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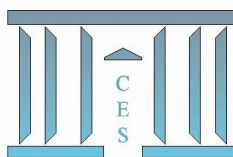
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FISSAR Model**

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ON THE PARAMETERS ESTIMATION OF THE SEASONAL FISSAR MODEL

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Abstract

In this paper we discuss the methods of estimating the parameters of the Seasonal FISSAR (Fractionally Integrated Separable Spatial Autoregressive with seasonality) model. First we implement the regression method based on the log-periodogram and the classical Whittle method for estimating memory parameters. To estimate the model's parameters simultaneously - innovation parameters and memory parameters- the maximum likelihood method, and the Whittle method based on the MCMC simulation are considered. We are investigated the consistency and the asymptotic normality of the estimators by simulation.

Keywords: Seasonal FISSAR, long memory, regression method, Whittle method, MLE method.

JEL Classification: C21; C51; C52.

1. Introduction

Spatial statistics are used for a variety of different types of fields; including meteorology (Lim et al., 2002 [25]), oceanography (Illig, 2006 [21]), agronomy (Whittle, 1986 [35]; Lambert et al., 2003 [24]), geology (Cressie, 1993 [12]), epidemiology (Marshall, 1991 [28]), image processing (Jain, 1981 [22]) or econometrics (Anselin, 1988 [2]). This large domain of applications is due to the richness of the modelling which associates a representation in time and in space. Cliff and Ord (1973) [11] introduce specific modellings including the STAR (Space-Time AutoRegressive) model and GSTAR (Generalized Space-Time AutoRegressive) model. Others models are

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the Simultaneous AutoRegression model, SAR, (Whittle, 1954 [34]), the Conditional AutoRegression model, CAR (Bartlett, 1971 [3] ; Besag, 1974 [7]), the moving average model (Haining, 1978 [19]) or the unilateral models (Basu and Keinsel, 1993 [6]).

Statistics for spatially referenced data become important as they are nowadays collected in large quantities in various areas of science. This requires specific probabilistic models and corresponding statistical methods. Boissy et al. (2005) [8] had extended the long memory concept from times series to the spatial context and introduced the class of fractional spatial autoregressive model. Shitan (2008) [33] used this model, called the Fractionally Integrated Separable Spatial Autoregressive (FISSAR) model to approximate the dynamics of spatial data when the autocorrelation function decays slowly. Cisse et al. (2016) [9] incorporated Seasonal patterns into the FISSAR, thus introduced the Seasonal FISSAR model and discussed the statistical properties of the model. In this paper, we investigate estimation procedures for the model's parameters.

Inference problems in spatial location or two-dimensional process have been studied by several authors. Hosking(1981) [20], Yajima (1985) [37], Yajima (1991) [36] and Fox and Taquq (1986) [16] discussed asymptotic estimation based on one-dimensional fractionally differenced autoregressive process. Sethuraman and Basawa (1995) [32] studied the asymptotic properties of the maximum likelihood estimators in two-dimensional lattice. The authors considered models which permit dependence between time points as well as within the group of individuals at each time point. Anderson (1978) [1], Basawa et al. (1984) [5] and Basawa and Billard (1989) [4] studied the case when the dependence is extended along the time axis.

Periodic and cyclical behaviours are present in many observations, and the Seasonal FISSAR model will have a lot of applications, including the modelling of temperatures, rainfalls when the data are collected during different seasons for different locations. In this paper, we consider the Seasonal FISSAR (Fractionally Integrated Separable Spatial Autoregressive model), introduced in Cisse et al. (2016)[9] and we assumed the same pattern for all locations for the seasonal parameter. We discuss how to estimate the parameters of this new model and examine the properties of the estimators. The main purpose of this work is to assess and compare the finite sample performance of several estimation methods for Seasonal FISSAR model. Several estimation methods have been proposed in the literature. We generally use semiparametric methods (GPH (1983) [17] , Robinson (1994, 1995a, 1995b), Hurvich and Beltrao (1993), Hurvich and Deo (1999)) and maximum likelihood methods (Exact MLE, Whittle (1951)). The first class estimate only the long memory parameters, which requires estimating the parameters in two steps which is expensive in computation time, while the last class estimates all the parameters simultaneously .

We first briefly describe the Seasonal FISSAR Model defined in Cisse et al. (2016) [9].

Let $\{X_{ij}\}_{i,j \in \mathbb{Z}_+}$ be a sequence of spatial observations on two dimensional regular lattices. The

Seasonal FISSAR process is defined as

$$(1 - \phi_{10}B_1 - \phi_{01}B_2 + \phi_{10}\phi_{01}B_1B_2) \left(1 - \psi_{10}B_1^s - \psi_{01}B_2^s + \psi_{10}\psi_{01}B_1^sB_2^s\right) \times (1 - B_1)^{d_1} (1 - B_1^s)^{D_1} (1 - B_2)^{d_2} (1 - B_2^s)^{D_2} X_{ij} = \varepsilon_{ij} \quad (1.1)$$

This equation (1.1) is equivalent to the following equations:

$$(1 - \phi_{10}B_1) (1 - \phi_{01}B_2) \left(1 - \psi_{10}B_1^s\right) \left(1 - \psi_{01}B_2^s\right) X_{ij} = W_{ij} \quad (1.2)$$

$$(1 - B_1)^{d_1} (1 - B_1^s)^{D_1} (1 - B_2)^{d_2} (1 - B_2^s)^{D_2} W_{ij} = \varepsilon_{ij} \quad (1.3)$$

where s is the seasonal period, B_1 and B_2 are the usual backward shift operators acting in the i^{th} and j^{th} direction, respectively. In addition, the parameters d_1 and d_2 are called memory parameters, D_1 and D_2 are fractional parameters and $\{\varepsilon_{ij}\}_{i,j \in \mathbb{Z}_+}$ is a two-dimensional white noise process, mean zero and variance σ_ε^2 . We denote $\{W_{ij}\}_{i,j \in \mathbb{Z}_+}$ the two-dimensional seasonal fractionally integrated white noise process.

The Seasonal FISSAR model can be rewritten as

$$\Phi(B_1, B_2) \Psi(B_1^s, B_2^s) X_{ij} = W_{ij} \quad (1.4)$$

or

$$\Phi(B_1, B_2) \Psi(B_1^s, B_2^s) (1 - B_1)^{d_1} (1 - B_1^s)^{D_1} (1 - B_2)^{d_2} (1 - B_2^s)^{D_2} X_{ij} = \varepsilon_{ij} \quad (1.5)$$

where

$$\Phi(B_1, B_2) = (1 - \phi_{10}B_1) (1 - \phi_{01}B_2), \quad (1.6)$$

$$\Psi(B_1^s, B_2^s) = (1 - \psi_{10}B_1^s) (1 - \psi_{01}B_2^s), \quad (1.7)$$

with W_{ij} given by (1.3).

