more volatile and increase rapidly until excess demand is absorbed and a new cycle starts.

Hence, despite its simplicity (there is no evolution of the adjacency structure of the network at this stage), the model displays a very rich taxonomy of behavior: general equilibrium, rationing, periodic "overproduction" crisis.

The key determinant of the transition between the stable general equilibrium phase and the unstable "excess demand" phase is the relative speed of price (τ_p) and technological (τ_w) adjustment. The faster the relative speed of price adjustment, the more stable the system is. Yet, the stability range increases as the absolute speed of price adjustment decreases. Moreover, the size of the stable region increases both as the elasticity of substitution decreases and as the firms optimal budget allocation for households increases (i.e. for larger values of the parameter α in the production function). There exists a critical value σ^* ($\sigma^* \sim 5/9$ for the parameter setting used in figure 2) such that for $\sigma \leq \sigma^*$, the unstable region disappears and the system converges to equilibrium independently of the speeds of price and technological adjustment.

These latter results are reminiscent of those obtained in Bonart et al. [2014]: the larger the intrinsic volatility of the system (in our setting, the higher the elasticity of substitution), the slower the adjustment processes shall be for the system to be stable. As for the formation of networks, these results confirm that in absence of changes in the adjacency structure, the characteristics of the production networks are completely determined by exogenous technological constraints (represented by the production functions in our setting).

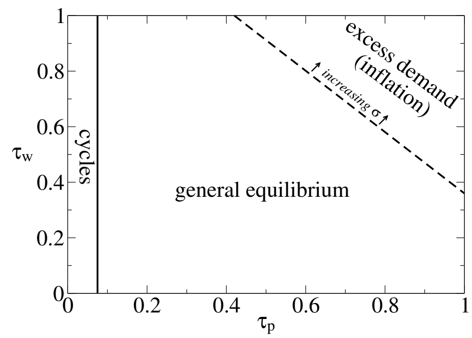

Figure 2: Phase diagram of the model with $\rho_{chg} = 0$ for $\sigma = 4/5$.

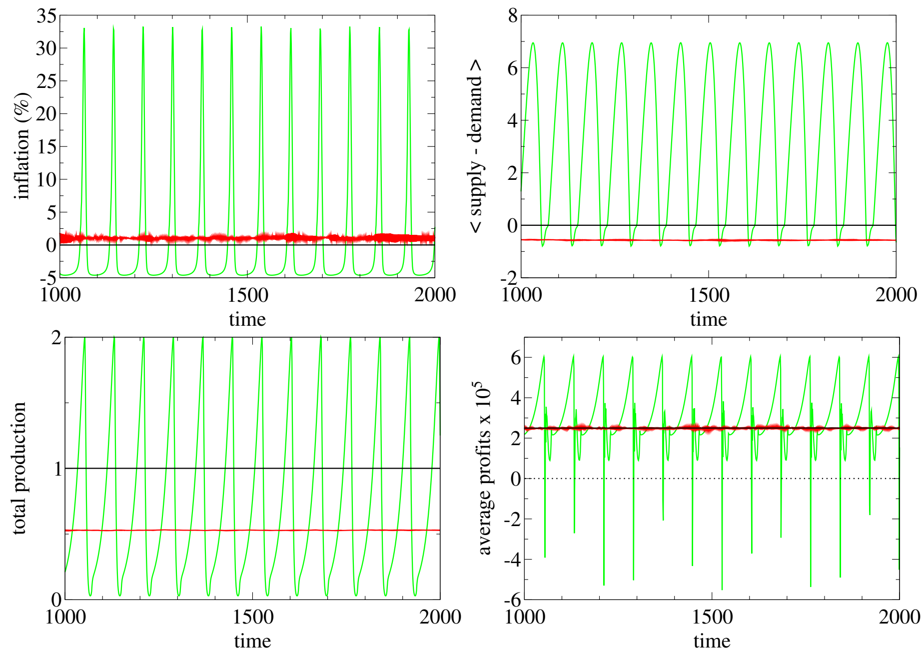

Figure 3: One time-step inflation rate, average mismatch between supply and demand, total production and average firms profits as a function of time for the basic model ($\rho_{chg} = 0$) and three different values of τ_p, τ_w corresponding to different regions of Fig. 2: $\tau_p = 0.5$ and $\tau_w = 0.5$ (black lines), $\tau_p = 0.9$ and $\tau_w = 0.9$ (red lines), $\tau_p = 0.05$ and $\tau_w = 0.5$ (green lines). Other parameters are: $\sigma = 4/5$, $\lambda = 0.05$, $M = 2000$.

