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# Analysis, Constructions & Diagrams in Classical Geometry\*

Marco Panza<sup>†</sup>

July 4, 2021

Mathematics is usually regarded as a body of knowledge. This can only mean that it is regarded as a body of contents that are the object of knowledge. In turn, this can be intended in two different ways: this can either be a body of contents to be known or a body of contents that are actually known. The former option often goes together with a Platonist perspective: the relevant contents are, then, intended as abstract objects, or facts about these objects, and the purpose of mathematics is conceived of describing these objects or facts. Acquiring mathematical knowledge is seen similarly as attaining a system or truths, which are there, as such, waiting to be known: some have been so, some others still not. The latter option suggests a quite different picture: acquiring mathematical knowledge is rather seen as the same as reaching a bunch of results, which are there only in force of their having been reached, and then, known. This option often goes together with a constructivist perspective: the relevant contents are intended as (outcomes of) constructions, and the purpose of mathematics is conceived of carrying out these constructions. Both options assign an important role to mathematical agents and their activity. However, whereas the former option conceives of this role as being external to mathematics itself, as pertaining to the act of tending towards it, the latter option conceives this role as being wholly internal to mathematics, as pertaining to the act of doing it.

Platonism and constructivism are not monolithic systems. To be precise, they are not systems at all. They are rather perspectives sheltering quite different views. The two different options that I have very generally described are neither inevitably embedded within these perspectives. Nothing prevents one, for example, from choosing the second option and still looking at mathematics as an activity dealing with abstract objects, both forged and inspected along with it (without any preclusion towards classical logic). This is, indeed, my

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\*Drawn from a talk presented at the Conference *Greek Geometrical Analysis. Problems and Prospects*, Clermont-Ferrand, March 4th-6th, 2020. I thank the audience for the stimulating discussion that pushed me to transform this talk in a more detailed paper. Special thanks to Sébastien Gandon and Gianluca Longa, which not only organized the conference and kindly invited me to present a talk, but also discussed it deeply with me, and gave me useful suggestions, and to Jeevan Acharya, for his careful editing and linguistic revisions.

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mindset, and makes me focus on the way mathematics is done, not only today, but also throughout its history.

The recent and quickly growing movement for a philosophy of mathematical practice (Mancosu [2008]; Carter [2019]) shares a similar focus, though implementing it in many different ways, looking not only at different aspects of mathematical activity, but also at each of them from different viewpoints. An aspect often considered within this movement is the use that this activity made and still makes of diagrams, in different forms and for different purposes. Using diagrams during mathematical activity is what I mean here as a diagrammatic (mathematical) gesture. This is both a practical and an intellectual gesture: since diagrams are both drawn and studied (or even merely imagined, but imagined as such, as something that might be drawn).

A common question that many works on mathematical diagrams tried, and still try to answer, is whether there are pieces of mathematical activity that indispensably require diagrams for reaching the relevant results (Giaquinto [2007]: Manders [2008a]; Manders [2008b]; Carter [2010]; DeToffoli [2017]). In my [2012], I advanced that this is the case of Euclid's Plane Geometry (EPG), though arguing that diagrams are indispensable for EPG not insofar as this cannot avoid drawing and inspecting them, but only insofar as it cannot avoid considering the geometrical objects it deals with as represented by them (which leaves, on the one side, open the possibility of merely imagining them, and makes them count, on the other side, not for the properties they actually have, but for those that are ascribed to them).

In the present paper, I take the conclusions I reached there for granted, and aim to add a new argument about the indispensability of diagrams (and of the corresponding gesture) for a classical form of geometrical analysis, which is largely present both in EPG and, more generally, in classical geometry: the geometrical practice that characterizes Greek, Arabic and Latin mathematics up to the early modern age, at least. The purpose of this argument is not merely to assert this indispensability. Were it so, the point would be nothing but a trivial reiteration of what I have already argued for, in my [2012]. Since, though extending it in several respects, classical geometry is essentially based on EPG, and inherits from it the indispensable use of diagrams. It would, then, be enough to remark that the relevant form of analysis is part of classical geometry to reach the same conclusion for free, so to say. My present purpose is rather to show that diagrams enter this form of analysis in a specific way, which has no correlation in other aspects of classical geometry. The point is, then, that a quite peculiar sort of diagrammatic gesture is at work in this form of analysis, which is absent in other aspects of classical geometry (and mathematics, in general).

To make my point, I need to distinguish two kinds of geometrical analysis: intra-configurational and trans-configurational. I advanced and justified this distinction in my [2007]. Here, I only need to illustrate it by means of examples<sup>1</sup>, and to stress the basic difference among the two sorts of analyses. Since it is only because of this difference that

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<sup>1</sup>Though I tried to choose quite simple examples, following them requires a minimum of mathematical skill. My presentation is however supposed to be self-containing (except, of course, for the most elementary notations and concepts). If some difficulties persist, they should be easily overcome through the consultation of any elementary textbooks in geometry and algebra.

intra-configurational, but not trans-configurational, analysis necessarily requires the mentioned sort of diagrammatic gesture.

## 1 Trans-configurational Analysis

Trans-configurational analysis is typical of the use of algebra in geometry<sup>2</sup>.

Clear examples are offered in Viète’s *Zeteticorum Libri* (Viète [1591b]; Viete [AAW]): a collection of problems<sup>3</sup>. One of these problems is the following:

*Zeteticum* II.17: Datâ differentiâ laterum & differentiâ Cuborum invenire latera.

Viète takes  $B$  to be the difference of sides (“*differentiâ laterum*”),  $D$  the difference of cubes (“*differentiâ cuborum*”) and  $E$  the sum of sides (“*summa laterum*”). In his notations, capital consonants denotes known quantities, and capital vowels unknown ones. Solving a problem requires making the latter known in terms of the former. In the present case, the problem requires making the sides known in terms of  $B$  and  $D$ . Though making their sum  $E$  known is still not enough for this final purpose, this makes the task quite easy, and is thus a crucial stage in the solution.

By adapting (for the sake of simplicity) Viète’s language and notation to our modern ones, his argument goes as follows. Let  $x$  and  $y$  be the two (unknown) sides, then:

$$\begin{aligned} x - y &= B & x^3 - y^3 &= D & x + y &= E \\ E + B &= 2x & E - B &= 2y \\ (E + B)^3 - (E - B)^3 &= 6E^2B + 2B^3 = 8(x^3 - y^3) = 8D \\ E^2 &= \frac{4D - B^3}{3B} \end{aligned}$$

This reduces the given problem to another already solved one:

*Zeteticum* I.1: Datâ differentiâ duorum laterum, & aggregato eorundem invenire latera

If the difference of sides is  $B$  and their sum (“*aggregatum eorundem*”) is  $\sqrt{\frac{4D-B^3}{3B}}$ , the two sides are of course

$$x = \frac{B + \sqrt{\frac{4D-B^3}{3B}}}{2} \quad \text{and} \quad y = \frac{\sqrt{\frac{4D-B^3}{3B}} - B}{2},$$

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<sup>2</sup>Within classical mathematics, algebra was in no way opposed to geometry as a different branch of mathematics. It was rather a common “art” for solving problems both in arithmetic and geometry, going together with a common language, admitting different interpretations in the two cases. A crucial aspect of the deep innovation promoted by Descartes’s *Géométrie* (Descartes [1637]) pertains to the introduction of a new way to interpret this language in geometry (also going together with a new and particularly performant notation).

<sup>3</sup>By adopting a calque from the Greek verb ‘ζητέω [to search]’ and adjective ‘ζητητικός, ή, όν [inclined to search]’, Viète calls each problem considered in his collection ‘zeteticum’.

which makes them known in terms of  $B$  and  $D$ .

An important distinction in classical geometry is that between problems and theorems. While both are traditionally called ‘propositions’, the former are in fact requests, and the latter are statements. A problem requires the construction of an object satisfying a certain condition. A theorem asserts that certain objects have certain properties or bear certain relations. Problems are solved. Theorems are proved<sup>4</sup>. In classical geometry, analysis was most applied to the solution of problems, but it also had less frequent applications to the proof of theorems. In the first case, it was called ‘problematic’, in the latter ‘theorematic’.

