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JEL Codes: C72; D82.

Keywords: Self-confirming equilibrium; Ambiguity aversion; Information feedback; Strategic uncertainty; Public good games; Volunteer dilemma



Feedback Design in Strategic-Form Games with Ambiguity Averse Players *

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March 10, 2023

Abstract

We use a notion of maxmin self-confirming equilibrium (MSCE) to study the optimal design of players' feedbacks about others' behavior in games with ambiguity averse players. Coarse feedbacks shape strategic uncertainty and can therefore modify players' equilibrium strategies in an advantageous way. We characterize MSCE and study the equilibrium implications of coarse feedbacks in various classes of games. We show how feedbacks should be optimally designed to improve contributions in generalized volunteer dilemmas and public good games with strategic substitutes, strategic complements, or more general production functions. We also study games with negative externalities and strategic substitutes such as Cournot oligopolies. In general, perfect and no feedbacks are suboptimal. Some results are extended to α -maxmin preferences.

KEYWORDS: self-confirming equilibrium; ambiguity aversion; information feedback; strategic uncertainty; public good games; volunteer dilemma.

JEL CLASSIFICATION: C72; D82.

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1 Introduction

A standard justification for Nash equilibrium in n -player strategic-form games is that a Nash equilibrium strategy profile corresponds to a steady state of a collective learning process at the population level. Whenever the game is played, each player $i = 1, \dots, n$ playing the game is drawn from a large population of individuals assigned to the role of player i . Each individual plays the game only once, and for each population, the proportion of individuals playing each available action in this population is perfectly observed by all individuals.

The above equilibrium justification relies on the assumption of perfect feedbacks: individuals in each population i observe the aggregate behavior of individuals in every other population $j \neq i$, so at the steady state, individuals' subjective theories about others' behavior are correct. However, in many situations, feedbacks about the behavior in each population may be imperfect. For example, in public good problems, individuals may only observe whether the contributions in past play reached or not some threshold level of contributions. Alternatively, suppose the strategic-form game represents a dynamic game. In that case, only the actual history of play in the past occurrences of the game might be observed, but not the strategies (complete plans of actions) used by individuals in each population. When individuals receive such coarse feedbacks, a whole range of subjective beliefs about the aggregate behavior in each population could be consistent with these feedbacks. If the only restriction imposed on individuals' subjective beliefs is feedback consistency, the appropriate equilibrium concept is the notion of self-confirming equilibrium (Battigalli, 1987, Fudenberg and Levine, 1993). Because actual behaviors are consistent with any feedback, a Nash equilibrium is always a self-confirming equilibrium, and the set of Nash and self-confirming equilibria coincide under perfect feedback.

With coarse feedbacks, players' subjective beliefs do not have the same nature as when they stem from perfect feedbacks, even if it turns out that these beliefs are correct. Under perfect feedbacks, beliefs are objective, so the environment is risky: when an individual in population i knows that the proportion of individuals playing action a_j in population j is $s_j(a_j)$, he knows that the probability that player j drawn from this population plays action a_j is equal to $s_j(a_j)$. Under coarse feedbacks, this probability is ambiguous, so the environment is uncertain. If individuals are ambiguity averse, they do not behave as if their belief about population j is summarized by some objective strategy s_j . Instead, they should take into account several possible scenarios about s_j . In the most extreme case of ambiguity aversion, each individual only considers the worst case scenario when deciding which strategy to use. A steady state behavior in such a setting corresponds to what has been called a partially specified equilibrium by Lehrer (2012) or a maxmin self-confirming equilibrium by Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2015, 2016).¹

In this paper, we study a notion of maximin self-confirming equilibrium adapted from Lehrer (2012) and Battigalli et al. (2015, 2016) to study how equilibrium behavior of ambiguity-averse individuals in strategic-form games is affected by general coarse feedbacks. We study the impact of coarse feedbacks from a design perspective. Contrary to standard Bayesian incentive problems (e.g., Myerson, 1982)

¹There are alternative definitions of equilibrium under strategic uncertainty in which uncertainty is represented by capacities; see, for example, Stauber (2019) and references therein. In some models, as in Bade (2011) or Riedel and Sass (2014), players are allowed to explicitly use non-additive or ambiguous randomization devices to select their strategies. In the online appendix, we discuss the comparison between MSCE and Ellsberg equilibrium in more detail. Another trend of the literature studies refinement of Nash equilibrium by imposing robustness properties to small strategic uncertainty (see, for example, Bich, 2019).

or information design settings (e.g., Kamenica and Gentzkow, 2011, Bergemann and Morris, 2016, Taneva, 2019), the designer here controls strategic uncertainty by designing players' feedbacks.² How should such a designer design feedbacks in order to induce favorable equilibrium behavior? For instance, in public good problems, is the designer able to increase voluntary contributions by exploiting individuals' aversion to ambiguity and providing only coarse feedbacks about individuals' behavior in past occurrences of the game? Is cooperation improved or deteriorated if players receive more or less ambiguous feedbacks, or if players are more or less ambiguity averse?

We study strategic form games with a finite number of players and an infinite set of strategies S_j for each player j . For finite games with a finite action set A_j for each player j , the set of strategies of player j is identified with the set of mixed actions $\Delta(A_j)$. Thus, we consider both games with infinite action sets, e.g., public good games with contributions $s_i \in [0, 1]$, and finite games such as volunteer dilemma games. Each player receives separable feedbacks about the strategy of each of his opponents. The feedbacks induce partitions of the opponents' strategy sets: if player $j \neq i$ chooses some strategy s_j , then player i only knows that the strategy played by player j is in the partition element that contains s_j .

We consider ambiguity sensitive players and define a notion of maxmin self-confirming equilibrium (MSCE) and α -maxmin self-confirming equilibrium (α -MSCE). In an MSCE, players are extremely ambiguity averse and maximize their worst-case expected utility with respect to any strategy profile of the opponents that they consider possible given their feedbacks. In an α -MSCE, players consider best and worst case scenarios and maximize a weighted sum of best and worst case expected utilities. If players receive perfect feedback, they perfectly know the equilibrium strategy profile, and MSCE and α -MSCE are equivalent to Nash equilibria.

First, we characterize conditions on the feedbacks that ensure the existence of an MSCE. Under standard conditions, if feedbacks satisfy a linearity condition, then an MSCE always exists.³ These conditions for equilibrium existence cover the framework of partially specified equilibrium of Lehrer (2012). Further, we compare our notion of MSCE to Lehrer (2012) and Battigalli et al. (2016). We illustrate that general feedback functions exist such that an MSCE cannot be implemented as a partially specified equilibrium of Lehrer (2012) or with the class of feedback functions considered in Battigalli et al. (2016).

In the second part of the paper, we apply MSCE and α -MSCE to various applications. First, we consider a generalized class of volunteer dilemmas in which each player decides whether or not to volunteer. The probability that the public good is produced is increasing in the number of volunteers, but volunteering is costly. If volunteering is not a strictly dominated strategy, we show that coarse feedbacks allow inducing MSCE outcomes with any number of volunteers higher than in Nash equilibrium outcomes. In particular, the number of volunteers that maximizes total welfare could be

²There are a few other approaches in which partial feedback is used as an instrument to affect the steady state of collective learning in games. For example, Esponda (2008) studies the effect of coarse feedbacks in first price auctions on the set of self-confirming equilibrium bidding strategies and revenues. Jehiel (2011) uses the notion of analogy-based expectation equilibrium (Jehiel, 2005) to study revenues in some classes of auction formats when bidders get partial feedback about the distribution of bids submitted in earlier auctions. There, feedbacks are represented by analogy partitions of decision nodes from which players construct their subjective beliefs about the average behavior at these nodes.

³In two player zero-sum games, we can relax the linearity condition. In the online appendix, we show that for any feedback, every Nash equilibrium is an MSCE. Thus, an MSCE always exists in zero-sum games with arbitrary feedbacks. However, an MSCE strategy profile may not be a Nash equilibrium.

implemented with coarse feedbacks. In addition, under some assumptions, symmetric feedbacks could be used to induce MSCE in which all players volunteer. We also show how coarse feedbacks could be used to avoid coordination problems when the designer would like to incentivize only one player to volunteer: in games such as the standard volunteer dilemma game in which the public good is produced whenever at least one player volunteers, we show that there exist simple coarse feedbacks such that the MSCE is unique and is such that exactly one specific player volunteers. More generally, if the Nash equilibria with the lowest number of volunteers are strict and there is no equilibrium with zero volunteer, then there exist feedbacks such that such a Nash equilibrium can be implemented as the unique MSCE.

Second, we study a class of continuous public good games with strategic substitutes, strategic complements, or more general forms of s-shaped production functions. We characterize feedbacks that increase cooperation and imply a higher average contribution than Nash equilibria. For public good games with strategic substitutes or s-shaped production functions, we can always construct feedbacks that increase cooperation compared to the Nash equilibrium. Further, for strategic substitutes or s-shaped production functions, more ambiguous feedbacks increase contributions, whereas more ambiguous feedbacks decrease contributions for strategic complements. For public good games with strategic substitutes, we generalize our results to α -MSCE. For ambiguity loving players, the highest contribution that can be implemented as an α -MSCE equals the Nash equilibrium contribution. However, as the degree of ambiguity aversion α increases, we can implement α -MSCE with a strictly higher contribution.

Finally, we study a class of games with strategic substitutes and negative externalities with linear best responses, such as Cournot oligopolies, and provide conditions and feedback functions that allow implementing the collusive outcome as an MSCE outcome.⁴

Organization of the paper Section 2 presents the model and defines our equilibrium concept with coarse feedbacks. Section 3 contains general results, examples, and comparisons with other equilibrium concepts in games with strategic ambiguity. Sections 4, 5, and 6 characterize equilibria and optimal feedbacks in some classes of games. The proofs not included in the main text are in the appendix. The online appendix contains additional examples, applications, and the comparison with Ellsberg equilibrium.

2 Model

Game A strategic-form game is denoted by $G = (N, S, u)$ where

- $N = \{1, \dots, n\}$ is the set of players;
- $S = \prod_{i \in N} S_i$ is the set of strategy profiles, where S_i is the set of strategies of player i ;
- $u = (u_i)_{i \in N}$, where $u_i : S \rightarrow \mathbb{R}$ is the utility function of player i .