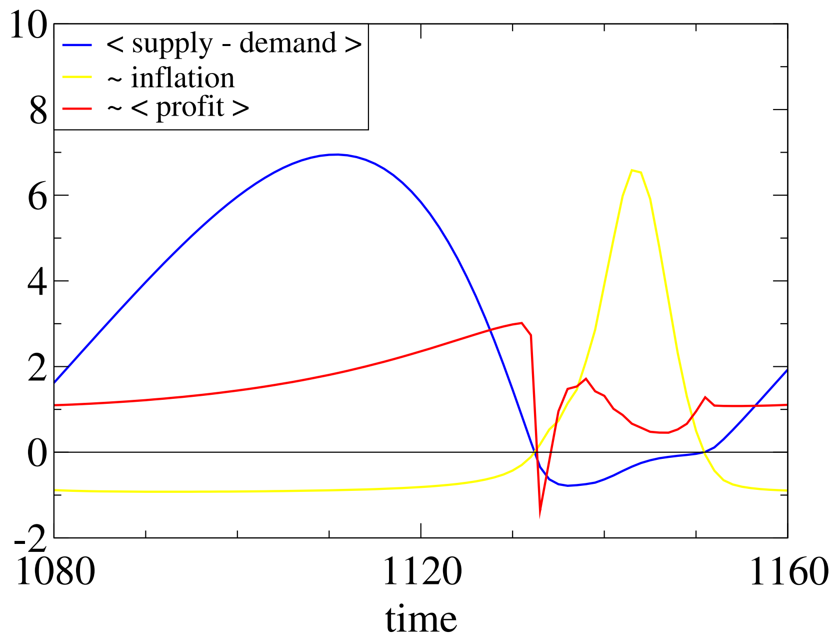

Figure 4: Average excess supply, inflation and average profit as a function of time for a cycle observed in Fig. 3 for $\tau_p = 0.05$ and $\tau_w = 0.5$. Inflation and profits are rescaled for illustrative purposes. Other parameters are as in Fig. 3.

3 The endogenous formation of production networks

3.1 Steady-state analysis

In this section, we account for the increased flexibility in production technologies implied by models of monopolistic competition on the market for intermediary goods à la Ethier [1982] and Romer [1990]. In other words, we consider that the adjacency structure evolves according to Equation (11).

A steady state of the system would consist in vectors of wealths $\tilde{w}$, prices $\tilde{p}$, productions $\tilde{q}$ and in an adjacency matrix $\tilde{\mathbb{A}}$ satisfying the following properties.

- First, according to equation 11, each firm i buys only from the cheapest suppliers (otherwise it would rewire). That is, one has for all $i \in N$:

$$\max_{j \in S_i} \tilde{p}_j \leq \min_{k \notin S_i} \tilde{p}_k \tag{14}$$

- Second, firms only differ in terms of their number of suppliers n_i and, at a steady state, the larger the number of suppliers of a firm, the more productive and the cheaper it is (otherwise it could adopt the same production technique than any firm with a smaller number of suppliers and improve upon it by diversifying marginally). That is one has for all $i, j \in M$:

$$n_i > n_j \Rightarrow \tilde{p}_i > \tilde{p}_j \tag{15}$$

- Therefrom, one can deduce that at a steady-state only the firms with the maximal number of suppliers (the more productive according to equation 15) actually have consumers (according to equation 14). More precisely, let us denote by V the set of active firms in the steady state (i.e these actually having consumers), by $M_\mu := \{i \in M \mid n_i = \mu\}$ the set of firms with exactly μ suppliers, by m_μ the number of such firms, by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$ the decreasing sequence of μs for which $M_\nu \neq \emptyset$ and let $\nu_i = \sum_{j=1}^{i} m_{\mu_j}$. One then has:

$$M_{\mu_i} \subset V \Rightarrow \text{card}\{\ell \in M \mid n_\ell \geq \nu_{i-1}\} \geq m_{\mu_i} \tag{16}$$

  That is to say, for a firm with μ_i suppliers to have at least one consumer, there must be a firm that requires more than ν_{i-1} suppliers because its first ν_{i-1} suppliers are these that are more productive and hence cheaper than the μ_i suppliers firm.