The present case is one of problematic analysis, of course, since the *Zeteticum* II.17 is just a problem. By its nature, problematic analysis supposes the problem solved and operates on the supposed solution, in order to find a way for actually solving it. In the present case, supposing the problem solved consists of expressing an unknown quantity (the sum of sides), whose knowledge makes the actual solution easy, through a given letter, namely ‘ $E$ ’. Though this is a vowel rather than a consonant, the difference is purely typographic and does not forbid operating on the former just as on the consonants expressing the quantities that are rather known. Doing it is just what operating on the supposed solution means, and this is what the analysis consists in. Though here, as in any other case of problematic analysis, the final purpose is that of helping synthesis to solve the problem<sup>5</sup> (namely indicating to it how to do it), the immediate task to be achieved is not properly that of suggesting a solution, but rather that of reducing the given problem (the *Zeteticum* II.17) to a simpler one: one, by the way, which had been previously solved (the *Zeteticum* I.1). Hence what the analysis does, properly, is supposing the former problem solved and operating on the supposed solution in order to modify the corresponding configuration of data in order to transform it into another, simpler, configuration, corresponding to the latter problem. This is just what makes it trans-configurational, in my language: it does not operate within a given configuration of data, but rather crucially transforms it into a new configuration.

There is no diagram here. But there is a quite codified formalism at work. When presented in the notation I used above, this formalism is not so dissimilar from the elementary algebraic one still used today. It is true that Viète uses another notation, quite unlike our present one, and justifies the formalism in a different way than we would do (and Descartes himself will do, only a few decades later). Though quite convenient for the purpose of trans-configurational analysis, this formalism is not indispensability required by it, however.

To see why, we need first to understand what makes it so convenient. This is not only its deductive agility; it is also, and even overall, its neutrality with respect to the possible interpretations of the problem at issue, and of the analysis itself. There is no need, indeed, to take Viète’s “*latera*”, and “*cubi*” just as geometrical segments and solids. One can cer-

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<sup>4</sup>The distinction comes back to ancient sources. While Euclid lets it implicit, by confining himself to use quite different codified formats to advance a problem and to state a theorem, respectively, it is made totally explicit both by Proclus’s in his *Commentary* to the first book of the *Elements* (Proclus [EEC]; Proclus [CEEM]), and by Pappus in the Seventh Book of the *Mathematical Collections* (Pappus [CH]; Pappus [C7J]).

<sup>5</sup>Analysis and synthesis were, indeed, considered as two successive stages of the same method (just called ‘method of analysis and synthesis’): the former was supposed to prepare the latter by suggesting a way to conduct it.

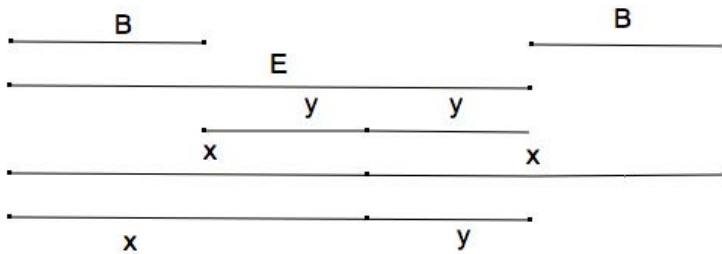


Diagram 1

tainly suppose they are so, but not necessarily, and Viète is quite explicit on it, in his *In artem analyticam Isagoge* (Viète [1591a]; Viete [AAW]), a sort of foundational essay that his *Zeteticorum Libri* openly depends on. *Latera* and *cubi* are just scalar quantities (Bos [2001], p. 143) admitting different possible instantiations: as numbers, or as geometrical objects, or even as other magnitudes sharing with those some crucial metric properties. Viète's formalism and notation are just conceived to make this generality and level of abstraction explicit.

At this general level, diagrams (or, at least, the usual geometrical diagrams, which are the only ones I'm considering here) can at most be useful for offering palpable illustrations of general arguments. However, they are certainly not indispensable as such, and rather potentially misleading. All that is relevant is the capacity to fix the relevant formalism in an appropriate way to make the required inferences licensed. But when one comes back from this general level to the particular case of geometrical segments and solids, the formalism that makes diagrams both dispensable and equivocal can be very well replaced by classical geometry's usual language without preventing trans-configurational analysis from working.

To show it, I will have recourse to a convent notation. However, this is only needed for the sake of brevity. The argument I'm going to present does in no way require this notation, and can perfectly be restated in the codified, but discursive language of Euclid's geometry by merely appealing to composition and decomposition of cubes and parallelepipeds, indispensably supplemented by appropriate diagrams (possibly only imagined, in fact, but not for this less indispensably operating as ingredients of the relevant intellectual activity).

Let  $B$ ,  $E$ ,  $x$  and  $y$  four segments respecting the same conditions as before, that is, such that

$$B = x - y \quad , \quad E = x + y$$

so that

$$E + B = (x + y) + (x - y) = 2x \quad , \quad E - B = (x + y) - (x - y) = 2y.$$

Let also these segments be represented by diagram 1<sup>6</sup>.

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<sup>6</sup>Notice that the diagram includes two distinct representations for  $B$  and three for  $x$  and  $y$ . This is possible because of the peculiar nature of the problem at issue, to which I shall longly come back later, namely of its being a purely quantitative problem. This means that what the problem is asking for is not (the construction of) a pair of particular segments, but rather the identification of the metric relations that

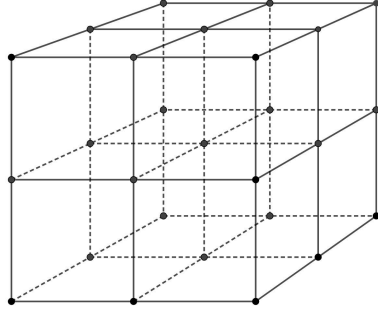


Diagram 2

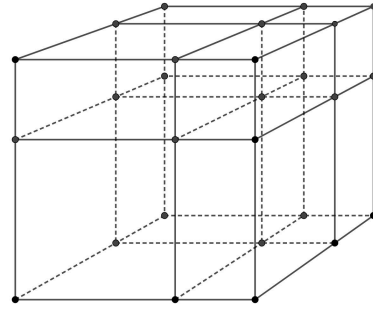


Diagram 3

For whatever segments  $a$ ,  $b$ , and  $c$ , let, moreover,  $\mathcal{C}(a)$  be the cube with side  $a$ , and  $\mathcal{P}(a, b, c)$  the parallelepiped with sides  $a$ ,  $b$ , and  $c$ . Hence, taking  $D$  to be the difference between  $\mathcal{C}(x)$  and  $\mathcal{C}(y)$ , from the elementary properties of cubes and parallelepipeds, immediately made manifest by diagrams 2 and 3, we'll have that

$$\begin{aligned}\mathcal{C}(E+B) &= \mathcal{C}(2x) = 8\mathcal{C}(x) \\ \mathcal{C}(E-B) &= \mathcal{C}(2y) = 8\mathcal{C}(y) \\ \mathcal{C}(E+B) - \mathcal{C}(E-B) &= 8[\mathcal{C}(x) - \mathcal{C}(y)] = 8D\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}(E+B) &= \mathcal{C}(E) + \mathcal{C}(B) + 3\mathcal{P}(E, B, B) + 3\mathcal{P}(E, E, B) \\ \mathcal{C}(E-B) &= \mathcal{C}(E) - \mathcal{C}(B) - 3\mathcal{P}(B, B, E-B) - 3\mathcal{P}(B, E-B, E-B) \\ \mathcal{P}(B, E-B, E-B) &= \mathcal{P}(E, E, B) - \mathcal{C}(B) - 2\mathcal{P}(B, B, E-B) \\ \mathcal{P}(B, B, E-B) &= \mathcal{P}(E, B, B) - \mathcal{C}(B)\end{aligned}$$

so that

$$\mathcal{C}(E-B) = \mathcal{C}(E) - 3\mathcal{P}(E, E, B) + 3\mathcal{P}(E, B, B) - \mathcal{C}(B)$$

and

$$\mathcal{C}(E+B) - \mathcal{C}(E-B) = 6\mathcal{P}(E, E, B) + 2\mathcal{C}(B) = 8D. \quad (*)$$

