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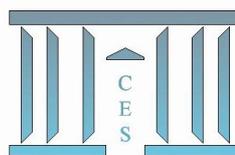
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Stability of marketable payoffs with re-trading

Jean-Marc BONNISSEAU, Achis CHÉRY

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Stability of marketable payoffs with re-trading

Jean-Marc Bonnisseau* and Achis Chéry†

September 18, 2018

Abstract

We consider a stochastic financial exchange economy with a finite date-event tree representing time and uncertainty and a financial structure with possibly long-term assets. We have addressed in Bonnisseau-Chéry [3] the question of the stability of financial structures thanks to a suitable Assumption **R** which is based only on the return of the assets independently of the price of the assets. However, Assumption **R** is never satisfied in the structures with re-trading (See [3]), which is a very common feature in many papers (See [5]). The main purpose of this new paper is to deepen the study of the stability of financial structures with long-term assets and especially to address the stability in cases where there is a re-trading of assets after their issuance at different nodes and different dates. We exhibit a newsufficient condition, which allows to have the equality between the kernel of payoffs matrices, and, thus the stability of the marketable payoffs. This sufficient condition is then used to get the stability result for the re-trading extension of a financial structure. We also show that the situation is more complex when all initial assets are not issued at the initial date and we then provide a stronger condition to cover this case.

Keywords: Incomplete markets, financial equilibrium, multi-period model, long-term assets, re-traded assets, financial structures with re-trading.

JEL codes: D5, D4, G1

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1 Introduction

In Bonnisseau-Chéry [3], we have studied the question of *stability of the set of marketable payoffs* with respect to the arbitrage free asset prices in presence of long-term asset. The stability means that the set-valued mapping of marketable payoffs from the set of non-arbitrage asset prices to the set of payoffs is lower semi-continuous and has a closed graph. This is a key property for the functioning of the financial markets since it implies that the demand for regular consumers is continuous. Then, we can get the existence of equilibrium prices under reasonable standard assumptions. Otherwise, we can observe a divergence of the trades under very small variations of the asset prices to get the same payoffs. Actually, the stability is a consequence of the equality of the kernel of the payoff matrix and the kernel of the full payoff matrix of the financial structure for all non-arbitrage asset prices. But this assumption cannot be checked on the fundamentals of the economy.

In [3], we provide a sufficient condition (called **R**) on the payoff matrix describing the returns of the assets. Under this condition we get the stability of marketable payoffs. Assumption **R** is stated in a framework without re-trading whereas in many contributions, (see, in particular, the seminal book of Magill and Quinzii [5]), an initial set of assets are issued at the initial node and are then re-traded at all successive date. Angeloni-Cornet ([1]) have shown that the re-trading can be equivalently described as the emission of a new asset with payoffs being the truncated payoffs of the initial asset. So, using the Angeloni-Cornet ([1]) transformation, we can check Assumption **R** in presence of re-trading. Unfortunately, it appears that is not satisfied, except on the very particular case where there is only short-term assets. Hence the need to introduce new Assumptions on the payoff matrix to address this issue of *stability of the set of marketable payoffs* of a financial structure with re-trading, which is the main purpose of this paper.

We start by working on a given financial structure and then we consider the case of the re-trading extension of a financial structure. Our objective is to provide sufficient conditions on the initial payoff matrix to get the stability for the re-trading extension.

We introduce a first assumption, Assumption **H** which implies that the kernel of the full payoff matrix is included in the kernel of the payoff matrix. However this inclusion can be strict as illustrated by a simple example. Note that we nevertheless get the equality in absence of redundant assets. Thanks to another Assumption **K**, we get the other inclusion. It should be noted that **H** and **K** are not incompatible and one does not implies the other. As a by-product, we prove that the set of arbitrage free prices is a cone under Assumption **H**. This result is always true in the two-period case. Cornet-Ranjan (see [4]) have obtained this result in a multi-period framework with another assumption, which also implies the convexity of the set of arbitrage free prices is a convex cone. But our Assumption **H** is not sufficient to have the convexity.

We pursue the study by providing a third assumption **K'**, which is stronger than **K** but is sufficient for the stability without **H**. In words, Assumption **K'** means that the distribution of returns of the assets issued at a given node is constant over time but the magnitude may vary.

The purpose of the last section is to apply these preliminary results to the case of an economy with re-trading. First we consider that all assets are issued at the same node, the initial one of the date-event tree representing the uncertainty. This is the usual case in the literature. we prove that Assumption **K'** on the initial payoff matrix is inherited by the re-trading extension when the assets have non-zero returns between their emission node and their terminal nodes.

We then remark that these conditions are not enough when assets are not issued at the same node. So we state a stronger condition, Assumption **K''**, which is then sufficient to get the stability of the re-trading extension. Assumption **K''** supposes that the magnitude leading the evolution of the payoffs of the assets over time is independent of the issuance node of the asset, whereas Assumption **K'** leaves the possibility to have different magnitudes for different issuance nodes.

After introducing notations and the model of a financial structure borrowed from [1, 5] in Section 2, we formulate our Assumptions on the payoff matrix $V(p)$ in Section 3 and we state the results on stability without re-trading. In Section 4, we introduce the re-trading extension of a financial structure and we prove the two results on stability with a unique node of issuance and with several nodes. Some proofs have been postponed in Appendix.

2 The T -period financial exchange economy

In this section, we present the model and the notations, which are borrowed from Angeloni-Cornet[1] and are essentially the same as those of Magill-Quinzii[5].

2.1 Time and uncertainty

We¹ consider a multi-period exchange economy with $(T + 1)$ dates, $t \in \mathcal{T} := \{0, \dots, T\}$, and a finite set of agents \mathcal{I} . The uncertainty is described by a date-event tree \mathbb{D} of length $T + 1$. The set \mathbb{D}_t is the set of nodes (also called date-events) that could occur at date t and the family $(\mathbb{D}_t)_{t \in \mathcal{T}}$ defines a partition of the set \mathbb{D} ; for each $\xi \in \mathbb{D}$, we denote by $t(\xi)$ the unique date $t \in \mathcal{T}$ such that $\xi \in \mathbb{D}_t$.

At date $t = 0$, there is a unique node ξ_0 , that is $\mathbb{D}_0 = \{\xi_0\}$. As \mathbb{D} is a tree, each node ξ in $\mathbb{D} \setminus \{\xi_0\}$ has a unique immediate predecessor denoted $pr(\xi)$ or ξ^- . The mapping pr maps \mathbb{D}_t to \mathbb{D}_{t-1} . Each node $\xi \in \mathbb{D} \setminus \mathbb{D}_T$ has a set of immediate

¹We use the following notations. A $(\mathbb{D} \times \mathcal{J})$ -matrix A is an element of $\mathbb{R}^{\mathbb{D} \times \mathcal{J}}$, with entries $(a_{\xi}^j)_{(\xi \in \mathbb{D}, j \in \mathcal{J})}$; we denote by $A_{\xi} \in \mathbb{R}^{\mathcal{J}}$ the ξ -th row of A and by $A^j \in \mathbb{R}^{\mathbb{D}}$ the j -th column of A . We recall that the transpose of A is the unique $(\mathcal{J} \times \mathbb{D})$ -matrix ${}^t A$ satisfying $(Ax) \bullet_{\mathbb{D}} y = x \bullet_{\mathcal{J}} ({}^t A y)$ for every $x \in \mathbb{R}^{\mathcal{J}}$, $y \in \mathbb{R}^{\mathbb{D}}$, where $\bullet_{\mathbb{D}}$ [resp. $\bullet_{\mathcal{J}}$] denotes the usual inner product in $\mathbb{R}^{\mathbb{D}}$ [resp. $\mathbb{R}^{\mathcal{J}}$]. We denote by $\text{rank} A$ the rank of the matrix A and by $\text{Vect}(A)$ the range of A , that is the linear sub-space spanned by the column vectors of A . For every subset $\tilde{\mathbb{D}} \subset \mathbb{D}$ and $\tilde{\mathcal{J}} \subset \mathcal{J}$, the matrix $A_{\tilde{\mathbb{D}}}^{\tilde{\mathcal{J}}}$ is the $(\tilde{\mathbb{D}} \times \tilde{\mathcal{J}})$ -sub-matrix of A with entries a_{ξ}^j for every $(\xi, j) \in (\tilde{\mathbb{D}} \times \tilde{\mathcal{J}})$. Let x, y be in \mathbb{R}^n ; $x \geq y$ (resp. $x \gg y$) means $x_h \geq y_h$ (resp. $x_h > y_h$) for every $h = 1, \dots, n$ and we let $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n : x \gg 0\}$. We also use the notation $x > y$ if $x \geq y$ and $x \neq y$. The Euclidean norm in the Euclidean different spaces is denoted $\|\cdot\|$ and the closed ball centered at x and of radius $r > 0$ is denoted $\bar{B}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$.

successors defined by $\xi^+ = \{\bar{\xi} \in \mathbb{D} : \xi = \bar{\xi}^-\}$.

For $\tau \in \mathcal{T} \setminus \{0\}$ and $\xi \in \mathbb{D} \setminus \cup_{t=0}^{\tau-1} \mathbb{D}_t$, we define $pr^\tau(\xi)$ by the recursive formula: $pr^\tau(\xi) = pr(pr^{\tau-1}(\xi))$. We then define the set of successors and the set of predecessors of ξ as follows:

$$\mathbb{D}^+(\xi) = \{\xi' \in \mathbb{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^\tau(\xi')\}$$

$$\mathbb{D}^-(\xi) = \{\xi' \in \mathbb{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^\tau(\xi)\}$$

If $\xi' \in \mathbb{D}^+(\xi)$ [resp. $\xi' \in \mathbb{D}(\xi) := \mathbb{D}^+(\xi) \cup \{\xi\}$], we shall use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. Note that $\xi' \in \mathbb{D}^+(\xi)$ if and only if $\xi \in \mathbb{D}^-(\xi')$ and similarly $\xi' \in \xi^+$ if and only if $\xi = (\xi')^-$.

In the example of the event tree of Figure 1,

$$\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}, \xi_{22}\},$$

$T = 2$, the length of \mathbb{D} is 3, $\mathbb{D}_2 = \{\xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}, \xi_{22}\}$, $\xi_1^+ = \{\xi_{11}, \xi_{12}, \xi_{13}\}$, $\mathbb{D}^+(\xi_2) = \{\xi_{21}, \xi_{22}\}$, $t(\xi_{11}) = t(\xi_{12}) = t(\xi_{13}) = t(\xi_{21}) = t(\xi_{22}) = 2$, $\mathbb{D}^-(\xi_{11}) = \{\xi_0, \xi_1\}$, $\mathbb{D}(\xi_2) = \{\xi_2, \xi_{21}, \xi_{22}\}$.

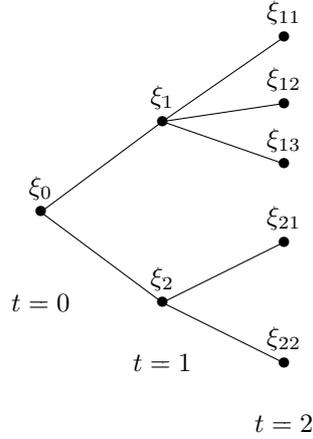


Figure 1: the tree \mathbb{D}

2.2 The financial structure without re-trading

The financial structure is constituted by a finite set of assets denoted $\mathcal{J} = \{1, \dots, J\}$. An asset $j \in \mathcal{J}$ is a contract issued at a given and unique node in \mathbb{D} denoted $\xi(j)$, called issuance node of j . Each asset is bought or sold only at its issuance node $\xi(j)$ and yields payoffs only at the successor nodes ξ' of $\mathbb{D}^+(\xi(j))$. To simplify the notation, we consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and we assume that it is zero if ξ is not a successor of the issuance node $\xi(j)$.

