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A model of anonymous influence with anti-conformist agents

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Abstract. We study a stochastic model of anonymous influence with conformist and anti-conformist individuals. Each agent with a 'yes' or 'no' initial opinion on a certain issue can change his opinion due to social influence. We consider anonymous influence, which depends on the number of agents having a certain opinion, but not on their identity. An individual is conformist/anti-conformist if his probability of saying 'yes' increases/decreases with the number of 'yes'- agents. In order to consider a society in which both conformists and anti-conformists co-exist, we investigate a generalized aggregation mechanism based on ordered weighted averages. Additionally, every agent has a coefficient of conformism which is a real number in $[-1, 1]$, with negative/positive values corresponding to anti-conformists/conformists. The two extreme values -1 and 1 represent a pure anti-conformist and a pure conformist, respectively, and the remaining values – so called 'mixed' agents. We consider two kinds of a society: without mixed agents, and with mixed agents who play randomly either as conformists or anti-conformists. For both cases of the model, we deliver a qualitative analysis of convergence, i.e., find all absorbing classes and conditions for their occurrence.

JEL Classification: C7, D7, D85

Keywords: influence, anonymity, anti-conformism, convergence, absorbing class

1 Introduction

This paper is devoted to a phenomenon of anti-conformism in the framework of opinion formation with anonymous influence. Despite the fact that anti-conformism plays a crucial role in many social and economic situations and can naturally explain human behavior and various dynamic phenomena, it has received little attention in the literature so far.

The seminal work of DeGroot (1974) and some of its extensions consider a non-anonymous influence in which agents update their opinions by using a weighted average of the opinions of their neighbors. We are interested in anonymous influence, which depends only on the number of individuals having a certain opinion and is not dependent on agents' identities. Förster et al. (2013) investigate anonymous social influence by using the ordered weighted averages (commonly called OWA operators, Yager (1988)) which are the unique anonymous aggregation functions. The authors departure from a general framework of influence based on aggregation functions (Grabisch and Rusinowska (2013)), where every individual updates his opinion by aggregating the agents' opinions which determines the probability that his opinion will be 'yes' in the next period. However,

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instead of allowing for arbitrary aggregation functions, Förster et al. (2013) consider a particular way of aggregating based on the OWA operators. Both frameworks of Grabisch and Rusinowska (2013) and Förster et al. (2013) cover only positive influence (imitation), since by definition aggregation functions are nondecreasing, and hence cannot model anti-conformism.

Our aim is to study opinion formation in societies consisting of both conformist and anti-conformist individuals. In order to consider the presence of anti-conformists and anonymous influence, we investigate a generalized aggregation mechanism. We also use the OWA operators, i.e., every agent is assumed to have a weight vector with weights assigned to ranks and not to agents. Additionally, every agent is characterized by a coefficient of conformism. Individuals are intrinsically either conformists or anti-conformists. The coefficient of conformism is a real number in $[-1, 1]$, where negative (positive) values correspond to anti-conformists (conformists). The extreme values -1 and 1 of this coefficient represent a pure anti-conformist and a pure conformist, respectively, and the remaining values describe 'mixed' agents. Each individual has an initial yes/no opinion on a certain issue and at every time step can update his opinion by taking into account how many individuals share a given opinion. The probability of saying 'yes' is monotonic with respect to the number of 'yes'-agents: it increases for conformists and decreases for anti-conformists. We consider two cases of a society: without mixed agents, and containing mixed agents. For both cases we provide a complete qualitative analysis of convergence, i.e., we identify all absorbing classes. The crucial information in determining the absorbing classes to which a society converges is the number of yes/no agents needed to influence an individual's opinion. It is determined by the number of left/right zeroes in the agents' weight vectors. If a weight vector of an agent is not symmetric with respect to the number of left and right zeros, then the agent is biased towards the 'yes' or 'no' answers.

There exist several different types of absorbing classes in the model, and conditions for their occurrence are determined by relations between the numbers of (anti-)conformists and the left/right zeroes in the weight vectors showing how easily conformists and anti-conformists are influenceable. An interesting observation is that the analysis for conformists and anti-conformists is not symmetric. In the long run, the opinion of a society with conformists and anti-conformists always converges to an absorbing class, but never to consensus, contrarily to a conformist society, where consensus can be reached. Moreover, a society containing only pure conformists and pure anti-conformists can be dichotomous, with one of the two groups saying 'yes' forever. Typically, a society with more conformists ends up in a dichotomy, while a society with more anti-conformists becomes more unstable, in the sense that the opinion of any of the two groups might be constantly changing over time. Such a society has also cycles as possible absorbing classes. If there exists at least one mixed agent which plays randomly either as a conformist or an anti-conformist, i.e., his weight vector is a convex combination of the weight vectors of conformists and anti-conformists, then the society cannot be dichotomous anymore, and cycles are not possible. However, periodic classes as well as intervals with agents oscillating between 'yes' and 'no', and unions of intervals are possible absorbing classes, both in a society without and with mixed agents. In the latter case, mixed agents are among the usual oscillating individuals. Their presence makes the society more irregular and unstable.

Our framework is suitable for many natural applications. It can explain various phenomena like stable and persistent shocks, large fluctuations, stylized facts in the industry of fashion, in particular its intrinsic dynamics, booms and burst in the frequency of surnames, etc. If fashion were only a matter of conformist imitation in an anonymous framework, there would be no trends over time. Anti-conformism and anti-coordination when individuals have an incentive to differ from what others do can also be detected, e.g., in organizational settings. For example, the choice of a firm to go compatible or not with other firms can be seen as a problem of anti-conformism. Anti-coordination can be optimal when adopting different roles or having complementary skills is necessary for a successful interaction or realization of a task in a team.

The rest of the paper is structured as follows. In Section 2 we introduce the model of anonymous influence with anti-conformist agents and distinguish between two kinds of a society: pure case (containing only pure anti-conformists and pure conformists) and mixed case (including also mixed agents). The convergence analysis for both cases is provided in Section 3. Section 4 contains some examples and simulations. In Section 5 we deliver a brief overview of the related literature. In Section 6 concluding remarks are presented. The proof of our main results on the possible absorbing classes in the model is given in the appendix.

2 The model

2.1 Basic assumptions

We consider a society N of $|N| = n$ agents, having to make a yes/no-decision on some issue. Each agent has a personal initial opinion on the issue, however, by knowing the opinion of the other agents or by some social interaction with them, the opinion of each agent may change due to mutual influence. Doing so, there is an evolution in time of the opinion of the agents, which may or may not stop at some stable state of the society.

We define the state of the society as the vector giving the opinion of each agent in N. Equivalently, the state of the society is determined by the set $S \subseteq N$ of agents whose opinion is 'yes'. Our fundamental assumption is that the evolution of the state is ruled by a homogeneous Markov chain, that is, the state evolves at discrete time steps, the state at time t depends only on the state at time $t-1$, and the transition matrix giving the probability of all possible transitions from a state S to a state T is constant over time.

These assumptions are basically those underlying (Grabisch and Rusinowska, 2013). As the number of states is 2^n, the size of the transition matrix is $2^n \times 2^n$. In order to avoid this exponential complexity, the latter reference uses a simple mechanism to generate the transition matrix, inspired by DeGroot (1974). Coding 'yes' and 'no' by 1 and 0, respectively, the probability $p_i(S)$ that an agent $i \in N$ says 'yes' at next time step, given the present state S (the set of agents saying 'yes'), is

$$p_i(S) = A_i(1_S), \tag{1}$$

where 1_S is the indicator function of S, i.e., $1_S(i) = 1$ if $i \in S$ and 0 otherwise, and A_i is a nondecreasing function from $[0,1]^n$ to $[0,1]$ satisfying $A_i(1_N) = 1$ and $A_i(1_\emptyset) = 0$ (called an *aggregation function*[1]). Supposing that the update of opinion is done independently,

[1] Traditionally, the domain of an aggregation is $[0,1]^n$ or any interval of $\mathbb{R}$ to the power n. In our study, however, only the vertices $\{0,1\}^n$ are used.

the probability of transition from a state S to a state T is

$$\lambda_{S,T} = \prod_{i \in T} p_i(S) \prod_{i \notin T} (1 - p_i(S)), \tag{2}$$

with $p_i(S)$ given by (1).

2.2 Anonymous influence

The most common example of aggregation function, used, e.g., in DeGroot (1974), is the weighted arithmetic mean

$$A_i(x) = \sum_{j=1}^{n} w_j^i x_j,$$

where $x = (x_1, \ldots, x_n)$ and the w_j^i's are weights on the entries, satisfying $w_j^i \geq 0$ and $\sum_{j=1}^n w_j^i = 1$. Here, w_j^i represents to which extent agent i puts confidence on the opinion of agent j. It depicts a situation where every agent knows the identity of every other agent, and is able to assess to which extent he trusts or agrees with the opinion or personal tastes of others.

In many situations however, like opinions and comments given on the internet, the identity of the agents is not known, or at least, there is no clue on the reliability or kind of personality of the agents. Therefore, agents can be considered as anonymous, and influence is merely due to the *number* of agents having a certain opinion, not their identity. The natural aggregation function for this situation is the *ordered weighted average (OWA)* (Yager, 1988):

$$\mathrm{OWA}_w(x) = \sum_{j=1}^{n} w_j x_{(j)}, \tag{3}$$

where the entries $x_1, \ldots, x_n$ are rearranged in decreasing order: $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$, and $w = (w_1, \ldots, w_n)$ is the weight vector defined as above. Hence, the weight w_j is not assigned to agent j but to rank j, and thus permits to model quantifiers. For example, taking $w_1 = 1$ and all other weights being 0 models the quantifier "there exists". Indeed, it is enough to have one of the entries being equal to 1 to get 1 as output. In our context, it means that only one agent saying 'yes' is enough to make your opinion being 'yes' for sure. Similarly, "for all" is modeled by $w_n = 1$ and all other weights being 0, and means that you need that all agents (including you) say 'yes' to be sure to continue to say 'yes'. Intermediate situations can of course be modeled as well: by letting $w_k = 1$ and $w_j = 0$ for all $k \neq j$, one obtains a model where k 'yes' among the n agents are needed to ensure that the concerned agent will say 'yes' at next time step. Moreover, soft or fuzzy quantifiers can be modeled as well: "approximately half" could be represented by the following weight vector (with $n = 10$):

$$w = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 \end{pmatrix}.$$

The above model using OWA as an aggregation function has been fully studied in Förster et al. (2013), in particular concerning convergence of opinion in the long run. Generally speaking (see, e.g. Kemeny and Snell (1976); Seneta (2006)), we recall that for a Markov chain with set of states E and transition matrix Λ and its associated digraph Γ, a *class* is a subset C of states such that for all states $e, f \in C$, there is a path in Γ from e to f, and C is maximal w.r.t. inclusion for this property. A class is *absorbing* or *absorbing* if for every $e \in C$ there is no arc in Γ from e to a state outside C. An absorbing class C is *periodic of period* k if it can be partitioned in blocks $C_1, \ldots, C_k$ such that for $i = 1, \ldots, k$, every outgoing arc of every state $e \in C_i$ goes to some state in C_{i+1}, with the convention $k + 1 = 1$. When each $C_1, \ldots, C_k$ reduces to a single state, one may speak of *cycle of length* k, by analogy with graph theory.

In our case, states are subsets of agents and therefore classes are collections of sets, which we denote by calligraphic letters, like $\mathcal{C}, \mathcal{B}$, etc. By definition, an absorbing class indicates the final state of opinion of the society. If an absorbing class reduces to a single state S, it means that in the long run, the society is dichotomous (unless $S = N$ or $S = \emptyset$, in which case consensus is reached): there is a set of agents S who say 'yes' forever, while the other ones say 'no' forever. Otherwise, there are endless transitions with some probability from one set $S \in \mathcal{C}$ to another one $S' \in \mathcal{C}$.

It is an obvious fact that for any type of aggregation function, $\emptyset$ and N are absorbing classes. Indeed, the conditions $A_i(1_N) = 1$ and $A_i(1_\emptyset) = 0$ for all $i \in N$ imply that $\lambda_{N,N} = 1$ and $\lambda_{\emptyset,\emptyset} = 1$, that is, once these states are reached, there is no possibility to escape from them. However, many other absorbing classes are possible. For the anonymous model, they are of two types:

(i) any single state $S \in 2^N$;
(ii) union of intervals of the type $[S, S \cup K]$, where $S, K \neq \emptyset, N$, with $[S, S \cup K] = \{T \in 2^N \mid S \subseteq T \subseteq S \cup K\}$.