The paper is organized as follows. The next section provides several techniques for estimating parameters of the Seasonal FISSAR model. We describe a semi parametric method based on the periodogram and the exact MLE method. The methods are illustrated in Section 3 by simulation experiments. In Section 4 we present others methods for estimating model's parameters- the MLE based on Whittle, the Whittle method and the MCMC Whittle method-. However, this approach results are will be developed in a companion paper. Conclusions are given in Section 5.

2. Estimation Procedures

In this section, we discuss the methods of estimation for the memory parameters and examine the properties of the estimators through a simulation study. The fundamental concept of parameter estimation is to determine optimal values for parameters from numerical model that predicts

dependent variable outputs of a function, process or phenomenon based on observations of independent variable inputs. A well-known estimation procedure is the Whittle Method. It is known that under fairly general assumptions, it leads to consistent and asymptotically normal estimates. More in the Gaussian case, it's asymptotically efficient. Here we present this method in spatial context and make a comparison with a regression method based on the log-periodogram methodology.

2.1 Log-periodogram Regression Method

In this paragraph, we investigate the regression method to estimate long memory parameters of the Seasonal FISSAR process defined in 1.1. Probably the most commonly applied semi parametric estimator in time series is the log-periodogram regression estimator introduced by Geweke and Porter-Hudak (1983) [17]. Following the work of Ghodsi and Shitan (2009) [18] we establish a regression method for estimating long memory parameters of the Seasonal FISSAR model. We briefly describe this method.

Let

$$H_{ij} = (1 - B_1)^{d_1} (1 - B_1^s)^{D_1} (1 - B_2)^{d_2} (1 - B_2^s)^{D_2} X_{ij}, \quad (2.8)$$

and $f_X(\lambda_1, \lambda_2)$ the spectral density function of the process defined in (1.1). Then $f_X(\lambda_1, \lambda_2)$ can be written as:

$$f_X(\lambda_1, \lambda_2) = |1 - e^{-i\lambda_1}|^{-2d_1} |1 - e^{-i\lambda_2}|^{-2d_2} |1 - e^{-is\lambda_1}|^{-2D_1} |1 - e^{-is\lambda_2}|^{-2D_2} f_H(\lambda_1, \lambda_2) \quad (2.9)$$

where

$$f_H(\lambda_1, \lambda_2) = \frac{\sigma_\varepsilon^2}{4\pi^2 |1 - \phi_{10} e^{-i\lambda_1}|^2 |1 - \phi_{01} e^{-i\lambda_2}|^2 |1 - \psi_{10} e^{-is\lambda_1}|^2 |1 - \psi_{01} e^{-is\lambda_2}|^2}. \quad (2.10)$$

Taking the logarithm in (2.9), we obtain:

$$\begin{aligned} \log f_X(\lambda_1, \lambda_2) &= -d_1 \log |1 - e^{-i\lambda_1}|^2 - d_2 \log |1 - e^{-i\lambda_2}|^2 - D_1 \log |1 - e^{-is\lambda_1}|^2 \\ &\quad - D_2 \log |1 - e^{-is\lambda_2}|^2 + \log f_H(\lambda_1, \lambda_2) \\ &= \log f_H(0, 0) - d_1 \log |1 - e^{-i\lambda_1}|^2 - d_2 \log |1 - e^{-i\lambda_2}|^2 - D_1 \log |1 - e^{-is\lambda_1}|^2 \\ &\quad - D_2 \log |1 - e^{-is\lambda_2}|^2 + \log \frac{f_H(\lambda_1, \lambda_2)}{f_H(0, 0)} \end{aligned} \quad (2.11)$$

Replacing λ_1 by $\omega_{1,j}$ and λ_2 by $\omega_{2,j}$ in (2.11), we obtain

$$\begin{aligned} \log f_X(\omega_{1,j}, \omega_{2,j}) &= \log f_H(0, 0) - d_1 \log |1 - e^{-i\omega_{1,j}}|^2 - d_2 \log |1 - e^{-i\omega_{2,j}}|^2 - D_1 \log |1 - e^{-is\omega_{1,j}}|^2 \\ &\quad - D_2 \log |1 - e^{-is\omega_{2,j}}|^2 + \log \frac{f_H(\omega_{1,j}, \omega_{2,j})}{f_H(0, 0)} \end{aligned} \quad (2.12)$$

Denote $N_1 \times N_2$ the size of the regular rectangle when the process $\{X_{ij}\}$ is defined, and define

$$I(\lambda_1, \lambda_2) = \frac{1}{4\pi^2 N_1 N_2} \left| \sum_{k=1}^{N_1} \sum_{l=1}^{N_2} X_{k,l} e^{-i(k\lambda_1 + l\lambda_2)} \right|^2 \quad (2.13)$$

to be the periodogram of the process. By adding $\log I(\omega_{1,j}, \omega_{1,j})$ in (2.12), we obtain:

$$\begin{aligned} \log I(\omega_{1,j}, \omega_{1,j}) &= \log f_H(0, 0) - d_1 \log |1 - e^{-i\omega_{1,j}}|^2 - d_2 \log |1 - e^{-i\omega_{2,j}}|^2 - D_1 \log |1 - e^{-is\omega_{1,j}}|^2 \\ &\quad - D_2 |1 - e^{-is\omega_{2,j}}|^2 + \log \frac{f_H(\omega_{1,j}, \omega_{2,j})}{f_H(0, 0)} + \log \frac{I(\omega_{1,j}, \omega_{1,j})}{f_X(\omega_{1,j}, \omega_{2,j})} \end{aligned} \quad (2.14)$$

If $\omega_{1,j}$ and $\omega_{2,j}$ are close to zero, then the term $\log \frac{f_H(\omega_{1,j}, \omega_{2,j})}{f_H(0, 0)}$ would be negligible when compared with the other terms of (2.14), and equation (2.14) reduces to:

$$\begin{aligned} \log I(\omega_{1,j}, \omega_{1,j}) &\approx \log f_H(0, 0) - d_1 \log |1 - e^{-i\omega_{1,j}}|^2 - d_2 \log |1 - e^{-i\omega_{2,j}}|^2 - D_1 \log |1 - e^{-is\omega_{1,j}}|^2 \\ &\quad - D_2 |1 - e^{-is\omega_{2,j}}|^2 + \log \frac{I(\omega_{1,j}, \omega_{1,j})}{f_X(\omega_{1,j}, \omega_{2,j})}. \end{aligned} \quad (2.15)$$