This provides a geometrical interpretation of the equality

$$(E+B)^3 - (E-B)^3 = 6E^2B + 2B^3 = 8D,$$

---

the search for segments must have with the given segment  $B$  and the given difference of cubes  $D$ . The metric nature of these relations makes the only relevant property of the given and searched for segments be their length, which in classical geometry was not conceived as a real number (as we do today), but as that property a segment shares with all other segments equal to it (on this matter, cf. my [2020], § 2, pp. 70-80). In modern terms, this means that what is relevant here are not single segments but appropriate equivalent classes of (equal) segments. Hence, what ' $B$ ', ' $x$ ' and ' $y$ ' properly denote are not single segments but circumstantial representatives of these equivalence classes.

which we had already got by using Viète's formalism. There is, however, no need to appeal to this last equality and to this formalism to go head and directly conclude from (\*) that

$$\mathcal{P}(E, E, B) = \frac{1}{3}[4D - \mathcal{C}(B)].$$

This conclusion does not directly correspond to that got through Viète's formalism, namely:

$$E^2 = \frac{4D - B^3}{3B}$$

since nothing allows decomposing  $\mathcal{P}(E, E, B)$  in the product of the square  $\mathcal{S}(E)$  and the segment  $B$ . However, this still makes the original problem reduced to a new simpler one, easily solvable by rule and compass.

Let us see what this new problem is, and how it can be solved. For the sake of simplicity, let us take the difference of cubes  $D$  to be a parallelepiped  $\mathcal{P}(B, B, a)$  constructed on the square  $\mathcal{S}(B)$ , where, according to *Elements*, XI.32 and VI.1,  $a$  is such that  $\mathcal{C}(B) : D = B : a$  (for Euclid's *Elements*, see Euclid [EEHS] and Euclid [EEH]). Insofar as  $B$  is given, admitting that  $D$  be also given is the same as admitting that the segment  $a$  is given. This makes  $4D$  equal to  $\mathcal{P}(B, B, 4a)$ ,  $4D - \mathcal{C}(B)$  equal to  $\mathcal{P}(B, B, 4a - B)$ , and  $\frac{1}{3}[4D - \mathcal{C}(B)]$  equal to  $\mathcal{P}(B, B, \frac{4a-B}{3})$ , all of which will also be given by quite easy constructions. Let this last parallelepiped be  $C$ . The new problem is, then, the following:

On a given side  $[B]$ , construct a parallelepiped with a square base  $[\mathcal{S}(E)]$  equal to a given parallelepiped  $[C]$ ,

where  $E$  is, of course, the unknown segment to be constructed. This is nothing but a simple particular case of *Elements*, XI.27, and can then be solved through the construction spelled out there.

The reduction of the given problem (*Zeteticum*, II.7), in its most natural geometrical interpretation, to this last problem (*Elements*, XI.27) is a clear case of trans-configurational analysis performed with no help of Viète's formalism (or of any other similar algebraic formalism). It shows, then, that this form of analysis does in no way require such a formalism.

Insofar as this is a form of analysis, it involves, of course, some work on unknown quantities. In this case, this work consists in dealing (or reasoning) in the same way both with (on) the unknown segments  $E$ ,  $x$  and  $y$ , and with (on) the given segment  $B$  and the given solid  $D$ , regardless to the fact that the former magnitudes are unknown, and the latter are given (and then known). This includes representing both the former and the latter magnitudes by appropriate diagrams involving no difference of nature between their respective representations. These diagrams surely play an indispensable role in the argument, perfectly in line with the indispensable role they play in the whole EPG, as accounted for in my [2012]. This role is quite marginal in this case, however, or better it is not made explicitly manifest. The most explicit roles these diagrams have, here, is keeping fixed the reference of the names of the relevant geometrical objects or equivalence classes and allowing to justify the basic results about the relevant segments, cubes and parallelepipeds that are appealed



to during the proof by merely considering the way these segments, cubes and parallelepipeds mutually compose and decompose.

More important for my purpose is what diagrams do not do here. This is immediately made manifest by the consideration of diagrams 2 and 3, representing a cube respectively decomposed in eight other cubes and in two other cubes and six (non-cubic) parallelepipeds. Though visually and intuitively different, these diagrams are, in fact, perfectly equivalent in the role they have within the previous argument: each of them can be eliminated by leaving its role played by the other. This is because they represent a way a cube is decomposed when all its sides are cut in two. All that the argument requires is the diagram(s) is to show how the cut of sides propagates on the whole solid, which is the same regardless if these sides are cut in equal or different parts, even if they are not all cut in the same proportion. The relevant information making one able to decide whether the parts composing the great cube are cubes, or rather (non-cubic) parallelepipeds, is neither provided, nor reflected by the diagrams, but rather by the assumptions of the problems (which only discursively stated).

In my [2012]'s language, this is an obvious consequence of the fact that equality is not a diagrammatic relation; that is, to be short (and even cursory), it is in no way accounted for in diagrammatic terms. Still, this consequence has a crucial effect on the logical nature of the argument: it allows avoiding to require that the relevant unknown segments be supposed to be constructed. It makes absolutely inessential, for the purpose of letting the diagrams play their role, whether these segments are known or not and actually satisfy the conditions of the problem; all that is relevant is that they are taken as sides of a cube or a parallelepiped, or as parts of such sides. In other terms, all that is relevant here is that we reason on cubes or parallelepipeds, as well as on their respective parts and sides, whatever they might be one to one another, both in magnitude, position and ratio (to use the three modalities in which geometrical objects can be given according to Euclid's *Data*: Euclid [EDHM]; Taisbak [2003]).

Such is what makes diagrams play their role in the argument. However, the argument does not only go ahead because of diagrams. It also depends on additive relations and rules of replacements proper to any sort of magnitudes, for example to the fact that

$$(\alpha + \beta) - (\alpha - \beta) = 2\beta \quad \text{and} \quad \left( \begin{array}{l} \alpha = \beta - \gamma - 3\delta - 3\varepsilon \\ \varepsilon = \zeta - \gamma - 2\delta \\ \delta = \eta - \gamma \end{array} \right) \Rightarrow \alpha = \beta - 3\zeta + 3\eta - \gamma,$$

whenever quantities  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$  might be. These relations and rules are essential for making analysis proceed, but they depend on no constructive clause, and diagrams have no role (or, at least, no direct role) in fixing them.

The (trans-configurational) analysis justifying the reduction of the given problem to another is then perfectly independent of the way the relevant segments, cubes and parallelepipeds are given or supposed to be given, provided they: enduringly belong to the same equivalence classes, bear the relevant additive relations, and obey the relevant substitution rules. There is no need to suppose that they have been constructed, since they are not identified in force of their actual or supposed construction, but rather of their staying to

each other in the relevant additive relations and of the fact that the segments are taken to be sides, or parts of sides, of the relevant cubes and parallelepipeds.

*Mutatis mutandis*, this happens in any instance of geometrical trans-configurational analysis.

Another example, essentially different in many respects from the one just considered, might help to understand this point. It concerns al-Khayyām’s *Algebra* (Woepcke [1851]; Rashed & Vahabzadeh [1999])—a treatise pre-dating Viète’s *Zeteticorum libri* by about five hundreds years. Here is the first of the twelve “trinomial”<sup>7</sup> problems it presents:

A square plus ten of its roots are equal to thirty-nine in number.

The most obvious way to conceive of this problem, in modern terms, is as a second degree equation. The numbers ten and thirty-nine are taken by one of Khwarizmi’s examples of the case where “Roots and Squares are equal to Numbers”, in which he states: “A Square and ten Roots are equal to thirty-nine Dirhems” (the reference is, of course, to Khwarizmi’s *Alegbra*: al-Khwarizmi [AMMR]; al-Khwarizmi [BAR]). In Khwarizmi’s explanation (as translated by Gerard of Cremona: Hughes [1986]), “a Root is anything which is to be multiplied by itself”, “a Square is whatever amount of the root multiplied by itself”, and “a Number is anything expressed by number words, without reference to Root or Square”. The case of “Roots and Squares are equal to Numbers” is, then, a general case which, as such, is not specifically concerned with numbers<sup>8</sup>. In the particular example at issue, the “Numbers” become, however, thirty-nine Dirhems, and this suggests to look at the problem as one in arithmetic, and it is just as an arithmetical that he solves it, though he later explains (or, better, justifies) this solution in geometrical terms.