We assume that for every player i , S_i is a nonempty, compact, and convex subset of a Euclidian space, and u_i is continuous and quasi-concave on S_i . For a finite game (N, A, u) where $A = \prod_{i \in N} A_i$

⁴In the online appendix, we also study Tullock auctions as a class of games with strategic substitutes, negative externalities, and non-linear best response.

and A_i is the finite set of actions of player i , we identify S_i with the set of mixed actions of player i , i.e., $S_i = \Delta(A_i)$, and utility functions are extended to mixed actions in the usual way:

$$u_i(s) = \sum_{a \in A} s(a)u_i(a), \text{ where } s(a) = \prod_i s_i(a_i).$$

The game (N, S, u) is then called the mixed extension of (N, A, u) .

Feedbacks Each player i gets feedbacks about the strategy of player j through a feedback function that depends on the strategy of player j . Formally, a *feedback function* for player i about player $j \neq i$ is given by a function

$$\phi_{ij} : S_j \rightarrow Y.$$

Let $\phi_i = (\phi_{ij})_{j \neq i}$ and $\phi = (\phi_i)_{i \in N}$. A game with feedbacks is denoted by (G, ϕ) . We say that player i gets *perfect feedback* about player j if ϕ_{ij} is injective, i.e., $\phi_{ij}(s_j) \neq \phi_{ij}(s'_j)$ for every $s_j \neq s'_j$. We say that player i gets *no feedback* about player j if ϕ_{ij} is constant, i.e., $\phi_{ij}(s_j) = \phi_{ij}(s'_j)$ for every s_j and s'_j . Suppose $S_i = S_j$ for all $i, j \in N$. Then, feedbacks are called *symmetric* if two conditions are satisfied. First, every player i receives the same feedback about every opponent, i.e., $\phi_{ij} = \phi_{ik}$ for every j and k . Second, two distinct players i and k receive the same feedback about every third player j , $\phi_{ij} = \phi_{kj}$ for every j and every i and k different from j .

The set of strategies of player j consistent with the feedbacks of player i given s_j is given by

$$\Sigma_{ij}(s_j) = \phi_{ij}^{-1}(\phi_{ij}(s_j)) = \{\tilde{s}_j \in S_j : \phi_{ij}(\tilde{s}_j) = \phi_{ij}(s_j)\}.$$

That is, the set $\Sigma_{ij}(s_j)$ is the set of player j 's strategies considered as possible by player i when the actual strategy of player j is s_j . Note that Σ_{ij} induces a partition over S_j and $s_j \in \Sigma_{ij}(s_j)$ for every s_j .

The *identification correspondence* of player i is given by

$$\Sigma_i : \prod_{j \neq i} S_j \rightrightarrows \prod_{j \neq i} S_j,$$

where for every s_{-i} , $\Sigma_i(s_{-i}) = \prod_{j \neq i} \Sigma_{ij}(s_j)$ is the set of strategy profiles of players other than i consistent with the feedbacks received by player i given the strategy profile s_{-i} . Note that for every game with feedbacks (G, ϕ) and $s_{-i} \in S_{-i}$ we have $s_{-i} \in \Sigma_i(s_{-i})$.

Equilibrium with coarse feedbacks and ambiguity aversion

Definition 1 A strategy profile s^* is a *maxmin self-confirming equilibrium* (MSCE) of (G, ϕ) iff for every $i \in N$ we have

$$s_i^* \in \arg \max_{s_i \in S_i} \inf_{s_{-i} \in \Sigma_i(s_{-i}^*)} u_i(s_i, s_{-i}).$$

In a maxmin self-confirming equilibrium, each player i considers, for each of his strategies s_i , the subjective theory about others' behavior consistent with his feedbacks, s_{-i} in $\Sigma_i(s_{-i}^*)$, that is the most pessimistic given s_i . It corresponds to extreme ambiguity aversion. A more general subjective theory

about others' behavior is obtained when each player puts weight $\alpha \in [0, 1]$ on the worse case scenario and weight $1 - \alpha$ on the best case scenario, which leads to the following definition:

Definition 2 A strategy profile s^* is an *α -maxmin self-confirming equilibrium* (α -MSCE) of (G, ϕ) iff for every $i \in N$ we have

$$s_i^* \in \arg \max_{s_i \in S_i} \left(\alpha \inf_{s_{-i} \in \Sigma_i(s_{-i}^*)} u_i(s_i, s_{-i}) + (1 - \alpha) \sup_{s_{-i} \in \Sigma_i(s_{-i}^*)} u_i(s_i, s_{-i}) \right).$$

The case $\alpha = 0$ can be interpreted as a self-confirming equilibrium when players are extreme ambiguity lovers or, alternatively, when players are wishful thinkers.⁵

Under perfect feedback we have $\Sigma_i(s_{-i}) = \{s_{-i}\}$, so the set of self-confirming equilibria of (G, ϕ) coincides with the set of Nash equilibria of G . The following simple example illustrates the difference between Nash equilibrium and α -MSCE and shows that players could benefit from coarse feedbacks. In the example, there is a unique Nash equilibrium, Pareto dominated by the unique α -MSCE for α large enough, when player 1 gets no feedback and player 2 gets perfect feedback.

Example 1 There are two players. Player 1 plays top (T) or bottom (B), and player 2 left (L) or right (R).

	L	R
T	2,4	3,7
B	-1,3	5,2

There is only one Nash Equilibrium: player 1 plays T with probability $\frac{1}{4}$ and player 2 plays L with probability $\frac{2}{5}$. The Nash equilibrium expected payoff is $2\frac{3}{5}$ for player 1 and $3\frac{1}{4}$ for player 2. Further, (T, R) is the unique MSCE if player 1 receives no feedback and player 2 receives perfect feedback. The MSCE (T, R) Pareto dominates the unique Nash equilibrium of this game. (T, R) is also the unique α -MSCE as long as $\alpha > \frac{2}{5}$. Otherwise, if $\alpha < \frac{2}{5}$, then the unique α -MSCE is (B, L) . \diamond

3 Existence and comparison with other concepts

This section characterizes conditions on feedbacks to ensure the existence of MSCE for general games with feedbacks. We also compare and illustrate the difference between MSCE and the notions of partially specified equilibrium of Lehrer (2012), and maxmin self-confirming equilibrium of Battigalli et al. (2016).

3.1 Existence

Consider a game with feedbacks (G, ϕ) . For every i , define the function $v_i : S \rightarrow \mathbb{R}$ as follows for every $s \in S$:

$$v_i(s) = \inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(s_i, \tilde{s}_{-i}).$$

⁵The implications of (common knowledge of) wishful thinking about others' strategies has been analyzed in Yildiz (2007) by using an epistemic approach.

Let $v = (v_i)_{i \in N}$. By definition, s^* is an MSCE of (G, ϕ) iff s^* is a Nash equilibrium of the auxiliary strategic form game (N, S, v) . Hence, the existence of an MSCE of (G, ϕ) is equivalent to the existence of a Nash equilibrium of (N, S, v) .

The usual assumptions for the existence of a Nash equilibrium of (N, S, v) are satisfied if, for every i , v_i is continuous and quasi-concave on S_i . Lemma 4 in the appendix shows that v_i is always quasi-concave on S_i . However, to get continuity of v_i , we need additional assumptions on the identification correspondences Σ_i . Precisely, v_i is continuous if for every i the identification correspondence Σ_i of player i is continuous (i.e., both upper and lower hemicontinuous) and compact-valued. Combined with Lemma 4, we get the following proposition:

Proposition 1 *If the identification correspondence Σ_i is continuous and compact-valued for every player i , then the game with feedbacks (G, ϕ) has at least one MSCE.*

Proof. See the appendix. ■

Example 1 in the online appendix illustrates that an α -MSCE may not exist even if the identification correspondences are continuous and compact-valued. The reason is that the analog of the quasi-concave property of v_i under α -maxmin is not satisfied for $\alpha < 1$.

If the feedback function ϕ_{ij} is continuous for every $j \neq i$, then the identification correspondence Σ_i of player i is upper hemicontinuous and compact-valued. However, continuity of the feedback functions is not sufficient for the identification correspondences to be lower hemicontinuous, which is crucial for the existence of an MSCE. In the following example, we consider a (mixed extension of a) 2-player 2-action game with continuous feedback functions in which the identification correspondence of player 1 is not lower hemicontinuous, and there is no MSCE.

Example 2 Consider again the game of Example 1. Let s_1 be the probability that player 1 plays T and s_2 the probability that player 2 plays L . Player 2 gets perfect feedback, and the identification correspondence of player 1 is:

$$\Sigma_{12}(s_2) = \begin{cases} [\underline{s}_2, \bar{s}_2] & \text{if } s_2 \in [\underline{s}_2, \bar{s}_2] \\ \{s_2\} & \text{if } s_2 \notin [\underline{s}_2, \bar{s}_2], \end{cases}$$

where $0 < \underline{s}_2 < \frac{2}{5} < \bar{s}_2$. This identification correspondence is upper but not lower hemicontinuous, and it can be induced by the following continuous feedback function:

$$\phi_{12}(s_2) = \begin{cases} s_2 & \text{if } s_2 \leq \underline{s}_2 \\ \underline{s}_2 & \text{if } s_2 \in [\underline{s}_2, \bar{s}_2] \\ s_2 - (\bar{s}_2 - \underline{s}_2) & \text{if } s_2 \geq \bar{s}_2. \end{cases}$$

Assume that (s_1, s_2) is an MSCE. Then we have:

- If $s_2 > \bar{s}_2$, then player 1 best responds to s_2 with $s_1 = 1$. Thus, player 2 best responds with $s_2 = 0$, a contradiction.
- If $s_2 < \underline{s}_2$, then player 1 best responds to s_2 with $s_1 = 0$. Thus, player 2 best responds with $s_2 = 1$, a contradiction.

- If $s_2 \in [\underline{s}_2, \bar{s}_2]$, then player 1 best responds to \bar{s}_2 with $s_1 = 1$. Thus, player 2 best responds with $s_2 = 0$, a contradiction.

