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**JEL Codes: D11, D90**

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# TOWARDS A DECOMPOSITION FOR THE FUTURE: CLOSENESS, REMOTENESS & TEMPORAL BIASES\*

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# ABSTRACT

This article builds an axiomatization of inter-temporal trade-offs that makes an explicit account of the distant future and thus encompasses motives related to sustainability, transmission to offsprings and altruism. The focus is on separable representations and the approach is completed following a decision-theory index based approach that is applied to utility streams. The introduction of some new axioms is shown to lead to the emergence of two distinct orders that respectively relate to the *distant future* and *close future* components of some utility stream. This enlightens the limits of the commonly used fat tail intensity requisites for the evaluation of utility streams these are superseded and replaced by an axiomatic approach to optimal myopia degrees. The argument is then augmented by time-dependent orders that are shown to lead to a recursive representations. It is first established how a multiple discounts version of *present bias* becomes available and directly builds upon the features of the close future order. An eventual argument reintroduces a role for the distant future order by establishing the scope for *distant future bias* where the time-dependent order is influenced by the distant future.

KEYWORDS: Axiomatization, Myopia, Temporal Order Decompositions, Distant future sensitivities, Present Bias, Distant Future Bias.

JEL CLASSIFICATION: D11, D90.

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# 1. INTRODUCTION

Even though the long-run concerns for sustainability, conservation and the well-being of the future generations of offsprings nowadays widely overstep the boundaries of the academic circles and promptly come into the fore for most public agendas, it is not the least surprising that there seems to have been limited efforts towards a penetrating understanding of the actual meaning of *having an unbounded horizon* or *accounting for the infinite*. The first endeavor towards an axiomatic approach to the topic was brought by Brown & Lewis [6] and explicitly anchored on *myopia*: it has nonetheless received the sparse echo that was due to what was perceived as a mere mathematical curiosity, *i.e.*, the identification of the *weight of the distant future*. This nevertheless raises a number of questions that may not have hitherto received sufficient attention. Is, together with most of the social welfare literature, an arbitrarily large finite future a satisfactory proxy for an unbounded horizon? Does the very fact of having some remote low orders *tail* for a stream of utils mean that it is negligible in not exerting any influence for finite dates? More precisely, are there some specificities attached to arbitrarily remote infinite horizon streams and is it reasonable to compare these through the same apparatus that is used for the *head* and finite parts of these streams? Otherwise stated, does order theory keep on being the appropriate apparatus for such elements and, assuming this is the case, how is it to be adapted to simultaneously accommodate finite and infinite elements? Finally, a large part of the recent evaluation of streams of utils has been dealing with the importance of *time perception paradoxes* and *temporal distances* that would modify over time and gives rise to various *present biases*: how does this relate to the above concerns and is there any specificity that is associated to infinite dimensional elements in this regard?

The purpose of this article is to provide an integrated picture of myopia and the valuation of inter-temporal utility streams as pictured by the properties of some index functions. The literature criteria for comparing inter-temporal utility streams commonly rest upon intuitive properties such as *completeness*, *monotonicity*, *continuity*, *positive homogeneity* and *constant additivity*. While an order satisfying these properties can be represented by an *index function*, it is worthwhile emphasizing that these properties can be preserved through usual operators, *e.g.*, *summation*, *maximum*, *minimum*, or by any convex combination of these. As a matter of illustration, one can consider these following examples of index functions satisfying these fundamental properties.

$$I_1(x) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t x_t \text{ for some } 0 < \delta < 1,$$

$$I_2(x) = \liminf_{t \rightarrow \infty} x_t.$$

The first index function represents an order which is *very myopic*, *i.e.*, the value of each utility stream is essentially defined, for  $0 < \delta < 1$ , through a finite number of dates or generations. In opposition to this, the second index-order belongs to another, *extremely non-myopic*, orders kind: the evaluation of the entire utility

stream would not vary if only the values of a *finite* number of dates got modified. As mentioned before, a convex combination of these two index functions can also be considered:

$$I(x) = (1 - \chi)I_1(x) + \chi I_2(x) \text{ for some } 0 < \chi < 1.$$

The evaluation is now decomposed into a convex combination of two components, a first relating to the *evaluation of the close future* and the *head* of the utility stream through the index  $I_1$  and a second pertaining to the *evaluation of the distant future* and the *tail* of the utility stream through the index  $I_2$ . Within this expression, the parameter  $\chi$  can be understood as the *degree of myopia* that measures the weight of the *distant future* and the importance of the tail of the utility stream. This study will aim at providing a clarified picture these concerns. It will proceed by avoiding the commonly hidden myopic negligible tail insensitivity requisites and by building an axiomatization of inter-temporal trade-offs that makes an explicit account of the distant future and encompasses motives such as sustainability, transmission to offsprings or altruism. It will also illustrate how this argument closely relates with the scope for *temporal paradoxes* and *present biases*.

From a general perspective, an elementary way of assessing the effect of the distant future proceeds by considering constant gains or losses in the asymptotic behaviour of some inter-temporal utility streams. Given an order as represented by some index function  $I$ , the weight of the distant future could, *e.g.*, be measured through two simple parameters, *i.e.*,  $\chi_g = \lim_{T \rightarrow \infty} I(0, 0, \dots, 0, 1, 1, 1, \dots)$ , and  $\chi_\ell = -\lim_{T \rightarrow \infty} I(0, 0, \dots, 0, -1, -1, -1, \dots)$ , both with  $T$  nil components, that would respectively depict *remote constant gains* and *remote constant losses*. Building on such coefficients, two ranges of questioning naturally arise. First, is a configuration where both the gains and losses distant future coefficients sum up to zero ( $\chi_g = \chi_\ell = 0$ ) associated with some *tail-insensitivity* property where the distant future becomes negligible? Second, is there some scope for systematically decomposing the evaluation of inter-temporal streams between its distant future value—the *tail* of the utility stream—and its close future value—the *head* of the utility stream—and, assuming this is the case, which form could such a function uncover? Surprisingly enough, the answer to the first question is negative and this can be checked, *e.g.*, by carefully considering the following index function  $I$  as:

$$I(x) = \min \left\{ \hat{\omega} \cdot x, \max \left\{ \omega \cdot x, \liminf_{s \rightarrow \infty} x_s \right\} \right\}, \text{ for } x \in \ell_\infty.$$

for  $\omega$  and  $\hat{\omega}$  two probability measures belonging to  $\ell_1$  and satisfying  $\omega \neq \hat{\omega}$ . The preceding form can be understood as a social welfare function for an economy with two agents evaluating inter-temporal utility streams. While the first agent is very myopic and only considers the close future evaluation and the *head* of the utility stream, the second one is partially myopic evaluates any utility stream using the maximum between its value in the close future and its distant future value, or the maximum between the head and the tail of the utility stream. The criterion of the social planner eventually maximizes the welfare of the least favored agent, along the classical maximin criterion of Rawls [22] or some more recent argument in

Chambers & Echenique [10]. For this example, letting the two coefficients  $\chi_g$  and  $\chi_\ell$  be reduced to zero keeps on preserving some scope for altering the evaluation of a sequence by the sole modification of its distant future components, whence some *tail sensitivity*. Otherwise stated, and as negligible as the value of the distant future may *seem* at first sight, it keeps on exerting some influence on the evaluation of the utility streams.<sup>1</sup>

In order to reach a thorough understanding of such a potentiality but also to introduce the scope for a systematic decomposition of the future and then answer to the second question, supplementary structures have to be superimposed on the preferences order relation. Two new axioms are presented. The first *distant future sensitivities* one states that given an utility streams and a constant stream, the decision maker can always say about her preference between the distant futures of these two streams. The second *close future sensitivities* axiom is similar but relates to a comparison that takes place between some close futures. The introduction of this structure is shown to lead to the emergence of two distinct orders that respectively relate to the *distant future* and *close future* or, respectively, to the *tail* and *head* components of some utility stream. Both the *distant future* and the *close future* orders satisfy fundamentals properties and respectively assume representations through some *distant future and close future index functions*. Moreover, and it is the main result of this article, the evaluation of an utility streams can be decomposed into a convex combination of its distant future and its close future components, the parameters of this convex combination changing as a function of the utility streams and lying between  $\chi_g$  and  $\chi_\ell$ . Interestingly, these two values hereby play a decisive role in the characterization of the eventual *optimal myopia degrees*. They are shown to depend on the utility streams and may represent two different sorts of behaviours in the consideration of the distant future that respectively relate to *optimism* and *pessimism*.

As a result from strong evidence documented by numerous laboratory experiments, a large set of the recent literature on the evaluation of utility streams has however been interested in the time inconsistencies and present biases that result from the consideration of time-dependencies in the preference order, the *quasi-hyperbolic* acceptance of inter-temporal utility starting to increasingly question the relevance of the standard discounted criterion. An eventual part of this study will be interested in the way the consideration of time-dependent preference orders could comfort the earlier decomposition between the close future and the distant future. Having assumed that the time dependent orders are not influenced by the *head* of the utility streams, at any given date, the agent completes the evaluation of this utility stream. It is first established that the properties of this time-dependent order unequivocally rests upon the ones of the earlier close future order. In parallel to this, it is shown that this evaluation then consists of a *recursive convex sum* between the utility level at that date, and the evaluation at the subsequent date of the utility stream, a multitude of choices remaining however admissible for the weight parameters of this convex sum. Some behaviour shall then be labelled as *present biased* when, accord-

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<sup>1</sup>The associated calculus are detailed in the main text of this article.

ing to the perception of a given agent, the *temporal distance* between two successive dates it decreasing over time. This more specifically means that the *optimal* discount factor is increasing and this is shown to result from an specific axiom that further constrains the range of admissible time-dependent orders. A limit to this argument however results from its unequivocal articulation with the close future order, the earlier decomposition between the close future order and the distant future order becoming in that sense *incomplete*. An eventual argument will remedy to this conceptual difficulty by introducing distant-future temporal biases and provide an enriched version of the decomposition for the future between closeness and remoteness. More explicitly, this study reintroduces some actual role for the distant future evaluation through the retainment of an axiom that can provide an account of a *distant future bias* in the evaluation of a given utility stream. The currently introduced *distant future bias* is to be understood on the account of the existence of some continuity between the time dependent orders and the distant future order. Otherwise stated and under further qualifications, there does exist some direct influences of the evaluation of an *infinite* distant future on the evaluations of some *finite* close future. Interestingly, it is therefore established that some well-defined given order may concomitantly have some *present bias* and *distant future bias* facets. The contribution with the closest concerns from the ones of this article is a recent work due to Lapiéd & Renault [17] who consider a decision maker facing alternatives that are defined on a very distant future, *i.e.*, a time horizon that exceeds his life-time horizon. They emphasize the emergence of an *asymptotic patience* property, meaning that, for some remote date, no time tradeoff between alternative any longer prevails. Mention should also be made of the recent and independently completed contribution of Gabaix & Laibson [13] with a subtle articulation between forecasting accuracy, discounting and myopia in an imperfect information environment that relies upon a recent literature on experiments. More generally, the way this study relates to the earlier literature on the evaluation of utility streams can easily be understood with the above discussion. Indeed, the notion of *strong myopia*, due to Brown & Lewis [6] coincides with the *upward myopia* notion of Saywer [23] and means, in its version presented by Becker & Boyd [4], that, for any  $x > y$ , one has for any  $z$ ,  $x > (y_0, y_1, \dots, y_T, z_{T+1}, z_{T+2}, \dots)$  for sufficiently large values of  $T$ . In the context of this article, these cases are equivalent to the *downward myopia* of by Saywer [23] where  $x > y$  implies that, for any  $z$ ,  $(x_0, x_1, \dots, x_T, z_{T+1}, z_{T+2}, \dots) > y$  for sufficient large values of  $T$ . This corresponds to an extreme occurrence where  $\chi_g = \chi_\ell = 0$ . Another extreme, the *completely patient and time invariant* preferences in Marinacci [19], the Banach limits<sup>2</sup> correspond to the case  $\chi_g = \chi_\ell = 1$ . In parallel to this, Araujo [1] proves that, in order for the set of non trivial Pareto allocations to exist, consumers must exhibit some impatience in their preferences. Otherwise stated, this excludes the possibility of preferences being represented by Banach limits, or this is equivalent to at least of one of the two values  $\chi_g, \chi_\ell$  to differ from 1. Following a very different approach and contemplating a social planner problem, Chichilnisky [9] associated charges to the *non-dictatorship of present* part of the social welfare criterion where the present would have no *per se* effect. Following parallel

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<sup>2</sup>For a careful definition of Banach limits, see page 55 in Becker & Boyd [4].



roads and the decision theory multiple priors axiomatizations of Gilboa & Schmeidler [15] but relying upon a different system of axioms based upon *time-variability aversion*, Wakai [24] has provided an insightful account of smoothing behaviours where the optimal discount assumes a maxmin recursive representation. Finally, and formerly related with the current study with an analysis completed over the set of bounded real sequences  $\ell_\infty$ , Chambers & Echenique [10] have recently put forth an axiomatic approach to multiple discounts. The current approach is complementary to theirs in focusing on myopia dimensions that *precede* discounting concerns and emerge as soon as the *tail insensitivity* of the utility sequences gets relaxed.

In parallel to this, the most influential temporal inconsistencies tradition dates back Phelps & Pollack [21] and got revitalized by the works of Laibson [18] and Frederick, Loewenstein & O'Donoghue [12] under the so-called *quasi-hyperbolic discounting hypothesis*. Numerous experiments having supported the accuracy of this formulation, Montiel Olea & Strzalecki [20] have completed an axiomatic approach to the *quasi-hyperbolic discounting* representation of Phelps & Pollack [21], Laibson [18] and, more generally, to *present biased* preferences. They suppose that, for any two equivalent future sequences, a patient one and an impatient one, pushing both of them towards the present will distort the preference towards the impatient choice. This article assumes the *present bias* notion for every date  $T$  and not only the initial one. The index functions  $I_T$  are determined from a set of multiple discount rates. Hence the notion *present bias* in this article must contain two parts, one part for the upper bound of discount rates, and one part for the lower bound of discount rates. The axiom 10 in Montiel Olea & Strzalecki [20] correspond to the second part of the axiom **B2**, or the second part of Lemma 4.2. Finally, Chakraborty [7] has just completed a generalized appraisal of present bias within the Fishburn & Rubinstein [11] approach where preferences are defined on the realization of a single outcome at a given date. Even though it builds from a from an approach that differs from the current *utility streams* appraisal, his *weak present bias* axiom **A4** shares some similarity with the current axiom **B2**.

This study is organized as follows. Section two will introduce some basic axioms and emphasize the role of the distant future in the evaluation of utility streams over time. Section three will emphasize the different facets of myopia and introduce a decomposition for the future that is based upon closeness versus remoteness. Section three will illustrate how the introduction of time-dependencies in the preference order will make more precise the above decomposition and provide a recursive representation that provides a new picture of temporal biases. Most of the technical details are gathered in a final appendix.