Equation (2.15) is a pseudo-regression model. If the pseudo-errors $\log \frac{I(\omega_{1,j}, \omega_{2,j})}{f_X(\omega_{1,j}, \omega_{2,j})}$ behave like iid random variables, then the regression estimator is a reasonable estimation procedure (GPH in temporal case). We can write (2.15) as a multiple regression equation

$$X_j = \beta_0 + \beta_1 Z_{1,j} + \beta_2 Z_{2,j} + \beta_3 Z_{3,j} + \beta_4 Z_{4,j} + \epsilon_j \quad (2.16)$$

where $X_j = \log I(\omega_{1,j}, \omega_{1,j})$, $\beta_0 = \log f_H(0, 0)$, $\beta_1 = -d_1$, $\beta_2 = -d_2$, $\beta_3 = -D_1$, $\beta_4 = -D_2$, $Z_{1,j} = \log |1 - e^{-i\omega_{1,j}}|^2$, $Z_{2,j} = \log |1 - e^{-i\omega_{2,j}}|^2$, $Z_{3,j} = \log |1 - e^{-is\omega_{1,j}}|^2$, $Z_{4,j} = \log |1 - e^{-is\omega_{2,j}}|^2$, and $\epsilon_j = \log \frac{I(\omega_{1,j}, \omega_{1,j})}{f_X(\omega_{1,j}, \omega_{2,j})}$. We denote $\omega_{1,j}$ and $\omega_{2,j}$ the harmonic ordinates:

$$\begin{aligned} \omega_{j,1}^i &= \frac{2\pi i}{s} + \frac{2\pi j}{N_1}, & i = 0, 1, \dots, \left\lfloor \frac{s}{2} \right\rfloor - 1, & j = 1, 2, \dots, m_1, \\ \omega_{j,2}^i &= \frac{2\pi i}{s} + \frac{2\pi j}{N_2}, & i = 0, 1, \dots, \left\lfloor \frac{s}{2} \right\rfloor - 1, & j = 1, 2, \dots, m_2, \end{aligned}$$

where m_1 and m_2 satisfying respectively the condition $(\frac{1}{m_1} + \frac{m_1}{N_1}) \rightarrow 0$ as $N_1 \rightarrow \infty$, $(\frac{1}{m_2} + \frac{m_2}{N_2}) \rightarrow 0$ as $N_2 \rightarrow \infty$.

Now d_1 , d_2 , D_1 and D_2 may be estimated as $-\widehat{\beta}_1$, $-\widehat{\beta}_2$, $-\widehat{\beta}_3$, and $-\widehat{\beta}_4$ respectively using last squares estimation as in the usual multiple regression. This is a natural extension of the non-seasonal method developped in Ghodsi and Shitan (2009) [18]. We note that different estimation methods for d_1 , d_2 , D_1 and D_2 may be obtained by appropriate choices of the harmonic frequencies and bandwidth m_1 and m_2 . Next we consider bandwidth $m_k = \left\lfloor \frac{N_k}{s} \right\rfloor - 1$, $k = 1, 2$.

The least squares optimality criterion minimizes the sum of squares of residuals between actual observed outputs and output values of the numerical model that are predicted from input observations. The least squares (LS) estimators of the coefficients in the linear prediction function are obtained by minimizing the following sum of squared deviations:

$$\begin{aligned} S(\beta) = S(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) &= \sum_{j=1}^T \left(X_j - (\beta_0 + \beta_1 Z_{1,j} + \beta_2 Z_{2,j} + \beta_3 Z_{3,j} + \beta_4 Z_{4,j}) \right)^2 \\ &= \sum_{j=1}^T \left(X_j - \sum_{k=0}^4 \beta_k Z_{k,j} \right)^2, \text{ where } Z_{0,j} = 1 \text{ for all } j. \end{aligned}$$

Similar to the procedure in finding the minimum of a function in calculus, the least squares estimate β can be found by solving the equation based on the first derivative of $S(\beta)$. Partial derivatives have the following form:

$$\frac{\partial S}{\partial \beta_i} = -2 \left[\sum_{j=1}^T X_j Z_{i,j} - \sum_{k=0}^4 \beta_k \left(\sum_{j=1}^T Z_{k,j} Z_{i,j} \right) \right] \quad (2.17)$$

The minimum of $S(\beta)$ is obtained by setting the derivatives of $S(\beta)$ in (2.17) equal to zero. We obtain the following system of equations:

$$\frac{\partial S}{\partial \beta_i} = 0, \text{ for } i = 0, 1, 2, 3, 4.$$

or equivalently,

$$\sum_{k=0}^4 \left(\sum_{j=1}^T Z_{k,j} Z_{i,j} \right) \beta_k = \sum_{j=1}^T X_j Z_{i,j}, \text{ for } i = 0, 1, 2, 3, 4. \quad (2.18)$$

Let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_T \end{bmatrix}; \quad Z = \begin{bmatrix} 1 & Z_{1,1} & Z_{2,1} & Z_{3,1} & Z_{4,1} \\ 1 & Z_{1,2} & Z_{2,2} & Z_{3,2} & Z_{4,2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & Z_{1,T} & Z_{2,T} & Z_{3,T} & Z_{4,T} \end{bmatrix}; \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

The system (2.18) becomes:

$$(Z'Z)\beta = Z'X, \quad (2.19)$$

where Z' is the transpose of the matrix Z . Finally the vector of the LS estimators is:

$$\hat{\beta} = (Z'Z)^{-1} Z'X \quad (2.20)$$

The asymptotic properties of $\hat{\beta}$ will be obtaining by simulations. The parameters estimations were carried out by writing R programs. The main steps of the algorithm are:

Step 1: Choice of variables. Choose the variable to be explained (X) and the explanatory variables (Z_0, Z_1, Z_2, Z_3, Z_4 where Z_0 is often the constant that always takes the value 1).

Step 2: Collect data. Collect T observations of X and of the related values of Z_0, Z_1, Z_2, Z_3, Z_4 and store the data of X in an $T \times 1$ vector and the data on the explanatory variables in the $T \times 5$ matrix Z .

Step 3: Compute the estimates. Compute the least squares estimates by the LS formula (2.20) using a regression package.

Our simulation results are presented for both the two-dimensional seasonal fractionally integrated white noise process and the Seasonal FISSAR model. For the two-dimensional seasonal fractionally integrated white noise model we have done the study for various values of d_1, d_2, D_1, D_2 and we have fixed $s_1 = s_2 = 4$.