For the second case, when the absorbing class is reduced to a single interval $[S, S \cup K]$, it depicts a situation in the long run where agents in S say 'yes', those outside $S \cup K$ say 'no', and those in K oscillate between 'yes' and 'no' forever. Interestingly, no periodic class can occur, although in general for arbitrary aggregation functions cycles can occur (Grabisch and Rusinowska, 2013).

2.3 Anti-conformism and conformism

As aggregation functions are nondecreasing in each argument, models of influence based on them are necessarily *conformist*: if more agents say 'yes', your probability of saying 'yes' cannot decrease, i.e., you are more or less inclined to follow the trend. However, it is often observed that some individuals are inclined to go against the trend by some reactive behavior, which can be modeled by an "anti"-aggregation function, i.e., being nonincreasing in each argument.

In this paper, we introduce such functions, but limit our study to anonymous models. In order to consider both conformist and anti-conformist agents in a society, we propose a generalization of the above mechanism defined by (1) and (3). To this end, we find more convenient to replace 1 and 0 by 1 and -1, respectively, for the coding of 'yes' and 'no'. As usual, cardinality of a set is denoted by the corresponding lower case, e.g., $s = |S|$.

The probability that agent i says 'yes' at next time step, given that S is the set of agents saying 'yes' at present time is now given by

$$p_i(S) = \frac{1}{2}\Big(1 + \alpha_i \mathrm{OWA}_{w^i}(1_S)\Big) = \frac{1}{2}\Big(1 + \alpha_i\Big(\sum_{j=1}^{s} w_j^i - \sum_{j=s+1}^{n} w_j^i\Big)\Big), \tag{4}$$

with $\alpha_i \in [-1, 1]$, w^i is the weight vector of agent i, and the OWA operator is given by (3). The coefficient α_i is called the *coefficient of conformism*. We easily observe the following.

(i) The values taken by p_i are comprised between $1/2(1 - \alpha_i)$ and $1/2(1 + \alpha_i)$.

(ii) If $\alpha_i > 0$, then p_i is a monotone function w.r.t. set inclusion, i.e., the bigger the set S, the higher the probability to say 'yes', which indicates a conformist attitude for agent i. This effect is maximum when $\alpha_i = 1$, and we say then that the agent is *purely conformist*. Note that in the latter case (4) is identical to (1) with the OWA operator. Hence, if $\alpha_i = 1$ for all $i \in N$, we recover the classical (conformist) anonymous model studied in Förster et al. (2013).

(iii) If $\alpha_i < 0$, then p_i is antimonotone w.r.t. set inclusion, i.e., the smaller S, the higher the probability to say 'yes', which means that the agent is anti-conformist. If $\alpha_i = -1$, then we call i a *purely anti-conformist* agent.

(iv) If $\alpha_i = 0$, then $p_i(S) = \frac{1}{2}$ for every S, that is, the agent tosses a coin whatever the situation is.

Example 1 Let us take $n = 4$ and the weight vector $(0, 0, \frac{1}{2}, \frac{1}{2})$, which can be interpreted as the soft quantifier "most of". The probability $p_i(S)$ is computed for various α_i and sizes of S in the table below.

$p_i(S)$	$S = \emptyset$	$\|S\| = 1$	$\|S\| = 2$	$\|S\| = 3$	$S = N$
$\alpha_i = 1$	0	0	0	0.5	1
$\alpha_i = 0.5$	0.25	0.25	0.25	0.5	0.75
$\alpha_i = 0$	0.5	0.5	0.5	0.5	0.5
$\alpha_i = -1$	1	1	1	0.5	0

One can see that conformist agents tend to say 'yes' if most of people do so, while anti-conformist agents tend to say 'no' in this situation.

To facilitate the analysis of the model, we distinguish between three types of agents and introduce some assumption and notation. We partition the society of agents into

$$N = N^c \cup N^a \cup N^m$$

where N^c is the set of (purely) conformist agents with $\alpha_i = 1$, N^a the set of (purely) anti-conformist agents ($\alpha_i = -1$), and N^m is the set of "mixed" agents with $\alpha_i \in]-1, 1[$. To avoid trivialities we consider that $N^c, N^a \neq \emptyset$. When $N^m = \emptyset$, we call it the *pure case*.

Consider a weight vector $w = \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}$ of an OWA operator. As our analysis will reveal, the relevant information in w is merely the number of right and left zeroes, not the precise value of the weights. We denote by l and r the number of left zeroes and right zeroes in w, respectively. Formally, $0 = w_1 = \cdots = w_l \neq w_{l+1}$ and $w_{n-r} \neq w_{n-r+1} = \cdots = w_n = 0$. For example, taking $w = \begin{pmatrix} 0 & 0.1 & 0.5 & 0 & 0.2 & 0 & 0.2 & 0 & 0 \end{pmatrix}$, we have $l = 1$ and $r = 2$. Observe that due to the normalization condition of w, we have $0 \leq l + r < n$.

We make the following assumption: agents in N^c may have different weight vectors, however the number of left and right zeroes is the same for all of them. We denote them by l^c, r^c. Similarly, l^a, r^a (respectively, l^m, r^m) denote the number of left and right zeroes in the weight vector of agents in N^a (respectively, N^m).

We end this section by giving an interpretation of the left and right zeroes. Let us consider a weight vector with l left zeroes and r right zeroes. We can see from the definition of OWA that these zeroes eliminate the first l 'yes' and the first r 'no'. Therefore, the decision of an agent with such a weight vector is based on the number of people saying 'yes' and 'no', after having eliminated the first l 'yes' and r 'no'. The number of left/right zeroes indicates how many people the agent needs in order to start being influenced towards the yes/no opinion. In particular, a non symmetrical weight vector w.r.t. the number of left and right zeroes means that the agent is biased towards the 'yes' or 'no' answer, i.e., he needs a different number of people to start being convinced to say 'yes' or 'no'.

2.4 Basic properties of transitions

We study in this section the properties of the transition matrix Λ, with entries $\lambda_{S,T}$, $S,T \in 2^N$. We recall that $\lambda_{S,T}$ is given by (2), with $p_i(S)$ given by (4).

Our aim is to find under which conditions one has a possible transition from S to T, i.e., $\lambda_{S,T} > 0$. From (2), we have:

$$\lambda_{S,T} > 0 \Leftrightarrow [p_i(S) > 0 \forall i \in T] \& [p_i(S) < 1 \forall i \notin T].$$

The pure case We start with the case $N^m = \emptyset$. We first observe that $p_i(\emptyset) = 1$ if $i \in N^a$ and 0 otherwise, and $p_i(N) = 1$ if $i \in N^c$ and 0 otherwise. Therefore we have in any case the sure transitions

$$\lambda_{\emptyset,N^a} = 1, \quad \lambda_{N,N^c} = 1.$$

Using (4), we find, for any $S \neq \emptyset, N$,

$$(i \in N^c) \qquad p_i(S) > 0 \quad \Leftrightarrow \quad \sum_{j=1}^{s} w_j^c > 0 \quad \Leftrightarrow \quad s > l^c \tag{5}$$

$$p_i(S) < 1 \quad \Leftrightarrow \quad \sum_{j=s+1}^{n} w_j^c > 0 \quad \Leftrightarrow \quad n - s > r^c \tag{6}$$

$$(i \in N^a) \qquad p_i(S) > 0 \quad \Leftrightarrow \quad \sum_{j=s+1}^{n} w_j^a > 0 \quad \Leftrightarrow \quad n - s > r^a \tag{7}$$

$$p_i(S) < 1 \quad \Leftrightarrow \quad \sum_{j=1}^{s} w_j^a > 0 \quad \Leftrightarrow \quad s > l^a. \tag{8}$$

Clearly, the above conditions depend only on the number of left and right zeroes of the weight vector. Therefore, as announced, the sole knowledge of the number of left and right zeroes is sufficient for the analysis of transitions, and thus of convergence, as far as we are not interested in computing the precise values of the transition matrix.

By combining these conditions and their negation in various ways, one can see that we can have transitions to $\emptyset, N, N^a, N^c$ and any of their subset or superset. Table 1 summarizes the possible transitions, adding also those from $S = \emptyset$ and $S = N$. Let us introduce $Z = (l^c, r^c, l^a, r^a)$ the vector giving the number of left and right zeroes in the weight vectors of conformist and anti-conformist agents (in this order), and let

	$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$0 \leq s \leq l^a$	N^a	$T \in [N^a, N]$	N
$l^a < s < n - r^a$	$T \in [\emptyset, N^a]$	$T \in 2^N$	$T \in [N^c, N]$
$n - r^a \leq s \leq n$	$\emptyset$	$T \in [\emptyset, N^c]$	N^c

Table 1. Possible transitions from $S \in 2^N$ in the pure case

us write p_i^Z to emphasize the dependency of p_i on these parameters (and similarly for $\lambda_{S,T}$). Equations (5) to (8) show striking symmetries when interchanging conformists and anti-conformists, as well as when interchanging left and right zeroes. Z being given, we introduce the reversal of Z, $Z^\partial := (r^c, l^c, r^a, l^a)$, which amounts to reversing the weight vectors, and the interchange of Z, $Z' = (l^a, r^a, l^c, r^c)$, which amounts to interchanging conformists with anti-conformists. Considering these operations, we observe the following symmetries:

(i) Interchange:

$$p_i^Z(S) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z'}(S) < 1 \text{ for } i \in N^a$$
$$p_i^Z(S) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z'}(S) > 0 \text{ for } i \in N^a$$

(idem with N^a, N^c exchanged)

(ii) Reversal:

$$p_i^Z(S) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z^\partial}(N \setminus S) < 1 \text{ for } i \in N^c$$
$$p_i^Z(S) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z^\partial}(N \setminus S) > 0 \text{ for } i \in N^c$$

(idem with N^a, N^c exchanged)

(iii) Interchange and reversal:

$$p_i^Z(S) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{(Z^\partial)'}(N \setminus S) > 0 \text{ for } i \in N^a$$
$$p_i^Z(S) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{(Z^\partial)'}(N \setminus S) < 1 \text{ for } i \in N^a$$

(idem with N^a, N^c exchanged)

The second case is of particular interest and leads to the following lemma.

Lemma 1 (symmetry principle) *Let $S, T \in 2^N$, and $Z = (l^c, r^c, l^a, r^a)$. The following equivalence holds:*

$$\lambda_{S,T}^Z > 0 \Leftrightarrow \lambda_{N \setminus S, N \setminus T}^{Z^\partial} > 0.$$

Proof. Letting $\lambda_{S,T}^Z > 0$ means that for every $i \in N \setminus T$, $0 \leq p_i^Z(S) < 1$, and for every $i \in T$, $0 < p_i^Z(S) \leq 1$. Using the equivalences in (ii), we find that for every $i \in N \setminus T$, $0 < p_i^{Z^\partial}(N \setminus S) \leq 1$ and for every $i \in T$, $0 \leq p_i^{Z^\partial}(N \setminus S) < 1$. But this means that $\lambda_{N \setminus S, N \setminus T}^{Z^\partial} > 0$.

The mixed case The mixed case can be easily analyzed provided the weight vector of mixed agents is a suitable convex combination of the weight vectors of conformist and anti-conformist agents. Suppose that every conformist agent has weight vector w^c and every anti-conformist agent has weight vector w^a. Consider a mixed agent $i \in N^m$ with $\alpha_i \in]-1, 1[$ and weight vector w^m given by

$$w^m = \frac{\alpha_i + 1}{2} w^c + \frac{1 - \alpha_i}{2} w^a. \tag{9}$$

Then one can check from (4) that

$$p_i(S) = \frac{\alpha_i + 1}{2} p_c(S) + \frac{1 - \alpha_i}{2} p_a(S),$$

where c, a are any conformist and anti-conformist agents, respectively. This can be interpreted as: a mixed player i plays randomly either as a conformist or an anti-conformist, with probability $\frac{1+\alpha_i}{2}$ for conformist. Under this assumption, we can easily derive the conditions for $p_i(S)$ to be 0 or 1, using (5) to (8):

$$(i \in N^m) \qquad p_i(S) = 0 \quad \Leftrightarrow \quad n - r^a \leq s \leq l^c \tag{10}$$
$$p_i(S) = 1 \quad \Leftrightarrow \quad n - r^c \leq s \leq l^a. \tag{11}$$

For all other cases, $0 < p_i(S) < 1$.

An important remark is that these conditions do not depend on the particular α of i: it means that under the assumption (9), the (qualitative) analysis of transitions can be done without knowing the α of each mixed player, and they can also be different for each player.