Khayyām’s attitude is different, however. He does not only directly tackle the problem as a geometrical one, but also makes its solution preceded through a trans-configurational (geometrical) analysis:

Let us suppose that the square  $AC^9$  [in diagram 4] plus ten of its roots equals thirty-nine in number. Let us suppose, on the other hand, that ten of its roots are equal to the rectangle  $CE$ ; the straight line  $DE$  is thus ten. Cut it in half at  $G$ . Since we have cut the straight line  $DE$  in half at  $G$ , and we have added  $AD$  on its prolongation, the product of  $EA$  and  $AD$ , which is equal to the rectangle  $BE$ , plus the square of  $DG$ , are equal to the square of  $GA$ . Now the square of  $DG$ , which is the half of the number of roots, is known, and the rectangle  $BE$ , which is the given number, is known; the square of  $GA$  is thus known, and the straight line  $GA$  is known. If one takes away  $GD$ , it remains  $AD$ , known.

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<sup>7</sup>Al-Khayyām’s takes this problem to be trinomial for it involves three terms: a square, ten roots (of this square), and a number.

<sup>8</sup>See footnote 2, above

<sup>9</sup>This is, of course, the square  $ABCD$ . Notice that referring to a polygon by letters indicating opposite vertices of it was a common practice in Greek and Arabic Geometry. The same practice is also applied below.

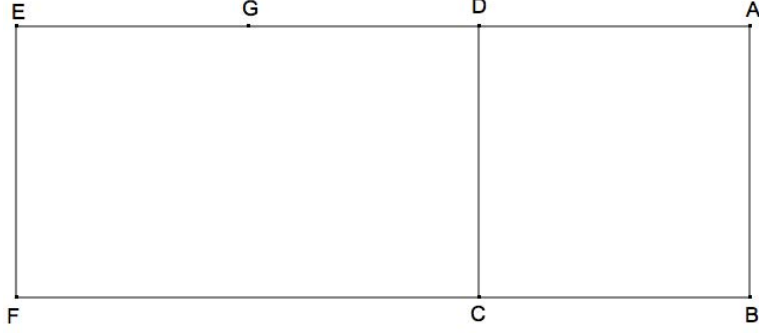


Diagram 4

This suggests looking at the numbers ten and thirty-nine as represented by segments (possibly measure by these segments with respect to a certain unitary segment, or directly as segments as such), and to generalize the problem as follows:

$$\mathcal{S}(x) + \mathcal{R}(a, x) = \mathcal{R}(b, c),$$

where  $a = DE$  is a given segment, which is, then, known,  $x = AD = AB$  is an unknown segment that is searched for, and  $\mathcal{R}(b, c)$  is a given rectangle, also known, which the (unknown) rectangle  $AF$  is supposed to be equal to.

The (trans-configurational) analysis goes, thus, as follows

Let us suppose that  $x$  be given. 1

Hence  $\mathcal{S}(x)$  is also supposed to be given. 2

Insofar as  $a$  is given, this is so also for  $\mathcal{R}(a, x)$  and  $\mathcal{S}(x) + \mathcal{R}(a, x)$ ,  
as well as for  $\frac{a}{2}$ ,  $\mathcal{S}\left(\frac{a}{2}\right)$ , and  $\mathcal{S}\left(x + \frac{a}{2}\right)$ . 3

By *Elements*, II.4, we know that  $\mathcal{S}\left(x + \frac{a}{2}\right) = \mathcal{S}(x) + \mathcal{R}(a, x) + \mathcal{S}\left(\frac{a}{2}\right)$ . 4

Hence, by the condition of the problem,  $\mathcal{S}\left(x + \frac{a}{2}\right) = \mathcal{R}(b, c) + \mathcal{S}\left(\frac{a}{2}\right)$ . 5

The synthesis is, then obvious: the searched for segment  $x$  results from subtracting the half of  $a$  from the side of a square equal to  $\mathcal{R}(b, c) + \mathcal{S}\left(\frac{a}{2}\right)$ .

This is essentially the method of the completion of the square<sup>10</sup>. Khayyām was not the first to use it. It also appears, in a slightly different form, appealing to *Elements*, II.6, rather than to *Elements*, II.4, in a short treatise by Thābit ibn-Qurra (Thābit [RPAR]), living about two centuries before Khayyām, and born about a quarter of century before Khwarizmi's death.

<sup>10</sup>In modern terms and notations, this method consists in solving a quadratic equation  $x^2 + ax + b = 0$  by observing that  $x^2 + ax + \frac{a^2}{4} = \left(x + \frac{a}{2}\right)^2$ , so that  $\left(x + \frac{a}{2}\right)^2 = \frac{a^2}{4} - b$ , and, then,  $x = \pm\sqrt{\frac{a^2}{4} - b} - \frac{a}{2}$ .

What was presumably new by Khayyām is the implicit extension of this sort of analysis to the case of problems involving cubes (or third-degree equations, in a modern understanding). I say 'implicit', since, in the case of these problems, Khayyām does not make the analysis explicit, though everything suggests that he had recourse to it to find his solutions.

Consider, for instance, the case of the other trinomial problem,

A cube plus some squares are equal to a number,

namely

$$\mathcal{C}(x) + \mathcal{P}(a, x, x) = \mathcal{P}(b, c, d).$$

Here is as he reasons in this case:

- |  |   |
|--|---|
| Let $\mathcal{P}(b, c, d) = \mathcal{C}(h)$ , then: $\mathcal{C}(x) + \mathcal{P}(a, x, x) = \mathcal{C}(h)$ ,<br>where $a$ and $h$ are given. | 1 |
| Let us suppose that $x$ be given.  | 2 |
| Adding $\mathcal{C}(x)$ and $\mathcal{P}(a, x, x)$ , we get $\mathcal{P}(a + x, x, x) = \mathcal{C}(h)$ .                                      | 3 |
| This is equivalent to the two proportions<br>$x : h = h : y$ and $h : y = y : a + x$ .   | 4 |

Also in this case the synthesis (which is, of course, made explicit by Khayyām) is obvious: these two proportions are respectively the *symptomata* of a hyperbola subtending  $\mathcal{S}(h)$ , and a parabola of *latus rectum*  $h$ , whose axis coincides with an asymptote of the hyperbola and whose vertex is at a distance  $a$  from the center of it<sup>11</sup>. Look at diagram 5: let **A** is the vertex of the parabola, **B** the center of the hyperbola and  $BE = h$ . It will follow that  $AB = a$ , and  $BC = CD = ED = h$ . The orthogonal projection **G** of the intersection point **F** of the two conics on the axis of the parabola is, then, such that  $x = BG$ ; to solve the problem is,

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<sup>11</sup>In the terminology of Apollonius's *Conics* (Apollonius of Perga [CH]), the *symptoma* of a conic is a proportion characterizing the conic with respect to a pair of intrinsic linear coordinates. The proportion  $x : h = h : y$  characterizes a hyperbola having two orthogonal asymptotes, each point of which is such that the rectangle  $\mathcal{R}(x, y)$ , constructed on its projections  $x, y$  on these asymptotes, is equal to the square  $\mathcal{S}(h)$ , where  $h$  is a given constant segment. Taking the intersection point of these asymptotes (i.e. the center of the hyperbola) as the origin of a system of Cartesian orthogonal (extrinsic) coordinates, whose axes are given by these very asymptotes, we have, then, that this hyperbola has equation  $xy = h^2$ . The proportion  $h : y = y : x$  characterizes instead a parabola whose focus and directrix are such that the chord passing through the former and parallel to the latter (namely its *latus rectum*, in Apollonius's terminology), is equal to  $h$ . Taking its vertex as the origin of a system of Cartesian orthogonal (extrinsic) coordinates, whose axes are given by its directrix and its axis, this parabola has equation  $y^2 = hx$ . Supposing that the axis and the directrix of the parabola be provided, respectively, by an asymptote of the hyperbola and by a parallel to the other asymptote (so as to make the variables  $x$  and  $y$  vary on the same axes of coordinates, in the two *symptomata*), we can imagine to translate the parabola along its axis, so as to place its vertex at a distance equal to  $a$  from the center of the hyperbola. The *symptoma* of the parabola in its new position will, then, be given by the proportion  $h : y = y : a + x$ , and its equation, with respect to the same system of coordinates to which we have referred the hyperbola, will be  $y^2 = h(a + x)$ . This should be enough to explain what follows.