## 2. SOME BASIC AXIOMS AND A ROLE FOR THE DISTANT FUTURE IN THE EVALUATION OF THE UTILITY STREAMS

### 2.1 FUNDAMENTALS, BASIC AXIOMS & THE CONSTRUCTION OF AN INDEX FUNCTION

This study contemplates an axiomatization approach to the evaluation of infinite utility streams, the whole argument being cast for discrete time sequences. In order to avoid any confusion, letters like  $x, y, z$  will be used for sequences (of utils) with values in  $\mathbb{R}$  while a notation  $c\mathbb{1}, c'\mathbb{1}, c''\mathbb{1}$  will be used for constant sequences, the notation  $\mathbb{1}$  being retained for the constant unitary sequence  $(1, 1, \dots)$ . In parallel to this, greek letters  $\lambda, \eta, \mu$  will be preferred for constant scalars.

Recall first that the space<sup>3</sup> of  $\ell_\infty$  is defined as the set of real sequences  $\{x_s\}_{s=0}^\infty$  such that  $\sup_{s \geq 0} |x_s| < +\infty$ . For every  $x \in \ell_\infty$  and  $T \geq 0$ , let  $x_{[0,T]} = (x_0, x_1, \dots, x_T)$  denote its  $T + 1$  first components,  $x_{[T+1, \infty[} = (x_{T+1}, x_{T+2}, \dots)$  its *tail* starting from date  $T + 1$  and, finally,  $(x_{[0,T]}, y_{[T+1, \infty[}) = (x_0, x_1, \dots, x_T, y_{T+1}, y_{T+2}, \dots)$  that considers the  $T + 1$  first elements of the sequence  $x$  and the  $T + 1$ -tail of the sequence  $y$ . Finally,  $(z_{[0,T]}, x) = (z_0, z_1, \dots, z_T, x_0, x_1, x_2, \dots)$ .

The following axiom introduces some fundamental properties for the order  $\geq$  on  $\ell_\infty$ .

AXIOM F. The order  $\geq$  satisfies the following properties:

- (i) *Completeness* For every  $x, y \in \ell_\infty$ , either  $x \geq y$  or  $y \geq x$ .
- (ii) *Transitivity* For every  $x, y, z \in \ell_\infty$ , if  $x \geq y$  and  $y \geq z$ , then  $x \geq z$ . Denote as  $x \sim y$  the case where  $x \geq y$  and  $y \geq x$ . Denote as  $x > y$  the case where  $x \geq y$  and  $y \not\geq x$ .
- (iii) *Monotonicity* If  $x, y \in \ell_\infty$  and  $x_s \geq y_s$  for every  $s \in \mathbb{N}$ , then  $x \geq y$ .
- (iv) *Non-triviality* There exist  $x, y \in \ell_\infty$  such that  $x > y$ .
- (v) *Archimedeanity* For  $x \in \ell_\infty$  and  $b\mathbb{1} > x > b'\mathbb{1}$ , there are  $\lambda, \mu \in ]0, 1[$  such that

$$(1 - \lambda)b\mathbb{1} + \lambda b'\mathbb{1} > x \text{ and } x > (1 - \mu)b\mathbb{1} + \mu b'\mathbb{1}.$$

- (vi) *Weak convexity* For every  $x, y, b\mathbb{1} \in \ell_\infty$ , and  $\lambda \in ]0, 1[$ ,

$$x \geq y \Leftrightarrow (1 - \lambda)x + \lambda b\mathbb{1} \geq (1 - \lambda)y + \lambda b\mathbb{1}.$$

All of the properties (i), (ii), (iii) and (iv) are standardly used in decision theory. The *Archimedeanity* property (v) ensures that the order is continuous in the sup-norm topology of  $\ell_\infty$ . The eventual *Weak convexity* property (vi) is admittedly less

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<sup>3</sup>Following the same arguments as Bewley [5], when one considers the *distant future* behavior of inter-temporal utility streams, it is not arbitrarily to impose that utility levels are generated by a bounded consumption streams, since the earth resources are limited.

immediate. It is referred to as *certainty independence* in the decision theory literature, contains the *positive homogeneity* property and ensures that direction  $\mathbb{1}$  is *comparison neutral*: following that direction, the comparison does not change between two sequences.

Under these conditions, the order  $\succeq$  can be represented by an index function which is homogeneous of degree 1 and constantly additive.

**LEMMA 2.1.** *Assume that the order  $\succeq$  satisfies axiom **F**. This order is represented by an index function  $I$  satisfying:*

- (i) For  $x \in \ell_\infty$ ,  $\lambda > 0$ ,  $I(\lambda x) = \lambda I(x)$ .
- (ii) For  $x \in \ell_\infty$ , constant  $b \in \mathbb{R}$ ,  $I(x + b\mathbb{1}) = I(x) + b$ .

Even though this directly compares with the conclusions reached in Gilboa & Schmeidler [15], and Ghirardato & al [14], it is worthwhile emphasizing that this article considers the total space  $\ell_\infty$  as opposed to the space of *simple acts*—these are equivalent to sequences in  $\ell_\infty$  which take a finite number of values—that was used by these authors.

## 2.2 NON-NEGLIGIBLE DISTANT FUTURE AND NON-NEGLIGIBLE CLOSE FUTURE

In the literature, the notions of *impatience*<sup>4</sup> or *delay aversion*<sup>5</sup> are generally understood through the convergence of  $\mathbb{1}_{\{T\}}$  to zero and as  $T$  tends to infinity. It is however to be stressed that such a property does not *per se* imply the convergence to zero of the effect of the associated tail, *i.e.*, some *constant distant future sequence*  $\mathbb{1}_{[T,+\infty[}$ . More generally, it is commonly assumed in the literature that the value of the distant future converges to zero when  $T$  converges to infinity. In the current framework and under Proposition 2.1, this is to mean that  $I(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[})$  and  $I(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[})$  are to converge to zero when  $T$  tends to infinity<sup>6</sup>. To check upon such this property in the current environment, it is first useful to introduce the two following coefficients:<sup>7</sup>

$$\begin{aligned}\chi_g &= \lim_{T \rightarrow \infty} I(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}), \\ \chi_\ell &= - \lim_{T \rightarrow \infty} I(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}) \\ &= 1 - \lim_{T \rightarrow \infty} I(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}).\end{aligned}$$

These two values  $\chi_g$  and  $\chi_\ell$ , which will be considered extensively in the course of this study, will further play an important role in the definition of the *myopia degrees*.

<sup>4</sup>See Koopmans [16].

<sup>5</sup>See Bastianello & Chateauneuf [2].

<sup>6</sup>Observe that these two properties are not equivalent.

<sup>7</sup>From the *monotonicity* property,  $I(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[})$  and  $1 - I(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[})$  are decreasing as a function of  $T$ , so these limits are well defined.

Under the above definition, the condition  $\chi_g = \chi_\ell = 0$  is similar to the usual *negligible-tail* or *tail-insensitivity* conditions of the literature. A natural conjecture hence formulates as the satisfaction, under this condition and for any  $x, z \in \ell_\infty$ , of:

$$\lim_{T \rightarrow \infty} I(x_{[0,T]}, z_{[T+1,\infty[}) = I(x),$$

*i.e.*, for sufficiently large values of  $T$ , the tail of the sequence  $z$  would become irrelevant and the whole evaluation of the utility stream would proceed from the sequence  $x$ . The following counter example will however provide an illustration where, in spite of a valuation of the distant future that could be nil,<sup>8</sup> this future remote component of the sequence could keep on exerting some influence on the evaluation of the whole sequence. This suggests a structure for the index function that could well be more complex than the one previously claimed on an intuitive intuitive basis.

**EXAMPLE 2.1.** *Consider two probability measures belonging to the set  $\ell_1$ , namely  $\omega$  and  $\hat{\omega}$ , and satisfying  $\omega \neq \hat{\omega}$ .<sup>9</sup> Define the index function  $I$  as:*

$$I(x) = \min \left\{ \hat{\omega} \cdot x, \max \left\{ \omega \cdot x, \liminf_{s \rightarrow \infty} x_s \right\} \right\}, \text{ for } x \in \ell_\infty.$$

*This representation can be understood as a social welfare function for an economy with two agents. While the first agent would be highly myopic and only consider the close future of the utility stream  $\hat{\omega} \cdot x$ , the second one would rely on a weaker form of myopia by considering the maximum between the close future value  $\omega \cdot x$  and the infimum limit of the distant future value of the stream. The criterion of the social planner would eventually maximize the welfare of the least favored agent along the classical maximin criteria of Rawls [22] or its more recent acceptance due to Chambers & Echenique [10].*

*It is readily checked that  $I$  satisfies the fundamental axiom **F**. Further observe that, for large enough values of  $T$ , both  $\omega \cdot (\mathbb{0}\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[})$  and  $\hat{\omega} \cdot (\mathbb{0}\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[})$  are bounded above by 1, which implies that the value of the index defined for asymptotically constant unitary gains, namely  $\chi_g$ , sums up to:*

$$I(\mathbb{0}\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}) = \min \left\{ \sum_{s=T+1}^{\infty} \hat{\omega}_s, 1 \right\},$$

*an expression that converges to zero. Likewise, the value of the index defined for asymptotically constant unitary losses satisfies:*

$$\lim_{T \rightarrow \infty} I(\mathbb{0}\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}) = 0.$$

*Remark that there however exist  $x, z \in \ell_\infty$  such that  $\lim_{T \rightarrow \infty} I(x_{[0,T]}, z_{[T+1,\infty[}) \neq I(x)$ . Indeed, the two sequences  $\hat{\omega}$  and  $\omega$  having been assumed to be distinct and both belonging*

<sup>8</sup>One can prove that  $\lim_{T \rightarrow \infty} I(\mathbb{0}\mathbb{1}_{[0,T]}, x_{[T+1,\infty[}) = 0$  for any  $x \in \ell_\infty$ .

<sup>9</sup> $\ell_1$  is the set of real sequences  $\{\omega_s\}_{s=0}^{\infty}$  such that  $\sum_{s=0}^{\infty} |\omega_s| < +\infty$ . For  $\omega \in \ell_1$  and  $x \in \ell_\infty$ , the scalar product  $\omega \cdot x = \sum_{s=0}^{\infty} \omega_s x_s$ . The word "probability" in the statement means that  $\omega_s, \hat{\omega}_s$  are non-negative for any  $s$  and  $\sum_{s=0}^{\infty} \omega_s = \sum_{s=0}^{\infty} \hat{\omega}_s = 1$ .

to  $\ell_1$ , there exists  $x \in \ell_\infty$  such that  $\hat{\omega} \cdot x > \omega \cdot x > \liminf_{s \rightarrow \infty} x_s$ . Considering now  $z$  satisfying  $\hat{\omega} \cdot x > \liminf_{s \rightarrow \infty} z_s > \omega \cdot x > \liminf_{s \rightarrow \infty} x_s$ . It is finally obtained that:

$$\lim_{T \rightarrow \infty} I(x_{[0,T]}, z_{[T+1,\infty[}) = \liminf_{s \rightarrow \infty} z_s,$$

i.e., the infimum of the asymptotic behaviour of the sequence  $z$ , that differs from  $I(x) = \omega \cdot x$ .

Along Example 2.1 where the sole occurrence to two nil values for the myopia parameters, i.e.,  $\chi_g = \chi_\ell = 0$  was not sufficient to ensure the negligibility of the distant future, the following example shows that, under a configuration  $\chi_g = \chi_\ell = 1$  where the two myopia parameters assume unitary values, the close future can keep on influencing on the evaluation of the distant future.

EXAMPLE 2.2. Consider an order being represented by the following index function

$$\hat{I}(x) = \min \left\{ \limsup_{s \rightarrow \infty} x_s, \max \left\{ \omega \cdot x, \liminf_{s \rightarrow \infty} x_s \right\} \right\},$$

with  $\omega$  a probability measure in  $\ell_1$ . Along the interpretation of Example , while the first agent in this economy is extremely non-myopic and evaluates utility streams by the sole consideration of the supremum of its asymptotic values, the second one is only partially myopic. Relying on the same arguments as in Example 2.2, it derives that:

$$\begin{aligned} \lim_{T \rightarrow \infty} \hat{I}(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}) &= 1, \\ - \lim_{T \rightarrow \infty} \hat{I}(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}) &= - \left( \lim_{T \rightarrow \infty} \hat{I}(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}) - \hat{I}(\mathbb{1}) \right) = 1 \end{aligned}$$

whence the obtention of unitary values for the two myopia parameters  $\chi_g = \chi_\ell = 1$ .<sup>10</sup> Considering again  $x, z$  satisfying  $\liminf_{s \rightarrow \infty} x_s < \omega \cdot x < \omega \cdot z < \limsup_{s \rightarrow \infty} x_s$ . It can be checked that

$$\lim_{T \rightarrow \infty} \hat{I}(z_{[0,T]}, x_{[T+1,\infty[}) = \omega \cdot z,$$

which differs from  $\hat{I}(x) = \omega \cdot x$ .

The consideration of Examples 2.1 and 2.2 urges the need for a deeper understanding of the problem at stake, i.e., the precise influence of the remote components of a utility stream. As this shall be argued in the next section, a clear picture becomes available when the preferences order is apprehended through appropriate complementary structures.

### 3. A DECOMPOSITION FOR THE FUTURE: CLOSENESS VS REMOTENESS

#### 3.1 DISTANT FUTURE ORDER

The following axiom assumes that there exists an *evaluation of the distant future components of the utility stream* which is independent from the starting components—the *close future*—of that utility stream.

<sup>10</sup>This implies for any  $x \in \ell_\infty$ ,  $\lim_{T \rightarrow \infty} \hat{I}(x_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}) = 0$ .

**AXIOM G 1.** For any  $x \in \ell_\infty$  and any constant  $d \in \mathbb{R}$ , either, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that for any  $z \in \ell_\infty$ , for every  $T \geq T_0(\epsilon)$ :

$$(z_{[0,T]}, x_{[T+1,\infty]}) \geq (z_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon\mathbb{1},$$

or, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that for any  $z \in \ell_\infty$ , for every  $T \geq T_0(\epsilon)$ :

$$(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) \geq (z_{[0,T]}, x_{[T+1,\infty]}) - \epsilon\mathbb{1}.$$

For any sequence  $x$  and a constant sequence  $d\mathbb{1}$ , the distant future component of the sequence  $x$  will either overtake the sequence  $(d - \epsilon)\mathbb{1}$  or be overtaken by the sequence  $(d + \epsilon)\mathbb{1}$ , and this is going to take place independently from the initial components—the *close future*—of the sequence  $z$ . Otherwise stated, either  $x$  or  $d\mathbb{1}$  dominates in the distant future. This *distant future sensitivities* axiom contradicts with the usual *negligible-tail* or *tail-insensitivity* axioms in the literature. Along these considerations, the simplest conceivable order satisfying both **F** and **G1** relates to the infimum limit of a sequence in  $\ell_\infty$  and is represented by  $I(x) = \liminf_{s \rightarrow \infty} x_s$ . It is associated with the occurrence of unitary values for both of the myopia parameters, *i.e.*,  $\chi_g = \chi_\ell = 1$ .