The payoff may depend upon the spot price vector² $p \in \mathbb{R}^{\mathbb{L}}$ and is denoted by $V_{\xi}^j(p)$. Formally, we assume that $V_{\xi}^j(p) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$.

For each consumer, $z_i = (z_i^j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ is called the portfolio of agent i . If $z_i^j > 0$ [resp. $z_i^j < 0$], then $|z_i^j|$ is the quantity of asset j bought [resp. sold] by agent i at the issuance node $\xi(j)$.

We assume that each consumer i is endowed with a portfolio set $Z_i \subset \mathbb{R}^{\mathcal{J}}$, which represents the set of admissible portfolios for agent i . For a discussion on this concept we refer to Angeloni-Cornet [1], Aouani-Cornet [2] and the references therein.

To summarize a financial structure $\mathcal{F} = (\mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi(j))_{j \in \mathcal{J}}, V)$ consists of

- a set of assets \mathcal{J} ,
- a collection of portfolio sets $(Z_i \subset \mathbb{R}^{\mathcal{J}})_{i \in \mathcal{I}}$,
- a node of issuance $\xi(j)$ for each asset $j \in \mathcal{J}$,
- a payoff mapping $V : \mathbb{R}^{\mathbb{L}} \rightarrow \mathbb{R}^{\mathbb{D} \times \mathcal{J}}$ which associates to every spot price $p \in \mathbb{R}^{\mathbb{L}}$ the $(\mathbb{D} \times \mathcal{J})$ -payoff matrix $V(p) = (V_{\xi}^j(p))_{\xi \in \mathbb{D}, j \in \mathcal{J}}$ and satisfies the condition $V_{\xi}^j(p) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$.

The price of asset j is denoted by q_j ; it is paid at its issuance node $\xi(j)$. We let $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ be the asset price vector.

The full payoff matrix $W(p, q)$ is the $(\mathbb{D} \times \mathcal{J})$ -matrix with the following entries:

$$W_{\xi}^j(p, q) := V_{\xi}^j(p) - \delta_{\xi, \xi(j)} q_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, given the prices (p, q) , the full flow of payoffs for a given portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is $W(p, q)z$ and the full payoff at node ξ is

$$\begin{aligned} [W(p, q)z](\xi) &:= W_{\xi}(p, q) \bullet_{\mathcal{J}} z = \sum_{j \in \mathcal{J}} V_{\xi}^j(p) z_j - \sum_{j \in \mathcal{J}} \delta_{\xi, \xi(j)} q_j z_j \\ &= \sum_{\{j \in \mathcal{J} \mid \xi(j) < \xi\}} V_{\xi}^j(p) z_j - \sum_{\{j \in \mathcal{J} \mid \xi(j) = \xi\}} q_j z_j. \end{aligned}$$

We assume that they are no participation constraints, i.e., for all $i \in \mathcal{I}$, $Z_i = \mathbb{R}^{\mathcal{J}}$.

We are now able to define the set of marketable payoffs for (p, q) as:

$$H(p, q) = \{w \in \mathbb{R}^{\mathbb{D}} \mid \exists z \in \mathbb{R}^{\mathcal{J}}, w = W(p, q)z\}$$

which is the range of the matrix $W(p, q)$.

We now recall that for a given spot price p , the asset price q is an arbitrage free price if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p, q)z > 0$. q is an arbitrage free price if and only if it exists a so-called state price vector $\lambda \in \mathbb{R}_{++}^{\mathbb{D}}$

² $\mathbb{L} = \mathbb{H} \times \mathbb{D}$ where \mathbb{H} is a finite set of divisible and physical goods exchanged at each node $\xi \in \mathbb{D}$.

such that ${}^tW(p, q)\lambda = 0$ (see, e.g. Magill-Quinzii [5]). Taken into account the particular structure of the matrix $W(p, q)$, this is equivalent to

$$\forall j \in \mathcal{J}, \lambda_{\xi(j)} q_j = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_{\xi} V_{\xi}^j(p).$$

In our financial structures there are **no trivial assets** i.e., there is no asset that gives zero payoffs to all nodes of the event tree \mathbb{D} .

In all our numerical examples, we assume that there is only one good at each node of the tree and the spot price of the unique good is equal to 1. Consequently, for the sake of simpler notations, we omit the price p and note the payoff matrix (resp. full payoff matrix) by V (resp. $W(q)$).

3 Stability of the set of marketable payoffs

In this section we will give conditions on the payoff matrices of our financial structures to get the stability of the marketable payoffs with respect to the arbitrage free asset prices. In [3], we have studied this question. The general abstract sufficient condition is the equality of kernels of payoff matrices. We show that this general condition is satisfied under Assumption **R** (See [3]). Assumption **R** means that the returns of the assets issued at a node ξ are not redundant³ with the returns of the assets issued at a predecessor node of ξ . So, the issuance of additional assets issued at ξ leads to a true financial innovation since the payoffs in the successors of ξ cannot be replicated by the payoffs of a portfolio built with the assets issued before ξ .

We recall that the continuity of the set of marketable payoffs, in mathematical terms, means that the correspondence $q \rightarrow H(p, q)$ has a closed graph and is lower semi-continuous for a given commodity price p . We show easily that in a multi-period economy, for each spot price p , the correspondence $H(p, \cdot)$ is lower semi-continuous (See [3]). This is a direct consequence of the fact that the matrix $W(p, q)$ depends continuously on q . We illustrate (See [3]) by means of a numerical example that the closedness of the graph of the set of marketable payoffs is not granted in a multi-period economy. Equality between the kernels of the payoffs matrices regardless of the non-arbitrage price is sufficient to get the closure of $H(p, \cdot)$ (See, Proposition 3.2 of Bonnisseau-Chéry [3]).

Assumption **R** is not suitable to study financial structures with re-trading. Indeed, in Angeloni and Cornet (2006), it is shown that a financial structure with re-trading is equivalent to a financial structure without re-trading by considering that a re-trade is equivalent to the issuance of a new asset. So we remark that if the financial structure has long-term assets with re-trading, then Assumption **R** may hold true without re-trading but not with re-trading. We give an example in the Section 3 of Bonnisseau-Chéry [3]). Hence the need to develop new Assumptions on the payoff matrix which will allow us to address

³Recall that an asset j_0 of a financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ is said to be redundant at spot price p if the column vector $V^{j_0}(p)$ representing its payoffs on \mathbb{D} is a linear combination column vectors representing the payoffs of other assets p i.e., if there exists $\alpha = (\alpha^j)_{j \in \mathcal{J} \setminus \{j_0\}} \in \mathbb{R}^{\mathcal{J} \setminus \{j_0\}}$ such that $V^{j_0}(p) = \sum_{j \in \mathcal{J} \setminus \{j_0\}} \alpha^j V^j(p)$.

the issue of *stability of the set of marketable payoffs* of a financial structure with re-trading. This will allow us to extend the domain of validity of the payoff matrices with continuous marketable payoffs.

Some additional notations

We first introduce some additional notations. For all $\xi \in \mathbb{D} \setminus \mathbb{D}_T$, $\mathcal{J}(\xi)$ is the set of assets issued at the node ξ , that is $\mathcal{J}(\xi) = \{j \in \mathcal{J} \mid \xi(j) = \xi\}$ and $\mathcal{J}(\mathbb{D}^-(\xi))$ is the set of assets issued at a predecessor of ξ , that is $\mathcal{J}(\mathbb{D}^-(\xi)) = \{j \in \mathcal{J} \mid \xi(j) < \xi\}$. For all $t \in \{0, \dots, T-1\}$, we denote by \mathcal{J}_t the set of assets issued at date t , that is, $\mathcal{J}_t = \{j \in \mathcal{J} \mid \xi(j) \in \mathbb{D}_t\}$.

Let (τ_1, \dots, τ_k) such that $0 \leq \tau_1 < \tau_2 < \dots < \tau_k \leq T-1$ be the dates at which there is at least the issuance of one asset, that is $\mathcal{J}_{\tau_\kappa} \neq \emptyset$. For $\kappa = 1, \dots, k$, let $\mathbb{D}_{\tau_\kappa}^e$ be the set of nodes at date τ_κ at which there is the issuance of at least one asset. $\mathbb{D}^e = \cup_{\kappa=1}^k \mathbb{D}_{\tau_\kappa}^e$ is the set of nodes at which there is the issuance of at least one asset. We remark that

$$\bigcup_{\tau \in \{0, \dots, T-1\}} \mathcal{J}_\tau = \bigcup_{\kappa \in \{1, \dots, k\}} \mathcal{J}_{\tau_\kappa} = \mathcal{J}$$

and for all $\tau \in \{\tau_1, \dots, \tau_k\}$, $\bigcup_{\xi \in \mathbb{D}_\tau} \mathcal{J}(\xi) = \mathcal{J}_\tau$.

3.1 Assumptions on $V_{\mathcal{F}}(p)$

We posit some assumptions on the payoff matrices.

Assumptions

Let $\mathcal{F} = \{\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V\}$ be a financial structure given.

Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price.

Assumption H:

$$\text{for all } j \in \mathcal{J}, V_{\xi}^j(p) = 0 \text{ if } \xi \in \mathbb{D}^e \cap \mathbb{D}(\xi(j)).$$

Assumption K:

$$\text{for each } \eta \in \mathbb{D}^e, \text{Ker}V_{\eta^+}^{\mathcal{J}(\eta)}(p) = \text{Ker}V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$$

Assumptions K': for each $\eta \in \mathbb{D}^e$ and for each $\xi \in \eta^+$, $\exists \alpha^\xi = (\alpha_{\xi'}^\xi)_{\xi' \in \mathbb{D}^+(\xi)} \in (\mathbb{R}_+)^{\mathbb{D}^+(\xi)}$ such that

$$\forall \xi' \in \mathbb{D}^+(\xi), V_{\xi'}^{\mathcal{J}(\eta)}(p) = \alpha_{\xi'}^\xi V_{\xi}^{\mathcal{J}(\eta)}(p)$$

Assumption **H** means that all assets give zero payoffs at the nodes at which there is the issuance of at least one asset i.e., at the nodes $\xi \in \mathbb{D}^e$. Assumption **K** means that if the portfolio of assets purchased today at node ξ gives zero payoff at the next date (i.e., at the nodes $\xi' \in \xi^+$) then this portfolio gives zero payoffs at all nodes of the event-tree \mathbb{D} . Assumption **K'** means that the array of returns of assets issued at the same node η at a node $\xi' \in \mathbb{D}^+(\eta)$ is proportional to the array of returns of these assets at the immediate successor node of η which is a predecessor of ξ' . In other words, the distribution of returns of assets issued at the same node is constant over time but the magnitude may vary.

Remark 3.1. Assumptions **K** and **H** are not incompatible and they do not imply Assumption **K'**. Indeed, consider the financial structure

$$\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$$

with $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10}\}$, $\mathcal{J} = \{j^1, j^2, j^3, j^4\}$ and that the first two assets are issued at node ξ_0 , The third asset is issued at node ξ_4 and the fourth is issued at node ξ_5 .