Hence, there is no MSCE. Note that the existence of an MSCE continues to fail even if we replace Σ_{12} by the following identification correspondence, which induces a finite partition of S_2 :

$$\Sigma_{12}(s_2) = \begin{cases} [0, \underline{s}_2) & \text{if } s_2 < \underline{s}_2 \\ [\underline{s}_2, \bar{s}_2] & \text{if } s_2 \in [\underline{s}_2, \bar{s}_2] \\ (\bar{s}_2, 1] & \text{if } s_2 > \bar{s}_2. \end{cases}$$

◇

To get lower hemicontinuous identification correspondences, we need stronger conditions on feedback functions. We introduce such conditions below, which generalize the framework of partially specified equilibrium of Lehrer (2012) and a class of feedback functions used in the maxmin self-confirming equilibrium of Battigalli et al. (2016) (see Section 3.2).

We say that the feedback function ϕ_{ij} is *linear* if it can be written as $\phi_{ij}(s_j) = Ms_j$, where M is a matrix of real numbers. Lemma 5 in the appendix shows that in games with linear feedbacks and in which the set of strategies of each player is a finite dimensional simplex, the identification correspondences are lower hemicontinuous. Therefore, we get the following proposition:

Proposition 2 *For every game with linear feedbacks (G, ϕ) such that S_i is a finite dimensional simplex for every $i \in N$, there exists an MSCE.*

Proof. Directly from Proposition 1 and Lemma 5 in the appendix. ■

3.2 Partially specified equilibrium and other notions of self-confirming equilibria

In the standard definition of maxmin self-confirming equilibrium in the literature (see, e.g., Battigalli et al., 2016), a feedback function is defined on the finite set of actions (pure strategies), not directly on the set of mixed strategies or on a continuum of strategies as in our setting. Precisely, given a finite game (N, A, u) , a (pure) feedback function for player i about player $j \neq i$ in Battigalli et al. (2016) is given by a mapping⁶

$$F_{ij} : A_j \rightarrow M,$$

where M is a finite set of messages. A mixed strategy s_j of player j induces a probability distribution $\hat{F}_{ij}(s_j) \in \Delta(M)$, with $\hat{F}_{ij}(m | s_j) = \sum_{a_j \in F_{ij}^{-1}(m)} s_j(a_j)$ for every $m \in M$.

The set of mixed strategies of player j consistent with the feedbacks of player i given s_j is then given by

$$\Sigma_{ij}(s_j) = \{\tilde{s}_j \in S_j : \hat{F}_{ij}(s_j) = \hat{F}_{ij}(\tilde{s}_j)\}.$$

Therefore, we get this setting with feedbacks on pure strategies as a particular case of our model by letting $\phi_{ij} : S_j \rightarrow \Delta(M)$ and $\phi_{ij} = \hat{F}_{ij}$. The next example illustrates that equilibria in our setting cannot always be induced by feedbacks defined on pure strategies.

⁶In Battigalli et al. (2016), the feedbacks of player i could also depend on the action profile of the other players (not only on the action of each other player separately) and on the strategy of player i . In the terminology of Battigalli et al. (2016), we always assume in our model “separable feedback” and “own-strategy independence of feedback”.

Example 3 Consider the following 2-player game. Player 2 has a strictly dominant strategy ($s_2 = \delta_L$), so he plays it in any equilibrium.

	L	R
T	3,1	-3,0
M	2,1	0,0
B	1,1	1,0

In the setting of Battigalli et al. (2016), there are only two relevant pure feedback functions for player 1 because player 2 has only two actions: perfect feedbacks, when $F_{12}(L) \neq F_{12}(R)$, and no feedback, $F_{12}(L) = F_{12}(R)$. In the first case, the unique MSCE strategy for player 1 is T ; in the second case, the unique MSCE strategy for player 1 is B .

Consider now the following class of general feedback functions for player 1, where $\underline{s} \in [0, 1]$:

$$\phi_{12}(s_2) = \begin{cases} s_2(L) & \text{if } s_2(L) \leq \underline{s} \\ \underline{s} & \text{if } s_2(L) \geq \underline{s}. \end{cases}$$

That is, player 1 learns the relative frequency of action L only until the threshold \underline{s} . The set of strategies of player 2 consistent with player 1's feedback is

$$\Sigma_{12}(s_2) = \begin{cases} \{s_2\} & \text{if } s_2(L) < \underline{s} \\ \{\tilde{s}_2 : \tilde{s}_2(L) \geq \underline{s}\} & \text{if } s_2(L) \geq \underline{s}. \end{cases}$$

Because in equilibrium we have $s_2^*(L) = 1$, player 1 knows that player 2's strategy belongs to $\Sigma_{12}(s_2^*) = \{s_2 : s_2(L) \geq \underline{s}\}$, and we have

$$\inf_{s_2 \in \Sigma_{12}(s_2^*)} u_1(a_1, s_2) = u_1(a_1, \underline{s}) = \begin{cases} 6\underline{s} - 3 & \text{if } a_1 = T \\ 2\underline{s} & \text{if } a_1 = M \\ 1 & \text{if } a_1 = B. \end{cases}$$

Hence, for $\underline{s} \in (\frac{1}{2}, \frac{3}{4})$, the unique MSCE is (M, L) , which is not an MSCE with any feedbacks on pure strategies. \diamond

Lehrer (2012) introduced a concept similar to MSCE, called partially specified equilibrium, in which for each player i and $j \neq i$, player i observes the expected values of random variables $(X_k)_k$ with $X_k : S_j \rightarrow \mathbb{R}$. Such feedbacks are obtained as special cases in our setting by letting

$$\phi_{ij}(s_j) = (E_{s_j}(X_k))_k.$$

As in the setting of Battigalli et al. (2016), if player j has only two actions, say L and R , such feedbacks reduce to either perfect feedback (if there exists k such that $X_k(R) \neq X_k(L)$) or no feedback (if $X_k(R) = X_k(L)$ for every k). Hence, whatever the feedback of player 1 in Example 3, (M, L) is never a partially specified equilibrium.

To see that pure feedback functions can be obtained as a particular case of Lehrer (2012), it suffices to observe that a pure feedback function F_{ij} induces a finite partition $\{H_k\}_k := \{F_{ij}^{-1}(m) : m \in M\}$,

and let $X_k = \mathbb{1}_{H_k}$, so that

$$\Sigma_{ij}(s_j) = \{\tilde{s}_j \in S_j : (E_{\tilde{s}_j}(X_k))_k = (E_{s_j}(X_k))_k\} = \{\tilde{s}_j \in S_j : \hat{F}_{ij}(\tilde{s}_j) = \hat{F}_{ij}(s_j)\}.$$

Finally, notice that our Proposition 2 implies that a partially specified equilibrium (and hence a maxmin self-confirming equilibrium with pure feedbacks) always exist because $\phi_{ij}(s_j) = (E_{s_j}(X_k))_k$ is linear in s_j .

Riedel and Sass (2014) use a different approach to induce strategic ambiguity. They define Ellsberg equilibria by allowing players to use ambiguous randomization devices to select their strategies. In their setting, strategies in an Ellsberg equilibrium can be described as a set of mixed strategies. In the online appendix, we show that for any mixed strategy profile s^* contained in an Ellsberg equilibrium strategy profile, we can construct feedbacks such that s^* is an MSCE. Thus, by constructing feedbacks in an appropriate way, a designer can always implement MSCE that are payoff-equivalent to Ellsberg equilibria. However, there could exist MSCE strategy profiles not in the support of any Ellsberg equilibrium.

4 Generalized volunteer dilemmas

In this section, we consider the following class of generalized symmetric volunteer dilemmas. The set of players $N = \{1, \dots, n\}$ is the set of potential volunteers and beneficiaries of a public good. The set of actions of each player i is $A_i = \{0, 1\}$, where action $a_i = 0$ corresponds to not volunteering, and action $a_i = 1$ corresponds to volunteering by exerting a costly effort. The probability $h(k) \in [0, 1]$ that the public good is produced is weakly increasing in the number $k = 0, 1, \dots, n$ of players who volunteer. We normalize the probability of producing the public good to 0 when there is no volunteer, i.e., $h(0) = 0$. The value of the public good is the same for each player and is normalized to 1. The individual cost of effort is $c \in (0, 1)$. The utility function of each player i is therefore given by

$$u_i(a) = h\left(\sum_{j=1}^n a_j\right) - a_i c.$$

In the standard volunteer's dilemma (Diekmann, 1985), the public good is produced if and only if at least one player volunteers, so $h(k) = 1$ for every $k \geq 1$. The more general case where the public good is produced if and only if at least $\underline{k} \geq 1$ players volunteers, i.e., $h(k) = 0$ for every $k < \underline{k}$ and $h(k) = 1$ for every $k \geq \underline{k}$, is a binary threshold public good game that has been studied extensively in problems of collective decisions such as lobbying, petitioning, getting vaccinated, and voting (see, e.g., Dziuda, Gitmez, and Shadmehr, 2021, Ginzburg, 2022, and references therein), as well as in the experimental literature (see, e.g., Goeree, Holt, and Smith, 2017). The case in which the public good is produced with some exogenous probability if the number of volunteers exceeds the threshold is also covered by the general class of volunteer dilemmas introduced above. For example, if the probability that a legislature supports a petition is equal to q if the minimum number of signatures required for the petition to be forwarded is \underline{k} , then $h(k) = 0$ for every $k < \underline{k}$ and $h(k) = q$ for every $k \geq \underline{k}$. The number of active players may also be uncertain, as in Hillenbrand and Winter (2018). For example, if the public good is produced if and only if at least one player volunteers but each player is inactive with probability ε , then $h(0) = 0$ and $h(k) = 1 - \varepsilon^k$ for every $k \geq 1$.

A total number of volunteers equal to k corresponds to a (pure strategy) Nash equilibrium outcome (i.e., an equilibrium a such that $\sum_{j=1}^n a_j = k$) iff

$$\begin{aligned} h(1) &\leq c, & \text{if } k = 0, \\ h(k+1) - h(k) &\leq c \leq h(k) - h(k-1), & \text{if } k \in \{1, \dots, n-1\}, \\ c &\leq h(n) - h(n-1), & \text{if } k = n. \end{aligned}$$

Observe that a pure strategy Nash equilibrium always exists, but equilibrium outcomes are not necessarily unique and mixed strategies equilibria might also exist. Also, observe that if

$$c \leq h(k) - h(k-1) \text{ for some } k \in \{1, \dots, n\}, \quad (1)$$

then there exists at least one pure strategy Nash equilibrium with at least one volunteer. Otherwise, volunteering is a strictly dominated strategy, so players' feedbacks have no impact on players' behavior. Therefore, we assume that condition (1) is satisfied in the rest of this section and denote by $\underline{k}^{NE} \geq 1$ the smallest number of volunteers among the set of pure strategy Nash equilibrium outcomes with at least one volunteer.