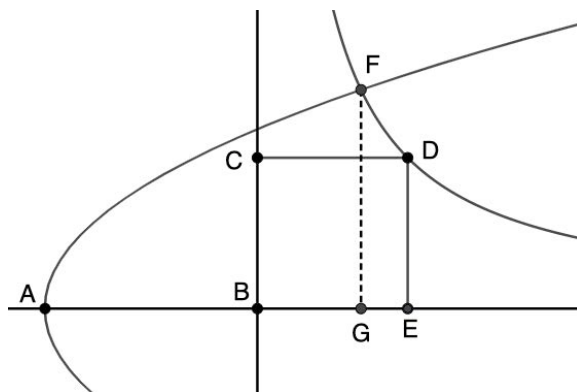


Diagram 5

then, enough to construct these conics, orthogonally project their intersection point on the axis of the parabola, and subtract  $a$  from the segment connecting the point so obtained to its vertex.

The way the diagram enters this synthesis is not relevant for my present point. It is enough to say that *mutatis mutandis* (because of the appeal to conics) it is in line with what I have described in my my [2012] for EPG.

What is relevant is that much less is required to diagrams by the analysis. This is made blatant by the fact that its reconstruction has required no (explicit) appeal to them. The reason is just the same already expounded above: the argument does not require that the relevant segments be supposed to be constructed. Differently than in the case of Viète's *Zeteticum*, which is essentially concerned with the way a cube can be decomposed in other cubes or parallelepipeds, each of Khayyām's problems concerns distinct figures: a square and two rectangles for the first considered above, and a cube and two parallelepipeds for the second. Still, these figures are not supposed to form a single system, whose different elements stay, to each other, in a particular position and are, then, linked by particular constructive relations. Each one of these figures enters the relevant problem in isolation, if not for the fact that some of their sides are supposed to be equal, or better belong to the same equivalence class. The problems are not about single constructive systems involving these figures at once, but, rather about the additive conditions that these figures, taken separately one from each other, are supposed to meet. Hence the relevant segments and figures are not identified in force of their actual or supposed construction, but rather in force of their bearing some additive relations, and of the fact that the segments are taken to be sides or parts of sides of the relevant figures. Again, there is no need that the unknown ones be supposed to have been constructed.

Up to here, I have only considered examples of problematic analysis. Still trans-configurational analysis can also be theorematic. By its nature, problematic analysis supposes the theorem proved, and operates on its statement, drawing appropriate consequences from it, in order to find a way for actually proving it.

An excellent example is provided by the alternative proofs of *Elements*, XIII.1-5, usually

attributed to Heron of Alexandria (Euclid [EEHS], vol. IV, pp. 198-204; Heiberg [1903], p. 58; Euclid [EEH], vol. III, p. 422), who probably lived in the first century AD. All these proofs are preceded by an explicit analysis, which is openly trans-configurational, indeed. Let us consider the case of *Elements*, XIII.1:

If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square of the half.

Using the previous notations this can be expressed this way:

$$(b + c = a \ \& \ a : b = b : c) \Rightarrow \mathcal{S}\left(b + \frac{a}{2}\right) = 5 \mathcal{S}\left(\frac{a}{2}\right)$$

The analysis goes, then, as follows

Let's suppose that	$\mathcal{S}\left(\frac{a}{2} + b\right) = 5 \mathcal{S}\left(\frac{a}{2}\right).$	1
For <i>Elements</i> , II.4	$\mathcal{S}(b) + \mathcal{R}(a, b) = \mathcal{S}\left(\frac{a}{2} + b\right) - \mathcal{S}\left(\frac{a}{2}\right).$	2
Hence	$\mathcal{S}(b) + \mathcal{R}(a, b) = 4 \mathcal{S}\left(\frac{a}{2}\right).$	3
For <i>Elements</i> , VI.17	$\mathcal{S}(b) = \mathcal{R}(a, c).$	4
Hence	$\mathcal{R}(a, c) + \mathcal{R}(a, b) = 4 \mathcal{S}\left(\frac{a}{2}\right).$	5
For <i>Elements</i> , II.2	$\mathcal{R}(a, c) + \mathcal{R}(a, b) = \mathcal{S}(a).$	6
Hence	$\mathcal{S}(a) = 4 \mathcal{S}\left(\frac{a}{2}\right).$	7

As this last equality is an obvious particular case of *Elements*, II.4, it can be taken as proved, and this suggests starting the proof of *Elements*, XIII.1 from this very equality and going ahead by merely reversing the analysis.

Here all the segments, squares and rectangles are given. So, the analysis does not work on any unknown objects as if it were known. It rather starts from a statement that is not (still) proved, and deduces an already proved theorem from it, suggesting that the required proof takes its start from it. It follows that diagrams can enter it, at most, insofar as they are required to perform the deduction or, indirectly, to prove the auxiliary theorems that the deduction has recourse to. As it happens for trans-configurational problematic analysis, also trans-configurational theorematic analysis does neither require that the unknown geometrical objects be supposed to have been constructed, nor makes any specific use of diagrams, apart from those that classical geometry generally requires to conduct its non-analytical arguments.

## 2 Intra-Configurational Analysis and the Purely Material Role of Diagrams

None of the previous arguments (both in their original form or in my reconstruction) has recourse to results that were not available to the Greek mathematicians of the Hellenistic

Period (such as Euclid, Archimedes and Apollonius, to mention only the main ones). Still, the way these mathematicians looked at similar problems was essentially different than that which emerges from these arguments.

To appreciate this crucial difference, let us consider Euclid’s solution of *Elements*, VI.29:

To a given [segment of] straight line to apply a parallelogram equal to a given rectilinear figure and exceeding by a parallelogrammatic figure similar to a given one.

A particular instance of this problem arises when the parallelogram to be constructed (or, in Euclid’s parlance, applied to the given segment) is required to be a rectangle, and both the given figures are supposed to be squares. If  $a$  were the given segment and the previous notation were adopted, the problem would consist in constructing a rectangle  $\mathcal{R}(a, x)$ , such that

$$\mathcal{R}(a + x, x) = \mathcal{S}(b)$$

where  $\mathcal{S}(b)$  is the given square. Insofar as, by *Elements*, II.2,  $\mathcal{R}(a + x, x) = \mathcal{S}(x) + \mathcal{R}(a, x)$ , the problem would, then fit with Khwarizmi’s case where “Roots and Squares are equal to Numbers”, in Khayyām’s geometrical interpretation, and under the condition that  $c = b$ . It would, then, admit a quite simple solution along the lines suggested by Khayyām’s analysis.

This is not the way Euclid looks at it, however. When Euclid’s solution is adapted to the particular case at issue, it does not properly show how to construct a segment  $x$  such that

$$\mathcal{S}(x) + \mathcal{R}(a, x) = \mathcal{S}(b).$$

He rather supposes that a particular segment  $AB$  be given, and shows how to construct another segment  $BQ$  through  $B$ , perpendicular to  $AB$ , such that the aggregate of the rectangle constructed on  $BQ$ ,  $AB$  and the square constructed on  $BQ$  be equal to a given square.

In modern terms, we might say the same this way: Euclid does not aim at solving the geometrical problem corresponding to a second degree equation, but rather at constructing a particular geometrical model meeting the condition expressed by such an equation; he does not look at the problem as to a purely quantitative one, concerned with equivalent classes of equal segments, but as to a problem requiring to construct a single segment meeting this condition, in a specified position with respect to the given one.

This model is easily represented by diagram 6, where  $BO$  is a square, and the segment  $BQ$  is to be so constructed that the rectangle  $AO$  be equal to the given square  $CD$ .