Now, from the above conditions it is easy to rewrite Table 1 for the mixed case. Finally,

	$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$0 \leq s \leq l^a$	$T \in [N^a, N^a \cup N^m]$	$T \in [N^a, N]$	N
$l^a < s < n - r^a$	$T \in [\emptyset, N^a \cup N^m]$	$T \in 2^N$	$T \in [N^c, N]$
$n - r^a \leq s \leq n$	$\emptyset$	$T \in [\emptyset, N^c \cup N^m]$	$T \in [N^c, N^c \cup N^m]$

Table 2. Possible transitions from $S \in 2^N$ in the mixed case

if $S = \emptyset$, then $\lambda_{S,T} > 0$ for every $T \in [N^a, N^a \cup N^m]$, and if $S = N$, then $\lambda_{S,T} > 0$ for every $T \in [N^c, N^c \cup N^m]$. Table 2 presents possible transitions from $S \in 2^N$ in the mixed case.

3 Convergence of the model

3.1 Convergence in the pure case ($N^m = \emptyset$)

This section is devoted to the study of absorbing classes. Unlike the case of a model with only conformist agents, their study appears to be extremely complex. We start the analysis with the pure case and then continue with the mixed case. We introduce some useful notation. We write $S \to T$ if a transition from S to T is possible, i.e., $\lambda_{S,T} > 0$, and $S \xrightarrow{1} T$ if $\lambda_{S,T} = 1$ (sure transition). We extend the latter notation to collections of sets: letting $\mathcal{S}, \mathcal{T}$ be two nonempty collections of sets in 2^N, we write

$$\mathcal{S} \xrightarrow{1} \mathcal{T} \quad \Leftrightarrow \quad \forall T \in \mathcal{T}, \exists S \in \mathcal{S} \text{ s.t. } \lambda_{S,T} > 0 \text{ and } \forall S \in \mathcal{S}, \forall T \notin \mathcal{T}, \lambda_{S,T} = 0.$$

We observe the following basic facts:

(F0) $\emptyset \xrightarrow{1} N^a$, $N \xrightarrow{1} N^c$ (as already observed).

(F1) If $\mathcal{S} \xrightarrow{1} \mathcal{T}$, $\mathcal{S}' \xrightarrow{1} \mathcal{T}'$ and $\mathcal{S} \subset \mathcal{S}'$, then $\mathcal{T} \subseteq \mathcal{T}'$.

(F2) Applying (F0) and (F1), we find that in a transition $\mathcal{S} \to \mathcal{T}$, $\emptyset \in \mathcal{S}$ implies $N^a \in \mathcal{T}$ and $N \in \mathcal{S}$ implies $N^c \in \mathcal{T}$.

(F3) Consider $\mathcal{S} \xrightarrow{1} \mathcal{T}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{T}_p$, with $p \geq 2$. If $\mathcal{S} \subseteq \mathcal{T}_1$, then $\mathcal{S} \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_p$.

(F4) 2^N is a possible absorbing class. Indeed, take $l^c = r^c = l^a = r^a = 0$. From Table 1 we immediately see that for any $S \neq \emptyset, N$ we have $S \xrightarrow{1} 2^N$. Since the power set of the set of states is the "default" absorbing class when no other can exist, we exclude it from our study and do not consider transitions to 2^N.

(F5) From Table 1, we see that we have to deal only with the sets $\emptyset, N^a, N^c, N$ and the intervals $[\emptyset, N^c], [\emptyset, N^a], [N^a, N], [N^c, N]$ (2^N being excluded by (F4)), i.e., only these can be constituents of an absorbing class. We put

$$\mathbb{B} = \{\{\emptyset\}, \{N^a\}, \{N^c\}, \{N\}, [\emptyset, N^c], [\emptyset, N^a], [N^a, N], [N^c, N]\}$$

the set of collections relevant to our study. Intervals not reduced to a singleton are called *nontrivial intervals*.

(F6) $\mathcal{S} \subseteq 2^N$ is an absorbing class if and only if $\mathcal{S} \xrightarrow{1} \mathcal{S}$ and $\mathcal{S}$ is *connected* (i.e., there is a path (chain of transitions) from S to T for any $S, T \in \mathcal{S}$).

(F6) will be our unique tool to find aperiodic absorbing classes, while periodic classes are of the form $\mathcal{S}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{S}_p$, with $\mathcal{S}_1, \ldots, \mathcal{S}_p \subset 2^N$ and being pairwise disjoint (no common set between $\mathcal{S}_i, \mathcal{S}_j$), and $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_p$ must be connected.

Since $N^m = \emptyset$, we have $n^a = n - n^c$, where $n^a = |N^a|$ and $n^c = |N^c|$. Hence, the model is entirely determined by $l^c, r^c, l^a, r^a, n^c, n$. We recall that these parameters must satisfy the following constraints:

$$\begin{aligned} &0 \leq l^a + r^a < n \\ &0 \leq l^c + r^c < n \\ &0 < n^c < n. \end{aligned}$$

Based on these facts, we can show the main result of this section.

Theorem 1 *Assume that $N^m = \emptyset$, $N^a \neq \emptyset$ and $N^c \neq \emptyset$. There are nineteen possible absorbing classes which are*[2]*:*

(i) Either one of the following singletons:

(1) N^a if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;

(2) N^c if and only if $n^c \geq (n - r^c) \vee (n - r^a)$;

(ii) or one of the following cycles and periodic classes:

[2] We use the standard notation $\vee$ and $\wedge$ to denote max and min, respectively.

(3) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$ *if and only* $n - l^c \leq n^c \leq r^a$*;*

(4) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$ *if and only if* $n - r^c \leq n^c \leq l^a$*;*

(5) $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$*;*

(6) $N^a \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^a$ *and* $r^c < n^c < n - l^c$*;*

(7) $N^c \xrightarrow{1} [N^a, N] \xrightarrow{1} N^c$ *if and only if* $n^c \leq r^c \wedge r^a \wedge l^a$ *and* $l^c < n^c < n - r^c$*;*

(8) $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ *if and only if* $r^c \vee l^c < n^c \leq r^a \wedge l^a \wedge (n - l^c - 1) \wedge (n - r^c - 1)$*;*

(9) $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ *if and only if* $n^c \leq r^c \wedge r^a \wedge l^c$ *and* $n^c \geq n - r^a$*;*

(10) $N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^c$ *and* $n^c \geq n - l^a$*;*

(iii) or one of the following intervals or union of intervals:

(11) $[\emptyset, N^a]$ *if and only if* $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$*;*

(12) $[N^c, N]$ *if and only if* $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$*;*

(13) $[\emptyset, N^a] \cup [\emptyset, N^c]$ *if and only if* $l^c \geq n - r^a$ *and* $n^c \in \big( ]r^c, n - l^c[\cap]l^a, n - r^c[\big) \cup \big(\big( ]l^a, n - r^a[\cup]l^c, n - r^c[\big) \cap]0, r^c] \big)$*;*

(14) $[N^a, N] \cup [N^c, N]$ *if and only if* $l^a \geq n - r^c$ *and* $n^c \in \big( ]l^c, n - r^c[\cap]r^a, n - l^c[\big) \cup \big(\big( ]r^a, n - l^a[\cup]r^c, n - l^c[\big) \cap]0, l^c] \big)$*;*

(15) $[\emptyset, N^a] \cup \{N^c\}$ *if and only if* $l^c + r^c = n - 1$*,* $r^a \geq r^c$ *and* $l^a < n^c \leq r^c \wedge l^c$*;*

(16) $[N^c, N] \cup \{N^a\}$ *if and only if* $l^c + r^c = n - 1$*,* $l^a \geq l^c$ *and* $r^a < n^c \leq r^c \wedge l^c$*.*

(17) $[\emptyset, N^c] \cup \{N^a\}$ *if and only if* $l^a + r^a = n - 1$*,* $l^c \geq l^a$*,* $n^c < n - r^c$ *and* $n^c \in]r^c, n - l^c[\cup]l^c, r^c]$*;*

(18) $[N^a, N] \cup \{N^c\}$ *if and only if* $l^a + r^a = n - 1$*,* $r^c \geq r^a$*,* $n^c < n - l^c$ *and* $n^c \in]l^c, n - r^c[\cup]r^c, l^c]$*.*

(19) 2^N *otherwise.*

Moreover, if $l^c + r^c = n - 1$*, then cases (6), (7), (8), (13) and (14) become impossible while (15) and (16) are specific to this case, and if* $l^a + r^a = n - 1$*, then cases (11) and (12) become impossible, while (17) and (18) are specific to this case.*

The proof of Theorem 1 can be found in the appendix.

According to Theorem 1, nineteen absorbing classes representing three different types are possible in the model. The first one is a singleton which means that in the long run the society is dichotomous: all anti-conformists end up in saying 'yes' (and all conformists in saying 'no', case (1)) or these are all conformists who say 'yes' forever (case (2)).

Note that cases (1) till (19) are not exclusive, which can already be seen when considering cases (1) and (2). Indeed, for a given society, which is represented by the set of parameters n, n^c, l^a, r^a, l^c and r^c, under some conditions both cases (1) and (2) are possible, and therefore two different absorbing classes might occur, N^a and N^c. However, the process will end up in only one of them.

The second type of possible absorbing classes corresponds to cycles and periodic classes which also have a natural interpretation. For instance, case (3) means that in the long run, anti-conformists say 'yes' at time t, then at time $t + 1$ nobody says 'yes', at time $t + 2$ again anti-conformists say 'yes', etc. Under case (6), at some step in the long run all anti-conformists say 'yes', in the following step they all say 'no' but a fraction of conformists might say 'yes', then in the next step again all anti-conformists say 'yes', etc. There exist also longer cycles, like the ones described in cases (9) and (10).

The absorbing class can also be an interval or a union of intervals. Case (11) corresponds to the situation where conformists say 'no' and anti-conformists oscillate between the two opinions. Notice that $[\emptyset, N^a]$ (case (11)) and $[N^c, N]$ (case (12)) can occur, but $[\emptyset, N^a] \cup [N^c, N]$ is never an absorbing class. Under the absorbing class (13), in one time step the process might be in $[\emptyset, N^a]$ (conformists say 'no' and anti-conformists oscillate) and in another step the process might be in $[\emptyset, N^c]$ (anti-conformists say 'no' and conformists oscillate). The cases (15) – (16) ((17) – (18), respectively) correspond to the situations in which conformists (anti-conformists, respectively) put the weight 1 on one rank. Under the absorbing class (15), in one time step the process might be in $[\emptyset, N^a]$ and in another step the process might be in N^c (all conformists say 'yes'). The absorbing class (19) means that at any time step the yes-coalition can be any subset of agents.

The analysis for conformists and anti-conformists is not symmetric. For example, $[\emptyset, N^a]$ is a possible absorbing class but not $[\emptyset, N^c]$. However, while there is no symmetry between "a" and "c" in this framework, there exists symmetry between S and $N \setminus S$ as pointed out in Lemma 1.

3.2 Convergence in the mixed case ($N^m \neq \emptyset$)

Next, we consider a society in which pure conformists, pure anti-conformists, and mixed agents co-exist, i.e., $n = n^a + n^c + n^m$, with $n^a > 0$, $n^c > 0$ and $n^m > 0$. We get the following result.