Consider the case in which a designer would like to induce a total number of volunteers equal to k^* . For example, if the designer's objective is to maximize total welfare, then k^* maximizes $nh(k^*) - k^*c$. Observe that, in this case, if k^{NE} is a Nash equilibrium outcome that does not maximize total welfare, then we necessarily have $k^{NE} < k^*$. To see this, assume by way of contradiction that k^* maximizes total welfare and k^{NE} is a Nash equilibrium outcome that does not maximize total welfare, but $k^{NE} > k^*$. Because k^{NE} is a Nash equilibrium, we have:

$$\begin{aligned} &h(k^{NE}) - h(k^{NE} - 1) \geq c, \\ \Rightarrow &h(k^{NE}) - h(k^*) \geq c, & \text{because } k^* \leq k^{NE} - 1 \text{ and } h \text{ is increasing,} \\ \Rightarrow &n(h(k^{NE}) - h(k^*)) \geq (k^{NE} - k^*)c, & \text{because } n \geq k^{NE} - k^*, \\ \Rightarrow &nh(k^{NE}) - k^{NE}c \geq nh(k^*) - k^*c, \end{aligned}$$

a contradiction with the assumption that k^* maximizes total welfare and k^{NE} does not maximize total welfare.

The next proposition shows that for every number k^* higher or equal to some pure strategy Nash equilibrium outcome with at least one volunteer (which exists under condition (1)), there exists feedback functions and an MSCE with exactly k^* volunteers. These feedbacks are asymmetric but simple: for each pair of players (i, j) , $i \neq j$, player i either gets perfect feedback or no feedback from player j .

Proposition 3 *Let $k^* \geq \underline{k}^{NE}$. Then there exist (pure and asymmetric) feedback functions and an MSCE such that the total number of volunteers is k^* .*

Proof. Consider the following action profile a^* : every player $i \leq k^*$ volunteers and every player $i \geq k^* + 1$ does not volunteer, i.e., $a_i^* = 1$ iff $i \leq k^*$. Consider the following feedbacks: every player $i \leq k^*$ gets perfect feedbacks about the strategies of all players in $\{k^* + 1, \dots, n\}$, perfect feedbacks about the strategies of $\underline{k}^{NE} - 1$ players (other than i) in $\{1, \dots, k^*\}$, and no feedback from

the other players. Every player $i \geq k^* + 1$ gets perfect feedbacks about the strategy of every player in $\{1, \dots, \underline{k}^{NE}\}$ and no feedback from every other player.

A maxmin player getting no feedback from player $j \neq i$ behaves as if player j does not volunteer. Hence, every player $i \leq k^*$ believes that exactly $\underline{k}^{NE} - 1$ other players volunteer. Because \underline{k}^{NE} is a Nash equilibrium outcome, it is a best response for these players to volunteer. Similarly, every player $i \geq k^* + 1$ believes that exactly \underline{k}^{NE} other players volunteer, so it is a best response for these players not to volunteer. We conclude that a^* is an MSCE with the above feedbacks. ■

To illustrate the proposition, consider the standard volunteer dilemma game but assume that each player is inactive with probability $\varepsilon \in (0, 1 - c)$. So, even if a player chooses to volunteer, he only contributes to the public good with probability $1 - \varepsilon$. Hence, $h(0) = 0$ and $h(k) = 1 - \varepsilon^k$ for every $k \geq 1$. The number k^{NE} of volunteers in a pure strategy Nash equilibrium is generically unique: if $c < \varepsilon^{n-1}(1 - \varepsilon)$, then $k^{NE} = n$; otherwise, if $c > \varepsilon^{n-1}(1 - \varepsilon)$ it is equal to the value $k \in \{1, \dots, n - 1\}$ such that $\varepsilon^k(1 - \varepsilon) < c < \varepsilon^{k-1}(1 - \varepsilon)$. Note that when the number of equilibrium volunteers is interior, it does not change as the total number of players, n , increases. On the contrary, the socially optimal number of volunteers $k^* \geq k^{NE}$ that maximizes $nh(k) - kc$ in k is increasing in n , and is strictly higher than k^{NE} for n large enough. From Proposition 3, there always exist feedback functions such that such a socially optimal number of volunteers constitutes an MSCE.

For example, assume that there are $n = 10$ players and the probability that each player is inactive is $\varepsilon = \frac{1}{2}$. If $c = \frac{1}{3}$, the Nash equilibrium number of volunteers is $k^{NE} = 1$ (because $\varepsilon^1(1 - \varepsilon) = \frac{1}{4} < c = \frac{1}{3} < \varepsilon^0(1 - \varepsilon) = \frac{1}{2}$), as in the standard volunteer dilemma with $\varepsilon = 0$, but the socially optimal number of volunteers is $k^* = 4$. An MSCE with $k^* = 4$ volunteers is obtained by providing (i) four players (the volunteers) no feedback about the strategies of the other volunteers, and perfect feedbacks about the remaining players, and (ii) perfect feedbacks from one of the volunteers and no feedback from the other players to the six remaining players who do not volunteer. If $c = \frac{1}{10}$, the Nash equilibrium number of volunteers is $k^{NE} = 3$ (because $\varepsilon^3(1 - \varepsilon) = \frac{1}{16} < c = \frac{1}{10} < \varepsilon^2(1 - \varepsilon) = \frac{1}{8}$), but the socially optimal number of volunteers is $k^* = 6$. An MSCE with $k^* = 6$ volunteers is obtained by providing (i) six players (the six volunteers) perfect feedbacks from two of them, perfect feedbacks from the four players not volunteering, and no feedbacks from the other players and (ii) perfect feedbacks from three of the volunteers and no feedback from the other players to the four remaining players not volunteering.

Proposition 3 can be extended to α -MSCE for α large enough, but the exact degrees of ambiguity aversion allowing to sustain an equilibrium with k^* volunteers depend on k^* and the shape of the function h . For instance, in the previous example there is an α -MSCE in which k^* players volunteer with the feedbacks used above iff

$$\alpha h(k^{NE}) + (1 - \alpha)h(k^*) - c \geq \alpha h(k^{NE} - 1) + (1 - \alpha)h(k^* - 1),$$

i.e.,

$$\alpha \geq \frac{c - (h(k^*) - h(k^* - 1))}{(h(k^{NE}) - h(k^{NE} - 1)) - (h(k^*) - h(k^* - 1))}.$$

Observe that this condition becomes stronger (i.e., requires more aversion to ambiguity) when the number of volunteers the designer would like to induce, k^* , increases. Going back to the numerical examples of the previous paragraph, if $c = \frac{1}{3}$, then there is an α -MSCE in which $k^* = 4$ players volunteer with the above feedbacks iff $\alpha \geq \frac{13}{21}$, and if $c = \frac{1}{10}$, then there is an α -MSCE in which $k^* = 6$

players volunteer with the above feedbacks iff $\alpha \geq \frac{27}{35}$.

The previous proposition implies that there always exist feedbacks and an MSCE in which all players volunteer (i.e., $k^* = n$). However, the constructed feedbacks are asymmetric if there is not already a Nash equilibrium in which all players volunteer. Under an additional condition, the next proposition shows that feedback functions can also be chosen symmetric to get an MSCE in which all players volunteer. In general, to construct such an MSCE, players must get partial (mixed) feedbacks from all the other players.

Similarly to the feedbacks analyzed in Example 2, consider a class of threshold feedback functions that induce an identification correspondance with the following property for every i and $j \neq i$, where the mixed strategy s_j of player j is identified with the probability that player j volunteers and $\underline{s} \in [0, 1]$:

$$\begin{aligned}\Sigma_{ij}(s_j) &= [\underline{s}, 1], \text{ for every } s_j \geq \underline{s} \text{ and} \\ \Sigma_{ij}(s_j) &= \{s_j\}, \text{ for every } s_j < \underline{s}.\end{aligned}$$

That is, if player j volunteers with probability s_j higher than \underline{s} , then player i only learns that player j volunteers with a probability higher than \underline{s} .

Proposition 4 *Assume a symmetric Nash equilibrium exists in which each player volunteers with strictly positive probability. Then, there exist symmetric threshold feedback functions and a symmetric MSCE such that all players volunteer with probability one.*

Proof. Assume that there exists a symmetric Nash equilibrium in which each player i volunteers with probability $s_i = s^M > 0$, and consider the symmetric threshold feedback functions such that $\Sigma_{ij}(s_j) = [s^M, 1]$ for every $s_j \geq s^M$. Consider the strategy profile s^* in which all players volunteer with probability 1, i.e., $s_i^* = 1$ for every i . Given the feedbacks above, every player believes that every other player volunteers with probability s^M . Because $s_i = s^M$ for every i is a mixed strategy Nash equilibrium in which all players volunteer with strictly positive probability, it is indeed a best response for every player to volunteer when believing that each other player volunteers with probability s^M . Hence, s^* is an MSCE with the above feedbacks. More generally, the construction above works for every symmetric threshold feedback function such that volunteering is a best response when all the other players volunteer with probability \underline{s} . ■

The existence of a symmetric Nash equilibrium in which each player volunteers with strictly positive probability is not guaranteed in every generalized volunteer dilemma. However, such an equilibrium is guaranteed to exist whenever $c < h(1)$. Indeed, in that case, not volunteering at all ($s_i = 0$ for every i) is not a Nash equilibrium, but a symmetric equilibrium must exist because the game is finite and symmetric. Hence, it should be a symmetric Nash equilibrium in which each player i volunteers with strictly positive probability. The standard volunteer dilemma ($h(k) = 1$ for every $k \geq 1$) satisfies this condition, and the symmetric Nash equilibrium is unique and given by

$$s_i = s^M = 1 - c^{\frac{1}{n-1}}, \text{ for every } i.$$

Then, every symmetric threshold feedback function with $\underline{s} \leq s^M$ induces an MSCE in which all players volunteer. Note that this MSCE, even if not socially efficient, leads to the same social welfare as the expected social welfare in the symmetric Nash equilibrium. Hence, under symmetry, a designer who

would like to maximize social welfare but is risk-averse would strictly prefer to provide such coarse feedbacks instead of perfect feedbacks.