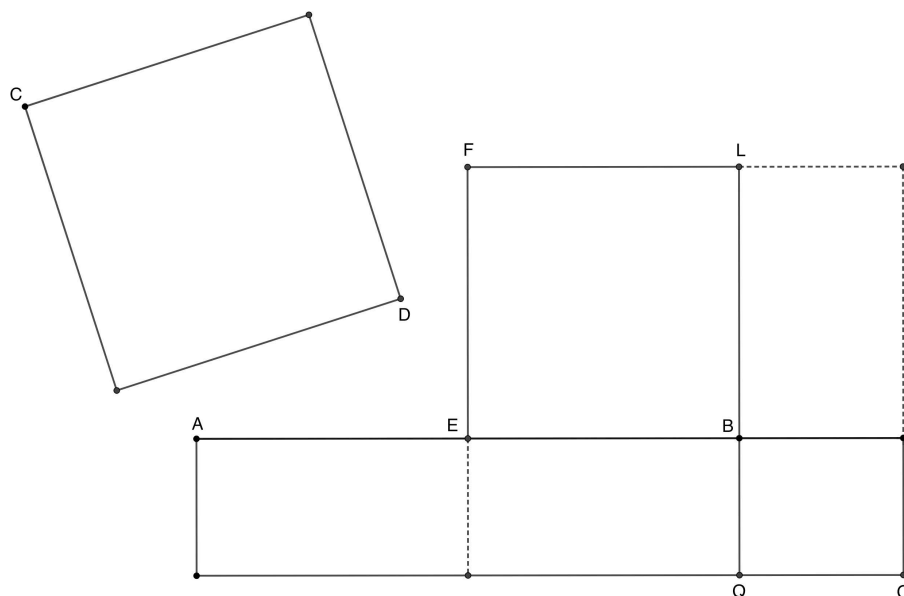


Diagram 6

When it is adapted to the particular case at issue, Euclid's solution goes as follows.

- 1 The segment  $AB$  is given. Construct its middle point  $E$ .
- 2 On the segment  $EB$ , construct the square  $EL$ .
- 3 Produce  $LB$  up to  $Q$  so that the square on  $LQ$ , equal to the square  $FO$ , be equal to the given square  $CD$  and the square  $EL$ , taken together.
- 4 The rectangle  $AO$  will be equal to the gnomon  $EOL$ , that is to the difference between the squares  $FO$  and  $EL$ , namely to the square  $CD$ .
- 5 The segment  $BQ$  is, then, the sought for segment  $x$ .

Euclid's equality between the square  $FO$  and the squares  $CD$  and  $EL$ , taken together, is clearly tantamount to step 5 of Khayyām's analysis, provided that  $b = c$  and the segment  $AB$  and the square  $CD$  be respectively taken to be  $a$  and  $\mathcal{S}(b)$ . This makes clear that the crucial mathematical point of the two arguments is just the same. What is different is the way this step is conceived of: while Euclid sees its equality as a synthetic constructive, one, concerned with the particular segment  $BQ$ , so constructed as to make the square  $FO$  equal to the sum of two given squares, Khayyām sees it as a general quantitative analytic equality, concerned with the unknown segment  $x$ , or better with the equivalence class it is required to belong to.

The difference is essential in relation to the development of mathematics, since it is only the advent of the second approach that made possible the birth and development of algebra (in the sense explained in footnote 2, above). Though diagrams enter Euclid's argument in



a much more prominent role than in Khayyām's, this is only because the former argument requires attention for the mutual position of the segment at issue, conceived in their singular individuality, and not to an essentially different relation to be established between geometrical objects and diagrams. The views advanced in my [2012] are, it seems to me, still appropriate to account for the role diagrams have in both sorts of arguments.

A crucial difference in the role of diagrams arises, instead, when Euclid's approach is at work in an analytical argument, rather than in a synthetic one, that is, when the constructions are not actually achieved by starting from given segments, but rather conditionally performed by starting from unknown segments, which are only supposed to be given. In this case, the analysis becomes intra-configurational, and it requires diagrams to play an essentially different role. This difference is due to the fact that, contrarily to trans-configurational one, intra-configurational has to suppose that the relevant unknown be constructed, and have, then, a specific constructive relation with the given ones. Since what is at issue here are no more equivalence classes of (equal) segments and the additive relation of the figures that have them as sides, but rather particular points or segments in an identified position, to be obtained by a particular construction. It is this construction that is to be disclosed since it is this construction that would make the sought for points or segments known, and that characterizes them as these very points and segments.

Consider a simple example: proposition II.3 of Archimedes's treatise *On the Sphere and Cylinder* (Archimedes [OOHS], vol. I, pp. 1-229; Archimedes [WH], pp. 1-90). This is a problem:

To cut a given sphere by a plane so that the surfaces of the [two spherical] segments [that are got in this way] have to one another a given ratio.

The analysis goes as follows.

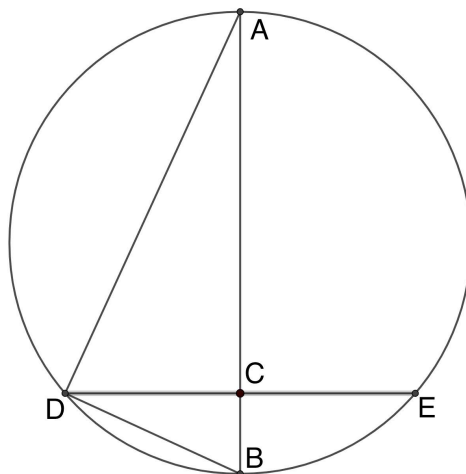


Diagram 7

Let  $ADBE$  (diagram 7) a great circle of the given sphere, and  $AB$  a diameter of it. 1

Suppose that the problem have been solved by cutting the sphere by the plane perpendicular to  $AB$ , which cuts this great circle in  $D$  and  $E$ , and the diameter itself in  $C$ . 2

The surface of the spherical segment  $DBE$  will, then, be to the surface of the spherical segment  $DAE$  in the given ratio. 3

According to *Sphere and Cylinder*, I.42-43, these surfaces are equal to the circles of radii  $DB$  and  $DA$ , respectively. 4

These circles are to one another as the squares  $\mathcal{S}(DB)$  and  $\mathcal{S}(DA)$  on these radii. 5

According to the Pythagorean theorem, these squares are to one another as the segments  $CB$  and  $AC$ . 6

Hence, these segments are to one another as the surfaces of spherical segments  $DBE$  and  $DAE$ . 7

The synthesis becomes, then, obvious: what has to be done is cutting the diameter  $AB$  in a point  $C$  (diagram 8), so that  $AC$  and  $CB$  be to one another in the given ratio. If this ratio is given insofar two segments  $Z$  and  $H$  are given, which are to one another in it, as Archimedes openly grants, then, this can be easily done by rule and compass by the construction of the fourth proportional between  $Z + H$ ,  $AB$  and  $Z$  (or  $H$ ). The construction is obvious, in agreement to *Elements*, I.2, I.11, I.31 and VI.9, and Archimedes does not detail it. Here is how it might go.

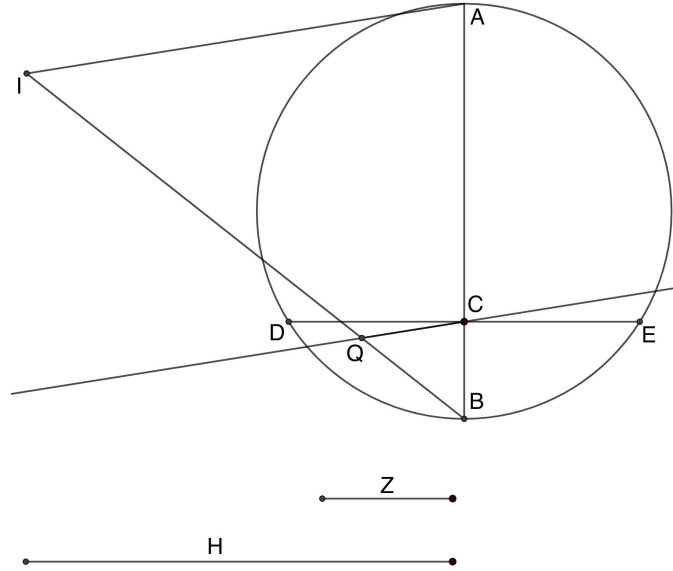


Diagram 8

In agreement to *Elements*, I.2, construct a segment BQ, starting from B and equal to Z; then, in agreement, again, to *Elements*, I.2, and to *Elements*, Post. I.2, construct another segment QI, starting from Q, collinear to BQ, and equal to H. In agreement to *Elements*, Post. I.1, trace the segment IA. In agreement to *Elements*, I.31, construct the parallel QC to IA passing through Q. The intersection point C of this parallel and the diameter AB cuts this diameter in such a way that AC is to CB as H to Z. In agreement to *Elements*, I.11, trace the perpendicular DE to AB through C. The plane perpendicular to AB through DE cuts the sphere so that the surface of the two spherical segments DBE and DAE are to one another in the given ratio of Z and H.