Theorem 2 *Assume that $N^m \neq \emptyset$, $N^a \neq \emptyset$ and $N^c \neq \emptyset$. Let $\overline{N}^a = N^a \cup N^m$ and $\overline{N}^c = N^c \cup N^m$. There are nineteen possible absorbing classes which are:*

(i) Either one of the following intervals:
- *(1) $[N^a, \overline{N}^a]$ if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;*
- *(2) $[N^c, \overline{N}^c]$ if and only if $n^c \geq (n - r^c) \vee (n - r^a)$;*
- *(3) $[\emptyset, \overline{N}^a]$ if and only if $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$;*
- *(4) $[N^c, N]$ if and only if $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$;*

(ii) or one of the following periodic classes:
- *(5) $[N^a, \overline{N}^a] \xrightarrow{1} \emptyset \xrightarrow{1} [N^a, \overline{N}^a]$ if and only $n - l^c \leq n^c \leq r^a - n^m$;*
- *(6) $[N^c, \overline{N}^c] \xrightarrow{1} N \xrightarrow{1} [N^c, \overline{N}^c]$ if and only if $n - r^c \leq n^c \leq l^a - n^m$;*
- *(7) $[N^a, \overline{N}^a] \xrightarrow{1} [N^c, \overline{N}^c] \xrightarrow{1} [N^a, \overline{N}^a]$ if and only if $n^c + n^m \leq l^c \wedge l^a \wedge r^c \wedge r^a$;*
- *(8) $[N^a, \overline{N}^a] \xrightarrow{1} [\emptyset, \overline{N}^c] \xrightarrow{1} [N^a, \overline{N}^a]$ if and only if $n^c + n^m \leq l^c \wedge l^a \wedge r^a$ and $r^c - n^m < n^c < n - l^c$;*
- *(9) $[N^c, \overline{N}^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [N^c, \overline{N}^c]$ if and only if $n^c + n^m \leq r^c \wedge r^a \wedge l^a$ and $l^c - n^m < n^c < n - r^c$;*
- *(10) $[\emptyset, \overline{N}^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, \overline{N}^c]$ if and only if $l^c \vee r^c < n^c + n^m \leq r^a \wedge l^a \wedge (n - (l^c + 1)) \wedge (n - (r^c + 1))$;*
- *(11) $\emptyset \xrightarrow{1} [N^a, \overline{N}^a] \xrightarrow{1} [N^c, \overline{N}^c] \xrightarrow{1} \emptyset$ if and only if $n^c + n^m \leq r^c \wedge r^a \wedge l^c$ and $n^c \geq n - r^a$;*
- *(12) $[N^a, \overline{N}^a] \xrightarrow{1} N \xrightarrow{1} [N^c, \overline{N}^c] \xrightarrow{1} [N^a, \overline{N}^a]$ if and only if $n^c + n^m \leq l^c \wedge l^a \wedge r^c$ and $n^c \geq n - l^a$;*

(iii) or one of the following unions of intervals:
- *(13) $[\emptyset, \overline{N}^a] \cup [\emptyset, \overline{N}^c]$ if and only if $l^c \geq n - r^a$ and*

$$n^c \in \big(]r^c - n^m, n - l^c[\cap]l^a, n - r^c[\big) \cup \big(\big(]l^a - n^m, n - r^a[\cup]l^c - n^m, n - r^c[\big) \cap]0, r^c - n^m]\big);$$

(14) $[N^a, N] \cup [N^c, N]$ *if and only if* $r^c \geq n - l^a$ *and*

$$n^c \in \big(]l^c - n^m, n - r^c[\cap]r^a, n - l^c[\big)\cup\big(\big(]r^a - n^m, n - l^a[\cup]r^c - n^m, n - l^c[\big)\cap]0, l^c - n^m]\big);$$

(15) $[\emptyset, \overline{N}^a] \cup [N^c, \overline{N}^c]$ *if and only if* $l^c + r^c = n - 1$, $r^c \leq r^a$ *and* $l^a - n^m < n^c \leq l^c \wedge r^c$;

(16) $[N^c, N] \cup [N^a, \overline{N}^a]$ *if and only if* $l^c + r^c = n - 1$, $l^c \leq l^a$ *and* $r^a - n^m < n^c \leq l^c \wedge r^c$;

(17) $[\emptyset, \overline{N}^c] \cup [N^a, \overline{N}^a]$ *if and only if* $l^a + r^a = n - 1$, $l^c \geq l^a$, *and either* $n^a \in]l^c - n^m, n - r^c[$ *and* $n^c < n - r^c$, *or* $n^a \geq n - r^c$ *and* $l^c - n^m < n^c < n - r^c$;

(18) $[N^a, N] \cup [N^c, \overline{N}^c]$ *if and only if* $l^a + r^a = n - 1$, $r^c \geq r^a$, *and either* $n^a \in]r^c - n^m, n - l^c[$ *and* $n^c < n - l^c$, *or* $n^a \geq n - l^c$ *and* $r^c - n^m < n^c < n - l^c$;

(19) 2^N *otherwise.*

The proof is similar to the one of Theorem 1 and is omitted here, but is available upon request.

Theorems 1 and 2 lead to clear conclusions concerning the comparison of absorbing classes in the pure and mixed cases. First of all, when mixed agents exist in a society, a dichotomy into two groups (one saying 'yes' and another one saying 'no', which was the case under $N^m = \emptyset$) is not possible anymore. However, under the same conditions as before (see cases (1) and (2) in Theorem 1), N^a and N^c are now replaced by $[N^a, \overline{N}^a]$ and $[N^c, \overline{N}^c]$. In other words, while anti-conformists (conformists, respectively) continue saying 'yes' and conformists (anti-conformists, respectively) say 'no' forever, the new type of individuals – mixed agents – oscillate between 'yes' and 'no'. In the pure case, two (simple) intervals $[\emptyset, N^a]$ and $[N^c, N]$ (cases (11) and (12) in Theorem 1) are possible absorbing classes. With the presence of mixed agents, we have the corresponding intervals $[\emptyset, \overline{N}^a]$ and $[N^c, N]$ (cases (3) and (4) in Theorem 2) under the same conditions as in the pure case. This means that while conformists do not change their behavior when mixed agents join the society and say either 'no' (absorbing class $[\emptyset, \overline{N}^a]$) or 'yes' (absorbing class $[N^c, N]$) forever, now besides anti-conformists also mixed agents oscillate. Another consequence of the presence of mixed agents on possible absorbing classes is that cycles (i.e., periodic classes with only single states, cases (3) - (5), (9) and (10) in Theorem 1) are not possible anymore. Instead, we have eight periodic classes with mixed agents oscillating (cases (5) till (12) in Theorem 2) that correspond to absorbing classes (3) - (10) of Theorem 1. The conditions for the existence of these periodic classes in the mixed case are the same as the ones for the corresponding 'pure' cases, but adjusted by the presence of n^m mixed agents. Finally, the unions of intervals in the mixed case (absorbing classes (13) till (18) in Theorem 2) correspond to the unions of intervals (13) till (18) in the pure case (Theorem 1), but again with mixed agents oscillating and the conditions taking into account n^m.

4 Examples

4.1 Absorbing classes in some specific situations

First, we consider two particular "symmetric" cases of the weight vectors. The first one concerns a kind of one-side symmetry across conformists and anti-conformists, in the sense that they ignore the same number of yes/no answers, i.e., they are equally influenceable. In this case we have $l^a = l^c$ and $r^a = r^c$ (see Example 2 below). The second case is related to symmetry within the population of conformists or anti-conformists, i.e., when weight vectors are symmetric with respect to the number of left and right zeros. Formally, this means that $l^a = r^a$ and $l^c = r^c$ (see Example 4). As already mentioned before, an interpretation of such symmetric weight vectors is that agents are not biased towards the answer 'yes' or 'no'. This assumption might be relevant for instance when voting for two candidates. However, it might not be relevant when saying 'yes' means 'adopting a new technology', where a bias towards a status quo or a bias towards technology adoption makes sense. The following examples follow directly from the results of Section 3.1.

Example 2 Assume that $N^a \neq \emptyset$, $N^c \neq \emptyset$, $N^m = \emptyset$, $l^a = l^c$ and $r^a = r^c$. If $l^a + r^a \neq n-1$ then the possible absorbing classes are:

- N^a if and only if $n^a \leq l^a$;
- N^c if and only if $n^a \leq r^a$;
- $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^c \leq l^a \wedge r^a$.
- 2^N otherwise.

If $l^a + r^a = n - 1$ then besides the absorbing classes listed above, two more absorbing classes are possible:

- $[\emptyset, N^c] \cup \{N^a\}$ if and only $n^a > r^a$ and $n^c \in]r^a, n - l^a[\cup]l^a, r^a]$;
- $[N^a, N] \cup \{N^c\}$ if and only if $n^a > l^a$ and $n^c \in]l^a, n - r^a[\cup]r^a, l^a]$.

Hence, if the number of anti-conformists does not exceed the number of left (right) zeroes in their weight vector, then in the long run the society might be dichotomous with all anti-conformists (conformists) saying 'yes'. The condition for a dichotomous society means that anti-conformists have negligible effect, since any agent needs more individuals than the total number of anti-conformists to start being influenceable. Note that both N^a and N^c might occur. For instance, depending on the initial conditions, if $n = 5$, $n^a = 1$, $n^c = 4$, $l^a = l^c = 2$, $r^a = r^c \in \{1, 2\}$, then either N^a or N^c will occur. These two absorbing classes exclude, however, the existence of $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ which might be the absorbing class if the number of conformists does not exceed the number of left zeroes nor the number of right zeroes, and means that in one time step all anti-conformists say 'yes' and in the following step all conformists say 'yes', etc. The condition for such a cycle means that conformists can never influence anti-conformists who need more individuals than the total number of conformists to start being influenceable. Moreover, under the condition $l^a + r^a = n - 1$, obviously N^a and N^c exclude the existence of $[\emptyset, N^c] \cup \{N^a\}$ and $[N^a, N] \cup \{N^c\}$.

Example 3 Assume that $N^a \neq \emptyset$, $N^c \neq \emptyset$, $N^m = \emptyset$, $l^a = r^a$ and $l^c = r^c$. If $2l^a \neq n - 1$ and $2l^c \neq n - 1$ then the possible absorbing classes are:

- N^a and N^c if and only if $n^a \leq l^a \wedge l^c$;
- $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^c \leq l^a \wedge l^c$;
- $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ if and only if $l^c < n^c \leq l^a \wedge (n - l^c - 1)$;
- $[\emptyset, N^a]$ and $[N^c, N]$ if and only if $l^a < n^a \leq l^c$ and $n^c \geq l^a + 1$;
- 2^N otherwise.

Hence, if there are not sufficiently many anti-conformists, and as a consequence they have no effect on any agent, in the long run the society might be dichotomous. On the other hand, if there are too few conformists and therefore they have no effect in the opinion formation process, the society might be cycling. If $2l^a = n-1$ or $2l^c = n-1$, then some of the absorbing classes listed above become impossible, but new possibilities appear. In particular, if $2l^a = 2l^c = n-1$ then the possible absorbing classes are:

- N^a and N^c if and only if $n^a \leq l^a$;
- $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^c \leq l^a$;
- $[\emptyset, N^c] \cup \{N^a\}$ and $[N^a, N] \cup \{N^c\}$ if and only if $n^a > l^a$ and $n^c \in]l^a, n - l^a[$;
- 2^N otherwise.

Note that while both N^a and N^c might occur, they exclude the existence of the remaining absorbing classes. Similarly, while both $[\emptyset, N^c] \cup \{N^a\}$ and $[N^a, N] \cup \{N^c\}$ might occur, other cases become then impossible.

Example 4 Assume again that $N^a \neq \emptyset$, $N^c \neq \emptyset$, and $N^m = \emptyset$. A particular case of symmetry is the one with highly influenceable individuals, in the sense that they start being influenceable already with one agent. Technically this means that there are no zeroes in the weight vectors. Such a situation excludes a dichotomy in the society. More precisely, if anti-conformists are highly influenceable, i.e., $l^a = r^a = 0$, then the possible absorbing classes are:

- $[\emptyset, N^a]$ if and only if $n^c \geq n - l^c$;
- $[N^c, N]$ if and only if $n^c \geq n - r^c$;
- 2^N otherwise.

The first two cases mean that there are sufficiently many conformists and too few anti-conformists. In this situation, conformists never change their opinion (they always say 'no'/'yes' in the first/second case), while anti-conformists do not have a fixed opinion and hesitate. On the other hand, if conformists are highly influenceable, i.e., $l^c = r^c = 0$, then the possible absorbing classes are:

- $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ if and only if $n^c \leq r^a \wedge l^a$;
- $[\emptyset, N^c] \cup \{N^a\}$ if and only if $l^a = 0$, $r^a = n-1$;
- $[N^a, N] \cup \{N^c\}$ if and only if $l^a = n-1$, $r^a = 0$;
- 2^N otherwise.

The first case means that conformists have no effect on anti-conformists. The second/third case means that anti-conformists are only sensitive to 'yes'/'no'. Finally, if $l^c = r^c = l^a = r^a = 0$, that is, if all individuals are highly influenceable, then only 2^N remains possible.

In the following example we analyze the impact of the relative number of conformists and anti-conformists on the possible absorbing classes, in particular, when combining it with the two kinds of symmetry considered in the previous examples.

Example 5 In a society with at least as many conformists as anti-conformists, the absorbing classes from (5) till (10), (15) and (16) listed in Theorem 1 become impossible. More precisely, cycles when the opinion of conformists might change and some unions of intervals are excluded.

Assume that there are strictly more conformists than anti-conformists and that the two kinds of symmetry hold, i.e., $l^a = l^c = r^a = r^c$. Then the possible absorbing classes are only N^a and N^c if $n^a \leq l^a$, and 2^N if $n^a > l^a$. The latter condition means that the agents are more easily influenceable, since they do not need many agents to start being influenced.