We have shown that coarse feedbacks allow us to induce new equilibria with more volunteers. We now show that coarse feedbacks may also be a relevant “full implementation” tool for the designer by avoiding coordination problems and adversarial equilibrium selection. For instance, in the standard volunteer dilemma, there are socially efficient Nash equilibria, but coordination to such asymmetric equilibria is difficult, and many other inefficient equilibria exist. The next proposition shows that in such a game, and more generally when the lowest Nash equilibrium is strict and has exactly $k^* > 0$ volunteers, then feedback functions exist such that the induced MSCE is unique (among all pure and mixed MSCE) and is such that exactly k^* players volunteer with probability 1.

Proposition 5 *Let k^* be the lowest Nash equilibrium number of volunteers, and assume that the equilibrium is strict and $k^* > 0$, i.e.,*

$$c > h(k^* + 1) - h(k^*), \tag{2}$$

$$c < h(k + 1) - h(k), \text{ for every } k = 0, 1, \dots, k^* - 1. \tag{3}$$

Then there exist (pure and asymmetric) feedback functions such that the MSCE is unique and is such that exactly k^ players volunteer.*

Proof. We consider the following feedbacks and show that the corresponding MSCE in which each player $j \in \{1, \dots, k^*\}$ volunteers and each other player $j \in \{k^* + 1, \dots, n\}$ does not volunteer is the unique MSCE.

Each player $j \in \{1, \dots, k^*\}$ receives perfect feedbacks from players in $\{1, \dots, k^*\} \setminus \{j\}$, and no feedback from players in $\{k^* + 1, \dots, n\}$. Hence, each player $j \in \{1, \dots, k^*\}$ knows the strategies of players in $\{1, \dots, k^*\}$ and believes that players in $\{k^* + 1, \dots, n\}$ do not volunteer. Thus, given condition (3), the unique best response of players in $\{1, \dots, k^*\}$ is to volunteer with probability 1 whatever the (pure and mixed) strategies of the other players.

Each player $j \in \{k^* + 1, \dots, n\}$ receives perfect feedbacks from players in $\{1, \dots, k^*\}$, and no feedback from players in $\{k^* + 1, \dots, n\} \setminus \{j\}$. Hence, in every MSCE, each player $j \in \{k^* + 1, \dots, n\}$ believes that each player in $\{1, \dots, k^*\}$ volunteers and that each player in $\{k^* + 1, \dots, n\} \setminus \{j\}$ does not volunteer. Thus, given condition (2), the unique best response of players in $\{k^* + 1, \dots, n\}$ is not to volunteer. ■

In this section, we have shown how coarse feedbacks could be used to improve equilibrium contributions in a general class of public good games with binary actions. In the next section, we show that similar insights are obtained in a general class of public good games with a continuous range of possible contributions for each player.

5 Public goods

We study a class of symmetric public good games in which the contribution of each player i is $s_i \in [0, 1] = S_i$, and the utility of player i depends on the average contribution of all players and the costs

cs_i of his own contribution:

$$u_i(s_i, s_{-i}) = f\left(\frac{\sum_{j=1}^n s_j}{n}\right) - cs_i,$$

where $f : [0, 1] \rightarrow \mathbb{R}$ is a production function that satisfies the following assumption.

Assumption 1 (i) $f(0) = 0$; (ii) f is strictly increasing in s_i for all $i \in N$; (iii) f is continuous and differentiable; (iv) $f(1) > c$ and $f'(1) \geq c$ (v) f is a sigmoid function: there exists a $t \in [0, 1]$ such that $f(x)$ is strictly convex for all $x < t$ and strictly concave for all $x > t$.

Assumptions (i)–(iii) are standard assumptions. Further, for simplicity, we assume that the total welfare is maximized if each player contributes one (Assumption 1 (iv)). With Assumption 1 (v), we consider s-shaped, convex, and concave production functions. If $t = 0$, the production function f is concave, and G is a public good game with strategic substitutes. If $t = 1$, the production function f is convex, and G is a public good game with strategic complements. Figure 1 depicts convex, concave, and s-shaped production functions.

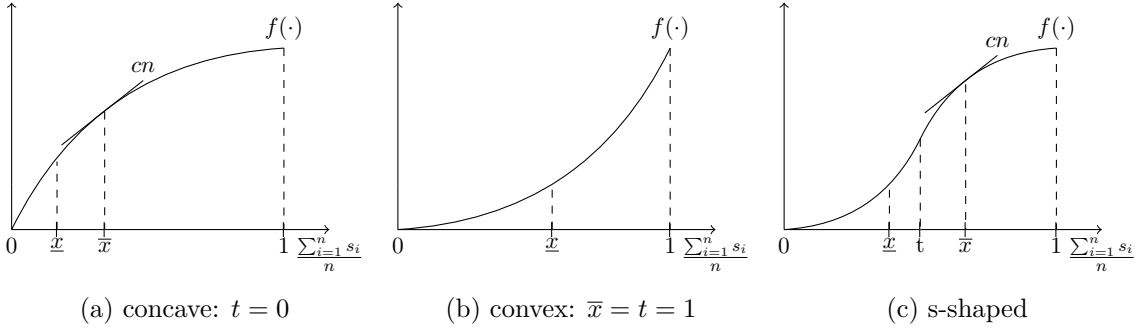


Figure 1: Production functions for $t = 0$, $t = 1$ and $t \in (0, 1)$.

Depending on the production function, there might exist multiple symmetric Nash equilibria. However, we can always characterize the symmetric Nash equilibrium with the highest contribution by \bar{x} with

$$\bar{x} := \begin{cases} x & \text{if } x \in [t, 1] \text{ and } f'(x) = cn, \\ 1 & \text{if } f'(x) \geq cn \forall x \in [t, 1], \\ t & \text{if } f'(x) \leq cn \forall x \in [t, 1]. \end{cases}$$

For concave parts of the production function, \bar{x} solves the first-order condition or equals a corner solution. For convex production functions, the above definition reduces to $\bar{x} = 1$. To ensure the existence of equilibria with positive contribution, we assume that costs are not too high:

Assumption 2 $c \leq \frac{1}{n} f'(\max\{0, \bar{x} - \frac{1}{n}\})$.

The next lemma characterizes the best response function for any production function satisfying Assumption 1 and Assumption 2. For convex parts of the production function, players only choose a positive contribution if they know that the contribution of others is high enough. We denote this threshold by \underline{x} . We characterize and show the uniqueness of \underline{x} in the appendix.

Lemma 1 *There exists a threshold $\underline{x} \leq \bar{x}$ such that the best response function of player i is given by*

$$BR_i(s_{-i}) = \begin{cases} 0 & \text{if } \sum_{j \neq i} s_j \leq n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq \sum_{j \neq i} s_j < n\bar{x} - 1, \\ n\bar{x} - \sum_{j \neq i} s_j & \text{if } n\bar{x} - 1 \leq \sum_{j \neq i} s_j < n\bar{x}, \\ 0 & \text{if } n\bar{x} \leq \sum_{j \neq i} s_j \end{cases} \quad (4)$$

Proof. See the appendix. ■

5.1 Nash equilibrium and MSCE

First, we show that the average contribution of any Nash equilibrium is indeed at most \bar{x} . Thus, $s_i^{NE} = \bar{x}$ is the symmetric Nash equilibrium with the highest average contribution.

Lemma 2 *The average contribution of every Nash equilibrium is smaller or equal \bar{x} .*

Proof. If f is convex, $\bar{x} = 1$ by definition, and the proof is trivial. Suppose f is concave or s-shaped and $(s_i^*)_{i \in N}$ is a Nash equilibrium with $\frac{\sum_{i \in N} s_i^*}{n} > \bar{x}$. Then, there exists a player i with $s_i^* > 0$ and $0 \leq \bar{s}_i < s_i^*$ such that $\frac{\sum_{j \neq i} s_j^* + s_i}{n} \geq \bar{x}$ for all $s_i \in [\bar{s}_i, s_i^*]$. By the definition of \bar{x} , $f(x)$ is concave for all $x \geq \bar{x}$ and

$$\frac{1}{n} f' \left(\frac{\sum_{j \neq i} s_j^* + s_i}{n} \right) \leq c$$

for all $s_i \in [\bar{s}_i, s_i^*]$. Thus, player i would prefer to deviate to \bar{s}_i , which contradicts the Nash equilibrium assumption. ■

The feedback of player i about j is called interval feedback if there exist \underline{s} and \bar{s} such that the identification correspondence is given by

$$\Sigma_{ij}(s_j) = \begin{cases} [\underline{s}, \bar{s}] & \text{if } s_j \in [\underline{s}, \bar{s}] \\ \{s_j\} & \text{if } s_j \notin [\underline{s}, \bar{s}]. \end{cases}$$

For example, this identification correspondence can be obtained with the following continuous feedback function

$$\phi_{ij}(s_j) = \begin{cases} s_j & \text{if } s_j \leq \underline{s} \\ \underline{s} & \text{if } s_j \in [\underline{s}, \bar{s}] \\ s_j - (\bar{s} - \underline{s}) & \text{if } s_j \geq \bar{s}. \end{cases}$$

The public good game is symmetric, and u_i decreases in s_j for $j \neq i$. Therefore, for any symmetric MSCE $(s_i^*)_i$, we can construct symmetric interval feedbacks such that $(s_i^*)_i$ is an MSCE given the symmetric interval feedbacks. Thus, we can focus without loss of generality on symmetric interval feedbacks. Further, the worst case of player i is the minimal contribution consistent with the identification correspondence, and for MSCE, we can set $\bar{s} = 1$ without loss of generality.