A corollary of equation 16 is that there exists a steady state only if there are at least m_{μ_r} firms that have more than ν_{r-1} suppliers as otherwise there would always be a firm (in M_r) without consumers. Such a firm would exit the market and be replaced by an entering firm, hence contradicting the fact that the system is at a steady state. To clarify this condition, let us consider the case where all the firms have a distinct number of suppliers (that is $r = m$ and $\forall i \; m_{\mu_i} = 1$). Then, there can be a steady state only if there exists a firm with exactly m suppliers, i.e a firm connected to every other firm. It is also worth noting that in this case the production network is a nested-split graph [see König et al., 2012, 2014] because every consumer of a firm in M_i also is a consumer of each of the cheaper firms (in M_j such that $j < i$).

This necessary condition for the existence of a steady-state is clearly extremely restrictive. It is not observed in simulations unless the system is initialized in a very peculiar state (e.g by letting all the firms exactly have the same number of suppliers). On the contrary, we generically observe sustained growth and decline of firms, entry and exit and changes in the micro-structure of the network. However, the system exhibits very robust distributional stylized facts that we investigate in the remaining of this paper.

3.2 Distribution of firms' growth rates

A first major stylized fact of firms' demographics is that the growth rates of firms are distributed according to a "tent-shaped" double-exponential distribution (see Bottazzi and Secchi [2006]). As illustrated in Fig. 5, our model generates exactly this type of Laplace distributions. For relatively short time intervals growth rates are indeed distributed according to $\sim \exp a|g - g_0|$ as found in empirical data. Following Arthur [1994], Bottazzi and Secchi put forward the fact that a Laplace type of distribution emerges because market success is cumulative or self-reinforcing. In their "island-based" model, this self-reinforcing process is hard-wired into the model: *"we model this idea using a process whereby the probability for a given firm to obtain new opportunities depends on the number of opportunities already caught."* In our setting, "self-reinforcing success" is also at play but it emerges endogenously. Indeed, the price-setting process (see equation 4) is such that whenever a firm gains a new consumer, its price increases (directly but also indirectly through the increase demand that it adresses to his own suppliers) and hence its competitiveness decreases. However, the larger the firm is the weaker the effect of an additional consumer is on its price and hence the more competitive it remains. Therefore, larger firms are more competitive and can seize more frequently new business opportunities. Hence, our model generates endogenously the self-reinforcing feedbacks introduced exogenously in Bottazzi and Secchi [2006] to generate a Laplacian distribution.

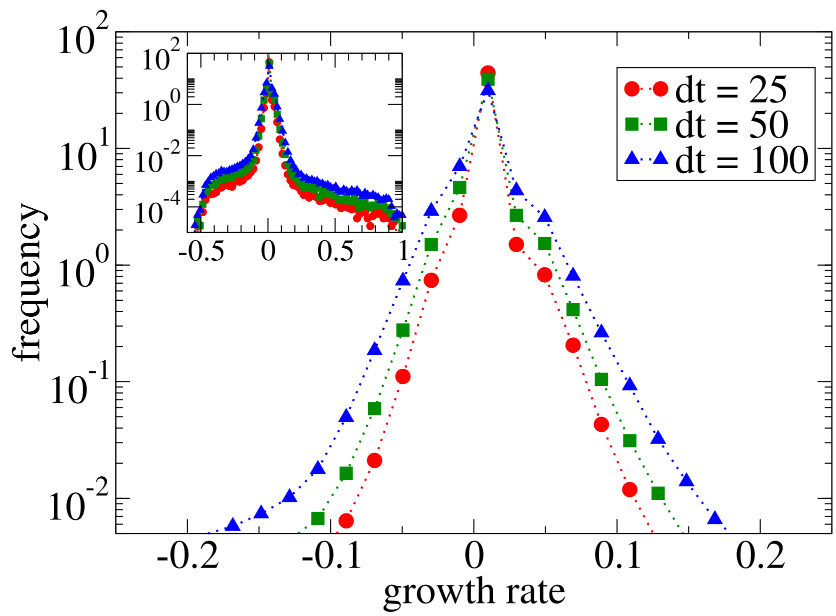

Figure 5: Distribution of firms' growth rates after 2 10^6 time steps for the model with $\rho_{chg} = \rho_{new} = 0.05$ and $\sigma = 1/2$. Different symbols / colors correspond to different time intervals to compute growth rates (for each firm we histogram only the last 30 rates). Other parameters are: $\tau_p = \tau_w = 0.8$ and $M = 10000$. Results are averaged over 20 realizations.