**DEFINITION 3.1.** Define the order  $\geq_d$  as, for any  $x, y \in \ell_\infty$ , the satisfaction of  $x \geq_d y$  if and only if, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$  and for every  $T \geq T_0(\epsilon)$ :

$$(z_{[0,T]}, x_{[T+1,\infty]}) \geq (z_{[0,T]}, y_{[T+1,\infty]}) - \epsilon\mathbb{1}.$$

The order in Example 2.2 also satisfies axioms **F** and **G1**.

**EXAMPLE 3.1.** Consider again the order represented by the index function  $I$  in Example 2.1.

$$I(x) = \min \left\{ \hat{\omega} \cdot x, \max \left\{ \omega \cdot x, \liminf_{s \rightarrow \infty} x_s \right\} \right\}, \text{ for } x \in \ell_\infty.$$

For any  $x \in \ell_\infty$  and some scalar  $d \in \mathbb{R}$ , consider first the configuration  $\liminf_{s \rightarrow \infty} x_s \geq d$ . Fixing any  $\epsilon > 0$ , select  $T_0(\epsilon)$  such that, for any  $T \geq T_0(\epsilon)$ :

$$\begin{aligned} \sum_{s=T+1}^{\infty} \omega_s x_s &\geq d \sum_{s=T+1}^{\infty} \omega_s - \epsilon, \\ \sum_{s=T+1}^{\infty} \hat{\omega}_s x_s &\geq d \sum_{s=T+1}^{\infty} \hat{\omega}_s - \epsilon. \end{aligned}$$

This translates, for any  $z \in \ell_\infty$  and any  $T \geq T_0(\epsilon)$ , as the satisfaction of:

$$\begin{aligned} I(z_{[0,T]}, x_{[T+1,\infty]}) &\geq \min \left\{ \hat{\omega} \cdot (z_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon, \right. \\ &\quad \left. \max \left\{ \omega \cdot (z_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon, \liminf_{s \rightarrow \infty} x_s - \epsilon \right\} \right\} \\ &\geq I(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon. \end{aligned}$$

Whence, and again for any  $T \geq T_0(\epsilon)$ , the behaviour described by Axiom **G1**:

$$(z_{[0,T]}, x_{[T+1,\infty[}) \geq (z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}) - \epsilon\mathbb{1},$$

a similar line of argument being available for the remaining configuration  $\liminf_{s \rightarrow \infty} x_s \leq d$ . Moreover, and even though  $\chi_g = \chi_\ell = 0$ , the order  $\geq_d$  is not trivial. Select indeed  $z^*$  satisfying  $\hat{\omega} \cdot z^* > 0 > \omega \cdot z^*$ . It derives that:

$$\begin{aligned} \lim_{T \rightarrow \infty} I(z_{[0,T]}^*, 0\mathbb{1}_{[T+1,\infty[}) &= \min\{\hat{\omega} \cdot z^*, \max\{\omega \cdot z^*, 0\}\} = 0, \\ \lim_{T \rightarrow \infty} I(z_{[0,T]}^*, \mathbb{1}_{[T+1,\infty[}) &= \min\{\hat{\omega} \cdot z^*, \max\{\omega \cdot z^*, 1\}\} = \min\{\hat{\omega} \cdot z^*, 1\} > 0. \end{aligned}$$

Whence  $\mathbb{1} \geq_d 0\mathbb{1}$  and  $0\mathbb{1} \not\geq_d \mathbb{1}$ , or  $\mathbb{1} >_d 0\mathbb{1}$ : the order  $\geq_d$  is not trivial.

Proposition 3.1 proves that it suffices for one of the two myopia parameters  $\chi_g$  and  $\chi_\ell$  to differ from zero for the order  $\geq_d$  to satisfy axiom **F**. This in its turn assumes as its most immediate consequence that there also exists an index function satisfying axiom **F**.

**PROPOSITION 3.1.** *Assume that the initial order  $\geq$  satisfies axioms **F** and **G1**.*

- (i) *The order  $\geq_d$  is complete.*
- (ii) *If at least one of the two values  $\chi_g, \chi_\ell$  differs from zero, the order  $\geq_d$  is non-trivial, satisfies axiom **F** and can be represented by an index function  $I_d$ , which is positively homogeneous, constantly additive, satisfying:*

$$I_d(z_{[0,T]}, x_{[T+1,\infty[}) = I_d(x) \text{ for any } x, z \in \ell_\infty, T \in \mathbb{N}.$$

Otherwise stated and from (ii), the value of the index function does not depend upon the starting components of the sequence  $z$ —the *close future*. More generally and, upon a change in a mere finite number of values of the inter-temporal stream, the distant future evaluation of that stream is left unmodified.

## 3.2 CLOSE FUTURE ORDER

In order to enable a decomposition between the *distant future* and the *close future*, consider a *close future sensitivities* axiom **G2**, that is to be understood as *the complement* of axiom **G1**. This axiom assumes that there exists an *evaluation of the close future components of the utility stream* which is independent from the tail—the *distant future*—of that utility stream.

**AXIOM G2.** For any  $x \in \ell_\infty$ , a constant  $c \in \mathbb{R}$ , either, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$  and for every  $T \geq T_0(\epsilon)$ ,

$$(x_{[0,T]}, z_{[T+1,\infty[}) \geq (c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) - \epsilon\mathbb{1},$$

or there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$  and for every  $T \geq T_0(\epsilon)$ ,

$$(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) \geq (x_{[0,T]}, z_{[T+1,\infty[}) - \epsilon\mathbb{1}.$$

This assumption reads as follows: for any sequence  $x$  and a constant sequence  $d\mathbb{1}$ , either the sequence  $x$  will overtake the sequence  $(c - \epsilon)\mathbb{1}$  or it will be dominated by the sequence  $(c + \epsilon)\mathbb{1}$ , both of these occurrences being defined whatever the behaviour in the distant future. Otherwise stated, either  $x$  or  $d\mathbb{1}$  dominates in the close future.

Usual conditions in the literature typically assume that the effect of the distant future converges to zero—*e.g.*, the *Continuity at infinity* of Chambers & Echenique [10], or the axioms ensuring insensitivity to the distant future, or some sort of *negligible tail* for the distribution. Remark that, in opposition to this, the *close future sensitivities* Axiom **G2** merely assumes that the distant future does not alter the evaluation of the close future.

As a basic illustration, consider the order represented by the index function  $I(x) = (1 - \delta) \sum_{s=0}^{\infty} \delta^s x_s$ , for some  $0 < \delta < 1$ . Such an order satisfies both **F** and **G2**, its myopia parameters being both nil  $\chi_g = \chi_\ell = 0$ . A more elaborated formulation is provided in the following example:

**EXAMPLE 3.2.** *Consider the order represented by the index function  $\hat{I}$  in Example 2.2:*

$$\hat{I}(x) = \min \left\{ \limsup_{s \rightarrow \infty} x_s, \max \left\{ \omega \cdot x, \liminf_{s \rightarrow \infty} x_s \right\} \right\}, \text{ for } x \in \ell_\infty.$$

*Fixing any  $x \in \ell_\infty$  and some constant  $c \in \mathbb{R}$ , consider first the configuration  $\omega \cdot x \geq c$ . For any given  $\epsilon > 0$ , select a date  $T_0(\epsilon)$  such that, for any  $T \geq T_0(\epsilon)$ , one has:*

$$\sum_{s=0}^T \omega_s x_s \geq c \sum_{s=0}^T \omega_s - \epsilon.$$

*For any sequence  $z \in \ell_\infty$  and any date  $T \geq T_0(\epsilon)$ , the value of the index  $\hat{I}$  satisfies:*

$$\begin{aligned} \hat{I}(x_{[0,T]}, z_{[T+1,\infty[}) &\geq \min \left\{ \limsup_{s \rightarrow \infty} z_s, \max \left\{ \omega \cdot (c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) - \epsilon, \liminf_{s \rightarrow \infty} z_s \right\} \right\} \\ &\geq \hat{I}(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) - \epsilon. \end{aligned}$$

*Whence, and for any  $T \geq T_0(\epsilon)$ , the behaviour described by Axiom **G2**:*

$$(x_{[0,T]}, z_{[T+1,\infty[}) \geq (c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) - \epsilon\mathbb{1}.$$

*The configuration  $\omega \cdot x \leq c$  could be understood following the same line of arguments. Moreover, and even though  $\chi_g = \chi_\ell = 1$ , the close order  $\geq_c$  is not trivial. Indeed, select  $z^* \in \ell_\infty$  such that  $\liminf_{s \rightarrow \infty} z_s^* < 0 < \limsup_{s \rightarrow \infty} z_s^*$ .*

$$\begin{aligned} \lim_{T \rightarrow \infty} \hat{I}(0\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}^*) &= \min \left\{ \limsup_{s \rightarrow \infty} z_s^*, \max \left\{ 0, \liminf_{s \rightarrow \infty} z_s^* \right\} \right\} = 0, \\ \lim_{T \rightarrow \infty} \hat{I}(\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}^*) &= \min \left\{ \limsup_{s \rightarrow \infty} z_s^*, \max \left\{ 1, \liminf_{s \rightarrow \infty} z_s^* \right\} \right\} = \min \left\{ \limsup_{s \rightarrow \infty} z_s^*, 1 \right\} > 0. \end{aligned}$$

*Whence  $\mathbb{1} \geq_c 0\mathbb{1}$  and  $0\mathbb{1} \not\geq_c \mathbb{1}$ , or  $\mathbb{1} >_c 0\mathbb{1}$ , the order  $\geq_c$  is not trivial.*



**DEFINITION 3.2.** Define the close future order  $\succeq_c$  as, for any  $x, y \in \ell_\infty$ , the satisfaction of  $x \succeq_c y$  if and only if for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for any sequence  $z \in \ell_\infty$  and for every date  $T \geq T_0(\epsilon)$ ,

$$(x_{[0,T]}, z_{[T+1,\infty]}) \succeq (y_{[0,T]}, z_{[T+1,\infty]}) - \epsilon \mathbb{1}.$$

**PROPOSITION 3.2.** Assume that the initial order  $\succeq$  satisfies axioms **F** and **G2**.

- (i) The close order  $\succeq_c$  is complete.
- (ii) If at least one of the two values  $\chi_g, \chi_\ell$  differs from 1, then the order  $\succeq_c$  is non-trivial, satisfies axiom **F** and be represented by an index function  $I_c$  which is positively homogeneous, constantly additive and satisfies:

$$\lim_{T \rightarrow \infty} I_c(x_{[0,T]}, z_{[T+1,\infty]}) = I_c(x) \text{ for any } x, z \in \ell_\infty.$$

The property (ii) illustrates the close future order recovers a *tail-insensitivity* related property, the corresponding distant future order of  $\succeq_c$  being indeed trivial.

The results in Propositions 3.1 and 3.2 as well as the characterizations in Examples 3.1 and 3.2, suggest the need for a more achieved characterization of the configurations where the two myopia parameters assume boundary values, *i.e.*,  $\chi_g = \chi_\ell = 0$ , or  $\chi_g = \chi_\ell = 1$ . The following statement aims at providing a clarified view of the way these relate with the triviality of either the distant or the future order:

**PROPOSITION 3.3.** Assume that the order  $\succeq$  satisfies axioms **F** and **G1, G2**.

- (i) If  $\chi_g = \chi_\ell = 0$ , then the order  $\succeq_d$  is trivial: for any  $x, y \in \ell_\infty$ ,  $x \sim_d y$ . Moreover,  $I(x) = I_c(x)$ .
- (ii) If  $\chi_g = \chi_\ell = 1$ , then the order  $\succeq_c$  is trivial: for any  $x, y \in \ell_\infty$ ,  $x \sim_c y$ . Moreover,  $I(x) = I_d(x)$ .

Otherwise stated, it is only in the case where the initial order satisfies *both* axioms **G1** and **G2**, *i.e.*, the one where the decomposition between the distant and the close components future is fully completed, that the boundary values for the myopia coefficients may result into the triviality of one of the two orders.

From now on, by convention, under axioms **F** and **G1, G2**, if  $\chi_g = \chi_\ell = 0$ , the distant future function will be defined as:  $I_d(x) = 0$  for any  $x \in \ell_\infty$ , and if  $\chi_g = \chi_\ell = 1$ , the close future function will be defined as:  $I_c(x) = 0$  for any  $x \in \ell_\infty$ .

### 3.3 A DECOMPOSITION BETWEEN THE DISTANT AND CLOSE FUTURE ORDERS

From the previous developments and under axioms **F, G1** and **G2**, one may surmise that there is some potential for the index function  $I$  to be decomposed into a convex sum of two index functions  $I_d$  and  $I_c$ , *e.g.*,

$$I(x) = (1 - \chi^*)I_c(x) + \chi^*I_d(x),$$

for some value  $\chi^* \in [0, 1]$ . First observe that, would the selected parameter  $\chi^*$  not modify over time, such a decomposition ought to imply that:

$$\lim_{T \rightarrow \infty} I(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty]}) + \lim_{T \rightarrow \infty} I(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) = 1,$$

which is equivalent to  $\chi_\ell = \chi_g$ , and therefore  $\chi^* = \chi_g = \chi_\ell$ . Remark however that, under axioms **F**, **G1** and **G2**, the satisfaction of such an equality cannot be guaranteed. In order to obtain this property, consider the following axiom, which is a strong version of *weak convexity* property. This axiom ensures that the comparison between two streams does not change by mixing them with a "direction" which is constant in close future and constant in distant future. A direct corollary of this property is, these "directions" are also *comparison neutral*.

**AXIOM A1.** Consider  $x, y \in \ell_\infty$  such that  $x \geq y$ , any constants  $c, d \in \mathbb{R}$ , any  $0 \leq \lambda \leq 1$ . For any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that for any  $T \geq T_0(\epsilon)$

$$(1 - \lambda)x + \lambda(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) \geq (1 - \lambda)y + \lambda(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon\mathbb{1}.$$

Under axiom **A1**, the two earlier myopia parameters  $\chi_g$  and  $\chi_\ell$  are equal. The index function satisfies a generalization of *constant additive* property. The evaluation of a given inter-temporal stream is decomposed as a convex combination of its close future and distant future values, the convex parameter being equal to the common myopia parameter.