Lemma 3 *For every public good game with symmetric interval feedback (G, ϕ) with $\bar{s} = 1$, there exists a symmetric MSCE, and all symmetric MSCE are characterized as follows:*

i) Suppose f is convex. Then, one symmetric MSCE is

$$s_i^* = \begin{cases} 1 & \text{if } \underline{s} \geq \frac{n\underline{x}-1}{(n-1)}, \\ 0 & \text{if } \underline{s} \leq \frac{n\underline{x}-1}{(n-1)}. \end{cases}$$

Further, if $f(\frac{1}{n}) < c$, then there exists a second symmetric MSCE with $s_i^* = 0$ for any symmetric interval feedback.

ii) Suppose f is concave. Then, the unique symmetric MSCE is

$$s_i^* = \begin{cases} 1 & \text{if } \underline{s} < \frac{n\bar{x}-1}{(n-1)}, \\ n\bar{x} - (n-1)\underline{s} & \text{if } \underline{s} \in \left[\frac{n\bar{x}-1}{(n-1)}, \bar{x} \right], \\ \bar{x} & \text{if } \underline{s} \geq \bar{x}. \end{cases}$$

iii) Suppose f is s-shaped. Then, one symmetric MSCE is

$$s_i^* = \begin{cases} 1 & \text{if } \underline{s} \in \left[\frac{n\underline{x}-1}{(n-1)}, \frac{n\bar{x}-1}{n-1} \right], \\ n\bar{x} - (n-1)\underline{s} & \text{if } \underline{s} \in \left[\frac{n\bar{x}-1}{(n-1)}, \bar{x} \right], \\ \bar{x} & \text{if } \underline{s} \geq \bar{x}. \end{cases}$$

Further, if $f(\frac{1}{n}) < c$, then there exists a second symmetric MSCE with $s_i^* = 0$ for any symmetric interval feedback.

Proof. See the appendix. ■

The next proposition compares the contribution in symmetric MSCE and Nash equilibria. Remember that full contribution is the first best solution for any production function satisfying Assumption 1. Therefore, we consider a designer who tries to implement the highest possible contribution.

Due to Assumption 2, full contribution is always a Nash equilibrium for convex production function. Thus, no feedback exists that implements an MSCE with a strictly larger contribution than the best Nash equilibrium.

However, for concave and s-shaped production functions, we can find symmetric interval feedback such that the contribution of the induced MSCE is strictly larger than the contribution of Nash equilibria. First, if the total contribution of the unique symmetric Nash equilibrium is below one, i.e., if $s_i^{NE} = \bar{x} < \frac{1}{n}$, we cannot implement full contribution. Even if player i believes that all opponents contribute zero, the best response would be smaller than one. Therefore, it is impossible to implement full contribution as an MSCE in this case. However, Proposition 6 shows that there still exist feedback functions such that the contribution of the MSCE is strictly larger than the symmetric Nash equilibrium. If the total Nash equilibrium contribution exceeds one, i.e., $s_i^{NE} = \bar{x} \geq \frac{1}{n}$, we can always construct feedbacks that implement full contribution and, therefore, the first best solution.

Proposition 6 *Suppose f is concave or s-shaped.*

i) Let $\bar{x} < \frac{1}{n}$. Then, the average contribution of any MSCE s^* of a public good game with interval feedback (G, ϕ) is smaller or equal, n times the maximal average contribution of a Nash equilibrium, i.e., $n\bar{x}$. The average contribution of a MSCE s^* of (G, ϕ) equals $n\bar{x}$ only if s^* is the symmetric MSCE $s_i^* = n\bar{x}$ for all $i \in N$. Further, if the players get no feedback, then $s_i^* = n\bar{x}$ for all $i \in N$ is an MSCE of (G, ϕ) .

ii) Let $\bar{x} \geq \frac{1}{n}$. Then, there exists feedback function, such that $s_i^* = 1$ for all $i \in N$ is a symmetric MSCE. Such feedback functions are characterized by $\underline{s} \in [\frac{n\bar{x}-1}{(n-1)}, \frac{n\bar{x}-1}{(n-1)}]$.

Proof. $\bar{x} < \frac{1}{n}$ implies $n\bar{x} - 1 < 0$. Therefore, for any s_{-i}^* , we have $n\bar{x} - 1 \leq \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j$ and any MSCE satisfies $s_i^* = n\bar{x} - \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j \leq n\bar{x}$ for all $i \in N$. Thus, the average contribution of any MSCE satisfies $\frac{\sum_{i \in N} s_i^*}{n} \leq \frac{n^2\bar{x}}{n} = n\bar{x}$. Further, in any MSCE, we have $s_i^* \leq n\bar{x}$. Then, the average contribution equals $n\bar{x}$ if and only if $s_i^* = n\bar{x}$.

If players receive no feedback, we have $\Sigma_{ij}(s_j^*) = [0, 1]$ for all $i, j \in N$. Thus, $\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = 0$ and $s_i^* = n\bar{x} - \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = n\bar{x}$ for all $i \in N$.

If $\bar{x} \geq \frac{1}{n}$, we have $n\bar{x} - 1 \geq 0$. Then, by Lemma 3 for concave and s-shaped f , the strategy profile $s_i^* = 1$ for all $i \in N$ is a MSCE if $(n-1)\underline{s} \in [n\underline{x} - 1, n\bar{x} - 1]$. Note, that for concave f , $\underline{x} < \frac{1}{n}$ and therefore, $n\underline{x} - 1 < 0$. ■

5.2 More ambiguous feedback

For interval feedbacks, we can define a simple notion of more ambiguous feedback. Remember that the identification correspondence of interval feedbacks is

$$\Sigma_{ij}(s_j) = \begin{cases} [\underline{s}, \bar{s}] & \text{if } s_j \in [\underline{s}, \bar{s}] \\ \{s_j\} & \text{if } s_j \notin [\underline{s}, \bar{s}]. \end{cases}$$

Further, since MSCE only consider the worst case, we can, without loss of generality, assume that $\bar{s} = 1$. If $s_j < \underline{s}$, player i receives perfect feedback about s_j . Thus, an interval feedback function is called more ambiguous if it induces a larger interval $[\underline{s}, 1]$.

Definition 3 Let ϕ_1 and ϕ_2 be two symmetric interval feedbacks with $\bar{s}_1 = \bar{s}_2 = 1$. We call ϕ_1 more ambiguous than ϕ_2 if $\underline{s}_1 \leq \underline{s}_2$.

Given the notion of more ambiguous feedback, we can study the impact of ambiguity on MSCE. As the following proposition shows, for concave and s-shaped production functions, more ambiguous feedback always induces a higher MSCE contribution.

Proposition 7 Let ϕ_1 be more ambiguous than ϕ_2 .

i) Suppose f is concave, or s-shaped and $f(\frac{1}{n}) \geq c$. Then, the average contribution of the unique MSCE of (G, ϕ_1) is higher than the average contribution of the unique MSCE of (G, ϕ_2) .

ii) Suppose f is convex. Then, the highest average contribution of an MSCE of (G, ϕ_1) is lower than the highest average contribution of an MSCE of (G, ϕ_2)

Proof. Consider a concave or s-shaped production function f . $f(\frac{1}{n}) \geq c$ implies $\underline{x} \leq \frac{1}{n}$. Therefore, Lemma 3 shows that the unique symmetric MSCE of a public good game with symmetric interval feedback (G, ϕ) with \underline{s} is s_i^* for all $i \in N$ with

$$s_i^* = \begin{cases} 1 & \text{if } (n-1)\underline{s} < n\bar{x} - 1 \\ n\bar{x} - (n-1)\underline{s} & \text{if } n\bar{x} - 1 \leq (n-1)\underline{s} < (n-1)\bar{x} \\ \bar{x} & \text{if } \underline{s} \geq \bar{x} \end{cases}$$

Further, $n\bar{x} - (n-1)\underline{s} \in [\bar{x}, 1]$ and $n\bar{x} - (n-1)\underline{s}$ is strictly decreasing in \underline{s} for all $\underline{s} \in [\frac{n\bar{x}-1}{n-1}, \bar{x}]$. Thus, the result follows since $\underline{s}_1 \leq \underline{s}_2$.

For convex production functions, Lemma 3 shows that for $\underline{x} \leq \frac{1}{n}$ the unique symmetric MSCE is $s_i^* = 1$ for all $i \in N$ for any threshold feedback. Thus, if $\underline{x} < \frac{1}{n}$, the unique MSCE of (G, ϕ_1) equals the unique MSCE of (G, ϕ_2) . If $\underline{x} > \frac{1}{n}$, the highest contribution of a symmetric MSCE of (G, ϕ) with symmetric interval feedback ϕ with \underline{s} is

$$s_i^* = \begin{cases} 1 & \text{if } (n-1)\underline{s} \geq n\underline{x} - 1 \\ 0 & \text{if } (n-1)\underline{s} < n\underline{x} - 1. \end{cases}$$

Then, the result follows from $\underline{s}_1 \leq \underline{s}_2$. ■

Note that the assumption $f(\frac{1}{n}) \geq c$ ensures the uniqueness of the MSCE for s-shaped production functions. However, even if we allow for multiple MSCE and compare the MSCE with the highest average contribution, Proposition 7 i) only holds if $f(\frac{1}{n}) \geq c$. Suppose $f(\frac{1}{n}) < c$, then $\underline{x} > \frac{1}{n}$ and we can find \underline{s}_1 and \underline{s}_2 such that $(n-1)\underline{s}_1 < n\underline{x} - 1 \leq (n-1)\underline{s}_2$. By Lemma 3, the unique MSCE of (G, ϕ_1) is $\hat{s}_i = 0$ for all $i \in N$. Further, (G, ϕ_2) has two MSCE: $s_i^* = 0$ and $\tilde{s}_i = 1$ for all $i \in N$. Thus, even if ϕ_1 is more ambiguous than ϕ_2 , the highest average contribution in an MSCE of (G, ϕ_1) is strictly smaller than the highest average contribution in an MSCE of (G, ϕ_2) .

Proposition 7 is in line with the results of Eichberger and Kelsey (2002). Eichberger and Kelsey (2002) study the impact of ambiguity in symmetric games with aggregated externalities. In contrast to our setting, they assume that players' beliefs about opponents' strategies are represented by capacities, i.e., non-additive probability measures. Similar to Proposition 7, they show that in settings with positive externalities and strategic substitutes, more ambiguous beliefs induce a larger equilibrium strategy. Further, they study a public good setting with a concave production function, n players, and a finite set of pure strategies. They show that maxmin preferences imply an equilibrium contribution of at least $n\tilde{s}$, where \tilde{s} is the lowest strategy played in a symmetric Nash equilibrium. In our setting, if f is concave and $\bar{x} < \frac{1}{n}$, then there exists a unique symmetric Nash equilibrium with $\tilde{s} = \bar{x}$. Further, by Proposition 6, there exists an MSCE with contribution $n\bar{x}$.