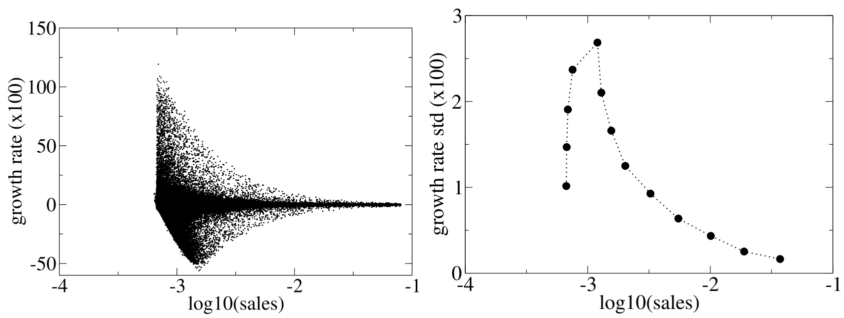

Figure 6: *Left*: scatter plot of firms growth rates versus firms (log) sales. *Right*: growth rates standard deviation as a function of firms (log) sales. Other parameters are as in Figures. 7 and **??**. We observe a small negative significant correlation (~ -0.008) between growth rate and (log) sales.

Figure 6 illustrates two other important stylized facts about firms' growth rates that the model captures very clearly [see e.g Coad, 2009]. There is a negative correlation between growth rate and size. There is also a negative relationship between growth rate variance and firm size.

3.3 Fat-tails in production networks

On a more structural level, key stylized facts about firms are the Zipf distribution of size [see Axtell, 2001] and the presence of fat-tails in the degree distribution of production networks [see Atalay et al., 2011, Acemoglu et al., 2012]. As illustrated in Fig. 7 both features are clearly matched in the long-run by our model. Both the distribution of firms' sizes and the in-degree distribution of the production network are characterized by a power-law tail with exponent close to 2.1. The power-law nature of the distributions are confirmed by Kolmogorv-Smirnov tests. More fundamentally, the emergence of these fat tails is completely independent of (i) the initial distribution of firms' sizes (ii) the initial structure of the network and (iii) of the elasticity of substitution σ.

3.4 A master-equation approach to the formation of production networks

As noted by Bottazzi and Secchi [2006], the emergence of a scale-free distribution of firms' size can not be explained by an exponential distribution of growth-rates. As a matter of fact, figure 5 shows that as the length of the time intervals used to compute the growth rates increase, the exponential character vanishes and fatter tails progressively appear.