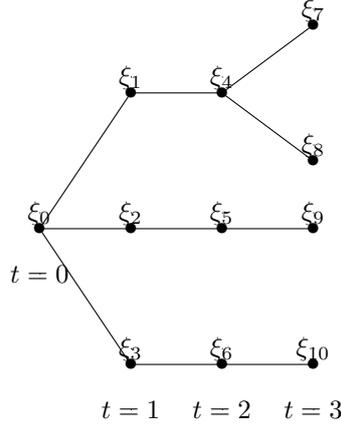


Figure 2: the tree \mathbb{D}

The payoff matrix is:

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 3 & 6 & 1 & 0 \\ -3 & 9 & 0 & 2 \\ 12 & -9 & 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{matrix}$$

$\mathbb{D}^e = \{\xi_0, \xi_4, \xi_5\}$, $\mathbb{D}^+(\xi_4) = \xi_4^+$, $\mathbb{D}^+(\xi_5) = \xi_5^+$ and $\mathbb{D}^+(\xi_1) = \{\xi_4, \xi_7, \xi_8\}$. We have $V_{\xi_0} = V_{\xi_4} = V_{\xi_5} = 0$ so Assumption **H** is satisfied in \mathcal{F} . We also remark that $\text{Ker}V_{\xi_0^+}^{\mathcal{J}(\xi_0)} = \text{Ker}V_{\mathbb{D}^+(\xi_0)}^{\mathcal{J}(\xi_0)} = \{0\}$, $\text{Ker}V_{\xi_4^+}^{\mathcal{J}(\xi_4)} = \text{Ker}V_{\mathbb{D}^+(\xi_4)}^{\mathcal{J}(\xi_4)} = \{0\}$, $\text{Ker}V_{\xi_5^+}^{\mathcal{J}(\xi_5)} = \text{Ker}V_{\mathbb{D}^+(\xi_5)}^{\mathcal{J}(\xi_5)} = \{0\}$ so Assumption **K** is satisfied in \mathcal{F} . Since regardless $\alpha \in \mathbb{R}_+$, we have $V_{\xi_4}^{\mathcal{J}(\xi_0)} = \begin{pmatrix} 1 & 2 \end{pmatrix} \neq \alpha V_{\xi_1}^{\mathcal{J}(\xi_0)} = \alpha \begin{pmatrix} 1 & 1 \end{pmatrix}$ so Assumption **K'** is not satisfied in \mathcal{F} .

Remark 3.2. Assumption **K'** is not stronger than Assumption **H**. Indeed, let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be financial structure such that $\mathcal{J} = \{j^1, j^2\}$,

$\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ and that the first asset is issued at node ξ_0 and the second asset is issued at node ξ_1 .



Figure 3: the tree \mathbb{D}

The payoff matrix is:

$$V = \begin{pmatrix} \mathbf{0} & 0 \\ 1 & \mathbf{0} \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Here we have: $\mathbb{D}^e = \{\xi_0, \xi_1\}$, $\xi_0^+ = \{\xi_1\}$, $\xi_1^+ = \{\xi_2\}$, $\xi_2^+ = \{\xi_3\}$, $\alpha^{\xi_1} = (1, 1)$, $\alpha^{\xi_2} = (1)$, so Assumption **K'** is satisfied in \mathcal{F} . Since $V_{\xi_1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$ so Assumption **H** is not satisfied in \mathcal{F} .

Remark 3.3. Assumption **K'** is equivalent to: for each $\eta \in \mathbb{D}^e$, for each $\xi \in \eta^+$, $\exists \alpha^\xi = (\alpha_{\xi'}^\xi)_{\xi' \in \mathbb{D}^+(\xi)} \in (\mathbb{R}_+)^{\mathbb{D}^+(\xi)}$ such that $\forall \xi' \in \mathbb{D}^+(\xi)$, $V_{\xi'}^j(p) = \alpha_{\xi'}^\xi V_\xi^j(p)$, $\forall j \in \mathcal{J}(\eta)$.

Remark 3.4. Assumption **K'** is satisfied by all financial structures consisting only of short-term assets. Indeed, it suffices to take $\alpha^\xi = 0$, for all $\xi \in \mathbb{D}^e$.

Lemma 3.1. All financial sub-structure⁴ of a financial structure which satisfies Assumption **K'** satisfies Assumption **K'**.

The proof of Lemma 3.1 is obvious.

Lemma 3.2. Assumption **K'** is stronger than Assumption **K**.

Proof of Lemma 3.2 is given in Appendix.

Remark 3.5. If there is only two dates i.e., if $T = 1$, Assumption **H** is satisfied.

If financial assets are issued at the same date, Assumption **H** is satisfied.

Assumption **K** is satisfied by the financial structures consisting only of short-term assets.

Assumption **K** is equivalent to each of the two following assumptions:

- for each $\eta \in \mathbb{D}^e$, $\text{Ker} V_{\eta^+}^{\mathcal{J}(\eta)}(p) = \text{Ker} V^{\mathcal{J}(\eta)}(p)$.
- for each $\eta \in \mathbb{D}^e$, $\text{Im}^t V_{\eta^+}^{\mathcal{J}(\eta)}(p) = \text{Im}^t V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$.

⁴Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a given financial structure. We call sub-structure of \mathcal{F} any financial structure $\mathcal{F}' = (\mathcal{J}', \mathbb{R}^{\mathcal{J}'}, (\xi(j))_{j \in \mathcal{J}'}, V')$ such that $\mathcal{J}' \subset \mathcal{J}$.

Lemma 3.3. *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K** at the spot price p . Then \mathcal{F} has no trivial assets⁵ if and only if $\forall \eta \in \mathbb{D}^e$ and $\forall j \in \mathcal{J}(\eta)$ there exists $\xi \in \eta^+$ such that $V_{\xi}^j(p) \neq 0$.*

Proof of Lemma 3.3 is given in Appendix.

3.2 Equality between the kernel of payoff matrices

The main purpose of this section is to provide a new sufficient conditions compatible with long term assets under which the kernels of the payoff and full payoff matrices are equal for every arbitrage free price. Hence the stability of the financial structure holds true because in Bonnisseau-Chéry [3], we have shown that the equality between kernels of payoffs matrices whatever the price of non arbitrage implies the stability in this financial structure.

Let $\mathcal{F} = \{\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V\}$ be a financial structure. For each spot price $p \in \mathbb{R}^{\mathbb{L}}$, we note by Q_p the set of arbitrage free prices with respect to p for \mathcal{F} where $Q_p = \{q \in \mathbb{R}^{\mathcal{J}}; \exists \lambda \in \mathbb{R}_{++}^{\mathbb{D}}$ satisfying ${}^t W(p, q)\lambda = 0\}$.

Under Assumption **H**, the following Proposition shows that the kernel the payoff matrix is always contains in the kernels of the full payoff matrices whatever the arbitrage free price. We get the other inclusion thanks to Assumption **K** (See Proposition 3.3). In addition, if there are no redundant assets, we have equality without Assumption **K**.

Proposition 3.1. *Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure and $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. If \mathcal{F} satisfies Assumption **H** at the spot price p , then*

$$\text{Ker}W(p, q) \subset \text{Ker}V(p), \forall q \in Q_p$$

and

$$\text{rank}V(p) \leq \text{rank}W(p, q), \forall q \in Q_p.$$

The proof of Proposition 3.1 is given after the following Corollary.

Corollary 3.1. *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **H** at the spot price p . If there is no redundant assets or, equivalently, $\text{rank}V(p) = \mathcal{J}$ then*

$$\text{Ker}W(p, q) = \text{Ker}V(p)$$

and

$$\text{rank}V(p) = \text{rank}W(p, q).$$

Proof of Proposition 3.1. Let $\lambda \in \mathbb{R}_{++}^{\mathbb{D}}$ be a state price and let q be the arbitrage free price associated with λ . For all $z \in \text{Ker}W(p, q)$ we have:

$0 = W(p, q)z = \sum_{j \in \mathcal{J}} W^j(p, q)z^j = \sum_{\eta \in \mathbb{D}^e} [\sum_{j \in \mathcal{J}(\eta)} W^j(p, q)z^j]$. This implies that

$$\forall \xi \in \mathbb{D}, 0 = \sum_{j \in \mathcal{J}} W_{\xi}^j(p, q)z^j = \sum_{\eta \in \mathbb{D}^e} \left[\sum_{j \in \mathcal{J}(\eta)} W_{\xi}^j(p, q)z^j \right] \quad (*)$$

⁵A trivial asset is an asset that gives zero returns at each node.

Thanks to the structure of the matrix $W(p, q)$ and Assumption **H**, for each $\xi \in \mathbb{D}$ we have

$$\sum_{j \in \mathcal{J}} W_{\xi}^j(p, q) z^j = \begin{cases} \sum_{j \in \mathcal{J}} V_{\xi}^j(p) z^j & \text{if } \xi \notin \mathbb{D}^e \\ -\sum_{j \in \mathcal{J}(\xi)} q_j z^j & \text{if } \xi \in \mathbb{D}^e \end{cases}$$

and we also have

$$\sum_{j \in \mathcal{J}} V_{\xi}^j(p) z^j = \begin{cases} \sum_{j \in \mathcal{J}} V_{\xi}^j(p) z^j & \text{if } \xi \notin \mathbb{D}^e \\ 0 & \text{if } \xi \in \mathbb{D}^e \end{cases}$$

Since for all $\xi \in \mathbb{D}$, $\sum_{j \in \mathcal{J}} W_{\xi}^j(p, q^{\lambda}) z^j = 0$ (See (*)) we have for all $\xi \in \mathbb{D}$, $\sum_{j \in \mathcal{J}} V_{\xi}^j(p) z^j = 0$ i.e., $z \in \text{Ker}V(p)$. Hence $\text{Ker}W(p, q) \subset \text{Ker}V(p)$. If moreover, $\text{rank}V(p) = \mathcal{J}$, we have

$$\{0\} \subset \text{Ker}W(p, q) \subset \text{Ker}V(p) = \{0\}.$$

and this implies that $\text{Ker}W(p, q) = \text{Ker}V(p)$. From the rank Theorem in linear algebra we have $\text{rank}V(p) \leq \text{rank}W(p, q)$. \square

Remark 3.6. The inclusion $\text{Ker}W(p, q) \subset \text{Ker}V(p)$ of Proposition 3.1 may be strict. Indeed, consider the financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ such that $\mathbb{D} = \{\xi, \xi_1, \xi_2\}$, $\mathcal{J} = \{j^1, j^2\}$ and that the first asset is issued at node ξ_0 and the second asset is issued at node ξ_1 .

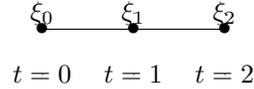


Figure 4: the tree \mathbb{D}

Taking $\lambda = (1, 1, 1)$

The payoff matrix and the full payoff matrix are:

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{matrix} \quad \text{and} \quad \mathbf{W}(\mathbf{q}) = \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{matrix}$$

It is clear that Assumption **H** is satisfied and we have: $\{0\} = \text{Ker}W(q) \neq \text{Ker}V = \{\alpha(1, -1); \alpha \in \mathbb{R}\}$.

The following proposition states that, under Assumption **H**, the set of arbitrage free prices is a cone. This result is always true in two-period case. Cornet and Ranjan (See [4]) had proved a similar result in multi-period economy under another Assumption. They have showed with their Assumption that the set of arbitrage free prices is a convex cone. But Assumption **H** is not sufficient to have the convexity.