Here the diagram enters the argument just as it enters Euclid's solution of *Elements*, VI.29. This is also the way it enters the synthesis of our previous examples of trans-configurational analysis, but with an essential difference. In these latter cases, the given segments are, in fact, whatever elements of the relevant equivalence class of equal segments, working as incidental representatives of these classes. Both here and in Euclid's solution of *Elements*, VI.29, the given segments (namely AB, both in the solution of *Elements*, VI.29, and in Archimedes's synthesis) are particular individual objects, working as incidental representatives or whatever segments, and they provide a basis for the construction of other particular segments and points forming a geometrical system, which works, as a whole, as an incidental representative or any other (topologically) equivalent geometrical system. In other terms, the focus, in these two last cases, is no more on equivalence classes of equal segments but on a(n equivalence class of) particular system(s) of geometrical objects.

What makes this (equivalence class of) system(s) represented by a diagram is that the diagram is drawn, step-by-step, in agreement to some constructive clauses belonging to a constructive canon which is, as such, an essential component of the theory within which the

argument is conducted. Even the given segments, square and circle are just given insofar as they are supposed to have been constructed in advance through a licensed construction, and this makes their diagrams intended, in turn, as drawn in agreement with this construction which is not detailed here (if this construction is not detailed here is simply because the focus is not on it, but on a further construction).

This is a crucial aspect of the view about the role of diagrams in EPG, which I have defended in my [2012]: their drawing is nothing but a (material) counterpart of a constructive argument, which go, step-by-step, together with the whole process of this drawing; each diagrammatic gesture that is here at issue obeys a constructive clause licensed by the relevant canon; and it is this, rather than the actual features of the material drawing, that makes the diagrams represent the relevant objects and play their role in the geometrical argument.

This is what happens in Archimedes's synthesis, however, not in his analysis. Since what makes the latter an analysis is just that its starting point—the act of taking  $C$  as a particular given point on  $AB$ —is not the outcome of a licensed construction. This makes the dot representing this point not drawn in agreement to any constructive clause, but rather freehand, so to say.

To better see the point, let us come back to Archimedes's synthesis more in detail.

The segments  $Z$  and  $H$  are supposed to be given. Diagrammatically, this is reflected by the supposition that the two rectilinear strokes representing them are drawn on the relevant support (be it a wax tablet, a papyrus, a portion of a sandy beach, a sheet of paper, or even an imaginary screen) in a certain given position, with respect to the circular stroke representing the given great circle and the dots on it representing the extremes of the relevant diameter. It is not important which position they have respectively to one another. What is important is that their respective position is supposed to be given (possibly as the outcome of a previous construction). The diagrammatic gesture consisting of drawing these strokes is, then, to be conceived of as having been performed in advance, and considered as external to the solution itself. If no previous construction is presented, the diagram must be drawn freehand, by arbitrarily choosing the respective positions of the relevant strokes and dots. Still its being drawn freehand, by obeying no specific constraint and independently of any constructive clause, is in this case perfectly immaterial, since this just reflects the supposition that the relevant geometrical objects are given, without any further condition about the respective position they must have.

However, once this initial diagram is made available, the situation changes. *Elements* I.2 provides precise instructions about how the segments  $BQ$  and  $QI$  are to be constructed, starting from the segments  $Z$  and  $H$  and point  $B$ . Following these instructions and performing the construction has a diagrammatic counterpart consisting in drawing the strokes representing the former segments in a certain position that is imposed by the initial position of the strokes and dot representing the latter segments and points. The position of the dot representing the point  $I$  is, then, also determined, so that of the stroke representing the segment  $IA$ , since that of the dot representing the point  $A$  is also given (notice that, if the segments  $BI$  and  $BA$  result to be collinear, a simple trick is to be used, but is not necessary here entering this detail). This makes also the position of the stroke representing the parallel

to this segment through  $Q$  determinate, and this is, then, so also for the dot representing the sought for point  $C$ . The geometrical construction of this point goes, then, together with drawing a diagram in a way that is entirely determined by the initial conditions and the constructive clauses applied along its construction.

This is not so for the analysis. Since this starts with the supposition that the point  $C$  be given independently of its construction, which is just what is to be determined. Hence, the diagrammatical counterpart of this supposition cannot but be a free diagrammatic act, consisting in placing the dot representing this point in whatever position one likes or judges intuitively conveniently, on the stroke representing  $BA$ . This is what I mean by saying that this dot is drawn freehand. It is only after this initial diagrammatic gesture is freely performed, that one can go ahead tracing the strokes representing the segments  $DE$ ,  $BD$  and  $AD$ , in agreement to appropriate constructive clauses by then reasoning on the last two segments as said above.

The diagrammatic gesture making the point  $C$  represented by a certain dot, and reflecting the starting stage of the analysis—the supposition that this point be given—is, then, essentially different, in nature, from the diagrammatic gestures accompanying any other step of these two arguments. While the latter gestures are not only codified, but also imposed by licensed constructive steps and enter the two arguments this way, the former gesture enters the analysis as a purely material act, so to say, obeying no constructive constraint (the obvious condition apart, of course, that the relevant dot be taken on the stroke representing the segment  $AB$ ). It is this material act—this free diagrammatic gesture, as it were—that makes the analysis possible, by allowing taking the sought for point as given. It is this act that triggers an effective (but conditional) construction, making all along the analytical argument itself other diagrams constructed in agreement with appropriate constructive clauses. To resume, we can say that (some) diagrams necessarily play intra-configurational analysis in a purely material role.

Whereas in a geometrical trans-configurational analysis, what makes possible to work on unknown objects is, as I have said above, the fact that the analytical argument does not properly require these objects be supposed to have been constructed since these objects are not identified in force of their actual or supposed construction, this is just what happens heres, as well as in any other instance of intra-configurational geometrical analysis: the point  $C$  is here properly supposed to have been constructed, and solving the problem, during the synthesis, consists in actually performing its construction. It is this construction that identifies this point as such. A crucial role of diagrams, is, then, here, that of supporting the very possibility of working on a non-yet-constructed point as if it were constructed, by displaying it in a discretionary position: a role that can only be played by diagrams drawn freehand, or, as it were, thanks to a free diagrammatic gesture, a purely material act.

It has been often argued that, though certainly present in the works of the Greek mathematicians of the Hellenistic Period, geometrical analysis is not so frequent there. It has been maintained, for instance, that no example of analysis is present in the *Elements*, interpolations apart, of course<sup>12</sup>. This depends on a quite restricted way of understanding

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<sup>12</sup>Secondary literature on Greek analysis—and synthesis—is huge; here I only signal some sources: Robin-

geometrical analysis, which I do not consider appropriate, conceptually speaking. If an analysis is an argument concerned with items that are not at hand, but are nevertheless supposed to be so, or with some statements that have not been justified, but are nevertheless taken as granted, then an instance of intra-configurational geometrical problematic analysis is at work any time a certain geometrical objects is supposed to be constructed, though not being constructible, and an instance of intra-configurational geometrical theorematic analysis is at work any time a certain geometrical statement is supposed to be proved, though not being provable. This happens, any time a *reduction ad absurdum* is performed in geometry by showing that a contradiction follows by the supposition that either a certain geometrical object is constructed or a certain geometrical statement is admitted. But, if this is so, then intra-configurational geometrical analysis, both problematic and theorematic, is largely present, if not pervasive, in the works of the Greek mathematicians of the Hellenistic Period, and also, of course, in the *Elements*.