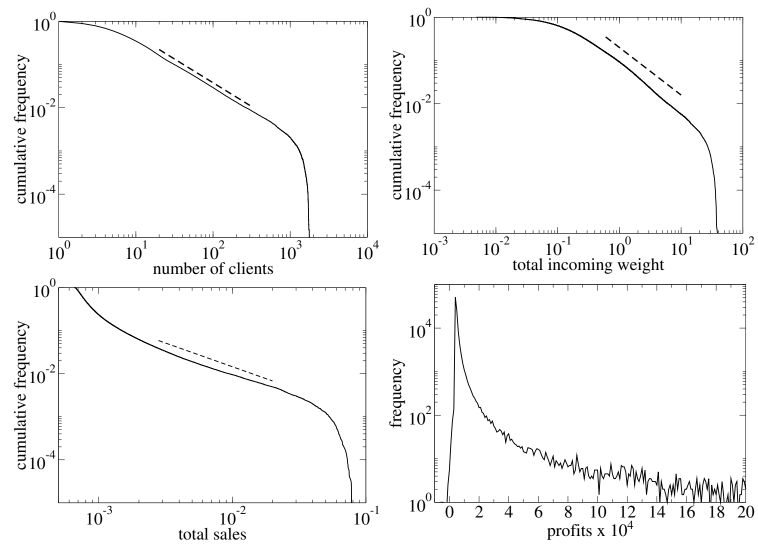

Figure 7: Basic firms statistics after 2 10^6 time steps for the model with $\rho_{chg} = \rho_{new} = 0.05$. *Top:* Cumulative frequency for the number of incoming links (clients) and total incoming weight (sum over all clients weights). *Bottom:* Cumulative frequency for total sales and profits histogram. In all graphs black solid lines are averages over 20 realizations. Dashed black lines are a guide for the eye and correspond to $f(x) \sim x^{-1.1}$. Other parameters are: $\sigma = 0.5$, $\tau_p = \tau_w = 0.8$ and $M = 10000$.

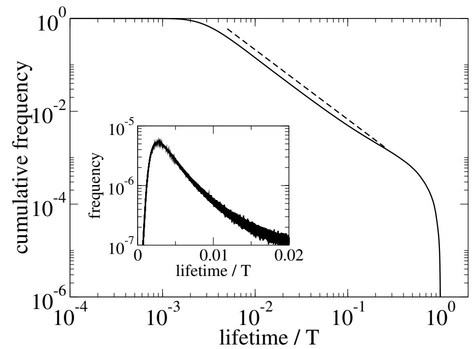

Figure 8: Other firms statistics after 2 10^6 time steps for the model with $\rho_{chg} = \rho_{new} = 0.05$ (as in Fig. 7). Frequency and cumulative frequency for firms lifetime. For longer lifetimes the distribution has an exponential decay (see inset) followed by a power law tail with exponent ~ 2.5. Results are averages over 20 realizations. Other parameters are: $\sigma = 0.5$, $\tau_p = \tau_w = 0.8$ and $M = 10000$.

To understand the emergence of scale-free production network and distribution of firm' size, one has to focus on the inner workings of competition. As hinted at in section 3.1, except in degenerate cases, there is permanent entry and exit of firms in our model. This competition shapes the structure of the production network and the distribution of firms' size. More precisely, the production structure that emerges must be consistent with the speeds at which challenger firms grow (when they are competitive) and incumbent firms shrink (when they are no longer competitive).

In particular, fat-tails in the degree distribution of the production network emerges in our setting because of two basic facts about the "economy" of suppliers' switches. On the one hand, the number of incoming business opportunities for a firm is independent of its size. On the other hand, the rate at which existing consumers may quit grows linearly with the size of the firm. Therefore, at (a statistical) equilibrium, the degree/size distribution must be Zipf in order to balance the flow of incoming and outgoing links. The remaining of this section provides a formal proof of this argument.

The natural approach to characterize the asymptotic properties of the degree distribution is to analyze the master equation[4] that specifies the evolution of the probability $P(k,t)$ to have a firm of degree k in the network at time t. Assuming that the time interval is chosen small enough so that there is a single link swap per period, this master equation is of the form:

$$P(k,t+1) = P(k-1,t)\rho_{k-1}(t) + P(k+1,t)\mu_{k+1}(t) - P(k,t)(\rho_k(t) + \mu_k(t)) \quad (17)$$

where $\rho_k(t)$ and $\mu_k(t)$ denote respectively the probability that a firm of degree k gains and loses a link at time t. Figure 9 represents the asymptotic behavior of a solution to this master-equation (with transition probabilities computed as described below). This behavior is perfectly in line with the one observed in the model: a power-law emerges with exponent close to 2. These simulation results have strong analytical counterparts. Namely, we obtain the following characterization of stationary distributions.

Proposition 1 *There exists a stationary solution* $(\overline{P}_k)_{k\in\mathbb{N}}$ *of Equation* (17) *such that, as* $k \to +\infty$:

$$\overline{F}_k := \sum_{\ell=1}^{k} \overline{P}_\ell \sim 1 - \frac{\overline{d}}{k} \quad (18)$$

where $\overline{d}$ *is the (fixed) mean degree in the network.*

Proof: *According to equation* 11, *the probability for a firm to gain or lose a link depends only on its position in the price ordering. More precisely, let us denote by* $i_j(t)$ *the id of the* j*th most expensive firm at time* t. *One then denotes*

[4]The master equation approach is seldom used in economics but is standard in the natural sciences, in particular in statistical physics [see e.g Van Kampen, 1992].