Proposition 3.2. *Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure and $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. If \mathcal{F} satisfies Assumption **H** at the spot price p , then, the set Q_p which is the set of arbitrage free prices associated with spot price p is a cone.*

Proof of proposition 3.2. Q_p is a cone if for all $q \in Q_p$ and $t \in]0, +\infty[$, $tq \in Q_p$. For each $q \in Q_p$, from the characterization of arbitrage free price, we have

$$\lambda_{\xi(j)} q_j = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_{\xi} V_{\xi}^j(p), \forall j \in \mathcal{J}.$$

Let $t \in]0, +\infty[$. So, to show that $tq \in Q_p$ it suffices to show that there exists $\lambda' = (\lambda'_{\xi})_{\xi \in \mathbb{D}} \in \mathbb{R}_{++}^{\mathbb{D}}$ such that:

$$\lambda'_{\xi(j)} (tq_j) = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda'_{\xi} V_{\xi}^j(p), \forall j \in \mathcal{J}.$$

Choose λ' according to the following manner: for each $\xi \in \mathbb{D}$ we have

$$\lambda'_{\xi} = \begin{cases} t\lambda_{\xi} & \text{if } \xi \notin \mathbb{D}^e \\ \lambda_{\xi} & \text{if } \xi \in \mathbb{D}^e \end{cases}$$

We easily verify that, for each $j \in \mathcal{J}$,

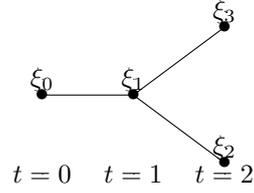
$$\begin{aligned} \lambda'_{\xi(j)} (tq_j) &= t(\lambda_{\xi(j)} q_j) = t \left(\sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_{\xi} V_{\xi}^j(p) \right) = \\ &= \sum_{\xi \in (\mathbb{D}^+(\xi(j)) \cap \mathbb{D}^e)} (t\lambda_{\xi}) V_{\xi}^j(p) + \sum_{\xi \in (\mathbb{D}^+(\xi(j)) \setminus \mathbb{D}^e)} (t\lambda_{\xi}) V_{\xi}^j(p) \end{aligned}$$

Assumption **H** implies that for all $\xi \in (\mathbb{D}^+(\xi(j)) \cap \mathbb{D}^e)$, $V_{\xi}^j(p) = 0$ hence we have $\sum_{\xi \in (\mathbb{D}^+(\xi(j)) \cap \mathbb{D}^e)} \lambda_{\xi} V_{\xi}^j(p) = 0$ and $\sum_{\xi \in (\mathbb{D}^+(\xi(j)) \cap \mathbb{D}^e)} \lambda'_{\xi} V_{\xi}^j(p) = 0$. So $\lambda'_{\xi(j)} (tq_j) = \sum_{\xi \in (\mathbb{D}^+(\xi(j)) \setminus \mathbb{D}^e)} (t\lambda_{\xi}) V_{\xi}^j(p) = \sum_{\xi \in (\mathbb{D}^+(\xi(j)) \setminus \mathbb{D}^e)} \lambda'_{\xi} V_{\xi}^j(p) = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda'_{\xi} V_{\xi}^j(p)$. \square

Remark 3.7. If Assumption **H** is not satisfied, Q_p may not be a cone even if there is equality between the kernels of the payoff matrices. Indeed, consider the financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ such that: $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$, $\mathcal{J} = \{j^1, j^2\}$ and that the first asset is issued at node ξ_0 and the second asset is issued at node ξ_1 .

The payoff matrix is:

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & 0 \\ -2 & \mathbf{0} \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Figure 5: the tree \mathbb{D}

Here Assumption **H** is not satisfied in \mathcal{F} and we have $\text{Ker}V = \text{Ker}W(q) = \{0\}$ for any asset price q . With $\lambda = (1, 1, 1, 1)$ we have $q^\lambda = (0, -1)$ while $2q^\lambda = (0, -2) \notin Q_p$.

In the following Proposition we show that, under Assumption **K**, the kernel of the payoff matrix is always included in the kernels of the full payoff matrices whatever the arbitrage free price.

Proposition 3.3. *Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure and $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. If \mathcal{F} satisfies Assumption **K** at the spot price p . Then*

$$\text{Ker}V(p) \subset \text{Ker}W(p, q), \forall q \in Q_p$$

and

$$\text{rank}W(p, q) \leq \text{rank}V(p), \forall q \in Q_p.$$

The proof of Proposition 3.3 is given below.

If the kernel of the payoff matrix of financial structure is included in the kernel of the full payoff matrix any arbitrage free price given, Proposition 3.4 states that, for each spot price $p \in \mathbb{R}^{\mathbb{L}}$ given, the set of arbitrage free prices associated with p is orthogonal at the kernel of the payoff matrix.

Proposition 3.4. *Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure and $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price vector. Suppose that for each $q \in Q_p$, $\text{Ker}V(p) \subset \text{Ker}W(p, q)$ then*

$$Q_p \subset (\text{Ker}V(p))^\perp$$

Proof of Proposition 3.4 is given in Appendix.

The following Corollary is deduced from Proposition 3.3 and Proposition 3.4.

Corollary 3.2. *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K** at the spot price p . We have*

$$Q_p \subset (\text{Ker}V(p))^\perp.$$

Now we give the proof of Proposition 3.3.

Proof of Proposition 3.3.

Let $z \in \mathbb{R}^{\mathcal{J}}$ such that $z \in \text{Ker}V(p)$.

- Let us show, first, the following claim :

Claim 3.1. $z \in \text{Ker}V(p)$ if and only if $[\forall \kappa \in \{1, \dots, k\}, \forall \eta \in \mathbb{D}_{\tau_\kappa}^e, z^{\mathcal{J}(\eta)} = (z^j)_{j \in \mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)]$.⁶

Proof of Claim 3.1

The implication $[\Leftarrow]$ is obvious. Let us show the other implication.

$$z \in \text{Ker}V(p) \Leftrightarrow \left[0 = \sum_{j \in \mathcal{J}} V^j(p)z^j = \sum_{\eta \in \mathbb{D}^e} \sum_{j \in \mathcal{J}(\eta)} V^j(p)z^j \right]$$

$$\Updownarrow$$

$$\forall \xi \in \mathbb{D}, 0 = \sum_{j \in \mathcal{J}} V_\xi^j(p)z^j = \sum_{\eta \in \mathbb{D}^e} \left[\sum_{j \in \mathcal{J}(\eta)} V_\xi^j(p)z^j \right]$$

Let us show that: for all $\eta \in \mathbb{D}_{\tau_1}^e, z^{\mathcal{J}(\eta)} = (z^j)_{j \in \mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$.

Let $\eta \in \mathbb{D}_{\tau_1}^e$. Given the structure of the matrix $V(p)$, noticing that $\tau_1 < \tau_\kappa$ for all $\kappa \neq 1$, we have for all $j \notin \mathcal{J}(\eta)$:

$$V_\xi^j(p) = 0, \forall \xi \in \eta^+$$

So, for each $\eta \in \mathbb{D}_{\tau_1}^e, \forall \xi \in \eta^+$ we have:

$$0 = \sum_{j \in \mathcal{J}} V_\xi^j(p)z^j = \sum_{j \in \mathcal{J}(\eta)} V_\xi^j(p)z^j$$

and this means that $z^{\mathcal{J}(\eta)} \in \text{Ker}V_{\eta^+}^{\mathcal{J}(\eta)}(p)$ and, Assumption **K** implies that $z^{\mathcal{J}(\eta)} \in \text{Ker}V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$. So, thanks to the structure of $V(p)$, we have: for all $\xi \notin \mathbb{D}^+(\eta), V_\xi^{\mathcal{J}(\eta)}(p) = 0$. This implies that $z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$.

Now, since for every $\eta \in \mathbb{D}_{\tau_1}^e, z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$, taking $z \in \text{Ker}V(p)$, we get

$$0 = \sum_{\eta \in \mathbb{D}^e} \left[\sum_{j \in \mathcal{J}(\eta)} V^j(p)z^j \right] = \sum_{\eta \in (\mathbb{D}^e \setminus \mathbb{D}_{\tau_1}^e)} \left[\sum_{j \in \mathcal{J}(\eta)} V^j(p)z^j \right]$$

$$\Updownarrow$$

$$\forall \xi \in \mathbb{D}, 0 = \sum_{j \in \mathcal{J}} V_\xi^j(p)z^j = \sum_{\eta \in (\mathbb{D}^e \setminus \mathbb{D}_{\tau_1}^e)} \left[\sum_{j \in \mathcal{J}(\eta)} V_\xi^j(p)z^j \right]$$

We do the same reasoning to get $z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$ for all $\eta \in \mathbb{D}_{\tau_2}^e$. Then for each (in increasing order) $\kappa \in \{2, \dots, k\}$ and for each node $\eta \in \mathbb{D}_{\tau_\kappa}^e$ and finally, we get that, for all $\kappa \in \{1, \dots, k\}$ and for all $\eta \in \mathbb{D}_{\tau_\kappa}^e, z^{\mathcal{J}(\eta)} = (z^j)_{j \in \mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$. \square

⁶ τ_κ is a date in which there is issuance of at least one asset and $\tau_1 < \dots < \tau_k$. $\mathbb{D}_{\tau_\kappa}^e$ is set of nodes where there is the issuance of at least one asset at date τ_κ .

- Let $\lambda \in \mathbb{R}_{++}^{\mathbb{D}}$ be a given state price and let q the arbitrage free price associated with λ .

Let $\kappa \in \{1, \dots, k\}$ and $\eta \in \mathbb{D}_{\tau_\kappa}^e$. Let us show that $z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p) \Rightarrow z^{\mathcal{J}(\eta)} \in \text{Ker}W^{\mathcal{J}(\eta)}(p, q)$. Indeed,

if $z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$ then, for each $\xi \in \mathbb{D}$, we have:

$$W_\xi^{\mathcal{J}(\eta)}(p, q) \cdot z^{\mathcal{J}(\eta)} = \sum_{j \in \mathcal{J}(\eta)} W_\xi^j(p, q) z^j = \begin{cases} \sum_{j \in \mathcal{J}(\eta)} V_\xi^j(p) z^j & \text{if } \xi \neq \eta \\ - \sum_{j \in \mathcal{J}(\eta)} q_j z^j & \text{if } \xi = \eta \end{cases}$$

Since q is the arbitrage free price associated with λ , we have

$$\begin{aligned} \sum_{j \in \mathcal{J}(\eta)} q_j z^j &= \sum_{j \in \mathcal{J}(\eta)} \left[\frac{1}{\lambda_\eta} \sum_{\xi' \in \mathbb{D}^+(\eta)} \lambda_{\xi'} V_{\xi'}^j(p) \right] z^j = \\ &= \frac{1}{\lambda_\eta} \sum_{\xi' \in \mathbb{D}^+(\eta)} \lambda_{\xi'} \left[\sum_{j \in \mathcal{J}(\eta)} V_{\xi'}^j(p) z^j \right] \end{aligned}$$

But $z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$ implies $\sum_{j \in \mathcal{J}(\eta)} V_{\xi'}^j(p) z^j = 0, \forall \xi' \in \mathbb{D}$ so

$$\sum_{j \in \mathcal{J}(\eta)} q_j z^j = \frac{1}{\lambda} \sum_{\xi' \in \mathbb{D}^+(\eta)} \lambda_{\xi'} [0] = 0.$$

Consequently, $W_\xi^{\mathcal{J}(\eta)}(p, q) \cdot z^{\mathcal{J}(\eta)} = 0$ for all ξ or, in other words, $z^{\mathcal{J}(\eta)} \in \text{Ker}W^{\mathcal{J}(\eta)}(p, q)$.