**THEOREM 3.1.** *Assume that the initial order  $\geq$  satisfies axioms **F** and **G1**, **G2** and **A1**.*

- (i) *The two myopia parameters are equal:  $\chi_g = \chi_\ell = \chi^*$ .*
- (ii) *For any  $x \in \ell_\infty$ , any constants  $c, d \in \mathbb{R}$ ,*

$$\lim_{T \rightarrow \infty} I(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty]})) = I(x) + (1 - \chi^*)c + \chi^*d.$$

- (iii) *For any  $x \in \ell_\infty$ ,*

$$I(x) = (1 - \chi^*)I_c(x) + \chi^*I_d(x).$$

While a convex combination between Examples 3.1 and 3.2 could have been conjectured to provide an interesting illustration of this decomposition, it is readily checked that such a formulation is inappropriate since it would satisfy neither Axiom **G1**, nor Axiom **G2**. The following illustration will however provide an elementary of the properties at stake:

**EXAMPLE 3.3.** *As a basic illustration, consider the order represented by the following index function:*

*For some  $0 < \chi^* < 1$ , the index function*

$$I(x) = (1 - \chi^*) \sum_{s=0}^{\infty} (1 - \delta)^s x_s + \chi^* \liminf_{s \rightarrow \infty} x_s,$$

is such that  $I_c(x) = \sum_{s=0}^{\infty} (1-\delta)\delta^s x_s$  and  $I_d(x) = \liminf_{s \rightarrow \infty} x_s$ . Fixing indeed any sequence  $x \in \ell_{\infty}$  and a constant  $c \in \mathbb{R}$ , any scalar  $\epsilon > 0$ . Consider the case  $\sum_{s=0}^{\infty} (1-\delta)\delta^s x_s \geq c$  and fix a date  $T_0(\epsilon)$  such that for any date  $T \geq T_0(\epsilon)$ , one has  $\sum_{s=0}^T (1-\delta)\delta^s x_s \geq c \sum_{s=0}^T \omega_s - \epsilon$ . This in turn implies that, for any  $z \in \ell_{\infty}$ , the following inequality is satisfied:

$$\begin{aligned} (1 - \chi^*) \left( \sum_{s=0}^T (1 - \delta) \delta^s x_s + \sum_{s=T+1}^{\infty} (1 - \delta) \delta^s z_s \right) + \chi^* \liminf_{s \rightarrow \infty} z_s \\ \geq (1 - \chi^*) \left( c \sum_{s=0}^T \omega_s + \sum_{s=T+1}^{\infty} \omega_s z_s \right) + \chi^* \liminf_{s \rightarrow \infty} z_s - \epsilon. \end{aligned}$$

It derives that, for any  $T \geq T_0(\epsilon)$  and  $z \in \ell_{\infty}$ , the index  $I$  satisfies:

$$I(x_{[0,T]}, z_{[T+1,\infty]}) \geq I(c \mathbb{1}_{[0,T]}, z_{[T+1,\infty]}) - \epsilon.$$

The remaining occurrence  $\sum_{s=0}^{\infty} (1-\delta)\delta^s x_s \leq c$  can be analyzed with a related argument. The order thus satisfies the close future sensitivities axiom **G2**, its close future order being represented by the function  $I_c(x) = \sum_{s=0}^{\infty} (1-\delta)\delta^s x_s$ . Relying on the same arguments, it is readily proved that this order also satisfies the distant future sensitivities axiom **G1**, its distant future index function being given by  $I_d(x) = \liminf_{s \rightarrow \infty} x_s$ .

As soon as a weaker form of convexity is retained and axiom **A1** is relaxed, results become more involving but also uncover interesting new facets. Indeed, and while the evaluation of a inter-temporal stream keeps on formulating as a convex combination of the its close future and distant future values, the decomposition parameter must now change as a function of the involved sequence  $x$ .

The configuration

$$\lim_{T \rightarrow \infty} I(\mathbb{1}_{[0,T]}, 0 \mathbb{1}_{[T+1,\infty]}) + \lim_{T \rightarrow \infty} I(0 \mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) \leq 1,$$

which is equivalent to the holding of  $\chi_g \leq \chi_l$ , can first be understood as a *pessimistic*, or a mainly *myopia-bending* occurrence: the value brought by the distant future is not sufficiently large to compensate the loss that is incurred in the close future.

Likewise, the configuration

$$\lim_{T \rightarrow \infty} I(\mathbb{1}_{[0,T]}, 0 \mathbb{1}_{[T+1,\infty]}) + \lim_{T \rightarrow \infty} I(0 \mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) \geq 1,$$

which is equivalent to the holding of  $\chi_g \geq \chi_l$ , can be understood as an *optimistic*, or an essentially *nonmyopia-bending* situation: the gain in the distant future is valued more than the lost that is incurred in the close future.

The following theorem will prove that there exists a multiplicity of *admissible myopia degrees*. This theorem also clarifies how it is the very choice of the *myopia degree*  $\chi$  that determines an optimal share between the close future and the distant future indexes.

**THEOREM 3.2.** *Assume that the initial order  $\geq$  satisfies axioms **F** and **G1**, **G2**.*

(i) For any  $x \in \ell_\infty$ , let  $c_x = I_c(x)$  and  $d_x = I_d(x)$ .

a) If  $c_x \leq d_x$ , then

$$I(x) = (1 - \chi_g)c_x + \chi_g d_x.$$

b) If  $c_x \geq d_x$ , then

$$I(x) = (1 - \chi_\ell)c_x + \chi_\ell d_x.$$

(ii) Let  $\underline{\chi} = \min\{\chi_g, \chi_\ell\}$ , and  $\bar{\chi} = \max\{\chi_g, \chi_\ell\}$ .

a) If  $\chi_g \leq \chi_\ell$ , then and for every  $x \in \ell_\infty$ ,

$$I(x) = \min_{\underline{\chi} \leq \chi \leq \bar{\chi}} [(1 - \chi)I_c(x) + \chi I_d(x)].$$

b) If  $\chi_g \geq \chi_\ell$ , then and for every  $x \in \ell_\infty$ ,

$$I(x) = \max_{\underline{\chi} \leq \chi \leq \bar{\chi}} [(1 - \chi)I_c(x) + \chi I_d(x)].$$

First remark that, from Theorem 3.2(i), the evaluation can be expressed as a function of the distant and close future values. The weight of the convex combination being provided by the remote gains myopia coefficient  $\chi_g$  for the case where the close future is less valued than the distant future and by the remote losses myopia coefficient  $\chi_\ell$  in the opposite case. Theorem 3.2(ii) is a direct consequence of Theorem 3.2(i). For  $\chi_g \leq \chi_\ell$ , the decision maker will always assign the highest possible parameter to the smallest value between  $I_c(x)$  and  $I_d(x)$ ; (s)he indeed always selects the minimum value of a convex combination whose weight is given by  $\chi$ .

For  $\chi_g \geq \chi_\ell$ , the decision maker exhibits an opposite behaviour, (s)he chooses the highest possible parameter to the bigger value between  $I_c(x)$  and  $I_d(x)$ ; (s)he always selects the maximum value of a convex combination whose weight is given by  $\chi$ .

One can take to another interpretation as follow: The case  $\chi_g \leq \chi_\ell$ , which is associated with the operator  $\min$ , represents the case where the increasing of utility coming from remote gain does not compensate the suffer caused by a remote loss of the same amount. The case  $\chi_g \geq \chi_\ell$ , which is associated with the operator  $\max$ , represents an opposite behavior.

It should finally be pointed out that neither the operator  $\min$  can prevail under  $\chi_g > \chi_\ell$ , nor the operator  $\max$  under  $\chi_g < \chi_\ell$ , the behaviour of decision maker being thus appropriately described by the comparison between  $\chi_g$  and  $\chi_\ell$ .

## 4. AN AUGMENTED DECOMPOSITION FOR TIME-DEPENDENT ORDERS

This Section will introduce time dependencies into the benchmark order and argue that this provides a new and integrated picture of the scope for *temporal biases* that is directly rooted on the closeness and remoteness concepts that were put forth in the earlier decomposition.

#### 4.1 TIME-DEPENDENT ORDERS & THE ROLE OF THE CLOSE FUTURE ORDER

The following axiom assumes that the decision maker can always evaluate the intertemporal stream starting from a certain date and that such an evaluation will take place independently from both the previous values of the stream and the distant future values of the stream.

**AXIOM B1.** Consider  $T \geq 1$ ,  $x \in \ell_\infty$  and a constant  $c \in \mathbb{R}$ . Either, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon) \geq T$  such that, for any  $z, z' \in \ell_\infty$ , any  $T' \geq T_0(\epsilon)$ ,

$$(z_{[0, T-1]}, x_{[T, T']}, z'_{[T'+1, \infty[}) \geq (z_{[0, T-1]}, c \mathbb{1}_{[T, T']}, z'_{[T'+1, \infty[}) - \epsilon \mathbb{1},$$

or, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon) \geq T$  such that, for any  $z, z' \in \ell_\infty$ , any  $T' \geq T_0(\epsilon)$ ,

$$(z_{[0, T-1]}, c \mathbb{1}_{[T, T']}, z'_{[T'+1, \infty[}) \geq (z_{[0, T-1]}, x_{[T, T']}, z'_{[T'+1, \infty[}) - \epsilon \mathbb{1}.$$

This axiom contemplates a variation of the classical *limited independence* condition of Koopmans [16] where the close future evaluation of some date  $T$  does neither depend on the origins nor on the distant future. Either the sequence  $x_{[T, +\infty[}$  dominates the constant sequence independently from the head and the tail of the utility sequences, or it is dominated by the constant sequence independently from the head and the tail of the utility sequences.

**LEMMA 4.1.** *Assume that the order  $\geq$  satisfies axioms **F**, **G1**, **G2**, and **B1**. Fix  $T \geq 1$ . For any  $x \in \ell_\infty$ , any constant  $c$ , either, for any  $z \in \ell_\infty$ :*

$$(z_{[0, T-1]}, x) \geq_c (z_{[0, T-1]}, c \mathbb{1}_{[T, \infty[}),$$

or, for any  $z \in \ell_\infty$ :

$$(z_{[0, T-1]}, c \mathbb{1}_{[T, \infty[}) \geq_c (z_{[0, T-1]}, x).$$

With Lemma 4.1, the analysis of the time-dependent order rests upon the one of the properties of the future order  $\geq_c$ .

**DEFINITION 4.1.** *For any  $x, y \in \ell_\infty$ , the temporal order  $\geq_T$  is defined as:  $x \geq_T y$  if and only if for any  $z_0, z_1, \dots, z_{T-1}$ ,*

$$(z_{[0, T-1]}, x) \geq_c (z_{[0, T-1]}, y).$$

Under axiom **B1**, the close future evaluation after some date  $T$  can be represented by the order  $\geq_T$  and an index function  $I_T$  satisfying axiom **F**.

**PROPOSITION 4.1.** *Assume that the order  $\geq$  satisfies axioms **F**, **G1**, **G2** and **B1**.*

(i) *For every  $T \geq 1$ , the order  $\geq_T$  is complete.*

(ii) *If at least one of the two values  $I_c(\mathbf{o}\mathbb{1}_{[0,T-1]}, \mathbb{1}_{[T,\infty[)})$  and  $-I_c(\mathbf{o}\mathbb{1}_{[0,T-1]}, -\mathbb{1}_{[T,\infty[)})$  differs from zero, then the order  $\succeq_T$  satisfies the axiom **F** and can be represented by an index function  $I_T$  satisfying positive homogeneity and constant additive properties:*

- a)  $I_T(\lambda x) = \lambda I_T(x)$ , for every  $\lambda \geq \mathbf{o}$ .
- b)  $I_T(x + c\mathbb{1}) = I_T(x) + c$ , for every constant  $c \in \mathbb{R}$ .

Define  $\chi_g^T$  and  $\chi_\ell^T$  as the parameters measuring the gain and the loss in the future of  $T$ .

$$\begin{aligned}\chi_g^T &= I_T(\mathbf{o}, \mathbb{1}), \\ \chi_\ell^T &= -I_T(\mathbf{o}, -\mathbb{1}) = \mathbf{1} - I_T(\mathbb{1}, \mathbf{o}\mathbb{1}).\end{aligned}$$

First remark that the properties of these parameters directly result from the ones of the close future index. Indeed,  $\chi_g^T > \mathbf{o}$  if and only if  $I_c(\mathbf{o}\mathbb{1}_{[0,T-1]}, \mathbb{1}_{[T,\infty[)}) > \mathbf{o}$  while  $\chi_\ell^T > \mathbf{o}$  if and only if  $-I_c(\mathbf{o}\mathbb{1}_{[0,T-1]}, -\mathbb{1}_{[T,\infty[)}) > \mathbf{o}$ . For the sake of simplicity, and by convention, in the case  $\chi_g^T = \chi_\ell^T = \mathbf{o}$ ,<sup>11</sup> the order  $\succeq_{T+1}$  becoming trivial, the temporal index function  $I_{T+1}$  is defined as:  $I_{T+1}(x) = \mathbf{o}$  for any  $x \in \ell_\infty$ .

**PROPOSITION 4.2.** *Assume that the order  $\succeq$  satisfies axioms **F**, **G1**, **G2** and **B1**.*

*Let  $\underline{\delta}_T = \min\{\chi_g^T, \chi_\ell^T\}$  and  $\bar{\delta}_T = \max\{\chi_g^T, \chi_\ell^T\}$ .*

(i) *If  $\chi_g^T \leq \chi_\ell^T$ , then for any  $x \in \ell_\infty$ :*

$$I_T(x_{[T,\infty[)}) = \min_{\underline{\delta}_T \leq \delta \leq \bar{\delta}_T} [(1 - \delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[)})].$$

(ii) *If  $\chi_g^T \geq \chi_\ell^T$ , then for any  $x \in \ell_\infty$ :*

$$I_T(x_{[T,\infty[)}) = \max_{\underline{\delta}_T \leq \delta \leq \bar{\delta}_T} [(1 - \delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[)})].$$

At any given date, the evaluation of a utility stream builds from a *recursive convex sum* between the utility level at that date, and the evaluation at the subsequent date of the utility stream. Interestingly, a multitude of choices are shown to be admissible for the weight parameters of this convex sum. The minimum solution hereby represents a configuration where the value of the future beyond some date  $T$  is not large enough to compensate the lost that is incurred in present, the maximum solution representing the opposite occurrence. It is finally to be stressed that, relying upon a system of axioms based upon *time-variability aversion*, Wakai [24] has provided an insightful account of smoothing behaviours with *gain/loss asymmetry* which explicitly builds upon a related recursive representation.

The scope for separability being however central to this study, it is of interest to emphasize its specificity with respect to its earlier acceptions in the literature. Indeed,

<sup>11</sup>This implies that the order  $\succeq_{T+1}$  is trivial, and for any  $T' \geq T$ ,  $\chi_g^{T'} = \chi_\ell^{T'} = \mathbf{o}$ .

both the classical approach of Koopmans [16] or the more recent axiomatization of quasi-hyperbolic discounting due to Montiel Olea & Strzalecki [20] assume that the first or the first and the second components of two utility streams can be compared independently from its future components. Together with *stationarity* or *quasi-stationarity* postulates on the preferences ordering, these imply the existence of unique discount rate for every day or every generation, such a discount rate being constant for any  $T \geq 0$  with *stationarity*, or constant from  $T = 1$  to infinity with a *quasi-stationarity*. In contradistinction with this, the approach of this article postulates that the components of two utility streams starting from a given date can be compared independently from their earlier past components, that gives rise to the possibility of multiple discount rates. The following example further proves that, for multiple discount rates, neither the *independence* nor the *extended independence* of Koopmans [16] are satisfied, the two approaches hence fundamentally differing, be it in their formulation or in their predictions.