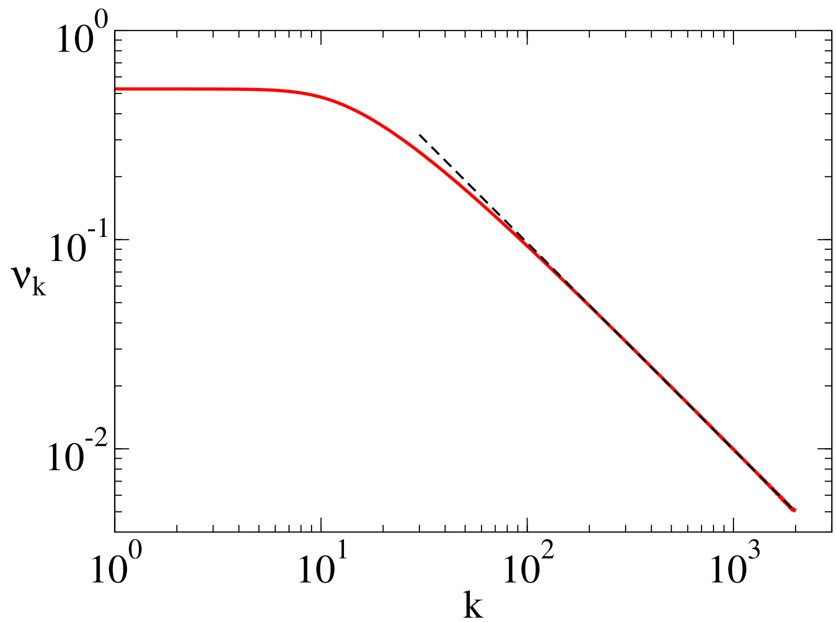

Figure 9: Numerical results for the stationary cumulative distribution of Eq. 17 with $n = 2000$ (red line). The exponent of the dashed black line is 0.989.

by $\pi_j^+(t)$ and $\pi_j^-(t)$ respectively the probability that firm $i_j(t)$ receives a new incoming link and loses an incoming link. One has:

$$\pi_j^+(t) = \frac{1}{n}\frac{n-j}{n-1} \tag{19}$$

$$\pi_j^-(t) = \frac{d_{j(t)}}{n\bar{d}}\frac{j-1}{n-1} \tag{20}$$

where $d_{j(t)}$ denotes the (in)degree of firm $j(t)$ an $n\bar{d}$ is the total number of links. One then has:

$$\rho_k(t) = \frac{1}{n(n-1)} \sum_{\{j|d_j(t)=k\}} n-j \tag{21}$$

$$\mu_k(t) = \frac{k}{(n-1)n\bar{d}} \sum_{\{j|d_j(t)=k\}} j-1 \tag{22}$$

We then focus on sufficient conditions for a degree distribution to be stationary, i.e to satisfy the following equation for all $k \in \mathbb{N}$:

$$P_{k-1}\rho_{k-1} + P_{k+1}\mu_{k+1} - P_k\rho_k - P_k\mu_k = 0 \tag{23}$$

A natural approach is to focus first on the "detailed balance" condition, which provides a simpler sufficient condition for stationarity of the distribution as:

$$P_k \mu_k = P_{k-1} \rho_{k-1} \tag{24}$$

or equivalently:

$$\frac{P_{k-1}}{P_k} = \frac{\mu_k}{\rho_{k-1}} \tag{25}$$

Now, it is clear from equations (19) *and* (20) *that at a stationary distribution, the position in the price ordering must decrease with the degree. Hence, if one denotes by* η_k *the number of firms of degree* k *at the stationary state and by* $\nu_k = \sum_{i=1}^{k} \eta_i$*, it must be that the* η_k *firms of degree* k *have positions* $n - \nu_{k-1}$ *to* $n - \nu_{k-1} - \eta_k + 1 = n - \nu_k + 1$ *in the price ordering. It is then standard calculus to check that:*

$$\sum_{\{j|d_j = k-1\}} n - j = \frac{1}{2}(2\nu_{k-2} + \eta_{k-1} - 1)\eta_{k-1} \tag{26}$$

$$\sum_{\{j|d_j = k\}} j - 1 = \frac{1}{2}(2n - 2\nu_{k-1} - \eta_k - 1)\eta_k \tag{27}$$