Remark 2.1. *The GPH estimator requires two important assumptions:*

H1: *For small frequencies $\omega_{1,j}$ and $\omega_{2,j}$ the term $\log \frac{f_H(\omega_{1,j}, \omega_{2,j})}{f_H(0,0)}$ is negligible.*

H2: *The term $\log \frac{I(\omega_{1,j}, \omega_{2,j})}{f_X(\omega_{1,j}, \omega_{2,j})}$ (pseudo-errors) is a sequence of iid random variables.*

Remark 2.2. *The regression method only estimates the long memory and short memory parameters. For inference purposes, however, estimation of the innovation parameter is required.*

2.2 Exact Maximum Likelihood method

The maximum likelihood methods are considered among the most important estimation methods to estimate the long memory parameter. There are the most effective methods to estimate all parameters simultaneously. Several researchers have used ML approach for estimating the statistical parameters in spatial models. On spatial linear models Mardia (1990) [26] studied ML estimators for Direct Representation (DR), Conditional AutoRegression (CAR) models and Simultaneous AutoRegression (SAR) models in Gaussian case. In addition, the author gives extension of the method in multivariate case, block data and missing value in lattice data. Earlier, Mardia and Marshall (1984) [27] described the maximum likelihood method for fitting the linear model when residuals are correlated and when the covariance among the residuals is determined by a parametric model containing unknown parameters. Pardo-Iguzquiza (1998) [30] used the maximum likelihood method for inferring the parameters of spatial covariances. The advantages of the ML estimation are discussed in Pardo-Iguzquiza (1998) [30] for the multivariate distribution of the data and spatial analysis.

In this section, the maximum likelihood (ML) method for inferring the parameters of the Seasonal FISSAR model introduced in Cisse et al. (2016) is examined. The causal moving average representation for the processes $\{X_{ij}\}$ in (1.1) and $\{W_{ij}\}$ in (1.2) are given in Cisse et al. (2016) [9], Proposition (2.1):

$$X_{ij} = \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \phi_{10}^k \phi_{01}^l \psi_{10}^m \psi_{01}^n W_{i-k-m s_1, j-l-n s_2}, \quad (2.21)$$

where

$$W_{ij} = \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \phi_k(d_1) \phi_l(d_2) \phi_m(D_1) \phi_n(D_2) \varepsilon_{i-k-ms_1, j-l-ns_2}, \quad (2.22)$$

with

$$\phi_k(d_1) = \begin{cases} \frac{\Gamma(k+d_1)}{\Gamma(k+1)\Gamma(d_1)} & \text{if } k \in \mathbb{Z}_+ \\ 0 & \text{if } k \notin \mathbb{Z}_+ \end{cases}; \quad \phi_l(d_2) = \begin{cases} \frac{\Gamma(l+d_2)}{\Gamma(l+1)\Gamma(d_2)} & \text{if } l \in \mathbb{Z}_+ \\ 0 & \text{if } l \notin \mathbb{Z}_+ \end{cases} \quad (2.23)$$

and

$$\phi_m(D_1) = \begin{cases} \frac{\Gamma(m+D_1)}{\Gamma(m+1)\Gamma(D_1)} & \text{if } m \in \mathbb{Z}_+ \\ 0 & \text{if } m \notin \mathbb{Z}_+ \end{cases}; \quad \phi_n(D_2) = \begin{cases} \frac{\Gamma(n+D_2)}{\Gamma(n+1)\Gamma(D_2)} & \text{if } n \in \mathbb{Z}_+ \\ 0 & \text{if } n \notin \mathbb{Z}_+ \end{cases} \quad (2.24)$$

$\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ and $\{\varepsilon_{ij}\}_{i,j \in \mathbb{Z}_+}$ is a two-dimensional white noise process.

The ML method is presented for both the two-dimensional seasonal fractionally integrated white noise process and the Seasonal FISSAR model. Asymptotic properties of maximum likelihood estimators for fractionally differenced AR model on a two-dimensional lattice were considered by Sethuraman and Basawa (1995) [32], who showed that the usual asymptotic properties of consistency and asymptotic normality are satisfied under several conditions. Same procedures in Sethuraman and Basawa (1995) [32] have been developed for obtaining maximum likelihood estimates of the parameters of the Seasonal FISSAR model and their asymptotic properties.

We consider a $\{W_{ij}\}$ process defined as (1.2). We suppose that the stationary conditions are satisfied. Under this assumption and the normality assumption of the bi-dimensional white noise, we first define the autocovariances of the process to compute the likelihood function simply amounts to expressing the autocovariances of the process. The covariance of $\{W_{ij}\}$ is given by:

$$\text{Cov}(W_{ij}, W_{i_1 j_1}) = \text{Cov} \left(\sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \phi_k(d_1) \phi_l(d_2) \phi_m(D_1) \phi_n(D_2) \varepsilon_{i-k-ms_1, j-l-ns_2}, \right. \\ \left. \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \phi_k(d_1) \phi_l(d_2) \phi_m(D_1) \phi_n(D_2) \varepsilon_{i_1-k-ms_1, j_1-l-ns_2} \right)$$

Let $\{Y_{ij}\}$ define by

$$(1 - B_1)^{d_1} (1 - B_2)^{d_2} Y_{ij} = \varepsilon'_{ij} \quad (2.25)$$

The process $\{W_{ij}\}$ can be rewritten as

$$W_{ij} = \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \varphi_k(d_1) \varphi_l(d_2) Y_{i-s_1k, j-s_2l} \quad (2.26)$$

Therefore

$$\begin{aligned} \text{Cov}(W_{ij}, W_{i_1j_1}) &= \text{Cov}\left(\sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \varphi_k(d_1) \varphi_l(d_2) Y_{i-s_1k, j-s_2l}, \sum_{k_1=0}^{+\infty} \sum_{l_1=0}^{+\infty} \varphi_{k_1}(d_1) \varphi_{l_1}(d_2) Y_{i_1-s_1k_1, j_1-s_2l_1}\right) \\ &= \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \varphi_k(d_1) \varphi_l(d_2) \sum_{k_1=0}^{+\infty} \sum_{l_1=0}^{+\infty} \varphi_{k_1}(d_1) \varphi_{l_1}(d_2) \text{Cov}(Y_{i-s_1k, j-s_2l}, Y_{i_1-s_1k_1, j_1-s_2l_1}) \\ &= \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{k_1=0}^{+\infty} \sum_{l_1=0}^{+\infty} \varphi_k(d_1) \varphi_l(d_2) \varphi_{k_1}(d_1) \varphi_{l_1}(d_2) \text{Cov}(Y_{i-s_1k, j-s_2l}, Y_{i_1-s_1k_1, j_1-s_2l_1}) \end{aligned} \quad (2.27)$$

Assuming that the process is Gaussian, the likelihood function of the sample $W = (W_{ij}, i = 1, \dots, n; j = 1, \dots, n)$ is equal to:

$$L_W = (2\pi\sigma_\varepsilon^2)^{-n^2/2} |\Sigma_W|^{-1/2} \exp\left[-\frac{1}{2\sigma_\varepsilon^2} W^T \Sigma_W^{-1} W\right],$$

where Σ_W is the variance-covariance matrix of X determined by (2.27).

The likelihood function of the Seasonal FISSAR $X = (X_{ij}, i = 1, \dots, n; j = 1, \dots, n)$ model defined in a $n \times n$ regular lattice will be defined by the same way.

$$L_X = (2\pi\sigma_\varepsilon^2)^{-n^2/2} |\Sigma_X|^{-1/2} \exp\left[-\frac{1}{2\sigma_\varepsilon^2} X^T \Sigma_X^{-1} X\right],$$

For the computational algorithm of the maximum likelihood method, the main steps of the algorithm are:

Step 1: To compute the likelihood function $L(\phi, D)$ and fixe a real parameters ϵ_0 .