If n is even and $n^a = n^c$, and the two kinds of symmetry hold, then 2^N becomes the only possible absorbing class.

Consider now a society with more anti-conformists than conformists. Without imposing additional conditions, we do not exclude any absorbing class. In other words, anti-conformists might make the society more 'unstable' in the sense that the opinion of any of these two groups of society might change over time. Under the two kinds of symmetry, the society cannot be dichotomous anymore and only two absorbing classes become possible: $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if $n^c \leq l^a$, and 2^N if $n^c > l^a$. Hence, in a society with a majority of anti-conformists it is possible that at some step in the long run all anti-conformists say 'yes' and in the following step all conformists say 'yes', etc. Such a situation would be impossible in a society with a majority of conformists.

4.2 Simulations of some absorbing classes

Almost all absorbing classes that can appear in the model, except N^a and N^c, exhibit a dynamic behavior, in the sense that neither consensus nor dichotomy is attained. In this section, we show simulations of some absorbing classes in the model with $n = 20$. In what follows, on the horizontal axis of the figures we put time periods, and on the vertical axis – the number of agents saying 'yes'. Moreover, we represent in green the number s^c of conformists saying 'yes', in red the number s^a of anti-conformists saying 'yes', and in purple the number s^m of mixed agents saying 'yes'. The total number of agents saying 'yes' $s = s^a + s^c + s^m$ is drawn in black. These variables are updated in every period.

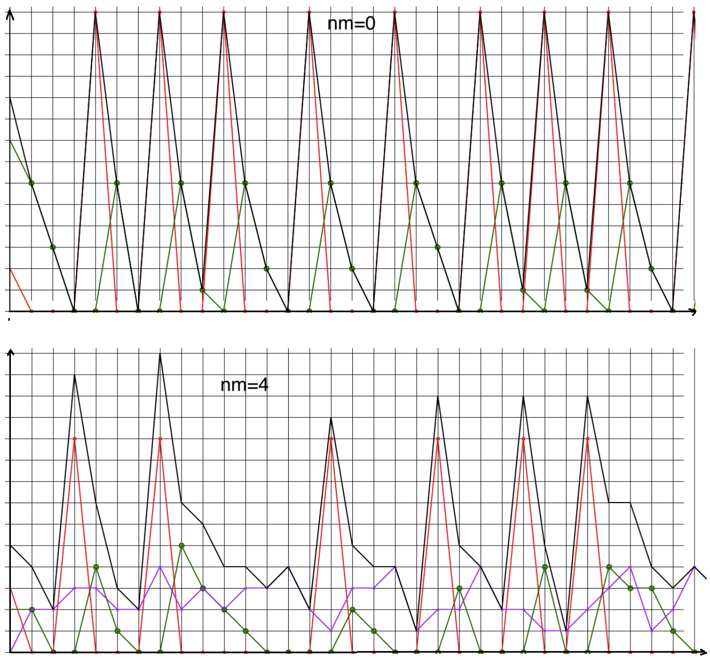
Example 6 Consider the following simulation which generates the absorbing class $[\emptyset, N^c] \cup \{N^a\}$ (case (17) in Theorem 1). It is run for $n^c = 6$, no mixed agents, and the weight vectors defined by:

$$w^a = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$w^c = \left(0, 0, 0, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, 0, 0\right)$$

The conditions stated in (17) of Theorem 1 for the occurrence of $[\emptyset, N^c] \cup \{N^a\}$ are satisfied. Note that in the proof of Theorem 1, when considering all possible transitions and checking connectedness for each candidate, in case (17) one must have either $N^a \xrightarrow{1} [\emptyset, N^c]$ (which happens if and only if $n^a \in]l^c, n - r^c[$) or $N^a \xrightarrow{1} N^c$ (which happens if and only if $n^a \geq n - r^c$). In this example, the first condition is satisfied.

The absorbing class $[\emptyset, \overline{N}^c] \cup [N^a, \overline{N}^a]$ (case (17) in Theorem 2) represented by the second simulation below, can be generated with n^c, w^a and w^c as chosen previously, and $n^m = 4$ with $\alpha_m = 0$. Hence, the number of anti-conformist agents decreases in the second simulation, as four of them are replaced by mixed agents.



As a rule of thumb, it seems that societies with mixed agents reach their absorbing class slower than their counterpart society in the pure case, and the presence of mixed agents can slow down the alternation of peaks. This is indeed what we can observe in this example. Moreover, the framework can be used to model irregular periods.

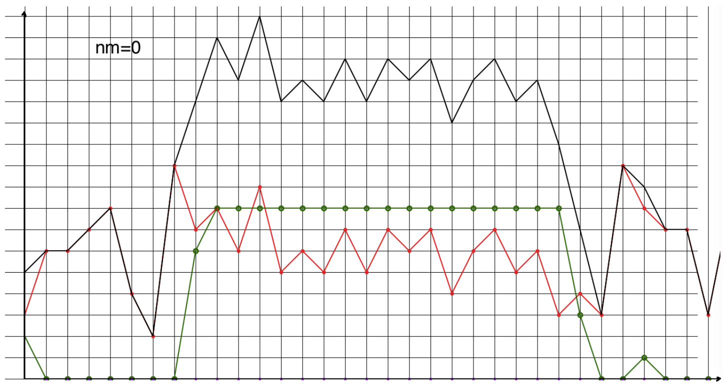
Example 7 The absorbing class 2^N can be generated with $n^c = 8$ and the weights given by

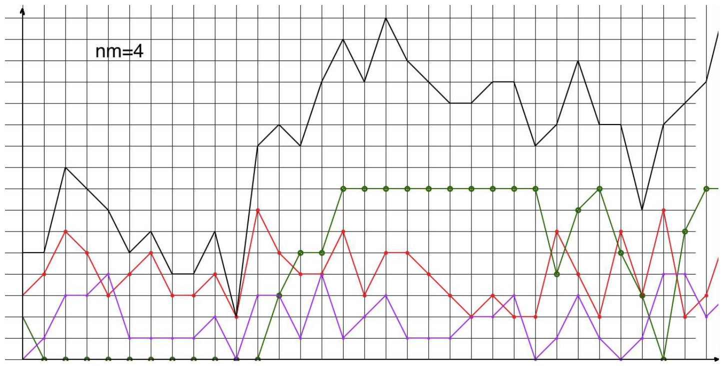
$$w^a = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$$

and

$$w^c = \left(0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\right).$$

In the first simulation $n^m = 0$, while in the second one $n^m = 4$ with $\alpha_m = -0.9$. Hence, again the number of anti-conformist agents decreases in the second simulation, as four of them are replaced by mixed agents.





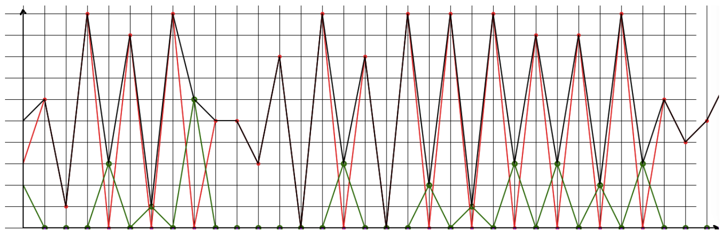
Here the possibility of shocks is related to the ratio $\frac{n^a}{n}$. Indeed, the high s^a (in period 7 in the first simulation for the pure case, and in period 11 in the second simulation for the mixed case) will lead to a small decrease of s^a in the next period, by far compensated by the high increase of s^c, leading to a strong overall increase of $s^a + s^c$ (in black). The expected value of s in the next period increases with s^a. In the mixed case, the effect is slowed down by the fact that anti-conformist agents are replaced by mixed agents with α_m close to -1, i.e., by agents having an anti-conformist behavior adulterated by a slight conformist behavior. Hence, the decrease of $s^a + s^m$ in the mixed case is slower than the decrease of s^a in the pure case. As a consequence, the increase in s is slower, which leads to the trend.

A challenge for econometricians would be to extract $q = (n^a, n^c, w^a, w^c)$ from a time series. In particular, we would have to disentangle one observable series into the sum of two series: the opinion of conformist agents and the opinion of anti-conformist agents. One notable application of such extraction would be to predict shocks, as can be seen in this example. Finally, depending of whether α_m is close to -1 or 1, the frequency of such shocks varies a lot. More generally, given that α_m determines whether mixed agents tend to follow more conformist or anti-conformist agents, from a qualitative point of view this value does matter a lot.

Example 8 The absorbing class $[\emptyset, N^a] \cup [\emptyset, N^c]$ (see case (13) in Theorem 1) can be generated with $n^c = 10$ and the weight vectors given by

$$w^a = \left(0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right)$$

$$w^c = \left(0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, 0, 0, 0, 0\right)$$



Our theorem on the nineteen absorbing classes is a qualitative result. Simulations show that, from a qualitative point of view, a society represented by a given absorbing class may exhibit a behavior similar to another society, sometimes temporarily. For example, the absorbing class $[\emptyset, N^a] \cup [\emptyset, N^c]$ with a high l^c can look like $\emptyset \xrightarrow{1} N^a \xrightarrow{1} \emptyset$.

One typical phenomenon observable in the model is the amplifying oscillations, due to the behavior of anti-conformist agents. Actually, amplifying oscillations are even a hint of the presence of such agents, and the amplitude of the oscillations can be used to evaluate the ratio $\frac{n^a}{n}$. With simulations we can observe that our model can also generate trends which remind of random walks, and alternation of low and high peaks in absorbing classes, like in $[\emptyset, N^a] \cup [\emptyset, N^c]$.

5 Related literature

A wide branch of the related literature concerns opinion conformity which, contrarily to anti-conformism, has been widely studied in various fields and settings, and by using different approaches; for surveys see, e.g., Jackson (2008); Acemoglu and Ozdaglar (2011); Förster et al. (2013). A subset of this literature focuses on various extensions of the DeGroot model (DeGroot (1974)), see e.g., DeMarzo et al. (2003); Jackson (2008); Golub and Jackson (2010); Büchel et al. (2014, 2015). Concerning an anti-conformist behavior, we are aware of only a few works related to this phenomenon. Grabisch and Rusinowska (2010a,b) address the problem of measuring negative influence in a social network but only in one-step (static) settings. Büchel et al. (2015) study a dynamic model of opinion formation, where agents update their opinion by averaging over opinions of their neighbors, but might misrepresent their own opinion by conforming or counter-conforming with the neighbors. Although their model is related to DeGroot (1974), it is very different from our framework of anonymous influence with conformist and anti-conformist agents. Moreover, the authors focus on the relation between an agent's influence in the long run opinion and network centrality, and on wisdom of the society, while we determine all possible absorbing classes and conditions for their occurrence. In particular, they show that an agent's social influence on the group opinion is increasing in network centrality and decreasing in conformity.

Konishi et al. (1997) present a setting completely different from the present paper but related to our definition of anti-conformist agents. They consider a non-cooperative anonymous game in which one of the assumptions on individuals' preferences is partial rivalry, implying that the payoff of every player increases if the number of players who choose the same strategy declines. The authors examine the existence of strong Nash equilibrium in pure strategies for such a game with a finite set of players, and then with continuum of players. There are several other works that study network formation and anti-coordination games, i.e., games where agents prefer to choose an action different from that chosen by their partners. Our approach is different from anti-coordination games, in particular, because we have an essential dissymmetry between agents. Bramoullé (2007) investigates anti-coordination games played on fixed networks. In his model, agents are embedded in a fixed network and play with each of their neighbors a symmetric anti-coordination game, like the chicken game. The author examines how social interactions interplay with the incentives to anti-coordinate, and how the social network affects choices in equilibrium. He shows that the network structure has a much stronger impact on the equilibria than in coordination games. Bramoullé et al. (2004) study anti-coordination games played on endogenous networks, where players choose partners as well as actions in coordination games played with their partners. They characterize (strict) Nash architectures and study the effects of network structure on agents' behavior. The authors show that both network structure and induced behavior depend crucially on the value of cost of forming links. López-Pintado (2009) extends the model of Bramoullé et al. (2004) which is one-sided to a framework in which the cost of link formation is not necessarily distributed as in the one- or two-sided models, but is shared between the two players forming the link. She introduces an exogenous parameter specifying the partition of the cost and characterizes the Nash equilibria depending on the cost of link formation and the cost partition. Kojima and Takahashi (2007) introduce the class of anti-coordination games and investigate the dynamic stability of the equilibrium in a one-population setting. They focus on the best response dynamic, where agents in a large population take myopic best responses, and the perfect foresight dynamic, where agents maximize total discounted payoffs from the present to the future. Cao et al. (2013) consider the fashion game of pure competition and pure cooperation. It is a network game with conformists ('what is popular is fashionable') and rebels ('being different is the essence') that are located on social networks (a spatial cellular automata network and small-world networks). The authors run simulations showing that in most cases players can reach a very high level of cooperation through the best response dynamic. They define different indices (cooperation degree, average satisfaction degree, equilibrium ratio and complete ratio) and apply them to measure players' cooperation levels.