Since for all $\kappa \in \{1, \dots, k\}$ and for all $\eta \in \mathbb{D}_{\tau_\kappa}^e$ (see Claim 3.1) we have $z^{\mathcal{J}(\eta)} \in \text{Ker}V^{\mathcal{J}(\eta)}(p)$, this implies that for all $\kappa \in \{1, \dots, k\}$ and for all $\eta \in \mathbb{D}_{\tau_\kappa}^e$ we have $z^{\mathcal{J}(\eta)} \in \text{Ker}W^{\mathcal{J}(\eta)}(p, q)$ and this implies that $z \in \text{Ker}W(p, q)$. Hence $\text{Ker}V(p) \subset \text{Ker}W(p, q)$. \square

Corollary 3.3. *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K** at the spot price p . Then the linear subspaces of the family $\{\text{Im}V^{\mathcal{J}(\xi)}(p)\}_{\xi \in \mathbb{D}^e}$ are linearly independent⁷ and⁸*

$$\text{Im}V(p) = \bigoplus_{\xi \in \mathbb{D}^e} \text{Im}V^{\mathcal{J}(\xi)}(p).$$

⁷Linear subspaces U_1, U_2, \dots, U_m (with $U_\ell \neq \{0\}$ for all ℓ) are **linearly independent** if from $u_1 + u_2 + \dots + u_m = 0$, where $u_\ell \in U_\ell$ it follows that $u_\ell = 0$ for all ℓ . Sum of the linearly independent linear subspaces is called a **direct sum** of these linear subspaces and is denoted by $\bigoplus_{\ell=1}^m U_\ell = U_1 \oplus \dots \oplus U_m$.

⁸Linear subspace U is said to be equal to $\bigoplus_{\ell=1}^m U_\ell = U_1 \oplus \dots \oplus U_m$ if any u from U can be represented as $u = u_1 + \dots + u_m, u_\ell \in U_\ell$ uniquely.

The proof of Corollary 3.3 is deduced from Claim 3.1.

The following Corollary are deduced from Corollary 3.3.

Corollary 3.4. *Let $p \in \mathbb{R}^L$ be a spot price. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K** at the spot price p . Then the two following conditions are equivalent:*

- (i) *for every $\xi \in \mathbb{D}^e$, the family $\{V^j(p)\}_{j \in \mathcal{J}(\xi)}$ is linearly independent;*
- (ii) *the family $\{V^j(p)\}_{j \in \mathcal{J}}$ is linearly independent.*

Remark 3.8. The inclusion $\text{Ker}V(p) \subset \text{Ker}W(p, q)$ of Proposition 3.3 may be strict. Indeed, let a financial structure with three dates such that $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}\}$, $\mathcal{J} = \{j^1, j^2, j^3\}$, $\xi(j^1) = \xi_0$, $\xi(j^2) = \xi_1$ and $\xi(j^3) = \xi_2$.

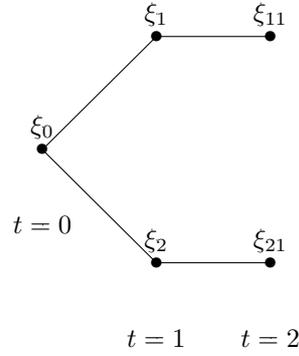


Figure 6: the tree \mathbb{D}

The payoff matrices are:

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & 0 & 0 \\ 1 & \mathbf{0} & 0 \\ -1 & 0 & \mathbf{0} \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{21} \end{matrix}$$

Assumption **K** is satisfied in our financial structure. Taking $\lambda = (1, 1, 1, 1, 1)$, the full payoff matrix is:

$$\mathbf{W}(\mathbf{q}) = \begin{bmatrix} \mathbf{0} & 0 & 0 \\ 1 & \mathbf{1} & 0 \\ -1 & 0 & -\mathbf{1} \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{21} \end{matrix}$$

We have: $\text{Ker}V = \{0\}$ because $\text{rank}V = 3$ and $z = {}^t(1, -1, -1) \in \text{Ker}W(q) \neq \{0\}$.

The following Corollary which is deduced of Proposition 3.3 and Proposition 3.1, gives a result of equality between the kernel the payoff matrix and the kernel full payoff matrice at any arbitrage free prices even with redundant assets.

Corollary 3.5. *Let $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K** and **H** at the spot price p . Then*

$$\text{Ker}V(p) = \text{Ker}W(p, q), \forall q \in Q_p$$

and

$$\text{rank}V(p) = \text{rank}W(p, q), \forall q \in Q_p.$$

The following Proposition shows that under Assumption **K'** the kernel of the payoff matrix of the financial structure is equal to the kernel of the full payoff matrix regardless the arbitrage free price. It is a complementary result since there are no implications between Assumptions **H** and **K'** (See Remarks 3.1 and 3.2).

Proposition 3.5. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. If the payoff matrix V satisfies Assumption **K'** at the spot price p , then for all arbitrage free price $q \in \mathbb{R}^{\mathcal{J}}$, $\text{Ker}V(p) = \text{Ker}W(p, q)$.*

The main difficulty of the proof of Proposition 3.5 relies on the fact that there is not necessarily emission of assets at each date $t \in \{0, \dots, T-1\}$ of the tree \mathbb{D} i.e., we have not necessarily: for all $t \in \{0, \dots, T-1\}$, $\mathbb{D}_t \cap \mathbb{D}^e \neq \emptyset$. And, even if t is a date where assets are issued, all nodes at date t are not issuers of assets i.e., if $\mathbb{D}_t \cap \mathbb{D}^e \neq \emptyset$ this does not imply that $\mathbb{D}_t = \mathbb{D}_t^e$.

Proof of Proposition 3.5. Let $q \in \mathbb{R}^{\mathcal{J}}$ be an arbitrage free price and let $\lambda \in \mathbb{R}_{++}^{\mathbb{D}}$ be a state price satisfying ${}^tW(p, q)\lambda = 0$.

Thanks to Lemma 3.2 Assumption **K'** is stronger than Assumption **K**. From Proposition 3.3 we have the inclusion $\text{Ker}V(p) \subset \text{Ker}W(p, q)$. It remains to show that $\text{Ker}W(p, q) \subset \text{Ker}V(p)$. Let $z = (z^j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ such that $z \in \text{Ker}W(p, q)$. This is equivalent to

$$\sum_{j \in \mathcal{J}} W^j(p, q)z^j = 0 \Leftrightarrow \left[\sum_{j \in \mathcal{J}} W_{\xi}^j(p, q)z^j = 0 \text{ for all } \xi \in \mathbb{D} \right]$$

Let us show that: for each $\xi \in \mathbb{D}^e$, $\sum_{j \in \mathcal{J}(\xi)} V_{\xi'}^j(p)z^j = 0$ for all $\xi' \in \mathbb{D}$, which implies $z \in \text{Ker}V(p)$.

Note that, thanks to the structure matrix $V(p)$, we always have

$$\text{for each } \xi \in \mathbb{D}^e, \sum_{j \in \mathcal{J}(\xi)} V_{\xi'}^j(p)z^j = 0 \text{ for all } \xi' \notin \mathbb{D}^+(\xi).$$

So it remains to show that

$$\text{for each } \xi \in \mathbb{D}^e, \sum_{j \in \mathcal{J}(\xi)} V_{\xi'}^j(p)z^j = 0 \text{ for all } \xi' \in \mathbb{D}^+(\xi).$$

The following Lemma will facilitate the understanding of the proof of Proposition 3.5.

Lemma 3.4. *Let $p \in \mathbb{R}^L$ be a given spot price vector. Let $\mathcal{F} = \{\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V\}$ be a financial structure. Let $\lambda = (\lambda_\xi)_{\xi \in \mathbb{D}} \in \mathbb{R}_{++}^{\mathbb{D}}$ be a given state price. Then for all arbitrage free price q associated with λ we have: $\forall \eta \in \mathbb{D}, \forall z \in \mathbb{R}^{\mathcal{J}}$,*

$$\sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{j \in \mathcal{J}} \lambda_\xi W_\xi^j(p, q) z^j \right] = \sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^-(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_\xi W_\xi^j(p, q) z^j \right]$$

The proof of Lemma 3.4 is given in Appendix.

Claim 3.2. *For all $\underline{\xi} \in \mathbb{D}_{\tau_1}^e$ we have $\sum_{j \in \mathcal{J}(\underline{\xi})} V_\xi^j(p) z^j = 0, \forall \xi \in \mathbb{D}^+(\underline{\xi})$.*

Proof of Claim 3.2. let $\underline{\xi} \in \mathbb{D}_{\tau_1}^e$. Let $\eta \in \underline{\xi}^+$ be given.

Thanks to Lemma 3.4, we have

$$\sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}} \lambda_\xi W_\xi^j(p, q) z^j \right) = \sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^-(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_\xi W_\xi^j(p, q) z^j \right]$$

Since $\eta^- = \underline{\xi} \in \mathbb{D}_{\tau_1}^e$, $\mathbb{D}^e \cap \mathbb{D}^-(\eta) = \{\underline{\xi}\}$, the sum above is equal to

$$\sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\underline{\xi})} \lambda_\xi W_\xi^j(p, q) z^j \right)$$

thanks to the structure of the full payoff matrix $W(p, q)$ we have

$$\sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\underline{\xi})} \lambda_\xi W_\xi^j(p, q) z^j \right) = \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\underline{\xi})} \lambda_\xi V_\xi^j(p) z^j \right)$$

But, since $z \in \text{Ker}W(p, q)$ we have

$$\sum_{j \in \mathcal{J}} W_\xi^j(p, q) z^j = 0 \text{ for all } \xi \in \mathbb{D}$$

So

$$0 = \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}} \lambda_\xi W_\xi^j(p, q) z^j \right) = \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\underline{\xi})} \lambda_\xi V_\xi^j(p) z^j \right)^9 (*)$$

Thanks to Assumption **K'** there exists $\alpha^\eta = (\alpha_\xi^\eta)_{\xi \in \mathbb{D}(\eta)} \in \mathbb{R}_+^{\mathbb{D}^+(\eta)}$ (with $\alpha_\eta^\eta = 1$) such that

⁹We remind that for each $j \in \mathcal{J}$, $-\lambda_{\xi(j)} q_j + \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi V_\xi^j(p) = 0$ because q_j is an arbitrage free price associated to the state price λ .

$$\begin{aligned} \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\xi)} \lambda_{\xi} V_{\xi}^j(p) z^j \right) &= \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\xi)} \lambda_{\xi} (\alpha_{\xi}^{\eta} V_{\eta}^j(p)) z^j \right) \\ &= \left(\sum_{\xi \in \mathbb{D}(\eta)} \lambda_{\xi} \alpha_{\xi}^{\eta} \right) \left(\sum_{j \in \mathcal{J}(\xi)} V_{\eta}^j(p) z^j \right) \end{aligned}$$

According to (*) we have

$$0 = \left(\sum_{\xi \in \mathbb{D}(\eta)} \lambda_{\xi} \alpha_{\xi}^{\eta} \right) \left(\sum_{j \in \mathcal{J}(\xi)} V_{\eta}^j(p) z^j \right)$$

But $\sum_{\xi \in \mathbb{D}(\eta)} \lambda_{\xi} \alpha_{\xi}^{\eta} > 0$ because $\alpha_{\xi}^{\eta} \geq 0, \forall \xi \in \mathbb{D}^+(\eta)$, $\alpha_{\eta}^{\eta} = 1$ and $\lambda_{\xi} > 0, \forall \xi \in \mathbb{D}(\eta)$, so

$$\sum_{j \in \mathcal{J}(\xi)} V_{\eta}^j(p) z^j = 0$$

and this implies that

$$\sum_{j \in \mathcal{J}(\xi)} \alpha_{\xi}^{\eta} V_{\eta}^j(p) z^j = 0, \forall \xi \in \mathbb{D}(\eta)$$

↓

$$\sum_{j \in \mathcal{J}(\xi)} V_{\xi}^j(p) z^j = \sum_{j \in \mathcal{J}(\xi)} \alpha_{\xi}^{\eta} V_{\eta}^j(p) z^j = 0, \forall \xi \in \mathbb{D}(\eta)$$

Since η is arbitrary in $\underline{\xi}^+$, we have so

$$\text{for each } \eta \in \underline{\xi}^+, \sum_{j \in \mathcal{J}(\xi)} V_{\xi}^j(p) z^j = 0, \forall \xi \in \mathbb{D}(\eta)$$

Consequently:

$$\sum_{j \in \mathcal{J}(\xi)} V_{\xi'}^j(p) z^j = 0, \forall \xi' \in \mathbb{D}^+(\underline{\xi}).$$

Since $\underline{\xi}$ is arbitrary in $\mathbb{D}_{\tau_1}^e$, we have:

$$\text{for each } \underline{\xi} \in \mathbb{D}_{\tau_1}^e, \sum_{j \in \mathcal{J}(\underline{\xi})} V_{\xi'}^j(p) z^j = 0, \forall \xi' \in \mathbb{D}^+(\underline{\xi}) \quad \square$$

If all assets are issued on the same date i.e., $\tau_k = \tau_1$, the proof is complete.