Step 2: Given ϕ_0 and a first estimation of D , we compute $\max_D L(\phi, D)$ to obtain $\hat{D}$.

Step 3: Given $\hat{D}$, we estimate ϕ by $\max_\phi L(\phi, \hat{D})$ and we obtain $\hat{\phi}$.

Step 4: Given $\hat{\phi}$, we estimate again a new value of D by $\max_D L(\hat{\phi}, D)$ and we obtain $\hat{\hat{D}}$. If $\hat{\hat{D}} - \hat{D} \leq \epsilon_0$ then $D = \hat{D}$, else

Step 5: Step 2 with $D = \hat{D}$.

3. Simulation Results

To show the performance of the proposed methods several experiments are performed. The Monte Carlo study is designed to check the variability of the Regression method in comparison with the MLE method. Simulation experiments have also aim to implement some theoretical results concerning the properties of these estimators. In other words, we try to study the validity of these properties in small and large samples. We perform an experiment of 500 replications for a Seasona FISSAR process with spatial Gaussian noise process for 3 different sample sizes ($N \times N = 50 \times 50$, $N \times N = 100 \times 50$ and $N \times N = 150 \times 150$). The simulation results give the average values, the corresponding sample standard deviation (sd) and the root mean square error (RMSE) of the estimation procedures based on MCMC replications. All computations are carried out with fixed seasonal period $s = 4$ of the following models for various sample sizes.

Parameters	innovation parameters				memory parameters			
	ϕ_{10}	ϕ_{01}	ψ_{10}	ψ_{01}	d_1	d_2	D_1	D_2
Model	0.10	0.25	0.10	0.25	0.10	0.10	0.10	0.10
50×50								
mean	—	—	—	—	0.1310	0.1354	0.1321	0.1505
bias	—	—	—	—	0.0310	0.0354	0.0321	0.0505
sd	—	—	—	—	0.1127	0.1044	0.1148	0.1043
RMSE	—	—	—	—	0.1169	0.1102	0.1192	0.1159
100×100								
mean	—	—	—	—	0.1107	0.1110	0.1108	0.1247
bias	—	—	—	—	0.0107	0.0110	0.0108	0.0247
sd	—	—	—	—	0.0634	0.0658	0.0634	0.0659
RMSE	—	—	—	—	0.0643	0.0657	0.0653	0.0704
150×150								
mean	—	—	—	—	0.1108	0.1096	0.1236	0.1193
bias	—	—	—	—	0.0108	0.0096	0.0236	0.0193
sd	—	—	—	—	0.0499	0.0528	0.0527	0.0503
RMSE	—	—	—	—	0.0511	0.0536	0.0577	0.0539

Table 1: Simulation results of estimating the Seasonal FISSAR model by regression method

The results of the application of the regression method are summarized in Table 1. The estimation method provides a reasonable approximation, for each parameter, the sample mean and the standard deviation of the estimated parameter parameter are reasonably close to the theoretical values. As expected, standard errors and estimated values of the parameter d_1, d_2, D_1 and D_2 become better as sample size increases. Figures 1 - 4 shows the influence of the sample size on the application of this method (when N increases the results are better).