Our setting can be applied to some existing models, like herd behavior and information cascades (Banerjee (1992); Bikhchandani et al. (1992)) which have been used to explain fads, investment patterns, etc.; see Anderson and Holt (2008) for a survey of experiments on cascade behavior. Although Bikhchandani et al. (1992) have already addressed the issue of fashion, the present model takes a different turn, since we assume no sequential choices and some agents are anti-conformists while others are conformists. In the model of herd behavior (Banerjee (1992)) agents play sequentially and wrong cascades can occur. Though it can be rational to follow the crowd, some anti-conformists may want to play a mixed-strategy: either following the crowd or not. This is particularly true under bounded rationality. Agents may not be able to know what is rational, for example because they lack information or do not have enough time or computational capacities. As a consequence, they may play according to rules of the thumb like counting how many people said 'yes' rather than computing bayesian probabilities. Chandrasekhar et al. (2016) show in a lab experiment that people tend to behave according to the DeGroot model rather than to Bayesian updating; see also Celen and Kariv (2004). This is also consistent with Anderson and Holt (1997) who show that counting is the most salient bias to explain departure from Bayesian updating.

6 Concluding remarks

In this paper we analyzed a process of opinion formation in a society with conformists and anti-conformists. We focused on anonymous influence, meaning that an individual can change his initial opinion but the change depends on the number of agents with a certain opinion and not on their identities. If the number of yes-agents increases, then the probability that a given agent says 'yes' increases if the agent is conformist and decreases if he is anti-conformist. Every individual has a coefficient of conformism which is a real number between -1 and 1. We assumed that pure conformists (agents with the coefficient of conformism being 1) as well as pure anti-conformists (the ones with the coefficient of conformism being -1) exist. First, we focused on a society without 'mixed' individuals. Next, we relaxed the assumption that a society consists of only pure conformists and pure anti-conformists, and allowed for the presence of mixed individuals whose weight vectors are a combination of w^a and w^c, and $\alpha_i \in]-1, 1[$. In both cases, we determined all possible absorbing classes and conditions for their occurrence.

We observe natural but essential differences between a society formed entirely by conformists and a society with both conformists and anti-conformists. First of all, while under anonymous influence a society of conformists can reach consensus as shown in Förster et al. (2013), no consensus is possible under anonymous influence in a society of conformists and anti-conformists, independently of the number of anti-conformists. Even if only one anti-conformist agent exists, the society cannot reach consensus anymore. On the other hand, while no periodic absorbing class can exist in a conformist society, the presence of anti-conformists makes cycles possible, and even a number of different cycles and periodic classes might exist in such a society. Typically, when the effect of anti-conformists is negligible, the society tends to a dichotomy, and when conformists are negligible, the opinion is cycling. Another interesting observation is that the effects of conformists and anti-conformists on opinion formation in a society are not symmetric, in the sense that we cannot simply replace "a" by "c" in our study, but what does hold is the symmetry principle as shown in Lemma 1.

Introducing mixed agents in a society has some effects on opinion formation. Mixed agents do not change the number of possible absorbing classes, but their presence "blurs" them, because the opinion of mixed agents always oscillate. The society with mixed individuals cannot be dichotomous anymore. They delay periods and convergence, and cause more irregularity. We notice that there always exists an absorbing class, since by virtue of Theorems 1 and 2, if none of cases (1) till (18) occurs, the absorbing class 2^N (case (19)) is possible. However, some of the conditions stated in Theorems 1 and 2 are not exclusive, and as a consequence, sometimes several different absorbing classes can be possible. For instance, if $N^m = \emptyset$ and $l^a = l^c = r^a = r^c$, then the possible absorbing classes are N^a, N^c and $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$, and which of them will occur in the society will depend on the initial conditions. On the other hand, not all natural situations can be present in a society with both conformists and anti-conformists. For instance, as already mentioned, consensus between all society members is impossible under the coexistence of conformists and anti-conformists. Another example of a final state of opinion of the society which cannot appear in this framework is the situation when all anti-conformists say 'no' forever and conformists oscillate between 'yes' and 'no'.

In our follow-up research on anti-conformism, we intend to relax the anonymity assumption and study a more general framework of anti-conformism. Moreover, we would like to link anti-coordination with anti-conformism.

7 Acknowledgments

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A Proof of Theorem 1

Our strategy is based on (F6): aperiodic absorbing classes are connected collections $\mathcal{S}$ such that $\mathcal{S} \stackrel{1}{\to} \mathcal{S}$. Periodic absorbing classes are of the form $\mathcal{S}_1 \stackrel{1}{\to} \cdots \stackrel{1}{\to} \mathcal{S}_p$ with all $\mathcal{S}_i$ pairwise incomparable, and $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_p$ is connected. Consequently, we study all possible kinds of transition $\mathcal{S} \stackrel{1}{\to} \mathcal{T}$, and check connectedness for each candidate. We distinguish between "simple" transitions of the type $\mathcal{B} \stackrel{1}{\to} \mathcal{B}'$ with $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, and "multiple" transitions $\mathcal{S} \stackrel{1}{\to} \mathcal{T}$, where $\mathcal{S}, \mathcal{T}$ are composed with several elements of $\mathbb{B}$, e.g., $[\emptyset, N^a] \cup [\emptyset, N^c]$.

A.1 Simple transitions

We focus on transitions of the type $\mathcal{B} \stackrel{1}{\to} \mathcal{B}'$, with $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, and look for conditions on the parameters of the model to obtain such transitions.

Observe that if $\mathcal{B}'$ is a nontrivial interval, it cannot be the union of other elements of $\mathcal{B}$. Therefore, $\mathcal{B} \stackrel{1}{\to} \mathcal{B}'$ if and only if for any $S \in \mathcal{B}$, $S \stackrel{1}{\to} \mathcal{B}''$ with $\mathcal{B}'' \in \mathbb{B}$ and $\mathcal{B}'' \subseteq \mathcal{B}'$, and there is at least one $S \in \mathcal{B}$ s.t. $S \stackrel{1}{\to} \mathcal{B}'$. Let us denote by $\mathcal{C}[\mathcal{B}]$ the conditions on $s = |S|$ to have a sure transition from S to $\mathcal{B}$, as given in Table 1. All these conditions are intervals.

Observe that all $\mathcal{B} \in \mathbb{B}$ are either singletons $\{B\}$ or nontrivial intervals $[\underline{B}, \overline{B}]$, and $\mathcal{B} \subset \mathcal{B}'$ if and only if $\mathcal{B} = \{\underline{B'}\}$ or $\{\overline{B'}\}$, with $\mathcal{B}' = [\underline{B'}, \overline{B'}]$. Hence:

$$\mathcal{B} \stackrel{1}{\to} \mathcal{B}' \Leftrightarrow \begin{cases} [\underline{b}, \overline{b}] \subseteq \mathcal{C}[\mathcal{B}'] \cup \mathcal{C}[\{\underline{B'}\}] \cup \mathcal{C}[\{\overline{B'}\}] \\ [\underline{b}, \overline{b}] \cap \mathcal{C}[\mathcal{B}'] \neq \emptyset, \end{cases} \tag{12}$$

with $\underline{b}, \overline{b}$ the cardinalities of $\underline{B}, \overline{B}$. Let us apply (12) to all possibilities. When $\{\mathcal{B}'\}$ is a singleton, the above condition reduces to $[\underline{b}, \overline{b}] \subseteq \mathcal{C}[\mathcal{B}']$, as given in Table 1. Otherwise,

(i) with $\mathcal{B}' = [\emptyset, N^a]$, we obtain $[\underline{b}, \overline{b}] \subseteq [0, l^c]$ and $[\underline{b}, \overline{b}] \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset$, which simplifies to

$$[\underline{b}, \overline{b}] \subseteq [0, l^c] \text{ and } [\underline{b}, \overline{b}] \cap]l^a, n - r^a[\neq \emptyset; \tag{13}$$

(ii) with $\mathcal{B}' = [\emptyset, N^c]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [n - r^a, n] \text{ and } [\underline{b}, \overline{b}] \cap]l^c, n - r^c[\neq \emptyset; \tag{14}$$

(iii) with $\mathcal{B}' = [N^c, N]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [n - r^c, n] \text{ and } [\underline{b}, \overline{b}] \cap]l^a, n - r^a[\neq \emptyset; \tag{15}$$

(iv) with $\mathcal{B}' = [N^a, N]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [0, l^a] \text{ and } [\underline{b}, \overline{b}] \cap]l^c, n - r^c[\neq \emptyset. \tag{16}$$

This yields Table 3. Observe that the table is symmetric w.r.t. its center by the symmetry principle (Lemma 1): just exchange r with l. The transitions being sure, all cases on each line are exclusive.

From Table 3, we can deduce absorbing classes reduced to singletons or intervals: they correspond to transitions $\mathcal{S} \xrightarrow{1} \mathcal{S}$ in the table, provided they are connected. We obtain:

(i) N^a, under the condition $n^c \geq (n - l^c) \vee (n - l^a)$;
(ii) N^c, under the condition $n^c \geq (n - r^c) \vee (n - r^a)$;
(iii) $[\emptyset, N^a]$, under the condition $n - l^c \leq n^c < n - l^a$;
(iv) $[N^c, N]$, under the condition $n - r^c \leq n^c < n - r^a$.

We check connectedness for (iii) ((iv) follows by symmetry). We see from Table 1 that every $S \in [\emptyset, N^a]$ with $s \leq l^a$ has a sure transition to N^a, while the other ones go to every set in the interval. Therefore, the interval is connected if and only if N^a has a possible transition to every set in the interval, i.e., we need $l^a < n^a < n - r^a$ and $n^a \leq l^c$, so the additional condition $n^a < n - r^a$ is needed. In summary:

(i) N^a is an absorbing class if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;
(ii) N^c is an absorbing class if and only if $n^c \geq (n - r^c) \vee (n - r^a)$;
(iii) $[\emptyset, N^a]$ is an absorbing class if and only if $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$;
(iv) $[N^c, N]$ is an absorbing class if and only if $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$.

In order to get (absorbing) cycles and periodic classes, we study chains of sure transitions of length 2: $\mathcal{S}_1 \xrightarrow{1} \mathcal{S}_2 \xrightarrow{1} \mathcal{S}_3$, with $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ being pairwise disjoint, except possibly $\mathcal{S}_1 = \mathcal{S}_3$. An inspection of Table 3 yields all such possible chains of length 2, summarized in Table 4. A second table can be obtained by symmetry.

From Table 4, we obtain the following candidates for absorbing cycles and periodic classes, after eliminating double occurrencies and using symmetry:

(i) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$, under the condition $n - l^c \leq n^c \leq r^a$;
(ii) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$, under the condition $n - r^c \leq n^c \leq l^a$;
(iii) $N^c \xrightarrow{1} N^a \xrightarrow{1} N^c$, under the condition $n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$;
(iv) $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$, under the condition $n^c \leq l^c \wedge l^a \wedge r^a, r^c < n^c < n - l^c$
(v) $[N^a, N] \xrightarrow{1} N^c \xrightarrow{1} [N^a, N]$, under the condition $n^c \leq r^c \wedge r^a \wedge l^a, l^c < n^c < n - r^c$
(vi) $[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$, under the condition $r^c \vee l^c < n^c \leq r^a \wedge l^a$.

It remains to check connectedness of (iv) and (vi) ((v) is obtained by symmetry). For (iv), we must check that N^a has a possible transition to every set in $[\emptyset, N^c]$. By Table 1, we must have $n^a \geq n - r^a$ and $l^c < n^a < n - r^c$, which is true by the conditions in (iv). We address (vi). We claim that under the conditions in (vi) $[N^a, N] \cup [\emptyset, N^c]$ is connected if and only if $N^a \xrightarrow{1} [\emptyset, N^c]$ and $N^c \xrightarrow{1} [N^a, N]$. Take any $S \in [\emptyset, N^c]$. Then S goes either to any set T in $[N^a, N]$ or only to N^a or only to N. In the first case, similarly, T goes either to any set $S' \in [\emptyset, N^c]$ (and we are done) or only to $\emptyset$ or only to N^c. If $T \xrightarrow{1} \emptyset$, then we have $T \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$ and we are done. Otherwise we have $T \xrightarrow{1} N^c \to N^a \xrightarrow{1} [\emptyset, N^c]$. Suppose now that $S \xrightarrow{1} N^a$, then N^a goes to any $S' \in [\emptyset, N^c]$ and we are done. Otherwise, $S \xrightarrow{1} N \xrightarrow{1} N^c \to N^a \xrightarrow{1} [\emptyset, N^c]$ and we are done. This proves sufficiency. Now suppose the condition is not fulfilled. This means that N^a goes to either $\emptyset$ or N^c (or similar condition for N^c). In fact, due to the conditions in (vi) and Table 1, we have that $N^a \xrightarrow{1} \emptyset$, but this yields the cycle $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$.