5.3 α -maxmin selfconfirming equilibrium

For simplicity, we restrict the analysis of α -MSCE to concave production functions and cases where full contribution is implementable as an MSCE. By Lemma 3, there exists a MSCE with full contribution if $f(\frac{1}{n}) \geq cn$, i.e., $\bar{x} \geq \frac{1}{n}$. Then, full contribution can be implemented as an α -MSCE as long as the degree of ambiguity aversion α is large enough.

Proposition 8 *There exist feedbacks such that full contribution, i.e., $s_i^* = 1$ for all $i \in N$, is an α -MSCE iff $\alpha \geq \frac{cn-f'(1)}{f'(\frac{1}{n})-f'(1)}$. Such feedback functions are characterized by an interval feedback function with $\bar{s} = 1$ and $\underline{s} \leq \tilde{s}_\alpha$ where $\tilde{s}_\alpha \geq 0$ satisfies*

$$f' \left(\frac{(n-1)\tilde{s}_\alpha + 1}{n} \right) = \frac{cn - (1-\alpha)f'(1)}{\alpha}. \quad (5)$$

Proof. Since f is concave, $s_i^* = 1$ is a best response to symmetric interval feedbacks with $\bar{s} = 1$ and \underline{s} iff

$$\alpha f' \left(\frac{(n-1)\underline{s} + 1}{n} \right) + (1-\alpha)f'(1) \geq cn. \quad (6)$$

Thus, by the definition of \tilde{s}_α and since f is concave, Equation (6) is satisfied for all $\underline{s} \leq \tilde{s}_\alpha$. Further, \tilde{s}_α is increasing in α and the assumption $\alpha \geq \frac{cn-f'(1)}{f'(\frac{1}{n})-f'(1)}$ ensures that $\tilde{s}_\alpha \geq 0$. ■

Note, that if $f'(1) \geq cn$, the unique Nash equilibrium is $s_i^* = 1$ for all $i \in N$ and $\frac{cn-f'(1)}{f'(\frac{1}{n})-f'(1)} < 0$. Then, for any $\alpha \in [0, 1]$, there exists an α -MSCE with full contribution. Further, \tilde{s}_α increases in α if full contribution is not a Nash equilibrium. Thus, for more ambiguity averse players, there exist more feedback functions implementing full contribution.

If $\alpha < \frac{cn-f'(1)}{f'(\frac{1}{n})-f'(1)}$, full contribution cannot be implemented as α -MSCE. However, we can still find the symmetric α -MSCE that maximizes the contribution. These α -MSCE are induced by interval feedback with $\underline{s} = 0$ and \bar{s} such that the symmetric α -MSCE is $s_i = \bar{s}$ for all $i \in N$ where \bar{s} satisfies

$$\alpha f' \left(\frac{\bar{s}}{n} \right) + (1-\alpha)f'(\bar{s}) = cn. \quad (7)$$

Since f is concave, $f'(x)$ is decreasing in x and there exists a unique \bar{s} satisfying Equation (7).

Further, \bar{s} increases as α increases. Thus, more ambiguity aversion leads to higher contribution in the α -MSCE. For $\alpha = 0$, players maximize their best case expected utility. Then, $\bar{s} = \bar{x}$ and the α -MSCE with $\alpha = 0$ coincides with the Nash equilibrium. For $\alpha = \frac{cn-f'(1)}{f'(\frac{1}{n})-f'(1)}$, Equation (7) with $\bar{s} = 1$ and Equation (5) with $\tilde{s}_\alpha = 0$ are equivalent. Thus, as α increases from 0 to $\frac{cn-f'(1)}{f'(\frac{1}{n})-f'(1)}$, the α -MSCE equilibrium contribution increases from the Nash equilibrium contribution to full contribution.

Similarly, we can extend the analysis of convex production functions to α -MSCE. For convex production functions, full contribution is always a Nash equilibrium and an MSCE if the interval feedback satisfies $\underline{s} \geq \frac{nx-1}{(n-1)}$ (Lemma 3). It can be shown that for α -MSCE, there exists a similar threshold for \underline{s} . However, this threshold is decreasing in α . Thus, for less ambiguity averse players, more ambiguous feedback still implements full contribution.

6 Strategic substitutes and negative externalities

In this section, we consider a class of symmetric games with strategic substitutes, negative externalities, and linear best responses, such as Cournot oligopolies with linear demand and symmetric constant

marginal costs. Precisely, let $N = \{1, \dots, n\}$, $S_i = [0, 1]$, and

$$u_i(s) = \gamma s_i - s_i^2 - d s_i \sum_{j \neq i} s_j,$$

where $\gamma, d > 0$. To avoid corner solutions we also assume $\gamma < 1 + d(n-1)$. The best response function of player i is

$$BR_i(s_{-i}) = \frac{\gamma - d \sum_{j \neq i} s_j}{2},$$

and the unique Nash equilibrium is symmetric and given by

$$s_i^{NE} = \frac{\gamma}{2 + d(n-1)}, \quad i \in N.$$

The symmetric collusive solution, which maximizes the sum of players' utility, is given by

$$s_i^C = \frac{\gamma}{2 + 2d(n-1)}, \quad i \in N.$$

Similar to the public good setting, the game is symmetric and u_i is decreasing in s_j for $j \neq i$. Therefore, we can focus without loss of generality on symmetric interval feedbacks, i.e., for every i and $j \neq i$:

$$\Sigma_{ij}(s_j) = \begin{cases} [\underline{s}, \bar{s}] & \text{if } s_j \in [\underline{s}, \bar{s}] \\ \{s_j\} & \text{if } s_j \notin [\underline{s}, \bar{s}]. \end{cases}$$

The next proposition shows that there always exist interval feedbacks such that the collusive outcome is an MSCE. The proposition that follows shows that this result also applies (but with different interval feedbacks) for α -MSCE as long as α is large enough.

Proposition 9 *There exist symmetric feedback functions such that the collusive strategy profile $(s_i^C)_i$ is an MSCE. Such feedback functions are characterized by interval feedback functions with $\underline{s} \leq s_i^C = \frac{\gamma}{2+2d(n-1)}$ and $\bar{s} = 2s_i^C = \frac{\gamma}{1+d(n-1)}$.*

Proof. Immediate from the observation that the best response of player i to $2s_{-i}^C$ is s_i^C . ■

Proposition 10 *There exist feedback functions such that the collusive strategy profile $(s_i^C)_i$ is an α -MSCE iff $\frac{\alpha}{1+\alpha} \geq s_i^C = \frac{\gamma}{2+2d(n-1)}$. Such feedback functions are characterized by interval feedback functions with $\underline{s} = s_i^C = \frac{\gamma}{2+2d(n-1)}$ and $\bar{s} = \frac{1+\alpha}{\alpha} s_i^C$.*

Proof. The collusive strategy profile is an α -MSCE iff for every $i \neq j$ the strategy of player j perceived by player i is $\alpha \bar{s} + (1-\alpha)\underline{s} = 2s_i^C$. Such an equilibrium exists only if it exists for $\underline{s} = s_i^C$, and therefore $\bar{s} = \frac{1+\alpha}{\alpha} s_i^C$. Since we must have $\bar{s} \leq 1$, the collusive outcome is an α -MSCE iff $\frac{\alpha}{1+\alpha} \geq s_i^C$. ■

Now assume that α is too small (players are too optimistic) for the collusive outcome to be implementable as an α -MSCE. That is, assume that $\frac{\alpha}{1+\alpha} < s_i^C$, i.e., $\alpha < \frac{s_i^C}{1-s_i^C} = \frac{\gamma}{2+2d(n-1)-\gamma}$. The best (symmetric) outcome is then obtained with the feedbacks that minimize the equilibrium

strategy s_i , i.e., $\bar{s} = 1$ and $\underline{s} = s_i$. The α -MSCE $(s_i)_i$ is then given by $s_i = BR_i(\tilde{s}_{-i})$ with $\tilde{s}_j = \alpha\bar{s} + (1 - \alpha)\underline{s} = \alpha + (1 - \alpha)s_j$, so in equilibrium

$$\begin{aligned} s_i &= \frac{\gamma - d(n-1)(\alpha + (1-\alpha)s_i)}{2} \\ \Leftrightarrow s_i &= \frac{\gamma - d(n-1)\alpha}{2 + d(n-1)(1-\alpha)}. \end{aligned}$$

It is immediate to check that the equilibrium strategy s_i (and hence, each player's actual equilibrium utility) decreases in α . For $\alpha = 0$, we get $s_i = s_i^{NE}$, and the best feedback to maximize players' welfare is to give perfect feedback. For $\alpha = \frac{s_i^C}{1-s_i^C}$ we get the first best $s_i = s_i^C$ as for $\alpha \geq \frac{s_i^C}{1-s_i^C}$, with $\underline{s} = s_i^C$ and $\bar{s} = 1$.

To conclude, for $\alpha \geq \frac{s_i^C}{1-s_i^C}$ we get the collusive outcome $s_i = s_i^C$ as an α -MSCE with the interval feedbacks $\underline{s} = s_i^C$ and $\bar{s} = \frac{1+\alpha}{\alpha}s_i^C$. For $\alpha < \frac{s_i^C}{1-s_i^C}$ the collusive outcome is not implementable as an α -MSCE, and the best α -MSCE is obtained with the interval feedbacks $\underline{s} = s_i$ and $\bar{s} = 1$. In that case, the best α -MSCE equilibrium strategy is increasing in α (going from s_i^{NE} when $\alpha = 0$ to s_i^C when $\alpha = \frac{s_i^C}{1-s_i^C}$).