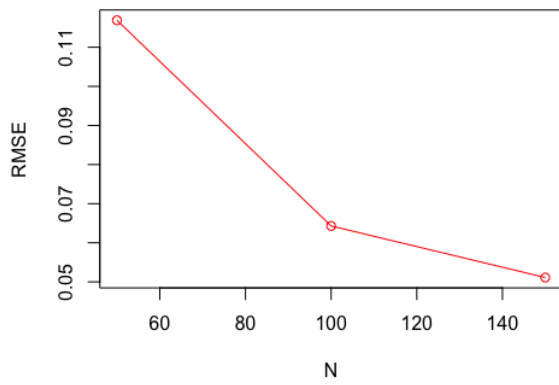


Figure 1: *RMSE plot of $\hat{d}_1$ by regression method*

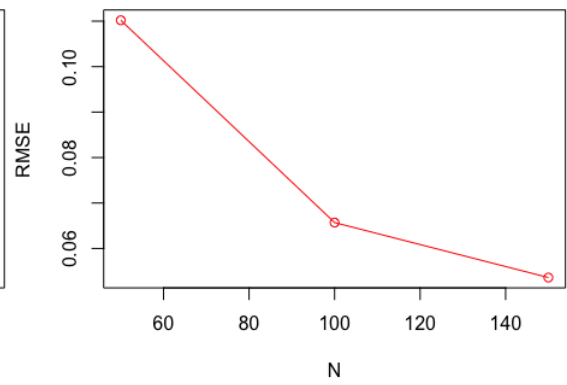


Figure 2: *RMSE plot of $\hat{d}_2$ by regression method*

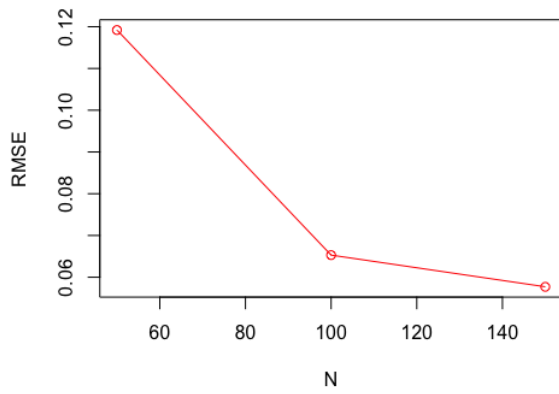


Figure 3: *RMSE plot of $\hat{D}_1$ by regression method*

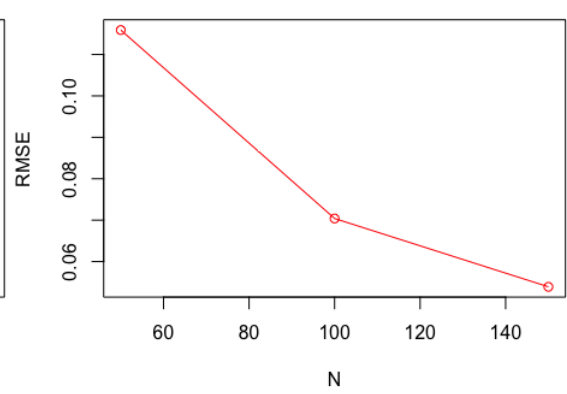


Figure 4: *RMSE plot of $\hat{D}_2$ by regression method*

Parameters	innovation parameters				memory parameters			
	ϕ_{10}	ϕ_{01}	ψ_{10}	ψ_{01}	d_1	d_2	D_1	D_2
Model ($s = 4$) 50×50	0.10	0.25	0.10	0.25	0.10	0.10	0.10	0.10
mean	0.0874	0.2431	0.1101	0.2411	0.1225	0.1440	0.1503	0.1439
bias	-0.0126	-0.0069	0.0101	-0.0089	0.0225	0.0440	0.0503	0.0439
sd	0.0830	0.0825	0.0632	0.0833	0.1042	0.1132	0.1050	0.1134
RMSE	0.0836	0.0824	0.0640	0.0833	0.1066	0.1214	0.1164	0.1216
100×100								
mean	0.1100	0.2706	0.1026	0.2871	0.1171	0.1339	0.1245	0.1110
bias	0.0100	0.0206	0.0026	0.0371	0.0171	0.0339	0.0245	0.0110
sd	0.0633	0.0649	0.0636	0.0650	0.0634	0.0660	0.0633	0.0661
RMSE	0.0641	0.0681	0.0633	0.0748	0.0656	0.0742	0.0679	0.0670
150×150								
mean	0.1114	0.2615	0.1108	0.2652	0.1133	0.1108	0.1179	0.1146
bias	0.0114	0.0114	0.0108	0.0152	0.0133	0.0108	0.0179	0.0146
sd	0.0514	0.0529	0.0500	0.0530	0.0527	0.0501	0.0527	0.0502
RMSE	0.0526	0.0542	0.0512	0.0551	0.0544	0.0513	0.0557	0.0522

Table 2: Simulation results of estimating the Seasonal FISSAR model by MLE method

Table 2 present the results concerned the maximum likelihood method for all parameters. Results reveal that estimates parameters are satisfactory in the sense that the Bias and the RMSE are very small. Figures 5 - 12 shows the plots of the RMSE of the differents estimated parameters and we show the influence of the sample size on the application of this method. Comparing the two methods, their results are fairly similar for large size. However, as the size increases the bias and RSME of d_1, d_2, D_1 and D_2 decreases substantially (see Figures 5 - 12), and these two methods become very competitive.