Otherwise, we iterate the process using the following claim:

Claim 3.3. For all $\widehat{\xi} \in \mathbb{D}_{\tau_2}^e$ we have $\sum_{j \in \mathcal{J}(\widehat{\xi})} V_{\xi}^j(p) z^j = 0, \forall \xi \in \mathbb{D}^+(\widehat{\xi})$.

Proof of Claim 3.3. let $\hat{\xi} \in \mathbb{D}_{\tau_2}^e$. Let $\eta \in \hat{\xi}^+$ be given.

Thanks to Lemma 3.4, we have

$$\sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right) = \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^-(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right)$$

Two possible cases: either $\mathbb{D}^e \cap \mathbb{D}^-(\eta) = \{\hat{\xi}\}$ or $\mathbb{D}^e \cap \mathbb{D}^-(\eta) = \{\underline{\xi}, \hat{\xi}\}$ (with $\underline{\xi} \in \mathbb{D}_{\tau_1}^e$).

First case: If $\mathbb{D}^e \cap \mathbb{D}^-(\eta) = \{\hat{\xi}\}$, thanks to the structure of the full matrix $W(p, q)$, we have

$$\begin{aligned} & \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^-(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right) = \\ & \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\hat{\xi})} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right) = \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\hat{\xi})} \lambda_{\xi} V_{\xi}^j(p) z^j \right) \end{aligned}$$

With the same reasoning as in the previous claim, we show that

$$\sum_{j \in \mathcal{J}(\hat{\xi})} V_{\xi}^j(p) z^j = 0 \text{ for all } \xi \in \mathbb{D}(\eta)$$

Second case If $\mathbb{D}^e \cap \mathbb{D}^-(\eta) = \{\underline{\xi}, \hat{\xi}\}$ (with $\underline{\xi} \in \mathbb{D}_{\tau_1}^e$), thanks to the structure of the full matrix $W(p, q)$, we have

$$\begin{aligned} & \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^-(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right) = \\ & \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\underline{\xi})} \lambda_{\xi} W_{\xi}^j(p, q) z^j + \sum_{j \in \mathcal{J}(\hat{\xi})} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right) = \\ & \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\underline{\xi})} \lambda_{\xi} V_{\xi}^j(p) z^j + \sum_{j \in \mathcal{J}(\hat{\xi})} \lambda_{\xi} V_{\xi}^j(p) z^j \right) \end{aligned}$$

According to Claim 3.2, $\sum_{j \in \mathcal{J}(\underline{\xi})} V_{\xi}^j(p) z^j = 0$ for all $\xi \in \mathbb{D}(\eta)$. Consequently

$$\sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right) = \sum_{\xi \in \mathbb{D}(\eta)} \left(\sum_{j \in \mathcal{J}(\hat{\xi})} \lambda_{\xi} V_{\xi}^j(p) z^j \right)$$

With the same reasoning as for Claim 3.2 we show that

$$\sum_{j \in \mathcal{J}(\hat{\xi})} V_{\xi}^j(p) z^j = 0 \text{ for all } \xi \in \mathbb{D}(\eta)$$

Since this is true for all $\eta \in \widehat{\xi}^+$, we have

$$\text{for each } \eta \in \widehat{\xi}, \sum_{j \in \mathcal{J}(\widehat{\xi})} V_{\xi}^j(p) z^j = \sum_{j \in \mathcal{J}(\widehat{\xi})} \alpha_{\xi}^{\eta} V_{\eta}^j(p) z^j = 0, \forall \xi \in \mathbb{D}(\eta)$$

Consequently:

$$\sum_{j \in \mathcal{J}(\widehat{\xi})} V_{\xi'}^j(p) z^j = 0, \forall \xi' \in \mathbb{D}^+(\widehat{\xi}).$$

Since this is true for all $\widehat{\xi} \in \mathbb{D}_{\tau_2}^e$, we have, for each $\widehat{\xi} \in \mathbb{D}_{\tau_2}^e$,

$$\sum_{j \in \mathcal{J}(\widehat{\xi})} V_{\xi'}^j(p) z^j = 0, \forall \xi' \in \mathbb{D}^+(\widehat{\xi}). \quad \square$$

If there were only two dates of issuance of asset i.e., $\tau_k = \tau_2$, the proof is complete thanks to Claim 3.3. Otherwise, we repeat the same reasoning for the next dates of emission to get at the end, for each $\xi \in \mathbb{D}^e$,

$$\sum_{j \in \mathcal{J}(\xi)} V_{\xi'}^j(p) z^j = 0, \text{ for all } \xi' \in \mathbb{D}.$$

So we deduce that: for all $z \in \text{Ker}W(p, q)$, $z \in \text{Ker}V(p)$. Hence $\text{Ker}W(p, q) \subset \text{Ker}V(p)$. \square

Corollary 3.6. *Given a spot price $p \in \mathbb{R}^{\mathbb{L}}$. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K'** at p , then for all arbitrage free price q , $\text{rank}V(p) = \text{rank}W(p, q)$.*

Remark 3.9. Corollary 3.6 is a generalization of Proposition 5.2. b) and c) in Angeloni-Cornet [1] and of Magill-Quinzii [5] where only short-term assets are considered.

Corollary 3.7 is a generalization of Corollary 3.6 when we restrict the portfolios to belong to a subspace G of $\mathbb{R}^{\mathcal{J}}$. The proof of Corollary 3.7 is given in Bonnisseau-Chéry [3], it suffices to replace Assumption **R** in the proof of Corollary 3.4 by each of the conditions of Corollary 3.7.

Corollary 3.7. *Given a spot price $p \in \mathbb{R}^{\mathbb{L}}$. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure given and let G be a linear subspace of $\mathbb{R}^{\mathcal{J}}$. Suppose that one of the following three conditions is satisfied:*

- i) Assumption **K'** at p
- ii) Assumption **K** and Assumption **H** at p
- iii) Assumption **H** at p and $\text{rank}V(p) = \mathcal{J}$.

Then for all arbitrage free price q , $\dim [W(p, q)G] = \dim(V(p)G)$.

4 Stability with re-trading

In this section, we introduce the notion of the *re-trading extension of the financial structure* as in Angeloni and Cornet [1]. Actually, it consists in considering that re-trade an asset after its date of issuance is as issuing a new asset with the payoffs being the truncated payoffs of the initial asset. We do not only consider the standard case where all assets are issued at the initial node, but we also present the extension of a financial structure when the assets of the initial structure are issued at different nodes and dates. The new financial structure thus constituted is called the *re-trading extension* of the initial financial structure.

When all assets are issued at the same node for a financial structure, we show that Assumption **K'** is inherited by the re-trading extension. We show that this is no more the case when the initial assets are issued at different periods. So we introduce a stronger assumption, Assumption **K''**. If the initial financial structure satisfies this condition, then the re-trading extension satisfies Assumption **K'**. So we obtain stability result or continuity of the *marketable payoffs* for re-trading extension of financial structure.

Definition 4.1. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure. The re-trading of asset $j \in \mathcal{J}$ at node $\xi' \in \mathbb{D}(\xi(j) \setminus \mathbb{D}_T)$, denoted $j_{\xi'}$, is the asset issued at ξ' , that is, $\xi(j_{\xi'}) = \xi'$, and whose flow of payoffs is given by

$$\tilde{V}_{\xi}^{j_{\xi'}}(p) = V_{\xi}^j(p), \text{ if } \xi \in \mathbb{D}^+(\xi');$$

$$\tilde{V}_{\xi}^{j_{\xi'}}(p) = 0 \text{ otherwise.}$$

Note that the asset $j_{\xi(j)}$ is just the asset j . We associate to \mathcal{F} its re-trading extension defined by:

$$\tilde{\mathcal{F}} = \left(\tilde{\mathcal{J}} = \bigcup_{\xi \in \mathbb{D}^e} \mathcal{J}(\xi) \times (\mathbb{D}(\xi) \setminus \mathbb{D}_T), \mathbb{R}^{\tilde{\mathcal{J}}}, (\xi(\tilde{j}))_{\tilde{j} \in \tilde{\mathcal{J}}}, \tilde{V} \right)$$

which consists of all assets $j \in \mathcal{J}$ and of all re-trading assets ($j_{\xi'}$) to all nodes $\xi' \in \mathbb{D}(\xi) \setminus \mathbb{D}_T$. Hence $\tilde{\mathcal{J}} = \bigcup_{\xi \in \mathbb{D}^e} [\mathcal{J}(\xi) \times (\mathbb{D}(\xi) \setminus \mathbb{D}_T)]$ and the $\mathbb{D} \times \tilde{\mathcal{J}}$ -matrix $\tilde{V}(p)$ has for coefficients $\tilde{V}_{\xi}^{j_{\xi'}}(p)$, as defined above.

We denote by $q_{j_{\xi'}}$ the price of asset $j_{\xi'}$ (i.e., the re-trading of asset j at node ξ'), which is sometimes also called the price of asset j at node ξ' . So, for the financial structure $\tilde{\mathcal{F}}$, both the asset price vector $q = (q_{j_{\xi'}})_{j \in \mathcal{J}, \xi' \in \bigcup_{\xi \in \mathbb{D}^e} (\mathbb{D}(\xi) \setminus \mathbb{D}_T)}$ and the portfolio $z = (z^{j_{\xi'}})_{j \in \mathcal{J}, \xi' \in \bigcup_{\xi \in \mathbb{D}^e} (\mathbb{D}(\xi) \setminus \mathbb{D}_T)}$ now belong to $\mathbb{R}^{\tilde{\mathcal{J}}}$.