EXAMPLE 4.1. Consider the configuration where for any  $T \geq 0$ ,  $\underline{\delta}_T = 0.5$ ,  $\bar{\delta}_T = 0.8$  and the operator is min. For any  $T$ ,

$$I_T(x_{[T,\infty[}) = \min_{0.5 \leq \delta \leq 0.8} [(1 - \delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})].$$

Consider these two utility streams  $x = (1, 0, 0, 0, \dots)$  and  $y = (0.5, 0.5, 0, 0, \dots)$ . Simple calculus give  $I_1(x_{[1,\infty[}) = 0$  and hence  $I_0(x) = 0.2$ . Similarly since  $I_1(y_{[1,\infty[}) = 0.1$ , one has  $I_0(y) = 0.18$ . This implies  $x > y$ .

Now consider  $x' = (1, 0, 0.5, 0.5, \dots)$  and  $y' = (0.5, 0.5, 0.5, 0.5, \dots)$ . The two sequences  $x$  and  $y$  are changed by keeping intact the first two components. One has  $I_2(x'_{[2,\infty[}) = 0.5$ . Calculus give  $I_1(x'_{[1,\infty[}) = 0.25$  and  $I_0(x') = 0.4$ . Obviously,  $I_0(y') = 0.5$ . This implies  $y' > x'$ . The extended stationarity property in Koopmans [16] is not satisfied.

## 4.2 PRESENT BIAS, DISTANT FUTURE BIAS & THE COMPLETENESS OF THE DECOMPOSITION FOR TIME-DEPENDENT ORDERS

### 4.2.1 A MULTIPLE DISCOUNT FORMULATION FOR PRESENT-BIASED PREFERENCES

*Present bias* is commonly understood in the literature in terms of a behaviour that is controlled by the discount rate and according to which, what happens today affects more the decision maker than it would were this to happen by some day in the future. A gain—equivalently, a loss—today causes more happiness—more unhappiness—than the same occurrence in the future. This is one of sources of *time inconsistency*: the decision maker may prefer some small amount today than a larger one tomorrow, but that same small amount tomorrow is less enjoyable than the same larger one after-tomorrow.

This section is organized in order to examine the scope for such a phenomenon while making use of the earlier temporal decomposition. The following axiom, building from two separate parts, is a move in that direction. The first part of the axiom says that the delay of a perpetual gain to the next day and at time  $T$

diminishes the happiness of the decision maker more than it would do at time  $T + 1$  or for other dates in the future of  $T$ . The second part introduces another behaviour: delaying a perpetual loss at date  $T$  makes the decision maker more happy than if this were to take place in the future of  $T$ .

AXIOM B2. For any  $T \geq 1$  and constants  $c \geq 0$ ,

(i) If there exists  $T'_0$  such that, for any  $T' \geq T'_0$ ,

$$(\mathbb{0}\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}) \succeq (\mathbb{0}\mathbb{1}_{[0,T-1]}, c\mathbb{1}_{[T,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}),$$

then there exists  $T'_1$  such that, for any  $T' \geq T'_1$ ,

$$(\mathbb{0}\mathbb{1}_{[0,T+1]}, \mathbb{1}_{[T+2,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}) \succeq (\mathbb{0}\mathbb{1}_{[0,T]}, c\mathbb{1}_{[T+1,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}).$$

(ii) If there exists  $T'_0$  such that, for any  $T' \geq T'_0$ ,

$$(\mathbb{0}\mathbb{1}_{[0,T-1]}, (-c)\mathbb{1}_{[T,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}) \succeq (\mathbb{0}\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}),$$

then there exists  $T'_1$  such that, for any  $T' \geq T'_1$ ,

$$(\mathbb{0}\mathbb{1}_{[0,T]}, (-c)\mathbb{1}_{[T+1,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}) \succeq (\mathbb{0}\mathbb{1}_{[0,T+1]}, -\mathbb{1}_{[T+2,T']}, \mathbb{0}\mathbb{1}_{[T'+1,\infty[)}).$$

The *supremum*—the greatest of the minorants—of the values of the parameter  $c$  in part (i) and part (ii) can both be used to figure out the *perception of the temporal distance* between date  $T$  and date  $T + 1$ . These *extremum* values are evaluated using the perception at date  $T$  of the two sequences  $(\mathbb{0}, \mathbb{1})$  and  $(\mathbb{0}, -\mathbb{1})$ . Axiom **B2** means that this *temporal distance* is decreasing as a function of  $T$ .

The following lemma provides a straightforward implication of axiom **B2**.

LEMMA 4.2. Assume that the order  $\succeq$  satisfies axioms **F**, **G1**, **G2** and **B1**, **B2**. Suppose that, for any  $T$ , at least one of the two values  $\chi_g^T$  and  $\chi_\ell^T$  differs from 1 and at least one of the two values  $\chi_g^T$  and  $\chi_\ell^T$  differs from zero. For any  $T \geq 0$  and a constant  $c \geq 0$ ,

(i) If  $(\mathbb{0}, \mathbb{1}) \sim_T c\mathbb{1}$ , then  $(\mathbb{0}, \mathbb{1}) \succeq_{T+1} c\mathbb{1}$ .

(ii) If  $(-c)\mathbb{1} \sim_T (\mathbb{0}, -\mathbb{1})$ , then  $(-c)\mathbb{1} \succeq_{T+1} (\mathbb{0}, -\mathbb{1})$ .

Delaying gain and loss affects the decision maker more at time  $T$  than at time  $T + 1$ . Indeed, at time  $T$ , delaying the gain for one day diminishes the welfare value from 1 to  $c$ . Delaying the same gain at time  $T + 1$  will diminish the welfare from 1 to some value  $c' \geq c$ . Similarly, delaying a loss at time  $T$  increases the welfare value from  $-1$  to  $-c$ , which is higher than to delay the same loss at time  $T + 1$ , which increases from  $-1$  to some  $(-c')$  smaller than  $-c$ .

Otherwise stated, the *temporal distance* that is perceived between dates  $T$  and  $T + 1$  is larger than the one that is perceived between dates  $T + 1$  and  $T + 2$ : at date  $T$ , the valuation of a constant sequence from tomorrow on is lower than its corresponding valuation at date  $T + 1$ . This intuition is presented in the following Proposition.



PROPOSITION 4.3. Assume that the initial order  $\succeq$  satisfies axioms **F**, **G1**, **G2** and **B1**, **B2**. The order  $\succeq$  is present-biased in that the two sequences  $\{\underline{\delta}_T\}_{T=0}^\infty$  and  $\{\bar{\delta}_T\}_{t=0}^\infty$  of the min or max occurrences in Proposition 4.2 are increasing according to  $\underline{\delta}_T \leq \underline{\delta}_{T+1}$  and  $\bar{\delta}_T \leq \bar{\delta}_{T+1}$  for any  $T$ .

The following *Quasi-stationarity* axiom, which is similar to the axiom 4 in Montiel Olea & Strzalecki [20], implies a generalization of *quasi-hyperbolic discounting*. The preferences satisfy the stationarity for the times  $T \geq 1$ .

AXIOM B3. For any  $x \in \ell_\infty$ , any constants  $c, \tilde{c} \in \mathbb{R}$ : for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$ , such that for any  $z \in \ell_\infty$ , any  $T \geq T_0(\epsilon)$ :

$$(z_0, \tilde{c}, x_{[0,T]}, z_{[T+2,+\infty)}) \succeq (z_0, \tilde{c}, c\mathbb{1}_{[0,T]}, z_{[T+2,+\infty)}) - \epsilon\mathbb{1},$$

if and only if for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that for any  $z \in \ell_\infty$ ,  $T \geq T_0(\epsilon)$ ,

$$(z_0, x_{[0,T]}, z_{[T+1,+\infty)}) \succeq (z_0, c\mathbb{1}_{[0,T]}, z_{[T+1,+\infty)}) - \epsilon\mathbb{1}.$$

The intuition of axiom **B3** can be represented in the following Lemma, which is the same statement as in Montiel Olea & Strzalecki [20].

LEMMA 4.3. Assume that the order  $\succeq$  satisfies axioms **F**, **G1**, **G2**, **B1**, and **B3**. For any constants  $a, b \in \mathbb{R}$ , utility streams  $x, y \in \ell_\infty$ ,

$$(a, b, x) \succeq_c (a, b, y) \text{ if and only if } (a, x) \succeq_c (a, y).$$

Under axiom **B3**, one can establish a generalization of *quasi-hyperbolic discounting* class of preferences.

PROPOSITION 4.4. Assume that the order  $\succeq$  satisfies axioms **F**, **G1**, **G2**, **B1**, and **B3**. Let  $\underline{\delta}_T = \min\{\chi_1^T, \chi_2^T\}$  and  $\bar{\delta}_T = \max\{\chi_1^T, \chi_2^T\}$ .

- (i) For any  $T \geq 1$ ,  $\underline{\delta}_T = \underline{\delta}_1$  and  $\bar{\delta}_T = \bar{\delta}_1$ .
- (ii) By adding axiom **B2**,  $\underline{\delta}_0 \leq \underline{\delta}_1$  and  $\bar{\delta}_0 \leq \bar{\delta}_1$ .

While, under Axiom **B1** and for each date  $T$ , there exists a set of possible discount rates, the quasi-stationarity axiom **B3** ensures that these sets are the same for any date  $T \geq 1$ . Moreover, and for any  $T \geq 1$ , the statement of Lemma 4.2 is strengthened to an equivalence condition. Finally, and as this is clarified in Proposition 4.4(ii), combining with the axiom **B2**, the set of discount rates associated with date  $T = 0$  assumes smaller lower and upper bounds than the subsequent sets of discount rates associated with  $T \geq 1$ .

REMARK 4.1. This result does provide an interesting generalization of the quasi-hyperbolic discounting of Phelps & Pollack [21] and Laibson [18]. Consider, e.g., the case where for

any  $T$ ,  $\underline{\delta}_T = \overline{\delta}_T$  with  $\delta_0 \leq \delta_1 = \delta$ ,<sup>12</sup> while the distant future is negligible. The comparison between two inter-temporal stream becomes:  $x \geq y$  if and only if

$$(1 - \delta_0)x_0 + \delta_0 \left( \sum_{s=0}^{\infty} (1 - \delta)\delta^s x_{1+s} \right) \geq (1 - \delta_0)y_0 + \delta_0 \left( \sum_{s=0}^{\infty} (1 - \delta)\delta^s y_{1+s} \right),$$

which is equivalent to

$$x_0 + \beta \left( \sum_{s=1}^{\infty} \delta^s x_s \right) \geq y_0 + \beta \left( \sum_{s=1}^{\infty} \delta^s y_s \right),$$

where  $\beta = (\delta_0/\delta) \times (1 - \delta)/(1 - \delta_0) \leq 1$ .

#### 4.2.2 DISTANT FUTURE BIAS & THE COMPLETENESS OF THE DECOMPOSITION FOR TIME-DEPENDENT ORDERS

The earlier section has provided a new and generalized picture of the much discussed and experimentally supported *present biases* that have been the object of numerous recent contributions. It was indeed clarified how this closely articulates with the properties of the close future order and the availability of a recursive representation for the time-dependent order. This argument unequivocally resting upon the properties of the close future order, an unattractive facet of these time-dependent orders is however that the decomposition is not any longer complete. This section will remedy to this conceptual difficulty by introducing distant-future temporal biases and argue how that this provides an enriched version of the decomposition for the future between closeness and remoteness. Exhibiting a future bias here means the emergence, for large values of  $T$ , of some sort of continuity between the temporal orders  $\geq_T$  and the distant order  $\geq_d$ . More explicitly, there does emerge a *range of influences that spring from the evaluation of the distant future*—the *infinite*—on the  $\geq_T$ -evaluations of the close future—the *finite*.

AXIOM B<sub>4</sub>. Take  $x, z \in \ell_\infty$  and some constants  $d, d' \in \mathbb{R}$ : if there exists  $T_0$  such that, for any  $T \geq T_0$ ,

$$(z_{[0,T]}, d \mathbb{1}_{[T+1,\infty[}) \geq (z_{[0,T]}, x_{[T+1,\infty[}) \geq (z_{[0,T]}, d' \mathbb{1}_{[T+1,\infty[}),$$

then, for any  $\epsilon > 0$ , there exist  $T'_0(\epsilon)$  such that, for any  $T \geq T'_0(\epsilon)$ , there exists  $s_0$  that depends on  $T$ , such that, for any large enough  $s \geq s_0$ ,

$$\begin{aligned} (z_{[0,T]}, d \mathbb{1}_{[T+1,\infty[}) &\geq (z_{[0,T]}, x_{[T+1,T+s]}, d \mathbb{1}_{[T+s+1,\infty[}) - \epsilon \mathbb{1}, \\ (z_{[0,T]}, x_{[T+1,T+s]}, d' \mathbb{1}_{[T+s+1,\infty[}) &\geq (z_{[0,T]}, d' \mathbb{1}_{[T+1,\infty[}) - \epsilon \mathbb{1}. \end{aligned}$$

This axiom means that, for a large finite time  $T$ , the evaluation of the sequence in a future that is close from  $T$  is influenced by the distant future evaluation. Even though its formulation is complex, the underlying intuition is rather simple and presented in the following proposition.

<sup>12</sup>This property can be obtained by adding the following assertion:  $x \geq_c y$  if and only if  $x + z \geq_c y + z$  for any  $z \in \ell_\infty$ .

PROPOSITION 4.5. Assume that the order  $\geq$  satisfies axioms **F**, **G1**, **G2**, **B1** and **B4**.

(i) For any  $x \in \ell_\infty$ ,  $d, d' \in \mathbb{R}$ , if

$$d\mathbb{1} \geq_d x \geq_d d'\mathbb{1},$$

then, for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for  $T \geq T_0(\epsilon)$ ,

$$(d + \epsilon)_{[T, \infty[} \mathbb{1} \geq_T x_{[T, \infty[} \geq_T (d' - \epsilon)_{[T, \infty[}.$$

(ii) For any  $x \in \ell_\infty$ ,  $\lim_{T \rightarrow \infty} I_T(x_{[T, \infty[}) = I_d(x)$ .

(iii) The sequence  $\{\bar{\delta}_T\}_{t=0}^\infty$  satisfies  $\limsup_{T \rightarrow \infty} \bar{\delta}_T = 1$ .

(iv) Adding axiom **B2**, the sequence  $\{\bar{\delta}_T\}_{t=0}^\infty$  further satisfies  $\lim_{T \rightarrow \infty} \bar{\delta}_T = 1$ .

Otherwise stated and in (i), the sequence  $x$  will either always dominate or be dominated by a constant sequence under the order  $\geq_T$  and for sufficiently large values of  $T$ . In these regards, a similitude should be pointed out with the former *distant future sensitivities* axiom **G1**, where a sequence  $x$  either always dominates or is always dominated by a constant sequence in the distant future. (ii), in its turn, enlightens the scope for some sort of continuity between  $I_T$  and  $I_d$  for sufficiently large values of  $T$ .