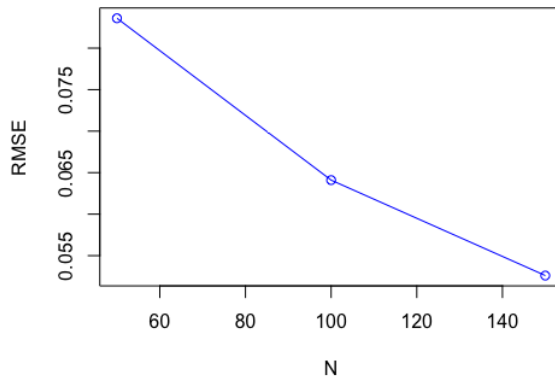


Figure 5: RMSE plot of $\hat{\phi}_{01}$ by MLE method

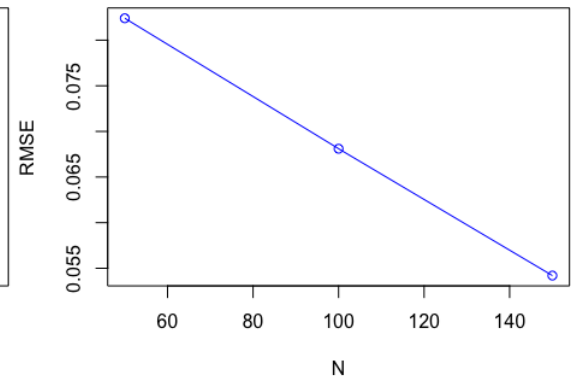


Figure 6: RMSE plot of $\hat{\phi}_{10}$ by MLE method

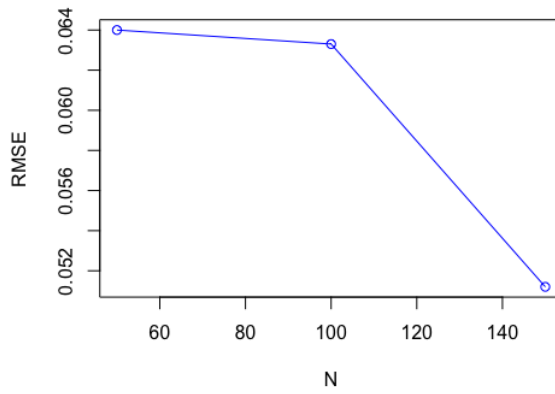


Figure 7: RMS plot of $\hat{\psi}_{01}$ by MLE method

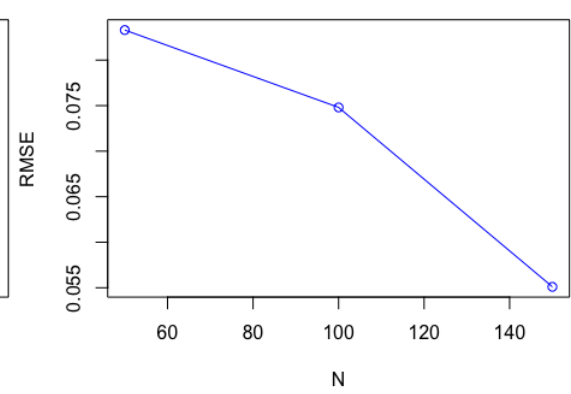


Figure 8: RMSE plot of $\hat{\psi}_{10}$ by MLE method

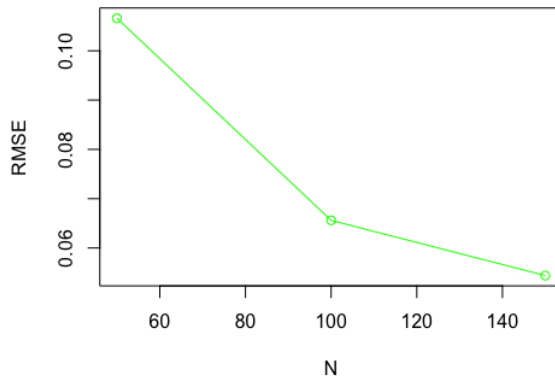


Figure 9: RMSE plot of $\hat{d}_1$ by MLE method

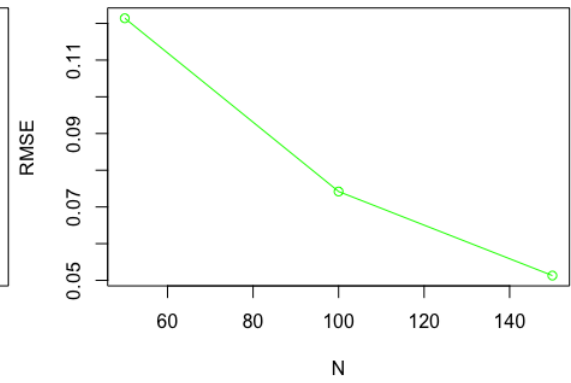


Figure 10: RMSE plot of $\hat{d}_2$ by MLE method

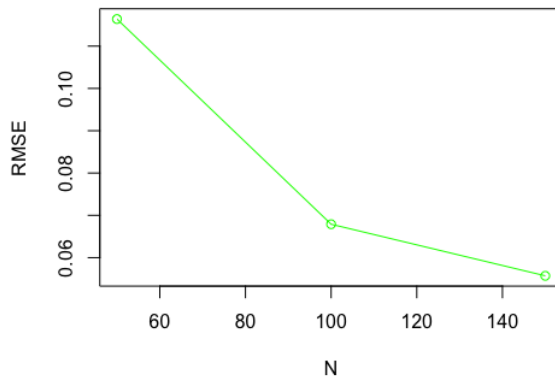


Figure 11: RMSE plot of $\hat{D}_1$ by MLE method

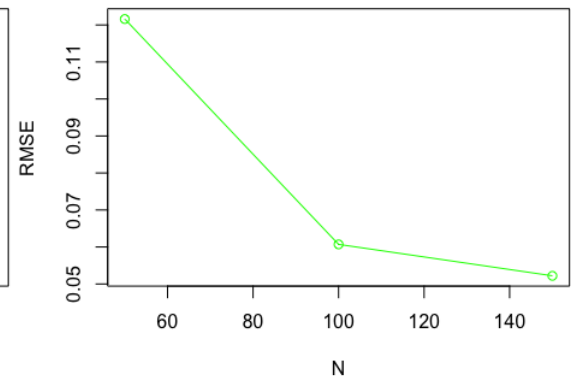


Figure 12: RMSE plot of $\hat{D}_2$ by MLE method

Which one to choose? MLE is of fundamental importance in the theory of inference and is a basis of many inferential techniques in statistics. The first motivation behind ML estimation is to estimate all parameters simultaneously, unlike our proposal regression method and Whittle method. On the other hand referring to the computational times the regression method based on Log-periodogram would be preferred. unfortunately the AR parameters are typically not accurately estimated. We can also observe that the results from the MLE procedure seems to perform better than those obtained by regression approach . However, the computational complexity of maximum likelihood (if there are $N \times N$ sampling points, then Σ is an $N \times N$ matrix, and the process can be slow if N is large) may be outweighed by its convenience as a very widely applicable method of estimation. Generally, it is suggested to try all proposal estimation approaches and to compare.

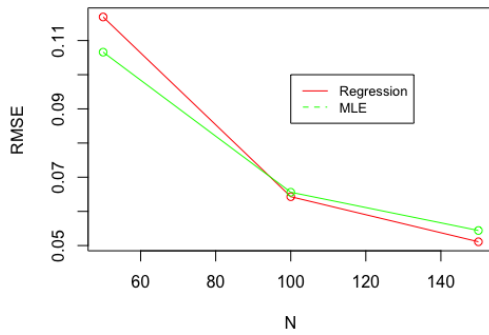


Figure 13: RMSE comparison of $\hat{d}_1$

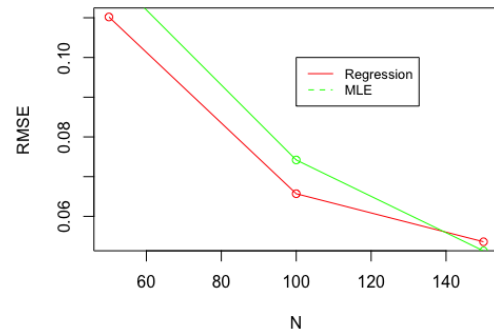


Figure 14: RMSE comparison of $\hat{d}_2$

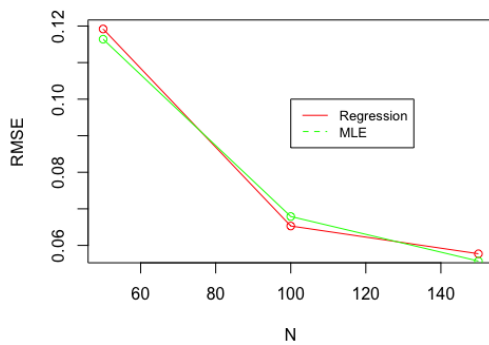


Figure 15: RMSE comparison of $\hat{D}_1$

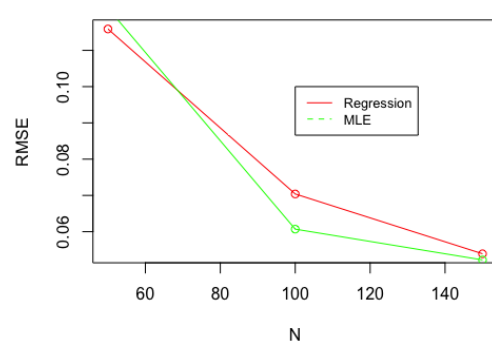


Figure 16: RMSE comparison of $\hat{D}_2$

4. Others methods

Beside these methods, there are several other estimation methods such as, for example: the MLE method based on Whittle function, the MCMC Whittle method, etc. These are all simple and easy to implement methods but in this section we will just give the methodology of some methods and calculation steps without simulations.

4.1 MLE method based on Whittle function

Here we consider the maximum likelihood method based on approximated Whittle function that usually gives a good estimator.

This estimator is a parametric procedure due to Whittle method with extension in the spatial context for the regular lattice. The estimator is based on the periodogram and it involves the function

$$L(\beta) = \int \int_{-\pi}^{\pi} \frac{I(\lambda_1, \lambda_2)}{f_X(\lambda_1, \lambda_2, \beta)} d\lambda_1 d\lambda_2, \quad (4.28)$$

where $f_X(\lambda_1, \lambda_2, \beta)$ is the known spectral density function at frequencies λ_1 and λ_2 and β denotes the vector of unknown parameters. The estimator is the value of β which minimizes the function $L(\cdot)$. For the Seasonal FISSAR process the vector β contains the parameter d_1, d_2, D_1, D_2 and also all the unknown autoregressive parameters. For computational purposes, the estimator is obtained by using the discret form of $L(\cdot)$:

$$L(\beta) = \frac{1}{4N_1 N_2} \sum_{j_1} \sum_{j_2} \left\{ \log f_X(\omega_{1,j_1}, \omega_{1,j_2}) + \frac{I(\omega_{1,j_1}, \omega_{1,j_2})}{f_X(\omega_{1,j_1}, \omega_{1,j_2}, \beta)} \right\}, \quad (4.29)$$

It is known that the maximum likelihood estimator of the long memory parameters in temporal case is strongly consistent, asymptotically normally distributed and asymptotically efficient in the Fisher sense (Dahlhaus, 1989 [13] and Yajima, 1985 [37]).