A simple example comes from Euclid's proof of *Elements*, I.7. Though this is just a theorem and requires a proof, the intra-configurational analysis involved in this proof is properly problematic, rather than theorematic, even if it concludes that proof, in fact. This is because what is to be proved is that a certain construction cannot be performed: a certain triangle is supposed to have been constructed, and it is shown that this supposition leads to a contradiction. What makes the analysis provide a proof is just a last meta-geometrical step, in which it is taken for granted that no licensed constructions can lead to a contradiction, and it is, then, concluded that no such construction could make the relevant triangle actually constructed.

Euclid's theorem is the following:

Given two [segments of] straight line constructed on a [segment of] straight line (from its extremities) and meeting in a point, there cannot be constructed on the same [segment of] straight line (from its extremity), and on the same side of it, two other [segments of] straight line[...] meeting in another point and equal to the former two respectively [...].

To prove it, Euclide takes three segments  $AB$ ,  $AC$ , and  $BC$  (diagram 9) to be given, so as to form a triangle  $ABC$ , and proves, by *reductio ad absurdum*, that no other point  $D$ , which is not symmetric to  $C$  with respect to  $AB$ , can be constructed, such that  $AD = AC$  and  $BD = BC$ . More generally, his argument actually proves that if whatever three distinct segments  $AB$ ,  $E$ ,  $Z$ , and a triangle  $ABC$  are given, such that  $AC = E$  and  $BC = Z$ , then, up to symmetry, no other triangle can be constructed on  $AB$ , whose other sides are respectively equal to  $E$  and  $Z$ . Hence, three segments being given, only one triangle, can be constructed, up to symmetry, such that its sides are equal to these segments.

Here is Euclid's argument.

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son [1936]; Mahoney [1968]; Hintikka & Remes [1974]; Knorr [1986]; Behboud [1994]; Mäaenpää [1997]; Panza [1997]; Netz [2000]; Fournarakis & Christianidis [2006]; Acerbi [2011]; Sidoli & Saito [2012]; Sidoli [2018].

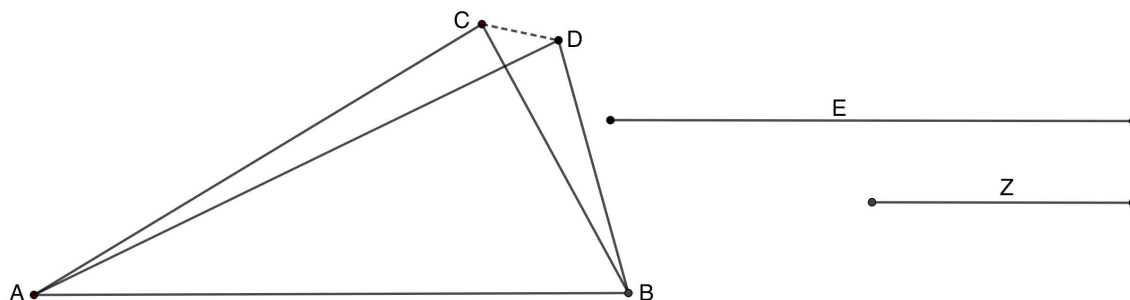


Diagram 9

Let a point D other than C supposed given  
 “on the same side” of AB than C, such that  
 $AD = AC$  and  $BD = BC$ . 1

In agreement to *Elements*, Post. I.1,  
 let the segment CD be traced. 2

The isosceles triangles ADC and BDC  
 are, then, given. 3

According to *Elements*, I.5,  
 the angles  $\widehat{ACD}$  and  $\widehat{ADC}$  are equal. 4

As the angle  $\widehat{BCD}$  is part of the angle  $\widehat{ACD}$ ,  
 in agreement to *Elements*, Com. Not. I.5,  
 the latter is greater than the former, 5  
 which makes also the angle  $\widehat{ADC}$   
 greater than the angle  $\widehat{BCD}$

As the angle  $\widehat{ADC}$  is part of the angle  $\widehat{BDC}$ ,  
 in agreement to *Elements*, Com. Not. I.5,  
 the latter is greater than the former, 6  
 which makes the angle  $\widehat{BCD}$   
 greater than the angle  $\widehat{BDC}$ .

But according to *Elements*, I.5,  
 these last two angles are rather equal. 7

Contradiction. 8

Hence, no point D other than C can be given  
 “on the same side” of AB than C, such that  
 $AD = AC$  and  $BD = BC$ . 9

This argument can be extended to prove that, up to symmetry, no triangle other than  $ABC$  can be constructed on  $AB$ , whose other sides are respectively equal to  $E$  and  $Z$ . How this can be done is obvious, and there is no need to detail it.

Steps 1-7 of this argument clearly form a problematic intra-configurational analysis. If, on the one side, this is made possible by the drawing of the diagram representing the two segments  $AD$  and  $BD$ , sharing the extremity  $D$ , which allows supposing that the triangles  $ADC$  and  $BDC$  be given, this same argument proves, on the other side, that these segments are not only supposed to be constructed without being actually so, but can not be constructed, in fact. This diagram is, then, not only traced freehand, but must be traced freehand. Tracing it is necessarily a free diagrammatic gesture, that is, as I have said below, a purely material act.

### 3 Conclusions

Diagrams indispensably enter classical geometry by playing different crucial roles in it. They can be either physically drawn on appropriate supports or simply imagined. Their being imagined does in no way make them count as abstract objects, however, since what is imagined are just physical drawings. Even if one wanted to argue that classical geometry is dealing with pure forms (which is not something I argued for, here), it would remain that these forms are not imagined diagrams, but geometrical objects represented by them. While doing classical geometry, imagining diagrams is not the same as conceiving of the very objects of this geometry, such as segments (of straight line), circles, or squares. *A fortiori* this is also the case for drawing diagrams. Whereas the latter is directly a physical gesture producing a physical object, the former is the imagination of such a physical gesture. By speaking above of a diagrammatic gesture, I referred to both this physical gesture and this act of imagination of such a physical gesture. In doing that, I defined neither diagrams in general, nor the particular diagrams entering classical geometry. I have simply taken for granted that the reader is familiar with them.

More than that, I have also confined myself to provide some examples displaying two different uses that diagrams have in classical geometry, both in analytic and synthetic arguments. By synthetic arguments, I mean arguments only concerned with given objects and actual constructions, as well as with proved statements or general liminary assumptions. By analytic arguments, I rather mean arguments also concerned with non-given objects, which are required to be constructed (in a subsequent synthesis), but are provisionally only supposed to have been so by a purported but still undiscovered construction, as well as with provisionally supposed statements. I have especially focused on analytical arguments, by distinguishing two sorts of geometrical analysis, at work in classical geometry. I have begun with some examples of trans-configurational analysis, whose purpose is to transform a certain configuration of data into another one to reduce a certain problem to another simpler one, or to deduce an already proved or disproved statement from another one still waiting for being proved or disproved. I have, then, offered examples of intra-configurational analysis, whose purpose is to enquire a supposed configuration of data to identify a way for constructing



the unknown (or unconstructed) elements of it starting from the given ones, or to show its actual impossibility.

By looking at the role of diagrams within these two sorts of geometrical analysis, I have showed that the second sort of them indispensably requires a kind of diagrammatic gesture which is not required by the former. This is the act of drawing or imagining to draw diagrams independently of any constructive clauses governing the actual construction of the objects that these diagrams are intended to represent, and simply consists, then, either in a freehand drawing, or in the imagination of such a drawing. It might appear strange, at first glance, that such a codified and rigorous practice as classical geometry not only have occasionally recourse to such sort of gesture, but have indispensably recourse to it while conducting a particular sort of its arguments. The main purpose of my paper was to explain that this is, in fact, far from strange but rather intrinsically inherent to the nature of these arguments.

Classical geometry is a limited and elementary fragment of mathematics, or better of its history. The way diagrams enter it is quite peculiar. However, this does not entail that diagrams do not indispensably enter other portions of mathematics, both less elementary and still practised by contemporary mathematicians. There would be no hope, however, to summarily account, in general, for a single role of diagrams in mathematics. Different sorts of diagrams play different sorts of roles in other portions of it. Accounting for these other roles, other examples and additional arguments would be needed. Many studies have been made and are made about this matter; others will surely follow. For my part, I cannot but stop here, for the time being.

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