The following illustrating example provides a final illustration of an order which concomitantly has *present bias* and *distant future bias* facets.

EXAMPLE 4.2. Consider discount rate sequences  $\{\underline{\delta}_T\}_{T=0}^\infty$  and  $\{\bar{\delta}_T\}_{T=0}^\infty$  such that: for  $T$  sufficiently large,  $\bar{\delta}_T = 1$  and for any  $T$ ,  $\underline{\delta}_T \leq \underline{\delta}_{T+1}$ . For  $s = 0$ , define the index functions  $I_T^0$  as:

$$I_T^0(x_{[T, \infty[}) = \liminf_{T \rightarrow \infty} x_t.$$

For each  $s \geq 0$ , let

$$I_T^{s+1}(x_{[T, \infty[}) = \min_{\underline{\delta}_T \leq \delta \leq \bar{\delta}_T} [(1 - \delta)x_T + \delta I_{T+1}^s(x_{[T+1, \infty[})].$$

For each  $T$ , let finally  $I_T(x_{[T, \infty[}) = \liminf_{s \rightarrow \infty} I_T^s(x_{[T, \infty[})$ . One can verify that for any  $T$ ,

$$I_T(x_{[T, +\infty[}) = \min_{\underline{\delta}_T \leq \delta \leq \bar{\delta}_T} [(1 - \delta)x_T + \delta I_{T+1}(x_{[T+1, +\infty[})].$$

By construction and using a recurrence argument, for  $T$  sufficiently big, for any  $s$ ,

$$I_T^s(x_{[T, \infty[}) \leq \liminf_{t \rightarrow \infty} x_t,$$

whence, for any  $T$  sufficiently big, the satisfaction of  $I_T(x_{[T, \infty[}) \leq \liminf_{t \rightarrow \infty} x_t$ . Further noticing that  $\liminf_{T \rightarrow \infty} I_T(x_{[T, \infty[}) \geq \liminf_{t \rightarrow \infty} x_t$  and finally taking  $I_d(x) = \liminf_{t \rightarrow \infty} x_t$ , it follows that

$$\lim_{T \rightarrow \infty} I_T(x_{[T, \infty[}) = I_d(x).$$

Define  $I_c(x) = I_0(x)$ . Consider the index function defined as

$$I(x) = \min_{\underline{\chi} \leq \chi \leq \bar{\chi}} [(1 - \chi)I_c(x) + \chi I_d(x)] \text{ for some } 0 < \underline{\chi} < \bar{\chi} < 1.$$

By the construction of  $\underline{\delta}_T$  and  $\bar{\delta}_T$ , the order  $\geq$  represented by  $I$  satisfies present bias property. Since  $\lim_{T \rightarrow \infty} I_T(x_{[T, \infty[}) = I_d(x)$ , this order therefore satisfies the future bias property.

## A. PROOFS FOR SECTION 2

### A.1 PROOF OF PROPOSITION 2.1

For  $x \in \ell_\infty$ , define  $b_x = \sup\{b \in \mathbb{R} \text{ such that } x \geq b\mathbb{1}\}$ . By the *Archimedeanity* property, it follows that  $x \sim b_x\mathbb{1}$ . Let then  $I(x) = b_x$ .

(i) First, consider that for any  $\lambda > 0$ ,  $x \geq y$  is equivalent to  $\lambda x \geq \lambda y$ . Indeed, for  $0 < \lambda \leq 1$ ,  $x \geq y$  is equivalent to  $\lambda x + (1 - \lambda)0\mathbb{1} \geq \lambda y + (1 - \lambda)0\mathbb{1}$ .

Considering then the case  $\lambda > 1$ ,  $\lambda x \geq \lambda y$  then prevails if and only if  $(1/\lambda)\lambda x \geq (1/\lambda)\lambda y$ , or  $x \geq y$ , also prevails. Hence for any  $\lambda \geq 0$ ,  $I(\lambda x) = \lambda I(x)$ .

(ii) Second, for any constant  $b \in \mathbb{R}$ ,  $x \geq y$  is equivalent to  $x + b\mathbb{1} \geq y + b\mathbb{1}$ . By the *weak convexity* property,  $x \geq y$  implies  $(1/2)x + (1/2)b\mathbb{1} \geq (1/2)y + (1/2)b\mathbb{1}$ . Multiplying the two sides by 2, it follows that  $x + b\mathbb{1} \geq y + b\mathbb{1}$ . Finally, and if  $x + b\mathbb{1} \geq y + b\mathbb{1}$ , then  $x + b\mathbb{1} + (-b\mathbb{1}) \geq y + b\mathbb{1} + (-b\mathbb{1})$ , or  $x \geq y$ . Hence  $I(x + b\mathbb{1}) = I(x) + b\mathbb{1}$ . QED

## B. PROOFS FOR SECTION 3

### B.1 PROOF OF PROPOSITION 3.1

(i) For  $x \in \ell_\infty$ , define  $D(x)$  as the set of values  $d$  such that for any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$ , for any  $T \geq T_0(\epsilon)$ , one has

$$(z_{[0, T]}, x_{[T+1, \infty[}) \geq (z_{[0, T]}, d\mathbb{1}_{[T+1, \infty[}) - \epsilon\mathbb{1}.$$

Define  $D(y)$  accordingly. Without loss of generality, suppose that  $\sup D(x) \geq \sup D(y)$  and first let  $\sup D(y) < +\infty$ . Then define  $d_y = \sup D(y)$ , that is finite. Fix any  $\epsilon > 0$ : since  $d_y + (\epsilon/2)\mathbb{1}$  does not belong to  $D(y)$  and  $d_y - (\epsilon/2)\mathbb{1}$  belongs to  $D(y)$ , there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$ , for  $T \geq T_0(\epsilon)$ :

$$\begin{aligned} \left( \left( z + \frac{\epsilon}{2}\mathbb{1} \right)_{[0, T]}, \left( d_y + \frac{\epsilon}{2}\mathbb{1} \right)_{[T+1, \infty[} \right) + \frac{\epsilon}{2}\mathbb{1} &\geq (z_{[0, T]}, y_{[T+1, \infty[}) \\ &\geq \left( \left( z - \frac{\epsilon}{2}\mathbb{1} \right)_{[0, T]}, \left( d_y - \frac{\epsilon}{2}\mathbb{1} \right)_{[T+1, \infty[} \right) - \frac{\epsilon}{2}\mathbb{1}. \end{aligned}$$

This implies, for  $T \geq T_0(\epsilon)$ , the satisfaction of:

$$(z_{[0, T]}, d_y\mathbb{1}_{[T+1, \infty[}) + \epsilon\mathbb{1} \geq (z_{[0, T]}, y_{[T+1, \infty[}) \geq (z_{[0, T]}, d_y\mathbb{1}_{[T+1, \infty[}) - \epsilon\mathbb{1}.$$

Since  $d_x \geq d_y$ , for every  $\epsilon > 0$  and  $z \in \ell_\infty$ , there exists  $T_0(\epsilon)$  such that

$$\begin{aligned} (z_{[0,T]}, x_{[T+1,\infty]}) &\geq (z_{[0,T]}, d_y \mathbb{1}_{[T+1,\infty]}) - \epsilon \mathbb{1} \\ &\geq (z_{[0,T]}, y_{[T+1,\infty]}) - 2\epsilon \mathbb{1}. \end{aligned}$$

This implies that  $x \geq_d y$ .

Consider now the case  $\sup D(y) = +\infty$ . This implies that  $\sup D(x) = +\infty$ . Take  $d > \sup_s y_s$ . Since  $d \in D(x)$ , for every  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$ , for  $T \geq T_0(\epsilon)$ :

$$\begin{aligned} (z_{[0,T]}, x_{[T+1,\infty]}) &\geq (z_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) - \epsilon \mathbb{1} \\ &\geq (z_{[0,T]}, y_{[T+1,\infty]}) - \epsilon \mathbb{1}. \end{aligned}$$

(ii) First, one must prove the existence of  $x, y \in \ell_\infty$  such that  $x >_d y$ . Chose by example  $\mathbb{1}$  and  $0\mathbb{1}$ . Obviously,  $\mathbb{1} \geq_d 0\mathbb{1}$  is first satisfied. Suppose now that  $0\mathbb{1} \geq_d \mathbb{1}$ . Consider first the case  $\chi_g > 0$ . Then, and for  $0 < \epsilon < \chi_g$ , there exists  $T_0(\epsilon)$  such that for  $T \geq T_0(\epsilon)$ ,

$$I(0\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty]}) \geq I(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) - \epsilon.$$

Letting  $T$  tend to infinity, it follows that  $0 \geq \chi_g - \epsilon$ , a contradiction. Consider then the case  $\chi_\ell > 0$ . For  $0 < \epsilon < \chi_\ell$ , there exists  $T_0(\epsilon)$  such that, for  $T \geq T_0(\epsilon)$ :

$$(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty]}) \geq I(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) - \epsilon.$$

Letting  $T$  tend to infinity, it follows that  $\epsilon \geq \chi_\ell$ , a contradiction. The distant order  $\geq_d$  is hence not trivial.

Further observe that, if  $x \geq_d d\mathbb{1}$ , then, for every  $d' \in \mathbb{R}$ ,  $x + d'\mathbb{1} \geq_d (d + d')\mathbb{1}$ . Indeed, for  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for any  $z \in \ell_\infty$ , for  $T \geq T_0(\epsilon)$ ,

$$((z - d'\mathbb{1})_{[0,T]}, x_{[T+1,\infty]}) \geq ((z - d'\mathbb{1})_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon \mathbb{1}.$$

From the *constantly additive* property, for  $T \geq T_0(\epsilon)$ ,

$$(z_{[0,T]}, (x + d'\mathbb{1})_{[T+1,\infty]}) \geq (z_{[0,T]}, (d + d')\mathbb{1}_{[T+1,\infty]}) - \epsilon \mathbb{1}.$$

Hence  $x + d'\mathbb{1} \geq_d (d + d')\mathbb{1}$ .

Then consider  $x \in \ell_\infty$  and a constant  $d$  such that,  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  with, for any  $z \in \ell_\infty$ , for  $T \geq T_0(\epsilon)$ ,

$$(z_{[0,T]}, x_{[T+1,\infty]}) \geq (z_{[0,T]}, d\mathbb{1}_{[T+1,\infty]}) - \epsilon \mathbb{1}.$$

Fix then any  $\lambda > 0$ . From axiom **G1**, there exists  $T'_0(\epsilon)$  such that, for  $T \geq T'_0(\epsilon)$ ,

$$\left( \left( \frac{1}{\lambda} z \right)_{[0,T]}, x_{[T+1,\infty]} \right) \geq \left( \left( \frac{1}{\lambda} z \right)_{[0,T]}, d\mathbb{1}_{[T+1,\infty]} \right) - \frac{1}{\lambda} \epsilon \mathbb{1},$$

that in its turn implies, for  $T \geq T'_0(\epsilon)$ ,

$$\left( z_{[0,T]}, (\lambda x)_{[T+1,\infty]} \right) \geq \left( z_{[0,T]}, \lambda d \mathbb{1}_{[T+1,\infty]} \right) - \epsilon \mathbb{1}.$$

Hence, for  $x \geq_d y$  and for every  $\lambda > 0$ , the occurrence of  $\lambda x \geq_d \lambda y$ .

Consider now  $x, y \in \ell_\infty$  such that  $x \geq_d y$ . For every  $0 < \lambda < 1$ , one has  $(1 - \lambda)x + \lambda d\mathbb{1} \geq_d (1 - \lambda)y + \lambda d\mathbb{1}$ .

The order  $\geq_d$  having been proved to be non trivial, the value  $d_x = \sup D(x)$  is finite and, for every  $d > d_x > d'$ , the relation  $d\mathbb{1} >_d x >_d d'\mathbb{1}$  is to hold. There thus obviously exists  $\lambda, \mu \in [0, 1]$  such that  $(1 - \lambda)d + \lambda d' > d_x > (1 - \mu)d + \mu d'$  and the order  $\geq_d$  satisfies the *Archimedeanity* property.

Since  $\geq_d$  satisfies **F**, there exists an index function  $I_d$  which is homogeneous and constantly additive. The last property is a direct consequence of the definition of the order  $\geq_d$ . QED

## B.2 PROOF OF PROPOSITION 3.2

(i) Using the same arguments as in the proof of Proposition 3.1, the order  $\geq_c$  is complete.

(ii) It can first be proved that  $\mathbb{1} >_c 0\mathbb{1}$ . Suppose the opposite and  $0\mathbb{1} \geq_c \mathbb{1}$  and consider the case  $\chi_g < 1$ . For  $0 < \epsilon < 1 - \chi_g$ , there exists  $T_0(\epsilon)$  such that, for  $T \geq T_0(\epsilon)$ ,

$$I(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) \geq I(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty]}) - \epsilon.$$

Letting  $T$  tend to infinity, one gets  $\chi_g \geq 1 - \epsilon$ : a contradiction. For the remaining case  $\chi_\ell < 1$ , make use of the same arguments. For the proof of the other properties in axiom **F**, follow the arguments developed for the proof of Proposition 3.1.

Consider any  $x \in \ell_\infty$  and fix a constant  $d$ . For every  $\epsilon > 0$  and for large enough values of  $T$ ,

$$I_c(c_x \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) + \epsilon \geq I_c(x_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) \geq I_c(c_x \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) - \epsilon.$$

Letting  $T$  tend to infinity and  $\epsilon$  converge to zero,

$$\lim_{T \rightarrow \infty} I_c(x_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) = I_c(x).$$

For every  $x, y \in \ell_\infty$ , fix then  $d \geq \sup_s y_s \geq \inf_s y_s \geq d'$ . Whence, for every  $T$ ,

$$I_c(x_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) \geq I_c(x_{[0,T]}, y_{[T+1,\infty]}) \geq I_c(x_{[0,T]}, d' \mathbb{1}_{[T+1,\infty]}).$$

Letting  $T$  tend to infinity, it eventually follows that  $\lim_{T \rightarrow \infty} I_c(x_{[0,T]}, y_{[T+1,\infty]}) = I_c(x)$ , that completes the proof. QED

## B.3 PROOF OF PROPOSITION 3.3

First consider the case  $\chi_g = \chi_\ell = 0$ . Define then the set  $D(x)$  as in the proof of Proposition 3.1. Recall that, for every  $x, y \in \ell_\infty$ ,  $x \geq_d y$  if and only if  $\sup D(x) \geq \sup D(y)$ . It is then to be proved that, for every  $x \in \ell_\infty$ ,  $\sup D(x) = +\infty$ . Observe that the following holds for any constants  $c$  and  $d$ :

$$\lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,\infty]}) = c.$$

Indeed, if  $c \leq d$ , since  $\chi_g = 0$ ,

$$\begin{aligned} \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) &= c + \lim_{T \rightarrow \infty} I(0 \mathbb{1}_{[0,T]}, (d-c) \mathbb{1}_{[T+1, \infty)}) \\ &= c + (d-c) \lim_{T \rightarrow \infty} I(0 \mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1, \infty)}) \\ &= c. \end{aligned}$$

For  $c \geq d$ , use the same argument.