A simple example

Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure such that $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$, $\mathcal{J} = \{j^1, j^2\}$, $\xi(j^1) = \xi_0$, $\xi(j^2) = \xi_1$,

and the payoff matrix is

$$V = \begin{pmatrix} \mathbf{0} & 0 \\ 2 & \mathbf{0} \\ 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Figure 7: the tree \mathbb{D}

We have $\tilde{\mathbb{D}}^e = \{\xi_0, \xi_1, \xi_2\}$, $\tilde{\mathcal{J}} = \{j_{\xi_0}^1, j_{\xi_1}^1, j_{\xi_1}^2, j_{\xi_2}^1, j_{\xi_2}^2\}$ (because the asset j_1 is retraded at nodes ξ_1 and ξ_2 and j_2 is retraded at node ξ_2). The payoff matrix of the re-trading extension financial structures of \mathcal{F} is:

$$\tilde{V} = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 0 \\ 2 & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 1 & 1 & 3 & \mathbf{0} & \mathbf{0} \\ 2 & 2 & 1 & 2 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

The full payoff matrix of the re-trading extension financial structures of \mathcal{F} at the asset price $q = (q_{\xi_0}^1, q_{\xi_1}^1, q_{\xi_1}^2, q_{\xi_2}^1, q_{\xi_2}^2)$ is:

$$\tilde{W}(q) = \begin{pmatrix} -q_{\xi_0}^1 & 0 & 0 & 0 & 0 \\ 2 & -q_{\xi_1}^1 & -q_{\xi_1}^2 & 0 & 0 \\ 1 & 1 & 3 & -q_{\xi_2}^1 & -q_{\xi_2}^2 \\ 2 & 2 & 1 & 2 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

Remark 4.1. Even if Assumption \mathbf{K}' is satisfied in the financial structures \mathcal{F} , it may not be satisfied in $\tilde{\mathcal{F}}$. Indeed, in the previous example, it is clear that \mathcal{F} satisfies Assumption \mathbf{K}' and in the re-trading extension $\tilde{\mathcal{F}}$ of \mathcal{F} Assumption \mathbf{K}' is not satisfied because regardless $\alpha \in \mathbb{R}$, $\tilde{V}_{\xi_3}^{\tilde{\mathcal{J}}(\xi_1)} \neq \alpha \tilde{V}_{\xi_2}^{\tilde{\mathcal{J}}(\xi_1)}$.

4.1 About Assumption \mathbf{K}' for re-trading

In this sub-section, we will make some remarks on Assumption \mathbf{K}' with respect to the stability with of the re-trading extension. The previous example shows that the re-trading extension may not satisfy Assumption \mathbf{K}' even if the initial financial structure satisfies it. We will show that Assumption \mathbf{K}' is satisfied by the re-trading extension of a financial structure when there is a unique node of issuance of all initial asset, Assumption \mathbf{K}' is satisfied by the initial financial structure and the payoffs of an asset do not vanish between the date of issuance and the expiration date of the asset. So, we get the stability of the re-trading extension under the same hypothesis.

Remark 4.2. Assumption \mathbf{K}' may not be inherited by the re-trading extension even if there is a signe asset issued at the initial node. Indeed, let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure such that $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$, $\mathcal{J} = \{j\}$ and $\xi(j) = \xi_0$,

Figure 8: the tree \mathbb{D}

and the payoff matrix is

$$V = \begin{pmatrix} \mathbf{0} \\ 1 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

It is clear that \mathcal{F} satisfies Assumption **K'**. In the re-trading extension $\tilde{\mathcal{F}}$ of \mathcal{F} we have $\tilde{\mathbb{D}}^e = \{\xi_0, \xi_1, \xi_2\}$ and the asset j is re-traded at nodes ξ_1 and ξ_2 . The payoff matrix of the re-trading extension financial structures of \mathcal{F}

$$\tilde{V} = \begin{pmatrix} \mathbf{0} & 0 & 0 \\ 1 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix}$$

We find that Assumption **K'** is not satisfied by $\tilde{\mathcal{F}}$ because $\tilde{V}_{\xi_2}^{j_{\xi_1}} = 0$ and that regardless $\alpha \in \mathbb{R}$, $\tilde{V}_{\xi_3}^{j_{\xi_1}} \neq \alpha \tilde{V}_{\xi_2}^{j_{\xi_1}}$.

4.2 Re-trading with a single issuance node

Suppose that $T \geq 2$ because if $T = 1$ there is no re-trading. Consider the financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ such that all the assets are issued at node ξ_0 . Suppose that each asset j , once issued, is re-traded at all succeeding nodes of the node ξ_0 except terminal nodes. In this case $\tilde{\mathbb{D}}^e = \mathbb{D}^- = \mathbb{D} \setminus \mathbb{D}_T$ and $\tilde{\mathcal{J}} = \mathcal{J} \times \mathbb{D}^-$.

Proposition 4.1 shows that Assumption **K'** is satisfied by the re-trading extensions if $\mathbb{D}^e = \{\xi_0\}$ and that the asset returns to non-terminal nodes are non-zero.

Proposition 4.1. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. Consider the financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j) = \xi_0)_{j \in \mathcal{J}}, V)$ such that $V(p)$ satisfies Assumption **K'** and for all $j \in \mathcal{J}$, if $V_{\xi}^j(p) = 0$, then $V_{\xi'}^j(p) = 0$ for all $\xi' > \xi$. Then the re-trading extension $\tilde{\mathcal{F}} = (\tilde{\mathcal{J}}, \mathbb{R}^{\tilde{\mathcal{J}}}, (\xi(\tilde{j}))_{\tilde{j} \in \tilde{\mathcal{J}}}, \tilde{V})$ of \mathcal{F} satisfies Assumption **K'** at price p .*

Proof of Proposition 4.1. Let $\eta \in \mathbb{D}^-$ and $\xi \in \eta^+$. If $\eta \in \mathbb{D}_{T-1}^-$ there is nothing to demonstrate because $\mathbb{D}^+(\xi) = \emptyset$. This implies that, if $T = 2$, Proposition 4.1 is obvious.

Now, if $T > 2$, let us take $\eta \in \mathbb{D}^- \setminus \mathbb{D}_{T-1}^-$, $\xi \in \eta^+$ and $\xi' \in \mathbb{D}^+(\xi)$. Then $\tilde{V}_{\xi}^{\tilde{\mathcal{J}}^{(\eta)}}(p) = V_{\xi}^{\mathcal{J}}(p)$ and $\tilde{V}_{\xi'}^{\tilde{\mathcal{J}}^{(\eta)}}(p) = V_{\xi'}^{\mathcal{J}}(p)$ because ξ and ξ' belong to $\mathbb{D}^+(\eta)$

(See Definition 4.1). Since $\xi' > \xi > \eta > \xi_0$, there exists a unique $\zeta \in \xi_0^+$ such that $\eta \geq \zeta$. Thanks to Assumption **K'** on \mathcal{F} there exist $\alpha_\xi^\zeta \in \mathbb{R}_+$ and $\alpha_{\xi'}^\zeta \in \mathbb{R}_+$ such that $V_\xi^{\mathcal{J}}(p) = \alpha_\xi^\zeta V_\zeta^{\mathcal{J}}(p)$ and $V_{\xi'}^{\mathcal{J}}(p) = \alpha_{\xi'}^\zeta V_\zeta^{\mathcal{J}}(p)$.

If $\alpha_\xi^\zeta > 0$, we let $\theta_{\xi'}^\xi = \frac{\alpha_{\xi'}^\zeta}{\alpha_\xi^\zeta}$, and we get $\tilde{V}_{\xi'}^{\tilde{\mathcal{J}}(\eta)}(p) = \theta_{\xi'}^\xi \tilde{V}_\xi^{\tilde{\mathcal{J}}(\eta)}(p)$.

If $\alpha_\xi^\zeta = 0$, then $V_\xi^{\mathcal{J}}(p) = 0$ and from our assumption, $V_{\xi'}^{\mathcal{J}}(p)$ also vanish since $\xi' \in \mathbb{D}^+(\xi)$. Then, let $\theta_{\xi'}^\xi = 0$, and we obviously get $0 = \tilde{V}_{\xi'}^{\tilde{\mathcal{J}}(\eta)}(p) = \theta_{\xi'}^\xi \tilde{V}_\xi^{\tilde{\mathcal{J}}(\eta)}(p)$.

Consequently the financial structure $\tilde{\mathcal{F}}$ satisfies Assumption **K'**. \square

From Proposition 4.1 and the results of the previous section, we deduce the following Corollary.

Corollary 4.1. *Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j) = \xi_0)_{j \in \mathcal{J}}, V)$ be a financial structure and $p \in \mathbb{R}^{\mathbb{L}}$ a spot price vector. such that $V(p)$ satisfies Assumption **K'** and for all $j \in \mathcal{J}$, if $V_\xi^{\mathcal{J}}(p) = 0$, then $V_{\xi'}^{\mathcal{J}}(p) = 0$ for all $\xi' > \xi$. Let $\tilde{\mathcal{F}} = (\tilde{\mathcal{J}}, \mathbb{R}^{\tilde{\mathcal{J}}}, (\xi(\tilde{j}))_{\tilde{j} \in \tilde{\mathcal{J}}}, \tilde{V})$ be the re-trading extension of \mathcal{F} . Then*

- $\text{Ker} \tilde{V}(p) = \text{Ker} W_{\tilde{\mathcal{F}}}(p, q)$ for all arbitrage free asset price q .
- the marketable payoffs correspondence defined by $q \rightarrow \tilde{H}(p, q) = \{\omega \in \mathbb{R}^{\mathbb{D}} | \exists z \in \mathbb{R}^{\tilde{\mathcal{J}}}, \omega = W_{\tilde{\mathcal{F}}}(p, q)z\}$ has a **closed graph**.

4.3 Re-trading with different dates of issuance

Consider now a financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ where assets may be issued at any non terminal node. Given that in this case Assumption **K'** does not allow to have stability, we exhibit the following Assumption **K''** (which is a slightly stronger than **K'**) to deal with the question of stability.

Assumption K'':

There is $\alpha = (\alpha_\ell)_{\ell \in \mathbb{D}_{t \geq 2}} \in \mathbb{R}_{++}^{\mathbb{D}_{t \geq 2}}$ such that for each $\eta \in \mathbb{D}^e$, for each $\xi \in \eta^+$, for each $\xi', \xi'' \in \mathbb{D}(\xi)$ such that $\xi'' \in \xi'^+$, $V_{\xi''}^{\mathcal{J}(\eta)}(p) = \alpha_{\xi''} V_{\xi'}^{\mathcal{J}(\eta)}(p)$.

Assumption **K''** means that the ratio of the returns at two successive nodes after the issuance node does not depend neither on the asset nor on the issuance node. It could be interpreted as if the returns of the assets are driven over time by a common financial index represented by the parameter α . The difference with Assumption **K** is that in this later case, the ratio could vary with respect to the issuance node.

Remark 4.3. It is clear that Assumption **K''** implies **K'**. It suffices to define the parameter $\alpha_{\xi'}^\xi$ of Assumption **K** as $\alpha_{\xi_1} \alpha_{\xi_2} \dots \alpha_{\xi'}$ where $(\xi_1, \xi_2, \dots, \xi')$ is the unique path connecting ξ to ξ' .

Proposition 4.2 shows that Assumption **K''** is satisfied by re-trading extensions.

Proposition 4.2. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. Consider a financial structure $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ such that $V(p)$ satisfies Assumption **K''**. Then the re-trading extension $\tilde{\mathcal{F}} = (\tilde{\mathcal{J}}, \mathbb{R}^{\tilde{\mathcal{J}}}, (\xi(j))_{j \in \tilde{\mathcal{J}}}, \tilde{V})$ of \mathcal{F} satisfies Assumption **K''** at price p .*

Proof of Proposition 4.2. Let $\eta \in \tilde{\mathbb{D}}^e$ and $\xi \in \eta^+$. If $\eta \in \mathbb{D}_{T-1}$ there is nothing to demonstrate because $\mathbb{D}^+(\xi) = \emptyset$. This implies that, if $T = 2$, Proposition 4.2 is obvious.