So in summary, candidates from (i) to (v) are all periodic classes under the specified conditions, and for (vi), the additional condition that $N^a \xrightarrow{1} [\emptyset, N^c]$ and $N^c \xrightarrow{1} [N^a, N]$ yields:

(vi') $[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$ under the condition $r^c \vee l^c < n^c \leq r^a \wedge l^a \wedge (n - l^c - 1) \wedge (n - r^c - 1)$.

For cycles and periodic classes of length 3, by combining the possible chains of length 2 of Table 4 with possible transitions of Table 3, we have only one candidate, all other being eliminated because the collections are not disjoint:

$$N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c.$$

Hence we find, taking into account the symmetry, two additional cycles:

(i) $N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$, under the condition $n^c \leq r^c \wedge r^a \wedge l^c, n^c \geq n - r^a$;
(ii) $N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} N^a$, under the condition $n^c \leq l^c \wedge l^a \wedge r^c, n^c \geq n - l^a$.

We now show that periodic classes of period greater than three cannot exist, which finishes the study of simple transitions.

Lemma 1. *There exists no periodic class of period $k \geq 4$.*

Proof. Let $\mathcal{S}$ be a periodic class. First, observe that if $\emptyset, N$ are not elements of $\mathcal{S}$, it is not possible to choose four distinct elements of $\mathbb{B} \setminus \{\{\emptyset\}, \{N\}\}$ such that these elements are pairwise disjoint. Hence, we suppose that there are transitions $\mathcal{B} \xrightarrow{1} \emptyset$ and/or $\mathcal{B} \xrightarrow{1} N$ in $\mathcal{S}$. From Table 3, we see that $\mathcal{B}$ is necessarily $\{N^a\}$ or $\{N^c\}$.

We claim that the cycle $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ is impossible. Indeed, by Table 4, we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N$ iff $n - l^a \leq n^c \leq r^c$ and $N \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ (its symmetric) iff $n - r^a \leq n^c \leq l^c$. This yields, respectively,

$$\begin{aligned} 2n^c &\geq 2n - l^a - r^a > n \\ 2n^c &\leq r^c + l^c < n, \end{aligned}$$

a contradiction.

Assume that we have a transition to $\emptyset$ (the case for N is obtained by symmetry). We have either $N^a \xrightarrow{1} \emptyset$ (which is discarded because it leads to the cycle $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$) or $N^c \xrightarrow{1} \emptyset$. Then, the only possible absorbing class of the form $N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} \mathcal{B}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{B}_p \xrightarrow{1} N^c$ is the cycle $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c \xrightarrow{1} \emptyset$, for, either $\mathcal{B}_1 = N$, and we obtain the impossible cycle in the claim above, or $\mathcal{B}_1$ contains N^a or N^c, which is impossible since elements in $\mathcal{S}$ should be pairwise disjoint.

A.2 Multiple transitions

We examine the case of transitions of the form $\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$, with $p \geq 2$, $\mathcal{S} \in 2^N$ and formed only from sets in $\mathbb{B}$, $\mathcal{B}_1, \ldots, \mathcal{B}_p \in \mathbb{B}$, and all $\mathcal{B}_1, \ldots, \mathcal{B}_p$ are pairwise incomparable by inclusion[3]. The analysis is done in the same way as for simple transitions: the above transition exists if and only if for every $S \in \mathcal{S}$, $S \xrightarrow{1} \mathcal{B}'$ with $\mathcal{B}' \in \mathbb{B}$ and $\mathcal{B}' \subseteq \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ and there exist distinct $S_1, \ldots, S_p \in \mathcal{S}$ such that $S_j \xrightarrow{1} \mathcal{B}_j$ for $j = 1, \ldots, p$, which readily shows that $\mathcal{S}$ cannot be a singleton. More explicitly, using previous notation and denoting by $\mathrm{supp}(\mathcal{S}) = \{|S| \ : \ S \in \mathcal{S}\}$ the support of $\mathcal{S}$, we get:

$$\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p \Leftrightarrow \begin{cases} \mathrm{supp}(\mathcal{S}) \subseteq \bigcup_{j=1}^{p} \mathcal{C}[\mathcal{B}_j] \cup \bigcup_{j=1}^{p} \mathcal{C}[\{\underline{B}_j\}] \cup \bigcup_{j=1}^{p} \mathcal{C}[\{\overline{B}_j\}] \\ \mathrm{supp}(\mathcal{S}) \cap \mathcal{C}[\mathcal{B}_j] \neq \emptyset, \quad j = 1, \ldots, p. \end{cases} \tag{17}$$

Let us investigate what the possible candidates for $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ are. We begin by restricting to nontrivial intervals and $p = 2$. From Table 1, we find:

(i) $[\emptyset, N^a] \cup [\emptyset, N^c]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup [n - r^a, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases} ; \tag{18}$$

(ii) $[\emptyset, N^a] \cup [N^c, N]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup [n - r^c, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [n - r^c, n] \neq \emptyset \end{cases} ; \tag{19}$$

(iii) $[N^a, N] \cup [\emptyset, N^c]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup [n - r^a, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [0, l^a] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases} ; \tag{20}$$

(iv) $[N^a, N] \cup [N^c, N]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup [n - r^c, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [0, l^a] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [n - r^c, n] \neq \emptyset \end{cases} , \tag{21}$$

the other combinations $[\emptyset, N^a] \cup [N^a, N]$ and $[\emptyset, N^c] \cup [N^c, N]$ being impossible as it can be checked. This readily shows that $p > 2$ with nontrivial intervals is impossible since a forbidden combination would appear in the list.

We consider now that singletons may appear. We begin by noticing that there is no absorbing class of the form $\{S_1, \ldots, S_p\}$ with $S_j \in \{\emptyset, N, N^a, N^c\}$ for all j and $p \geq 2$. Indeed, Table 3 shows that transitions from a set S can only lead to a single T, with no possibility of multiple transition. Hence, such collections would never be connected.

[3] The "$\cup$" is understood at the level of collections of sets, i.e., $\mathcal{B}_1 \cup \mathcal{B}_2 = \{S \in 2^N \mid S \in \mathcal{B}_1 \text{ or } S \in \mathcal{B}_2\}$.

Let us examine the case $\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \{S\}$, where $\mathcal{B}_1$ is a nontrivial interval. With $[\emptyset, N^a] \cup \{N\}$ we obtain:

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup ([0, l^a] \cap [n - r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^c] \cap]l^a, n - r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [n - r^c, n] \neq \emptyset \end{cases},$$

which is impossible. With $[\emptyset, N^a] \cup \{N^c\}$ we obtain

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup ([n-r^a, n] \cap [n-r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^c] \cap]l^a, n - r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [n - r^c, n] \cap [n - r^a, n] \neq \emptyset \end{cases}, \tag{22}$$

which is possible. Similarly, we find that $[\emptyset, N^c] \cup \{N\}$, $[N^a, N] \cup \{\emptyset\}$ and $[N^c, N] \cup \{\emptyset\}$ are impossible, while the following are possible:

(i) $\mathcal{S} \xrightarrow{1} [\emptyset, N^c] \cup \{N^a\}$ iff

$$\mathrm{supp}(\mathcal{S}) \subseteq [n-r^a, n] \cup ([0, l^a] \cap [0, l^c]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [n - r^a, n] \cap]l^c, n - r^c[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [0, l^c] \neq \emptyset \end{cases}, \tag{23}$$

(ii) $\mathcal{S} \xrightarrow{1} [N^a, N] \cup \{N^c\}$ iff

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup ([n-r^a, n] \cap [n-r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap]l^c, n - r^c[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [n - r^a, n] \cap [n - r^c, n] \neq \emptyset \end{cases}, \tag{24}$$

(iii) $\mathcal{S} \xrightarrow{1} [N^c, N] \cup \{N^a\}$ iff

$$\mathrm{supp}(\mathcal{S}) \subseteq [n-r^c, n] \cup ([0, l^a] \cap [0, l^c]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [n - r^c, n] \cap]l^a, n - r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [0, l^c] \neq \emptyset \end{cases}. \tag{25}$$

This shows that transitions of the form $\mathcal{S} \xrightarrow{1} \mathcal{B} \cup \{S_1\} \cup \{S_2\}$ are not possible since a forbidden configuration would appear.

We are now in position to study aperiodic absorbing classes.

(i) With $\mathcal{S} = [\emptyset, N^a] \cup [\emptyset, N^c]$, we find from (18) that

$$[0, n^a \vee n^c] \subseteq [0, l^c] \cup [n - r^a, n] \text{ and } \begin{cases} [0, n^a \vee n^c] \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ [0, n^a \vee n^c] \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases}$$

which is equivalent to

$$n^a \vee n^c > l^c \geq n - r^a. \tag{26}$$

We check connectedness. We begin by a simple observation. We have $\emptyset \xrightarrow{1} N^a$, therefore we must forbid the transitions $N^a \xrightarrow{1} \emptyset$ and $N^a \xrightarrow{1} N^a$. Using Table 1 and (26), we find that $n^a \in]l^a, n - r^a[\cup]l^c, n[$. Suppose that $n^a \in]l^a, n - r^a[$. From Table 1, we obtain that $N^a \xrightarrow{1} [\emptyset, N^a] \xrightarrow{1} [\emptyset, N^a]$, hence no connection to $[\emptyset, N^c]$ is obtained. Therefore we are forced to consider $n^a \in]l^c, n[$, which with (26) leads to

$$n^a > l^c \geq n - r^a. \tag{27}$$

From Table 1 again, this implies $N^a \xrightarrow{1} [\emptyset, N^c]$ when $n^a \in]l^c, n - r^c[$, or $N^a \xrightarrow{1} N^c$ when $n^a \in [n - r^c, n[$. We distinguish the two cases.

1. Suppose $n^a \in]l^c, n - r^c[$, so we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$. In order to connect $\emptyset, N^a$ to any set in $]\emptyset, N^a[$, there must exist $S \in [\emptyset, N^c]$ such that $S \xrightarrow{1} [\emptyset, N^a]$, i.e., $s \in]l^a, n - r^a[\cap [0, l^c] =]l^a, n - r^a[$ by (27). This is possible iff $n^c > l^a$. Let us check whether N^c is connected to any set in the class. From Table 1 and the condition $n^c > l^a$, we see that there is a possible transition to $\emptyset$, which suffices to prove that N^c is connected to any set in the class, except if $n^c \in [n - r^c, n]$ in which case $N^c \xrightarrow{1} N^c$. Therefore, we must ensure the following condition:

$$n^c \in]l^a, n - r^c[\,. \tag{28}$$

We check similarly whether any other set in the class is connected with the rest. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, there will be either a possible transition to $\emptyset$ or to N^a, so that S is connected to any set in the class. If $s > l^c$, S behaves like N^a and we are done. Take now $S \in]\emptyset, N^c[$. If $s \leq l^a$, then $S \xrightarrow{1} N^a$ and we are done. If $s \in]l^a, l^c]$, S has a possible transition to $\emptyset$ and we are done. Finally, if $s \in]l^c, n - r^c[$, S behaves like N^c. In conclusion, (28) summarizes the condition for connectedness in Case 1.

2. Suppose $n^a \in [n - r^c, n[$, so we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$. We must ensure that N^c is connected to any set in the class. In order to avoid $N^c \xrightarrow{1} N^c$ and the transitions $N^c \xrightarrow{1} N^a$ and $N^c \xrightarrow{1} \emptyset$ which would lead to cycles, we are left with the cases $n^c \in]l^a, n - r^a[$ (yielding $N^c \xrightarrow{1} [\emptyset, N^a]$) and $n^c \in]l^c, n - r^c[$ (yielding $N^c \xrightarrow{1} [\emptyset, N^c]$). We examine both cases.