This implies that, for every  $d, d'$  in  $\mathbb{R}$  and for any  $\epsilon > 0$ , there exists a large enough  $T(\epsilon)$  such that

$$(c \mathbb{1}_{[0,T]}, d' \mathbb{1}_{[T+1, \infty)}) \geq (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) - \epsilon \mathbb{1}.$$

Fix now any  $x, z \in \ell_\infty$ ,  $d \in \mathbb{R}$  and any  $\epsilon > 0$ . Define  $c_z = I_c(z)$ , that is finite since neither  $\chi_g$  nor  $\chi_\ell$  is equal to zero. Finally fix any  $d'$  such that  $d' \leq \inf_s x_s$ .

There then exists some  $T_0(\epsilon)$  such that, for  $T \geq T_0(\epsilon)$ ,

$$\begin{aligned} (z_{[0,T]}, x_{[T+1, \infty)}) &\geq (c_z \mathbb{1}_{[0,T]}, x_{[T+1, \infty)}) - \frac{\epsilon}{3} \mathbb{1} \\ &\geq (c_z \mathbb{1}_{[0,T]}, d' \mathbb{1}_{[T+1, \infty)}) - \frac{\epsilon}{3} \mathbb{1}. \end{aligned}$$

Since, for large enough values of  $T$ ,

$$(c \mathbb{1}_{[0,T]}, d' \mathbb{1}_{[T+1, \infty)}) \geq (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) - \frac{\epsilon}{3} \mathbb{1},$$

for such values of  $T$ , the following also holds:

$$(z_{[0,T]}, x_{[T+1, \infty)}) \geq (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) - \frac{2\epsilon}{3} \mathbb{1}.$$

But, by the very definition of  $c_z$  and for large enough values of  $T$ ,

$$(c_z \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) \geq (z_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) - \frac{\epsilon}{3} \mathbb{1},$$

that implies

$$(z_{[0,T]}, x_{[T+1, \infty)}) \geq (z_{[0,T]}, d \mathbb{1}_{[T+1, \infty)}) - \epsilon \mathbb{1}.$$

Hence, for every  $x, y \in \ell_\infty$ ,  $\sup D(x) = \sup D(y) = +\infty$ , or  $x \sim_d y$ . Finally and for the remaining case  $\chi_g = \chi_\ell = 1$ , making use of the same arguments, for every  $x, y \in \ell_\infty$ , the holding  $x \sim_c y$  is eventually established. QED

## B.4 PROOF OF THEOREM 3.1

First, observe that for any  $x \in \ell_\infty$ , any constants  $c, d \in \mathbb{R}$ , the following holds:

$$\lim_{T \rightarrow \infty} I(x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, +\infty)})) = I(x) + \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1, +\infty)}).$$

Indeed, take  $b_x = I(x)$ , which is equivalent to  $x \sim b_x \mathbb{1}$ . Fix any  $\epsilon > 0$ . Take  $\lambda = \frac{1}{2}$ , by axiom **A1**, for  $T$  sufficiently big,

$$\frac{1}{2}x + \frac{1}{2}(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) \geq \frac{1}{2}b_x \mathbb{1} + \frac{1}{2}(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) - \frac{\epsilon}{2} \mathbb{1}.$$

By positive homogeneity property, this implies

$$x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) \geq b_x \mathbb{1} + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) - \epsilon.$$

Let  $T$  converges to infinity and next, let  $\epsilon$  converges to zero,

$$\liminf_{T \rightarrow \infty} I(x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[})) \geq b_x + \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}).$$

Observe that the limit of  $I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[})$  exists. Indeed, for  $c \leq d$ , this term is decreasing as a function of  $T$ . For the case  $c \geq d$ , this term is increasing as a function of  $T$ .

Using the same arguments, by changing the role between  $x$  and  $b_x \mathbb{1}$ ,

$$b_x + \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) \geq \limsup_{T \rightarrow \infty} I(x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[})).$$

Hence the limit of  $I(x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}))$  exists and:

$$b_x + \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) = \lim_{T \rightarrow \infty} I(x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[})).$$

i. Fix  $T_0 \geq 0$ . Observe that for any  $T \geq T_0$ ,

$$(o \mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}) + (\mathbb{1}_{[0,T]}, o \mathbb{1}_{[T+1,+\infty[}) \geq \mathbb{1}.$$

Hence for any  $T_0$ ,

$$\begin{aligned} I(o \mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}) + (1 - \chi_\ell) &= I(o \mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}) + \lim_{T \rightarrow \infty} I(\mathbb{1}_{[0,T]}, o \mathbb{1}_{[T+1,+\infty[}) \\ &= \lim_{T \rightarrow \infty} ((o \mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}) + (\mathbb{1}_{[0,T]}, o \mathbb{1}_{[T+1,+\infty[})) \\ &\geq 1. \end{aligned}$$

Let  $T_0$  converges to infinity, this implies  $\chi_g + (1 - \chi_\ell) \geq 1$ , which is equivalent to  $\chi_g \geq \chi_\ell$ .

Similarly,

$$I(o \mathbb{1}_{[0,T_0]}, -\mathbb{1}_{[T_0+1,+\infty[}) + \lim_{T \rightarrow \infty} I(-\mathbb{1}_{[0,T]}, o \mathbb{1}_{[T+1,+\infty[}) \leq -1.$$

Let  $T_0$  converges to infinity, one gets  $-\chi_\ell + (\chi_g - 1) \leq -1$ , or  $\chi_g \leq \chi_\ell$ .

ii. Define  $\chi^* = \chi_g = \chi_\ell$ . For  $c \leq d$ , one gets

$$\begin{aligned} \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty[}) &= c + \lim_{T \rightarrow \infty} I(o \mathbb{1}_{[0,T]}, (d - c) \mathbb{1}_{[T+1,+\infty[}) \\ &= c + (d - c) \lim_{T \rightarrow \infty} I(o \mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,+\infty[}) \\ &= c + (d - c) \chi^* \\ &= (1 - \chi^*)c + \chi^* d. \end{aligned}$$



For  $c \geq d$ , using the same arguments,

$$\begin{aligned} \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty]}) &= (c - d) \lim_{T \rightarrow \infty} I(\mathbb{1}_{[0,T]}, \mathbf{0} \mathbb{1}_{[T+1,+\infty]}) + d \\ &= (1 - \chi^*)(c - d) + d \\ &= (1 - \chi^*)c + \chi^*d. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{T \rightarrow \infty} I(x + (c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty]})) &= I(x) + \lim_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty]}) \\ &= I(x) + (1 - \chi^*)c + \chi^*d. \end{aligned}$$

iii. Define  $c_x = I_c(x)$ ,  $d_x = I_d(x)$ . First, observe that

$$I(x) = \lim_{T \rightarrow \infty} I(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty]}).$$

Indeed, from the definition of  $c_x$  and  $d_x$ , for large enough values of  $T$ ,

$$\begin{aligned} x &= (x_{[0,T]}, x_{[T+1,+\infty]}) \\ &\geq (c_x \mathbb{1}_{[0,T]}, x_{[T+1,+\infty]}) - \epsilon \mathbb{1} \\ &\geq (c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty]}) - 2\epsilon \mathbb{1}. \end{aligned}$$

Therefore

$$I(x) \geq \limsup_{T \rightarrow \infty} I(c \mathbb{1}_{[0,T]}, d \mathbb{1}_{[T+1,+\infty]}) - 2\epsilon \mathbb{1}.$$

This inequality being further true for any arbitrary  $\epsilon > 0$ ,

$$I(x) \geq \limsup_{T \rightarrow \infty} I(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty]}).$$

Likewise,

$$I(x) \leq \liminf_{T \rightarrow \infty} I(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty]}).$$

Therefore

$$\begin{aligned} I(x) &= \lim_{T \rightarrow \infty} I(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty]}) \\ &= (1 - \chi^*)c_x + \chi^*d_x. \end{aligned}$$

## B.5 PROOF OF THEOREM 3.2

(i) First suppose that  $\chi_g \leq \chi_\ell$ , define  $c_x = I_c(x)$  and  $d_x = I_d(x)$  and fix  $\epsilon > 0$ .

First consider the configuration  $c_x \leq d_x$  or, equivalently,  $I_c(x) \leq I_d(x)$ . As  $d_x - c_x \geq 0$ , it is obtained that:

$$\begin{aligned} I(x) &= \lim_{T \rightarrow \infty} I(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty]}) \\ &= c_x + \lim_{T \rightarrow \infty} I(\mathbf{0} \mathbb{1}_{[0,T]}, (d_x - c_x) \mathbb{1}_{[T+1,+\infty]}) \\ &= c_x + (d_x - c_x) \lim_{T \rightarrow \infty} I(\mathbf{0} \mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,+\infty]}) \\ &= (1 - \chi_g)c_x + \chi_g d_x \\ &= (1 - \chi_g)I_c(x) + \chi_g I_d(x). \end{aligned}$$

For the case  $I_c(x) \geq I_d(x)$ , using similar arguments,

$$I(x) = (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

(ia) Consider first the configuration  $\chi_g \leq \chi_\ell$ . This implies  $\underline{\chi} = \chi_g, \bar{\chi} = \chi_\ell$ .

For the case  $I_c(x) \leq I_d(x)$ , for any  $\underline{\chi} \leq \chi \leq \bar{\chi}$ ,

$$I(x) = (1 - \chi_g)I_c(x) + \chi_g I_d(x) \leq (1 - \chi)I_c(x) + \chi I_d(x).$$

As for the remaining case  $I_c(x) \geq I_d(x)$ , and making use of the same arguments

$$\begin{aligned} I(x) &= (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x) \\ &\leq (1 - \chi)I_c(x) + \chi I_d(x), \end{aligned}$$

for any  $\chi \in [\underline{\chi}, \bar{\chi}]$ . Whence, finally

$$I(x) = \min_{\underline{\chi} \leq \chi \leq \bar{\chi}} [(1 - \chi)I_c(x) + \chi I_d(x)].$$

(iib) For the other configuration  $\chi_g \geq \chi_\ell$  and making use of the same line of arguments, it is similarly obtained that:

$$I(x) = \max_{\underline{\chi} \leq \chi \leq \bar{\chi}} [(1 - \chi)I_c(x) + \chi I_d(x)],$$

where  $\underline{\chi} = \chi_\ell, \bar{\chi} = \chi_g$ .

(iii) This is a direct consequence of Propositions 3.1, 3.2 and 3.3. QED

## C. PROOFS FOR SECTION 4

### C.1 PROOF OF LEMMA 4.1

Consider the first case. For any  $\epsilon > 0$ , there exists  $T_0(\epsilon) \geq T$  such that, for any  $z, z' \in \ell_\infty$ , any  $T' \geq T_0(\epsilon)$ ,

$$(z_{[0, T-1]}, x_{[T, T']}, z'_{[T'+1, \infty]}) \geq (z_{[0, T-1]}, c \mathbb{1}_{[T, T']}, z'_{[T'+1, \infty]}) - \epsilon \mathbb{1}.$$

Let  $T'$  converges to infinity and  $\epsilon$  converges to zero, by observing that the two streams have the same distant future, one gets

$$I_c(z_{[0, T-1]}, x) \geq I_c(z_{[0, T-1]}, c \mathbb{1}).$$

For the remaining case, use the same argument.

### C.2 PROOF OF PROPOSITION 4.1

(i) Define  $C_T(x)$  the set of values  $c$  such that, for any  $z_0, z_1, \dots, z_{T-1}$ ,

$$(z_{[0, T-1]}, x) \geq_c (z_{[0, T-1]}, c \mathbb{1}_{[T, \infty]}).$$

Define the order  $\geq_T$  as the holding of  $x \geq_T y$  if and only if  $\sup C(x) \geq \sup C(y)$ . Fix  $x, y \in \ell_\infty$  and suppose that for any  $z_0, z_1, \dots, z_{T-1}$ ,

$$(z_{[0, T-1]}, x) \geq_c (z_{[0, T-1]}, y).$$

This implies that  $C_T(y) \subset C_T(x)$ , or  $x \geq_T y$ . First consider the case  $\sup C_T(y) < +\infty$  and take  $c_y^T = \sup C_T(y)$ . It is readily checked that  $C_T(y)$  is closed, whence the satisfaction of  $c_y^T \in C_T(y) \subset C_T(x)$ . Further and from the definition of  $c_y^T$ , which is finite, for any  $z_0, z_1, \dots, z_{T-1}$ ,

$$(z_{[0, T-1]}, x) \geq_c (z_{[0, T-1]}, c_y^T \mathbb{1}_{[T, \infty[}) \geq_c (z_{[0, T-1]}, y).$$

Secondly consider the case  $\sup C_T(y) = +\infty$ , that implies the holding of  $\sup C_T(x) = +\infty$ . Whence, for any  $c \geq \sup_s y_s$  and for any  $z_0, z_1, \dots, z_{T-1}$ :

$$(z_{[0, T-1]}, x) \geq_c (z_{[0, T-1]}, c \mathbb{1}_{[T, \infty[}) \geq_c (z_{[0, T-1]}, y).$$

(ii) For the *transitivity*, *monotonicity* and *weak convexity* properties, replicate the arguments used for the proof of Proposition 3.1.

(iii) Suppose that at least one of two values  $I_c(\mathbf{o} \mathbb{1}_{[0, T-1]}, \mathbb{1}_{[T, \infty[})$  and  $-I_c(\mathbf{o} \mathbb{1}_{[0, T-1]}, -\mathbb{1}_{[T, \infty[})$  differs from zero. It is to be proved that the order  $\geq_T$  satisfies the technical *non-triviality* property. The *Archimedeanity* property would then follow as a direct corollary.

Assume that  $I_c(\mathbf{o} \mathbb{1}_{[0, T-1]}, \mathbb{1}_{[T, \infty[}) > \mathbf{o}$ . Then  $I_c(\mathbf{o} \mathbb{1}_{[0, T-1]}, \mathbb{1}) > I_c(\mathbf{o} \mathbb{1}_{[0, T-1]}, \mathbf{o} \mathbb{1})$ . This implies  $\mathbb{1} >_T \mathbf{o} \mathbb{1}$ . Use the same argument for the case  $-I_c(\mathbf{o} \mathbb{1}_{[0, T-1]}, -\mathbb{1}_{[T, \infty[}) > \mathbf{o}$ , which implies  $\mathbf{o} \mathbb{1} >_T -\mathbb{1}$ .