4.2 Whittle Method

The Whittle method was used by many authors in order to estimate the long memory parameters and it is based on evaluation of the Whittle likelihood function in temporal context. See for instance Kluppelberg and Mikosch (1993) [23], Kluppelberg and Mikosch (1994) [10], Mikosch et al. (1995) [29], Embrechts et al. (1997) [15]. Assuming the process $\{X_{ij}\}_{i,j \in \mathbb{Z}_+}$ is a stationary Seasonal FISSAR model defined in (1.5), the Whittle likelihood function may be expressed as

$$\sigma^2(\beta) = \frac{1}{N_1 N_2} \sum_{j_1} \sum_{j_2} \frac{I(\omega_{1,j_1}, \omega_{1,j_2})}{g(\omega_{1,j_1}, \omega_{1,j_2})}, \quad (4.30)$$

where $\beta = (d_1, d_1, D_1, D_2, \phi_{10}, \phi_{01}, \psi_{10}, \psi_{01})$ is the parameter vector of interest, $I(\omega_{1,j_1}, \omega_{1,j_2})$ is the periodogram define as

$$I(\lambda_1, \lambda_2) = \frac{1}{4\pi^2 N_1 N_2} \left| \sum_{k=1}^{N_1} \sum_{l=1}^{N_2} X_{k,l} e^{i(k\lambda_1 + l\lambda_2)} \right|^2 \quad (4.31)$$

and $g(\omega_{1,j_1}, \omega_{1,j_2}) = \frac{4\pi^2}{\sigma_\varepsilon^2} f(\omega_{1,j_1}, \omega_{1,j_2})$ and $f(\cdot, \cdot)$ is the spectral density defined in (??).

Hence, an estimation procedure of β is to minimize (4.30) over β . The Whittle estimator is given by:

$$\hat{\beta} = \text{Argmin} \{ \sigma^2(\beta) \} \quad (4.32)$$

4.3 MCMC Whittle Method

The MCMC Whittle method was studied in Diongue et al. (2008) [14] in temporal context to estimate the long-memory parameters of stable ARFISMA model. It is based on evaluation of the Whittle likelihood function, using Markov Chains Monte Carlo (MCMC, in short) method. We briefly review it here.

Assuming the process $\{X_{ij}\}_{i,j \in \mathbb{Z}_+}$ is a stationary Seasonal FISSAR model defined in (1.5), the Whittle likelihood function may expressed as

$$\sigma^2(\beta) = \int \int_{-\pi}^{\pi} \frac{\tilde{I}(\lambda_1, \lambda_2)}{h(\lambda_1, \lambda_2, \beta)} d\lambda_1 d\lambda_2, \quad -\pi \leq \lambda_1, \lambda_2 \leq \pi, \quad (4.33)$$

where $\beta = (d_1, d_1, D_1, D_2, \phi_{10}, \phi_{01}, \psi_{10}, \psi_{01})$ is the parameter vector of interest. The functions $h(\lambda_1, \lambda_2, \beta)$ and $\tilde{I}(\lambda_1, \lambda_2)$ represent respectively the power transfer function and the normalized periodogram and they are defined by:

$$h(\lambda_1, \lambda_2, \beta) = \left| \Phi(e^{-i\lambda_1}, e^{-i\lambda_2}, \beta) \Psi(e^{-is\lambda_1}, e^{-is\lambda_2}, \beta) \right|^{-2} f_W(\lambda_1, \lambda_2), \quad -\pi \leq \lambda \leq \pi \quad (4.34)$$

and

$$\tilde{I}(\lambda) = \left(\sum_{k=1}^{N_1} \sum_{l=1}^{N_2} X_{k,l}^2 \right)^{-1} \left| \sum_{k=1}^{N_1} \sum_{l=1}^{N_2} X_{k,l} e^{i(k\lambda_1 + l\lambda_2)} \right|^2, \quad -\pi \leq \lambda \leq \pi. \quad (4.35)$$

Let C be a constant such that $C = \int \int_{-\pi}^{\pi} \tilde{I}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2$. Thus the Whittle likelihood function can be written as:

$$\sigma^2(\beta) = C \mathbb{E}_f \left(\frac{1}{h(\lambda_1, \lambda_2, \beta)} \right), \quad (4.36)$$

where $f(\lambda_1, \lambda_2) = \frac{1}{C} \tilde{I}(\lambda_1, \lambda_2)$ is a density on $[-\pi, \pi]$ that is known up to a multiplicative factor. We can approximate the expectation in (4.36) by the empirical average

$$\bar{\sigma}^2(\beta) = \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N \frac{1}{h(\lambda_i, \lambda_j, \beta)}, \quad (4.37)$$

where M and N are taken large enough from the strong law of large number. The sequences λ_i, λ_j , where $i = 1, \dots, M, j = 1, \dots, N$ is generated using a Metropolis-Hastings algorithm. Using an ergodic Markov chain with stationary distribution f , we can obtain $\lambda_1, \dots, \lambda_N \sim f$ without directly simulating from f . Consequently, to generate the sample $(\lambda_1, \dots, \lambda_M) \times (\lambda_1, \dots, \lambda_N)$, we can fix M and N and use a Metropolis-Hastings algorithm which will be computed as follows:

- Given λ_t where $t = 1, \dots, M, 1, \dots, N$
- Generate Y_t from a uniform $\mathcal{U}_{[-\pi, \pi]}$ and denote the value obtained by y_t .
- Take

$$\lambda_{t+1} = \begin{cases} y_t & \text{with probability } \rho(\lambda_t, y_t), \\ \lambda_t & \text{with probability } 1 - \rho(\lambda_t, y_t), \end{cases}$$

where $\rho(\lambda_t, y_t) = \min \left\{ \frac{f(y_t)}{f(\lambda_t)}, 1 \right\} = \min \left\{ \frac{\tilde{f}(y_t)}{\tilde{f}(\lambda_t)}, 1 \right\}$. For more details we can refer to Robert and Casella (2005) [31]). Hence, an estimation procedure of β is to minimize (4.37).

5. Conclusion

The paper discusses and investigates the procedures estimation for the Seasonal FISSAR model. The classical Whittle and a regression method are used to estimate the memory parameters. For estimating the innovation parameters and memory parameters simultaneously, the maximum likelihood method and the Whittle method based on the MCMC simulation are considered. Two parameter estimation techniques have been considered for simulation study, regression and MLE. From the results, we see that all methods perform very well as the Bias, sd and RMSE are in most cases small. In general, the estimates parameters from the MLE approach are better than those given by regression method. Thus, it is suggested that the proposed methods can be applied to real data, empirical data, and real-world situations applications for a variety of data processes such as in economics, finance, agriculture, hydrology, environmental, etc. These issues should be addressed in future research. Another interesting issue for future research will focus on analyzing the asymptotics properties of the regression and MLE methods.

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