Now, if $T > 2$, let us take $\eta \in \tilde{\mathbb{D}}^e \setminus \{\mathbb{D}_{T-1} \cup \mathbb{D}_T\}$ and $\xi \in \eta^+$. For each ξ' , $\xi'' \in \mathbb{D}(\xi)$ with $\xi'' \in \xi'^+$ and for each $j_\eta \in \tilde{\mathcal{J}}(\eta)$ there exists $\alpha_{\xi''} \in \mathbb{R}_{++}$ (thanks to Assumption **K''** on \mathcal{F}) (See Definition 4.1) such that $\tilde{V}_{\xi''}^{j_\eta}(p) = V_{\xi''}^{j_\eta}(p) = \alpha_{\xi''} V_{\xi'}^{j_\eta}(p) = \alpha_{\xi''} \tilde{V}_{\xi'}^{j_\eta}(p)$. \square

We deduce the following results on kernels equality of payoffs matrices in the extension of financial structures where assets are issued at different dates and different nodes. The proofs of the following Corollaries are deduced from Remark 4.3 and Proposition 3.5.

Corollary 4.2. *Given a spot price $p \in \mathbb{R}^{\mathbb{L}}$. Let $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V)$ be a financial structure satisfying Assumption **K''**. Let $\tilde{\mathcal{F}} = (\tilde{\mathcal{J}}, \mathbb{R}^{\tilde{\mathcal{J}}}, (\xi(j))_{j \in \tilde{\mathcal{J}}}, \tilde{V})$ be the re-trading extension of \mathcal{F} . Then*

$$\text{Ker} \tilde{V}(p) = \text{Ker} W_{\tilde{\mathcal{F}}}(p, q) \text{ for all arbitrage free asset price } q.$$

5 Appendix

We recall that for each $\xi \in \mathbb{D}$, we denote by:

$\mathbb{D}^+(\xi)$ the set of successors of ξ ;

$\mathbb{D}^-(\xi)$ the set of predecessors of ξ with $\mathbb{D}^-(\xi_0) = \emptyset$;

$\mathbb{D}(\xi) := \mathbb{D}^+(\xi) \cup \{\xi\}$.

Proof of Lemma 3.2. Let $\mathcal{F} = \{\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j \in \mathcal{J}}, V\}$ be a financial structure and $p \in \mathbb{R}^{\mathbb{L}}$ be a spot price given. We assume that \mathcal{F} satisfies Assumption **K'** at the spot price p .

Let us show that for each $\eta \in \mathbb{D}^e$, $\text{Ker} V_{\eta^+}^{\mathcal{J}(\eta)}(p) = \text{Ker} V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$. Let $z \in \text{Ker} V_{\eta^+}^{\mathcal{J}(\eta)}(p)$. We have $\sum_{j \in \mathcal{J}(\eta)} V_{\eta'}^j(p) z^j = 0, \forall \eta' \in \eta^+$.

If $\mathbb{D}^+(\eta) = \eta^+$ we have $z \in \text{Ker} V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$. Otherwise, thanks to Assumption **K'**, for each $\xi \in (\mathbb{D}^+(\eta) \setminus \eta^+)$ there exists $\eta' \in \eta^+ \cap \mathbb{D}^-(\xi)$ and $\alpha_{\xi}^{\eta'} \in \mathbb{R}_+$ such

that: $\forall j \in \mathcal{J}(\eta)$, $V_\xi^j(p) = \alpha_\xi^{\eta'} V_{\eta'}^j(p)$; so

$$\sum_{j \in \mathcal{J}(\eta)} V_\xi^j(p) z^j = \sum_{j \in \mathcal{J}(\eta)} \alpha_\xi^{\eta'} V_{\eta'}^j(p) z^j = \alpha_\xi^{\eta'} \left(\sum_{j \in \mathcal{J}(\eta)} V_{\eta'}^j(p) z^j \right)$$

Since $\eta' \in \eta^+$ and $z \in \text{Ker} V_{\eta^+}^{\mathcal{J}(\eta)}(p)$, we have so $\forall j \in \mathcal{J}(\eta)$, $V_\xi^j(p) = 0$. Consequently, $\sum_{j \in \mathcal{J}(\eta)} V_\xi^j(p) z^j = 0, \forall \xi \in \mathbb{D}^+(\eta)$ i.e., $z \in \text{Ker} V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$. Hence $\text{Ker} V_{\eta^+}^{\mathcal{J}(\eta)}(p) \subset \text{Ker} V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p)$.

It is easy to see that for each $\eta \in \mathbb{D}^e$, $\text{Ker} V_{\mathbb{D}^+(\eta)}^{\mathcal{J}(\eta)}(p) \subset \text{Ker} V_{\eta^+}^{\mathcal{J}(\eta)}(p)$ because $\forall \eta \in \mathbb{D}^e, \eta^+ \subset \mathbb{D}^+(\eta)$. \square

Proof of Lemma 3.3. [\Leftarrow]. It is clear that if for all $j \in \mathcal{J}$, there exists $\xi \in \xi^+(j)$ such that $V_\xi^j(p) \neq 0$ then \mathcal{F} contains no trivial assets.

[\Rightarrow]. By contradiction. Suppose that there exists an asset $j_0 \in \mathcal{J}$ such that: $\forall \xi \in \xi^+(j_0)$, $V_\xi^{j_0}(p) = 0$. We assume without any loss of generality that the column $V^{j_0}(p)$ of $V(p)$ is ranked in such a way that it is the first column of the submatrix $V^{\mathcal{J}(\xi(j_0))}(p)$. Let $z = (z^j)_{j \in \mathcal{J}(\xi(j_0))} \in \text{Ker} V_{\xi(j_0)^+}^{\mathcal{J}(\xi(j_0))}(p)$. For all $\zeta \in \mathbb{R}$, $z' = (\zeta, (z^j)_{j \in \mathcal{J}(\xi(j_0)) \setminus \{j_0\}}) \in \text{Ker} V_{\xi(j_0)^+}^{\mathcal{J}(\xi(j_0))}(p)$. Assumption **K** implies that $z' \in \text{Ker} V_{\mathbb{D}^+(\xi(j_0))}^{\mathcal{J}(\xi(j_0))}(p)$. This implies that $\forall \xi \in \mathbb{D}(\xi(j_0))$, $V_\xi^{j_0}(p) = 0$, thus $V^{j_0}(p) = 0$, i.e., j_0 is a trivial asset. It is a contradiction. \square

Proof of Lemma 3.4. Let $\eta \in \mathbb{D}$ and $z \in \mathbb{R}^{\mathcal{J}}$ be given. We have

$$\sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{j \in \mathcal{J}} \lambda_\xi W_\xi^j(p, q) z^j \right] = \sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{\beta \in \mathbb{D}^e} \sum_{j \in \mathcal{J}(\beta)} \lambda_\xi W_\xi^j(p, q) z^j \right]$$

Note by $\mathbb{D}^\cup(\eta)$ The union of η with all its predecessors and his successors i.e., $\mathbb{D}^\cup(\eta) = \mathbb{D}^-(\eta) \cup \mathbb{D}(\eta)$.

- First given the structure of the matrix $W(p, q)$, for all $\beta \notin \mathbb{D}^\cup(\eta)$, and for all $\xi \in \mathbb{D}(\eta)$ we have

$$W_\xi^j(p, q) = 0 \text{ for all } j \in \mathcal{J}(\beta)$$

So

$$\sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{j \in \mathcal{J}} \lambda_\xi W_\xi^j(p, q) z^j \right] = \sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^\cup(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_\xi W_\xi^j(p, q) z^j \right]$$

- Second, for all $\beta \in \mathbb{D}^e \cap \mathbb{D}^\cup(\eta)$ we have

$$\sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_\xi W_\xi^j(p, q) z^j \right] = 0$$

Indeed,

$$\sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_\xi W_\xi^j(p, q) z^j \right] =$$

$$\sum_{\xi \in (\mathbb{D}(\eta) \setminus \mathbb{D}(\beta))} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right] + \sum_{\xi \in \mathbb{D}(\beta)} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right].$$

Thanks to the structure of the matrix $W(p, q)$, for all $\xi \in (\mathbb{D}(\eta) \setminus \mathbb{D}(\beta))$ we have $W_{\xi}^j(p, q) = 0$ for all $j \in \mathcal{J}(\beta)$ so

$$\sum_{\xi \in (\mathbb{D}(\eta) \setminus \mathbb{D}(\beta))} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right] = 0.$$

Now

$$\text{sum}_{\xi \in \mathbb{D}(\beta)} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right] = \sum_{j \in \mathcal{J}(\beta)} \left[\sum_{\xi \in \mathbb{D}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) \right] z^j$$

For each $j \in \mathcal{J}(\beta)$, $W_{\xi}^j(p, q) = -q_j$ if $\xi = \beta$ and $W_{\xi}^j(p, q) = V_{\xi}^j(p)$ if $\xi \neq \beta$. Since q is the arbitrage free price associated with λ , for each $j \in \mathcal{J}(\beta)$, $\lambda_{\beta} q_j = \sum_{\xi \in \mathbb{D}^+(\beta)} \lambda_{\xi} V_{\xi}^j(p)$. So, for each $j \in \mathcal{J}(\beta)$ we have $\sum_{\xi \in \mathbb{D}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) = 0$ hence $\sum_{\xi \in \mathbb{D}(\beta)} \left[\sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right] = 0$.

Consequently we have

$$\sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{j \in \mathcal{J}} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right] = \sum_{\xi \in \mathbb{D}(\eta)} \left[\sum_{\beta \in (\mathbb{D}^e \cap \mathbb{D}^-(\eta))} \sum_{j \in \mathcal{J}(\beta)} \lambda_{\xi} W_{\xi}^j(p, q) z^j \right]$$

□

Proof of Proposition 3.4. Let $z \in \text{Ker}V(p)$, since by Assumption $\text{Ker}V(p) \subset \text{Ker}W(p, q)$, we have $W(p, q)z = 0$. Thanks to the structures of the matrices $V(p)$ and $W(p, q)$,

$$\forall \xi \in \mathbb{D},$$

$$0 = V_{\xi}(p) \cdot z = \sum_{j \in \mathcal{J}} V_{\xi}^j(p) z^j = \sum_{j \in \mathcal{J}(\mathbb{D}^-(\xi))} V_{\xi}^j(p) z^j$$

and

$$W_{\xi}(p, q) \cdot z = \begin{cases} \sum_{j \in \mathcal{J}(\mathbb{D}^-(\xi))} V_{\xi}^j(p) z^j & \text{if } \xi \notin \mathbb{D}^e \\ \sum_{j \in \mathcal{J}(\mathbb{D}^-(\xi))} V_{\xi}^j(p) z^j - \sum_{j \in \mathcal{J}(\xi)} q_j z^j & \text{if } \xi \in \mathbb{D}^e \end{cases}$$

Since $W(p, q)z = 0$, one has $\sum_{j \in \mathcal{J}(\xi)} q_j z^j = 0, \forall \xi \in \mathbb{D}^e$. So

$$q \cdot z = \sum_{j \in \mathcal{J}} q_j z^j = \sum_{\xi \in \mathbb{D}^e} \left(\sum_{j \in \mathcal{J}(\xi)} q_j z^j \right) = 0.$$

□

References

- [1] L. Angeloni and B. Cornet. Existence of financial equilibria in a multi-period stochastic economy. *Mathematical Economics*, 8:1–31, 2006.
- [2] Z. Aouani and B. Cornet. Existence of financial equilibria with restricted participations. *Journal of Mathematical Economics*, 45:772–786, 2009.
- [3] J-M. Bonnisseau and A. Chéry. Stability of marketable payoffs with long-term assets. *Annals of Finance*, 10:523–552, 2014.
- [4] B. Cornet and A. Ranjan. A remark on the set of arbitrage-free prices in a multi-period model. *International Journal of Economic Theory*, 9:3543, 2013.
- [5] M. Magill and M. Quinzii. *Theory of Incomplete Markets*. Cambridge, 1996.