(iv) Suppose that at least one of two critical values is different from zero. From (i), (ii) and (iii),  $\sup C_T(x) < +\infty$  for any  $x$  and the order  $\geq_T$  satisfies every property in axiom **F**. The index function  $I_T(x) = \sup C(x)$  therefore satisfies every property listed in Proposition 2.1. QED

### C.3 PROOF OF PROPOSITION 4.2

Fix  $x \in \ell_\infty$ , let  $c = I_{T+1}(x_{[T+1, \infty[})$  and consider the case  $x_T \leq c$ . From Proposition 4.1 and as  $d - x_T \geq \mathbf{o}$ ,

$$\begin{aligned} I_T(x_{[T, \infty[}) &= I_T(x_T, c \mathbb{1}) \\ &= x_T + I_T(\mathbf{o}, (c - x_T) \mathbb{1}) \\ &= x_T + (c - x_T) I_T(\mathbf{o}, \mathbb{1}) \\ &= (\mathbf{1} - I_T(\mathbf{o}, \mathbb{1})) x_T + I_T(\mathbf{o}, \mathbb{1}) c. \end{aligned}$$

Likewise and for  $x_T \geq d$ :

$$\begin{aligned} I_T(x_{[T, \infty[}) &= I_T(x_T, c \mathbb{1}) \\ &= I_T(x_T - c, \mathbf{o} \mathbb{1}) + d \\ &= (x_T - c) I_T(\mathbf{1}, \mathbf{o} \mathbb{1}) + d \\ &= I_T(\mathbf{1}, \mathbf{o}) x_T + (\mathbf{1} - I_T(\mathbf{1}, \mathbf{o} \mathbb{1})) c. \end{aligned}$$

First suppose that  $\chi_g^T \leq \chi_\ell^T$ , or  $I_T(\mathbf{o}, \mathbb{1}) + I_T(\mathbf{1}, \mathbf{o}\mathbb{1}) \leq \mathbf{1}$ , and let  $\underline{\delta}_T = \chi_g^T = I(\mathbf{o}, \mathbb{1})$  and  $\bar{\delta}_T = \chi_\ell^T = \mathbf{1} - I(\mathbf{1}, \mathbf{o}\mathbb{1})$ . It follows that  $\mathbf{o} < \underline{\delta}_T \leq \bar{\delta}_T < \mathbf{1}$  and

$$I_T(x_{[T, \infty[}) = \min_{\underline{\delta}_T \leq \delta \leq \bar{\delta}_T} [(1 - \delta)x_T + \delta I_{T+1}(x_{[T+1, \infty[})].$$

Consider the remaining case  $\chi_g^T \geq \chi_\ell^T$ , or  $I_T(\mathbf{o}, \mathbb{1}) + I_T(\mathbf{1}, \mathbf{o}\mathbb{1}) \geq \mathbf{1}$  and let  $\underline{\delta}_T = \chi_\ell^T = \mathbf{1} - I_T(\mathbf{1}, \mathbf{o}\mathbb{1})$ ,  $\bar{\delta}_T = \chi_g^T = I_T(\mathbf{o}, \mathbb{1})$ . It follows that  $\mathbf{o} < \underline{\delta}_T \leq \bar{\delta}_T < \mathbf{1}$  and

$$I_T(x_{[T, \infty[}) = \max_{\underline{\delta}_T \leq \delta \leq \bar{\delta}_T} [(1 - \delta)x_T + \delta I_{T+1}(x_{[T+1, \infty[})],$$

which establishes the statement. QED

#### C.4 PROOF OF LEMMA 4.2

(i) Consider the constant  $c$  such that  $(\mathbf{o}, \mathbb{1}) \sim_T c\mathbb{1}$ . From Proposition 4.1, this implies that:

$$(\mathbf{o}\mathbb{1}_{[0, T]}, \mathbb{1}_{[T+1, \infty[}) \sim_c (\mathbf{o}\mathbb{1}_{[0, T-1]}, c\mathbb{1}_{[T, \infty[}).$$

The order  $\succeq_c$  further satisfying every property in axiom **F**, for any  $c' < c$ , the following is to hold:

$$(\mathbf{o}\mathbb{1}_{[0, T]}, \mathbb{1}_{[T+1, \infty[}) \succ_c (\mathbf{o}\mathbb{1}_{[0, T-1]}, c'\mathbb{1}_{[T, \infty[}).$$

From Proposition 3.2, there then exists a large enough  $T_0$  such that, for  $T' \geq T_0$ ,

$$(\mathbf{o}\mathbb{1}_{[0, T]}, \mathbb{1}_{[T+1, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}) \succeq_c (\mathbf{o}\mathbb{1}_{[0, T-1]}, c'\mathbb{1}_{[T, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}).$$

The left hand side and the right hand side of the above equation assuming the same distant future valuation, this implies that:

$$(\mathbf{o}\mathbb{1}_{[0, T]}, \mathbb{1}_{[T+1, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}) \geq (\mathbf{o}\mathbb{1}_{[0, T-1]}, c'\mathbb{1}_{[T, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}).$$

From axiom **B2**, this can be strengthened to:

$$(\mathbf{o}\mathbb{1}_{[0, T+1]}, \mathbb{1}_{[T+2, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}) \geq (\mathbf{o}\mathbb{1}_{[0, T]}, c'\mathbb{1}_{[T+1, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}),$$

which is equivalent to

$$(\mathbf{o}\mathbb{1}_{[0, T+1]}, \mathbb{1}_{[T+2, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}) \geq_c (\mathbf{o}\mathbb{1}_{[0, T]}, c'\mathbb{1}_{[T+1, T']}, \mathbf{o}\mathbb{1}_{[T'+1, \infty[}).$$

Letting  $T'$  tends to infinity, it follows that:

$$(\mathbf{o}, \mathbb{1}) \geq_{T+1} c'\mathbb{1}.$$

As this is true for any  $c' < c$ , letting  $c'$  converge to  $c$ , it derives that:

$$(\mathbf{o}, \mathbb{1}) \geq_{T+1} c\mathbb{1}.$$

(ii) Follow the same line of arguments as for (i). QED

## C.5 PROOF OF PROPOSITION 4.3

First observe that, for any  $T$ ,

$$\begin{aligned}\underline{\delta}_T &= \min\{I_T(\mathbf{o}, \mathbb{1}), 1 - I_T(\mathbf{1}, \mathbf{o}\mathbb{1})\}, \\ \overline{\delta}_T &= \max\{I_T(\mathbf{o}, \mathbb{1}), 1 - I_T(\mathbf{1}, \mathbf{o}\mathbb{1})\}.\end{aligned}$$

But and from Lemma 4.2, both of the two sequences  $\{I_T(\mathbf{o}, \mathbb{1})\}_{T=0}^\infty$  and  $\{1 - I_T(\mathbf{1}, \mathbf{o}\mathbb{1})\}_{T=0}^\infty$  are increasing. This in its turn implies that the two sequences  $\{\underline{\delta}_T\}_{T=0}^\infty$  and  $\{\overline{\delta}_T\}_{T=0}^\infty$  are also increasing.

## C.6 PROOF OF LEMMA 4.3

Fix any  $a, b \in \mathbb{R}$  and  $x, y \in \ell_\infty$  such that  $(a, b, x) \geq_c (a, b, y)$ . For any  $\epsilon > 0$ , there exists  $T(\epsilon)$  such that for any  $z \in \ell_\infty$ , one gets

$$(a, b, x_{[0,T]}, z_{[T+2,+\infty[)}) \geq (a, b, y_{[0,T]}, z_{[T+2,+\infty[)}) - \epsilon \mathbb{1},$$

which implies for  $T$  big enough

$$(a, x_{[0,T]}, z_{[T+1,+\infty[)}) \geq (a, y_{[0,T]}, z_{[T+1,+\infty[)}) - \epsilon \mathbb{1}.$$

Let  $T$  converges to infinity and  $\epsilon$  converges to zero. Since the two streams have the same distant future, one gets

$$(a, x) \geq_c (a, y).$$

For the case  $(a, x) \geq_c (a, y)$ , by using the same arguments,  $(a, b, x) \geq_c (a, b, y)$ .

## C.7 PROOF OF PROPOSITION 4.4

Fix any  $T \geq 1$ . Suppose that  $(\mathbf{o}, \mathbb{1}) \sim_{T+1} c\mathbb{1}$ , which is equivalent to  $(\mathbf{o}\mathbb{1}_{[0,T+1]}, \mathbb{1}) \sim_c (\mathbf{o}\mathbb{1}_{[0,T]}, c\mathbb{1})$ . By Lemma 4.3, this implies  $(\mathbf{o}\mathbb{1}_{[0,T]}, \mathbb{1}) \sim_c (\mathbf{o}\mathbb{1}_{[0,T-1]}, c\mathbb{1})$ , which is equivalent to  $(\mathbf{o}, \mathbb{1}) \sim_T c\mathbb{1}$ . Hence  $\chi_g^T = \chi_g^{T+1}$ . Use the same arguments, one gets  $\chi_\ell^T = \chi_\ell^{T+1}$ . Hence for any  $T \geq 1$ ,  $\underline{\delta}_T = \underline{\delta}_1$  and  $\overline{\delta}_T = \overline{\delta}_1$ .

The second part is direct consequence of *present bias* property.

## C.8 PROOF OF PROPOSITION 4.5

(i) Suppose that  $x \in \ell_\infty$ ,  $d, d' \in \mathbb{R}$  such that

$$d\mathbb{1} \geq_d x \geq_d d'\mathbb{1}.$$

Fix  $z \in \ell_\infty$  and  $\epsilon > 0$ . From axiom **G1**, there exists  $T_0(\epsilon, z)$  such that, for  $T \geq T_0(\epsilon, z)$  it holds that:

$$(z_{[0,T]}, (d + \epsilon)\mathbb{1}_{[T+1,+\infty[)}) \geq (z_{[0,T]}, x_{[T+1,+\infty[)}) \geq (z_{[0,T]}, (d' - \epsilon)\mathbb{1}_{[T+1,+\infty[)}).$$

From axiom **B4**, there then exists some date  $T'_0(\epsilon, z)$  such that, for any  $T \geq T'_0(\epsilon, z)$  and for any  $s \geq s_0$  there exists  $s_0(T)$  such that, for  $s \geq s_0(T)$ ,

$$\begin{aligned}(z_{[0,T]}, (d + \epsilon)\mathbb{1}_{[T+1,+\infty[)}) &\geq (z_{[0,T]}, x_{[T+1, T+s]}, (d + \epsilon)\mathbb{1}_{[T+s+1,+\infty[)}), \\ (z_{[0,T]}, x_{[T+1, T+s]}, (d' - \epsilon)\mathbb{1}_{[T+s+1,+\infty[)}) &\geq (z_{[0,T]}, (d' - \epsilon)\mathbb{1}_{[T+1,+\infty[)}).\end{aligned}$$

From axiom **B1** and for any  $T \geq T_0(\epsilon, z)$ , letting  $s$  tends to infinity, it holds that:

$$(d + \epsilon)\mathbb{1}_{[T, \infty[} \geq_T x_{[T, \infty[} \geq_T (d' - \epsilon)\mathbb{1}_{[T, \infty[}.$$

(ii) Take  $d = I_d(x)$ . For any  $\epsilon > 0$ , there exists  $T_0(\epsilon)$  such that, for  $T \geq T_0(\epsilon)$ ,

$$(d + \epsilon)\mathbb{1}_{[T, \infty[} \geq_T x_{[T, \infty[} \geq_T (d - \epsilon)\mathbb{1}_{[T, \infty[},$$

which is equivalent to

$$d + \epsilon \geq I_T(x_{[T, \infty[}) \geq d - \epsilon.$$

Letting  $\epsilon$  converge to zero, it follows that:

$$\lim_{T \rightarrow \infty} I_T(x_{[T, \infty[}) = I_d(x).$$

(iii) Take any sequence  $\{T_k\}_{k=0}^{\infty} \subset \mathbb{N}$  converging to infinity and satisfying, for any  $k$ ,  $T_{k-1} < T_k - 1$ . Then define  $\hat{x}$  as  $\hat{x}_T = 1$  for any  $T \notin \{T_k\}_{k=0}^{\infty}$  and  $\hat{x}_{T_k} = 0$  for any  $k$ . It follows that  $0 \leq I_d(\hat{x}) \leq 1$ . From Proposition 4.2 and for every  $T$ , there exists  $\delta_T \in [\underline{\delta}_T, \bar{\delta}_T]$  such that:

$$I_T(\hat{x}_{[T, \infty[}) = (1 - \delta_T)\hat{x}_T + \delta_T I_{T+1}(\hat{x}_{[T+1, \infty[}).$$

Recall then that  $\lim_{T \rightarrow \infty} I_T(\hat{x}_{[T, \infty[}) = I_d(\hat{x})$  and consider the case  $I_d(\hat{x}) = 0$ . For any  $k$  and as  $T_{k-1} < T_k - 1$ ,  $\hat{x}_{T_{k-1}} = 1$ , one has, for any  $k$ :

$$\begin{aligned} I_{T_{k-1}}(\hat{x}_{[T_{k-1}, \infty[}) &= (1 - \delta_{T_{k-1}})\hat{x}_{T_{k-1}} + \delta_{T_{k-1}} I_{T_k}(\hat{x}_{[T_k, \infty[}) \\ &= (1 - \delta_{T_{k-1}})\hat{x}_{T_{k-1}} + \delta_{T_{k-1}} I_{T_k}(\hat{x}_{[T_k, \infty[}). \end{aligned}$$

Letting  $k$  tends to infinity,  $I_{T_{k-1}}(\hat{x}_{[T_{k-1}, \infty[})$  and  $I_{T_k}(\hat{x}_{[T_k, \infty[})$  converge to  $I_d(\hat{x})$ , which sums up to zero. This implies that  $1 - \delta_{T_{k-1}}$  converges to zero, or that  $\delta_{T_{k-1}}$  converges to 1. As  $\delta_{T_{k-1}} \leq \bar{\delta}_{T_{k-1}} \leq 1$ , it derives that  $\bar{\delta}_{T_{k-1}}$  converges to 1.

Considering now the case  $I_d(\hat{x}) > 0$  and for any  $k$ ,

$$\begin{aligned} I_{T_k}(\hat{x}_{[T_k, \infty[}) &= (1 - \delta_{T_k})\hat{x}_{T_k} + \delta_{T_k} I_{T_{k+1}}(\hat{x}_{[T_{k+1}, \infty[}) \\ &= \delta_{T_k} I_{T_{k+1}}(\hat{x}_{[T_{k+1}, \infty[}). \end{aligned}$$

Letting  $k$  tends to infinity, both  $I_{T_k}(\hat{x}_{[T_k, \infty[})$  and  $I_{T_{k+1}}(\hat{x}_{[T_{k+1}, \infty[})$  do converge to  $I_d(\hat{x})$ , which is strictly positive, whence the convergence to 1 of  $\delta_{T_k}$  and  $\bar{\delta}_{T_k}$ .

(iv) Adding axiom **B2**, the sequence  $\{\bar{\delta}_T\}_{T=0}^{\infty}$  becomes increasing, whence and from (iii), the satisfaction of:

$$\lim_{T \rightarrow \infty} \bar{\delta}_T = 1,$$

which establishes the statement. QED

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