2.1. Suppose $n^c \in]l^a, n - r^a[$, then we have $N^c \xrightarrow{1} [\emptyset, N^a]$. It remains to ensure that there exists $S \in]\emptyset, N^a[$ which is connected with $[\emptyset, N^c]$. We must have $s \in]l^c, n - r^c[$, always possible under Case 2. So we have established that $\emptyset, N^a, N^c$ are connected with the rest of the class. It remains to check if this is true for the other sets in the class. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, a transition to $\emptyset$ of N^a is possible, and so we are done. If $s \in]l^c, n[$, then $S \to N^c$, and we are done. Take now $S \in]\emptyset, N^c[$. Then $s \in]0, n - r^a[$, so that $S \to N^a$ and we are done. As a conclusion, connectedness holds when $n^c \in]l^a, n - r^a[$.

2.2. Suppose $n^c \in]l^c, n - r^c[$, then $N^c \xrightarrow{1} [\emptyset, N^c]$. It remains to connect some set S in $]\emptyset, N^c[$ to $[\emptyset, N^a]$, which is possible iff $s \in]l^a, n - r^a[$. This is possible under Case 2, so N^c is connected to any set in the class. We check for the remaining sets. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, a connection is possible to N^a or $\emptyset$ so we are done. Otherwise, a connection to N^c is possible and we are done. For $S \in]\emptyset, N^c[$, it works exactly the same.

In conclusion of Case 2, connectedness is ensured iff $n^c \in]l^a, n - r^a[\cup]l^c, n - r^c[$.

There does not seem to be a simple way to write the final condition. Here is one possible: connectedness holds iff $l^c \geq n - r^a$ and

$$n^c \in \big( ]r^c, n - l^c[\cap]l^a, n - r^c[\big) \cup \Big(\big( ]l^a, n - r^a[\cup]l^c, n - r^c[\big) \cap]0, r^c] \Big).$$

(ii) Similarly, using (19), $\mathcal{S} = [N^a, N] \cup [N^c, N]$ is an absorbing class if and only if $l^a \geq n - r^c$ and $n^c \in \big( ]l^c, n - r^c[\cap]r^a, n - l^c[\big) \cup \big(( ]r^a, n - l^a[\cup]r^c, n - l^c[\big) \cap]0, l^c] \big)$.

(iii) With $\mathcal{S} = [\emptyset, N^c] \cup [N^a, N]$ we find from (20) the condition $l^c \vee r^c < n^c \leq l^a \wedge r^a$. Let us check connectedness. Starting from $\emptyset$, we have $\emptyset \xrightarrow{1} N^a$, and by Table 1 and the above condition we have $N^a \xrightarrow{1} [\emptyset, N^c]$ if $n^a > l^c$, and $N^a \xrightarrow{1} \emptyset$ otherwise. Clearly, the latter must be forbidden otherwise a cycle occurs. Therefore, we must have $n^a > l^c$. Moreover, we have $N^c \xrightarrow{1} [N^a, N]$ if $n^c < n - r^c$ and $N^c \xrightarrow{1} N$ otherwise. Since $N \xrightarrow{1} N^c$, the latter must be forbidden to avoid a cycle. Therefore, we must have $n^c < n - r^c$. Under these condition, from $\emptyset$ or N^a or N^c, any set can be attained. Now, taking $S \in]\emptyset, N^c[$, we have $S \xrightarrow{1} N^a$ or $S \xrightarrow{1} [N^a, N]$ so that $S \to N^a$ and we are done. Lastly, taking $S \in]N^a, N[$, we have $S \xrightarrow{1} N^c$ or $[\emptyset, N^c]$ and we are done. As a conclusion, the condition is $l^c \vee r^c < n^c \leq l^a \wedge r^a$ and $n^c < (n - l^c) \wedge (n - r^c)$, but then we obtain the periodic absorbing class studied before. Indeed, we see from the proof that we have necessarily $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$.

(iv) With $\mathcal{S} = [\emptyset, N^a] \cup [N^c, N]$, using (19), we find that $l^a \vee r^a < n^a \leq l^c \wedge r^c$. However, under these conditions, $\mathcal{S}$ cannot be connected. Indeed, starting from N^a, we have from Table 1 that for any set $S \in [\emptyset, N^a]$, we have either $S \xrightarrow{1} N^a$, or $S \xrightarrow{1} [\emptyset, N^a]$ or $S \xrightarrow{1} \emptyset$. Therefore, $[\emptyset, N^a]$ is not connected with every set in $\mathcal{S}$.

(v) We show that $[\emptyset, N^a] \cup \{N^c\}$ cannot be connected when $l^c + r^c \neq n - 1$. Indeed, we must have $N^c \xrightarrow{1} [\emptyset, N^a]$, which implies by Table 1 the condition $n^c \leq l^c$. However, by (22) and the condition $l^c + r^c \neq n - 1$, supp($\mathcal{S}$) must be in two disjoint intervals, implying that $[0, n^a] \subseteq [0, l^c]$ and $n^c \in [n - r^c, n]$, a contradiction.

We suppose now $l^c + r^c = n - 1$ and $n - r^a \leq n - r^c$, so that in (22) the first condition reduces to the void condition supp($\mathcal{S}$) $\subseteq [0, n]$, while the second becomes: $l^c > l^a$ and either $n^a \geq n - r^c$ (case 1), or $n^c \geq n - r^c$ and $n^a \in]l^a, n - r^a[$ (case 2). We check connectedness. N^c must be connected to $[\emptyset, N^a]$ or $\emptyset$ or N^a, which implies $n^c \leq l^c$, contradicting case 2. Therefore only case 1 is possible, so that $n^a \geq n - r^c$ and $n^c \leq l^c$. Note that this implies $N^a \xrightarrow{1} N^c$, so that we must ensure $N^c \xrightarrow{1} [\emptyset, N^a]$, implying $l^a < n^c \leq l^c$. Finally, for any $S \in [\emptyset, N^a]$, either $S \xrightarrow{1} N^c$ or $S \xrightarrow{1} [\emptyset, N^a]$ or $S \xrightarrow{1} N^a$, hence connectedness holds. In summary, this class exists iff $l^c + r^c = n - 1$, $n - r^a \leq n - r^c$, $n^a \geq n - r^c$ and $l^a < n^c \leq l^c$.

(vi) With $[\emptyset, N^c] \cup \{N^a\}$, we find from (23) and the assumption $l^a + r^a \neq n - 1$ that supp($\mathcal{S}$) must be in two disjoint intervals, which forces $n - r^a \leq n^a < n - r^c$ and $n^c \leq l^a \wedge l^c$. We know already that $[\emptyset, N^c] \xrightarrow{1} N^a$ is a periodic class. Let us show that this is the only possibility. Indeed, otherwise there should exist $S \in [\emptyset, N^c]$ such that $S \xrightarrow{1} [\emptyset, N^c]$. This would imply that $l^c < s < n - r^c$, which is impossible by the condition $n^c \leq l^c$.

Let us consider now that $l^a + r^a = n - 1$ and $l^c \geq l^a$, so that in (23) the first condition simply reduces to the void condition supp($\mathcal{S}$) $\subseteq [0, n]$, while the second becomes: either $n^a \in]l^c, n - r^c[$ or $n^c > l^c$. Let us check connectedness. We must have $N^a \xrightarrow{1} [\emptyset, N^c]$ or $N^a \xrightarrow{1} N^c$. The first case happens iff $n^a \in]l^c, n - r^c[$. Then observe that without further condition on n^c, any set in $[\emptyset, N^c]$ is connected to either N^a, $\emptyset$, $[\emptyset, N^c]$ or N^c. It suffices then to forbid the transition $N^c \xrightarrow{1} N^c$, i.e., $n^c < n - r^c$. The second case happens iff $n^a \geq n - r^c$, which forces $n^c > l^c$. To ensure that N^c is connected to $[\emptyset, N^c]$, we must have $l^c < n^c < n - r^c$. In summary, this class exists iff $l^a + r^a = n - 1$, $l^c \geq l^a$, and either $n^a \in]l^c, n - r^c[$ and $n^c < n - r^c$, or $n^a \geq n - r^c$ and $l^c < n^c < n - r^c$.

(vii) The case of $[N^a, N] \cup \{N^c\}$ is similar to its symmetric $[\emptyset, N^c] \cup \{N^a\}$. The class exists iff $l^a + r^a = n - 1$, $n - r^c \leq n - r^a$, and either $n^c \in]l^c, n - r^c[$ and $n^a > l^c$, or $n^c \leq l^c$ and $l^c < n^a < n - r^c$.

(viii) The case of $[N^c, N] \cup \{N^a\}$ is similar to its symmetric $[\emptyset, N^a] \cup \{N^c\}$. The class exists iff $l^c + r^c = n - 1$, $l^a \geq l^c$, $n^c \leq l^c$ and $n - r^c \leq n^a < n - r^a$.

It remains to study the existence of periodic classes. Since the collections must be pairwise disjoint, the only possibility is the periodic class $[\emptyset, N^a] \cup [\emptyset, N^c] \xrightarrow{1} N \xrightarrow{1} [\emptyset, N^a] \cup [\emptyset, N^c]$. But we know that the second transition is impossible since a singleton cannot lead to a multiple transition. Hence, there are no such periodic absorbing classes.

$\mathcal{S}\backslash\mathcal{T}$	$\emptyset$	N^a	$[\emptyset, N^c]$	$[\emptyset, N^a]$	$[N^c, N]$	$[N^a, N]$	N^c	N
$\emptyset$	$\times$	always	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
N^a	$n - l^c \leq n^c \leq r^a$	$n^c \geq n - l^c$ $n^c \geq n - l^a$	$n^c \leq r^a$ $r^c < n^c < n - l^c$	$n^c \geq n - l^c$ $r^a < n^c < n - l^a$	$n^c \leq r^c$ $r^a < n^c < n - l^a$	$n^c \geq n - l^a$ $r^c < n^c < n - l^c$	$n^c \leq r^c \wedge r^a$	$n - l^a \leq n^c \leq r^c$
$[\emptyset, N^c]$	$\times$	$n^c \leq l^c \wedge l^a$	$\times$	$l^a < n^c \leq l^c$	$\times$	$l^c < n^c \leq l^a$	$\times$	$\times$
$[\emptyset, N^a]$	$\times$	$n^c \geq n - l^c$ $n^c \geq n - l^a$	$\times$	$n - l^c \leq n^c$ $n^c < n - l^a$	$\times$	$n - l^a \leq n^c$ $n^c < n - l^c$	$\times$	$\times$
$[N^c, N]$	$\times$	$\times$	$n - r^a \leq n^c$ $n^c < n - r^c$	$\times$	$n - r^c \leq n^c$ $n^c < n - r^a$	$\times$	$n^c \geq n - r^c$ $n^c \geq n - r^a$	$\times$
$[N^a, N]$	$\times$	$\times$	$r^c < n^c \leq r^a$	$\times$	$r^a < n^c \leq r^c$	$\times$	$n^c \leq r^c \wedge r^a$	$\times$
N^c	$n - r^a \leq n^c \leq l^c$	$n^c \leq l^c \wedge l^a$	$n^c \geq n - r^a$ $l^c < n^c < n - r^c$	$n^c \leq l^c$ $l^a < n^c < n - r^a$	$n^c \geq n - r^c$ $l^a < n^c < n - r^a$	$n^c \leq l^a$ $l^c < n^c < n - r^c$	$n^c \geq n - r^c$ $n^c \geq n - r^a$	$n - r^c \leq n^c \leq l^a$
N	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	always	$\times$

Table 3. Conditions for sure transitions $\mathcal{S}$ to $\mathcal{T}$

$N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$	$n - l^c \leq n^c \leq r^a$
$N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a$	$n - r^a \leq n^c \leq l^c$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} [N^c, N]$	$n^c \leq r^c$ $r^a < n^c < n - l^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$	$n^c \leq r^c \wedge r^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} N$	$n - l^a \leq n^c \leq r^c$
$N^c \xrightarrow{1} N^a \xrightarrow{1} \emptyset$	$n - l^c \leq n^c \leq l^c \wedge l^a \wedge r^a$
$[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$	$n^c \leq l^c \wedge l^a \wedge r^a$ $r^c < n^c < n - l^c$
$N^c \xrightarrow{1} N^a \xrightarrow{1} N^c$	$n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$
N^c or $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} N$	$n - l^a \leq n^c \leq l^a \wedge l^c \wedge r^c$
$N^a \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} N^a$	$n^c \leq l^a \wedge l^c \wedge r^a$ $r^c < n^c < n - l^c$
$[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$	$l^c \vee r^c < n^c \leq r^a \wedge l^a$

Table 4. Conditions for chains of length 2 potentially yielding periodic classes