Using equations 26 *and* 27 *and the fact that* ${}^{P_k}/_{P_{k-1}} = {}^{\eta_k}/_{\eta_{k-1}}$*, the sufficient condition becomes:*

$$\frac{k}{\overline{d}} \frac{(2n - 2\nu_{k-1} - \eta_k - 1)\eta_k}{(2\nu_{k-2} + \eta_{k-1} - 1)\eta_{k-1}} = \frac{\eta_{k-1}}{\eta_k} \tag{28}$$

or equivalently

$$\frac{(2n - 2\nu_{k-1} - \eta_k - 1)}{(2\nu_{k-2} + \eta_{k-1} - 1)} = \frac{\overline{d}}{k}(\frac{\eta_{k-1}}{\eta_k})^2 \tag{29}$$

which eventually yields after division by n :

$$\frac{(2 - 2F_{k-1} - P_k - {}^1/_n)}{(2F_{k-2} + P_{k-1} - {}^1/_n)} = \frac{\overline{d}}{k}(\frac{P_{k-1}}{P_k})^2 \tag{30}$$

where $F_k = \sum_{\ell=1}^{k} P_\ell$ *is the probability of having a firm with degree less than* k.

Let us then conjecture the existence of a slowly decaying solution, that is $(\overline{P}_k)_{k\in\mathbb{N}}$ *such that:*

$$\lim_{k\to+\infty} \frac{\overline{P}_k}{\overline{P}_{k-1}} = 1. \tag{31}$$

It is then clear that $\overline{P}_k$ *is negligible with respect to* $1 - \overline{F}_k = \sum_{\ell=k+1}^{+\infty} \overline{P}_\ell$ *as* $k \to +\infty$*. As moreover* $\lim_{k\to+\infty} \overline{F}_k = 1$*, equation* (30) *yields that as* $k \to +\infty$:

$$1 - \overline{F}_k \sim \frac{\overline{d}}{k} \tag{32}$$

Therefrom, one can deduce that as $k \to \infty$:

$$F_k \sim 1 - \frac{\bar{d}}{2k} \tag{33}$$

Taking a continuous approximation and differentiating, it is clear that $\overline{P}_k$ asymptotically follows a power-law with exponent -2, consistently with the conjecture. This ends the proof.

The proof of the proposition highlights the fact that the asymptotic properties of the degree distribution are independent of the elasticity of substitution and of any other parameter of the model. Hence the result is extremely robust. The proof also provides a deeper insight about the driving force towards the emergence of a scale-fee production network. As anticipated at the beginning of this section, it is the fact that firms lose links proportionally to their degree whereas they gain link at a constant rate that generates the scale-free structure. Two processes are at play. On the one hand the most competitive firms tend to attract links and hence there is a tendency towards concentration on the most competitive firms. On the other hand, large firms are the most affected by competition from a new entrant because their chance to lose a customer is proportional to their size. This second process can be seen as a from of inverted preferential attachment process [see Barabási and Albert, 1999] where asymptotically large firms lose connections proportionally to their degree (whereas in the standard preferential attachment model finite-size firms gain new links proportionally to their degree).

4 Conclusion

The model of monopolistic competition on the markets for intermediate goods is central both to the international trade and the endogenous growth literature [Ethier, 1982, Romer, 1990]. In this paper, we develop a simple dynamic extension of this model in order to investigate the endogenous formation of production networks. The model subsumes the standard general equilibrium approach and robustly reproduces key stylized facts of firms' demographics, which are beyond the scope of general equilibrium theory. Firms' growth rates are negatively correlated with size and follow a double-exponential distribution. Firms' size and production network are power-law distributed. These properties emerge because of competitive forces. The continuous inflow of new firms shifts away the model from a steady state to a "dynamic equilibrium" in which firms get scaled according to their resistance to competitive forces.

An originality of our approach is to consider that technology is embedded within the production network. Further developments in this perspective that account for more acute forms technological innovation seems to be an interesting avenue for future research.

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