So far, we have studied feedback functions that implement the collusive outcome, i.e., in the Cournot oligopoly context, MSCE that maximize producers' surplus. In contrast, we could study MSCE that improve consumers' surplus compared to the Nash equilibrium. However, there does not exist feedbacks and MSCE that induce a higher consumer surplus than the Nash equilibrium. Maximizing the consumer surplus is equivalent to maximizing the equilibrium strategies s_i^* . Suppose, there exist feedback functions and a symmetric MSCE with $s_i^* > s_i^{NE}$. Remember that the true strategy profile of the opponents is always one of the strategy profiles that player i considers, i.e., $s_{-i}^* \in \Sigma_i(s_{-i}^*)$. Further, the worst case belief of player i is the highest possible production of his opponents. Thus, in the worst case, player i believes that his opponents play a strategy greater or equal than $s_j^* > s_j^{NE}$. The best response function is decreasing in the strategies of the opponents. Therefore, the best response of player i has to be smaller than the Nash equilibrium strategy. Thus, there is no symmetric MSCE with strategies higher than the Nash equilibrium strategy. These results are in line with Eichberger and Kelsey (2002) who showed that for strategic substitutes and negative externalities more ambiguity implies lower equilibrium strategies.

This section analyzes symmetric games with strategic substitutes, negative externalities and linear best response. In the online appendix we also study a class of Tullock contests with strategic substitutes and negative externalities but non-linear best responses. To implement the collusive outcome as MSCE in the Tullock contests, we need an additional assumption on the prize players can win. If the prize is too high, a positive bid is always optimal independently of the opponents' strategy, and the collusive outcome cannot be implemented. We characterize conditions on the prize and feedback such that the collusive outcome is a MSCE. Further, if the prize is too high, we can still characterize feedback function that implement MSCE with a smaller bid than the Nash equilibrium bid.

A Appendix

A.1 Proofs of Section 3

Lemma 4 *For every i , v_i is quasi-concave on S_i .*

Proof. Fix any $i \in N$, $s_{-i} \in S_{-i}$ and $y \in \mathbb{R}$. To prove that v_i is quasi-concave on S_i , we must show that

$$\{s_i \in S_i : v_i(s_i, s_{-i}) \geq y\},$$

is convex. That is, we must show that for every $s_i, s'_i \in S_i$ and $\beta \in [0, 1]$,

$$v_i(s_i, s_{-i}) \geq y \text{ and } v_i(s'_i, s_{-i}) \geq y \tag{8}$$

implies

$$v_i(\beta s_i + (1 - \beta)s'_i, s_{-i}) \geq y. \tag{9}$$

(8) is equivalent to

$$\inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(s_i, \tilde{s}_{-i}) \geq y \text{ and } \inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(s'_i, \tilde{s}_{-i}) \geq y,$$

i.e.,

$$u_i(s_i, \tilde{s}_{-i}) \geq y \text{ and } u_i(s'_i, \tilde{s}_{-i}) \geq y, \text{ for every } \tilde{s}_{-i} \in \Sigma_i(s_{-i}).$$

From the assumption that u_i is quasi-concave on S_i we get

$$u_i(\beta s_i + (1 - \beta)s'_i, \tilde{s}_{-i}) \geq y \text{ for every } \tilde{s}_{-i} \in \Sigma_i(s_{-i}).$$

Hence,

$$\inf_{\tilde{s}_{-i} \in \Sigma_i(s_{-i})} u_i(\beta s_i + (1 - \beta)s'_i, \tilde{s}_{-i}) \geq y,$$

which gives (9). ■

Proof of Proposition 1 From Berge's theorem, v_i is continuous because the identification correspondence Σ_i is continuous and compact-valued. From Lemma 4, v_i is quasi-concave on S_i . Hence, the auxiliary game (N, S, v) has at least one Nash equilibrium, i.e., the game with feedbacks (G, ϕ) has a least one MSCE. ■

Lemma 5 *Assume that S_i is a finite dimensional simplex for every $i \in N$. If ϕ_{ij} is linear for every $j \neq i$, then the identification correspondence Σ_i of player i is lower hemicontinuous.*

Proof. To simplify the notations, we fix i and j and remove the subscripts referring to players i and j from the notations. We must show that if $\phi(s) = Ms$, then the correspondence Σ defined by

$$\Sigma(s) = \phi^{-1}(\phi(s)) = \{\tilde{s} \in S : \phi(\tilde{s}) = \phi(s)\},$$

is lower hemicontinuous, where $S = \Delta(A)$. That is, we must show that for all sequences (s_k) in S with $s_k \rightarrow s$, and for all $\tilde{s} \in \Sigma(s)$, there exist a subsequence $(s_{k(m)})$ and a sequence (\tilde{s}_m) in S such that $\tilde{s}_m \rightarrow \tilde{s}$ and $\phi(\tilde{s}_m) = \phi(s_{k(m)})$ for all m . For $m = 1, 2, \dots$, let

$$s^m := \left(1 - \frac{1}{m}\right)\tilde{s} + \frac{1}{m}s.$$

Notice that $s^m \rightarrow \tilde{s}$ when $m \rightarrow \infty$, and by linearity of ϕ we have $\phi(s^m) = \phi(\tilde{s}) = \phi(s)$ for all m . Let

$$z_k^m := s_k - s + s^m.$$

By linearity of ϕ we also have $\phi(z_k^m) = \phi(s_k)$ for all m and k .

Claim 1 *For every m , and for k large enough, we have $z_k^m \in S$.*

Proof. For every $z \in S$, denote by $z[l]$ the l th coordinate of z . First, observe that

$$\sum_l z_k^m[l] = \sum_l s_k[l] - \sum_l s[l] + \sum_l s^m[l] = 1 - 1 + 1 = 1,$$

because $s_k, s, s^m \in S$. It remains to show that $z_k^m[l] \geq 0$ for all l and for k large enough. If $s^m[l] > 0$, then we have $z_k^m[l] = s_k[l] - s[l] + s^m[l] > 0$ for k large enough because $s_k[l] \rightarrow s[l]$. If $s^m[l] = 0$, then $s[l] = 0$, so $z_k^m[l] = s_k[l] \geq 0$. This completes the proof of the claim. ■

Hence, for all m , there is $k(m)$ such that $\tilde{s}_m := z_{k(m)}^m \in S$, $\phi(\tilde{s}_m) = \phi(s_{k(m)})$ for all m , and $\tilde{s}_m \rightarrow \tilde{s}$. This completes the proof of the lemma. ■

A.2 Proofs of Section 5

Proof of Lemma 1 Suppose f is concave, i.e., $t = 0$. We set $\underline{x} = 0$ and Equation (4) reduces to

$$BR_i(s_{-i}) = \begin{cases} 1 & \text{if } \sum_{j \neq i} s_j < n\bar{x} - 1, \\ n\bar{x} - \sum_{j \neq i} s_j & \text{if } n\bar{x} - 1 \leq \sum_{j \neq i} s_j < n\bar{x}, \\ 0 & \text{if } n\bar{x} \leq \sum_{j \neq i} s_j. \end{cases}$$

By the definition of \bar{x} and Assumption 1, if $\sum_{j \neq i} s_j < n\bar{x} - 1$ we have

$$f' \left(\frac{\sum_{j \neq i} s_j + s_i}{n} \right) \geq cn$$

for all $s_i \in [0, 1]$. Thus $BR_i(s_{-i}) = 1$. Similarly, if $n\bar{x} \leq \sum_{j \neq i} s_j$ we have

$$f' \left(\frac{\sum_{j \neq i} s_j + s_i}{n} \right) \leq cn$$

for all $s_i \in [0, 1]$ and $BR_i(s_{-i}) = 0$. If $n\bar{x} - 1 \leq \sum_{j \neq i} s_j < n\bar{x}$, then there exists a $BR_i(s_{-i}) \in [0, 1]$ that solves the FOC

$$\begin{aligned} f' \left(\frac{\sum_{j \neq i} s_j + BR_i(s_{-i})}{n} \right) &= cn \\ \Leftrightarrow \frac{\sum_{j \neq i} s_j + BR_i(s_{-i})}{n} &= \bar{x} \\ \Leftrightarrow BR_i(s_{-i}) &= n\bar{x} - \sum_{j \neq i} s_j. \end{aligned}$$

Suppose f is convex, i.e., $t = 1$. By definition of \bar{x} we have $\bar{x} = 1$ and the best response function reduces to

$$BR_i(s_{-i}) = \begin{cases} 0 & \text{if } \sum_{j \neq i} s_j < n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq \sum_{j \neq i} s_j, \end{cases}$$

where

$$\underline{x} := \begin{cases} 0 & \text{if } f(x) - f\left(x - \frac{1}{n}\right) > c, \forall x \in [0, 1], \\ 1 & \text{if } f(x) - f\left(x - \frac{1}{n}\right) < c, \forall x \in [0, 1], \\ x & \text{if } x \in [0, 1] \text{ and } f(x) - f\left(x - \frac{1}{n}\right) = c. \end{cases}$$

Note that \underline{x} is unique and well defined for convex production functions f because $g(x) := f(x) - f\left(x - \frac{1}{n}\right)$ is increasing in x . Further, by the monotonicity of $g(x)$, the best response is $BR_i(s_{-i}) = 1$ if and only if

$$\begin{aligned} f \left(\frac{\sum_{j \neq i} s_j + 1}{n} \right) - c &\geq f \left(\frac{\sum_{j \neq i} s_j}{n} \right) \\ \Leftrightarrow f \left(\frac{\sum_{j \neq i} s_j + 1}{n} \right) - f \left(\frac{\sum_{j \neq i} s_j}{n} \right) &\geq c \\ \Leftrightarrow \frac{\sum_{j \neq i} s_j + 1}{n} &\geq \underline{x} \\ \Leftrightarrow \sum_{j \neq i} s_j &\geq n\underline{x} - 1. \end{aligned}$$

Now, suppose f is s-shaped, i.e. $t \in (0, 1)$. If $\sum_{j \neq i} s_j \in [0, nt - 1]$ (or $\sum_{j \neq i} s_j \in [nt, n - 1]$), then f is convex (or concave) for all $s_i \in [0, 1]$ and the best response function can be derived as above. Therefore, consider a strategy profile s_{-i} such that $\sum_{j \neq i} s_j \in [nt - 1, nt]$.

Case 1. Suppose $nt \leq n\bar{x} - 1$. Then, $\sum_{j \neq i} s_j \in [nt - 1, nt]$ implies that $\sum_{j \neq i} s_j \leq n\bar{x} - 1$. Therefore, there are two possible solutions of the maximization problem: $BR_i(s_{-i}) = 0$ and $BR_i(s_{-i}) = 1$. $BR_i(s_{-i}) = 1$ is a global maximum if and only if

$$f \left(\frac{\sum_{j \neq i} s_j + 1}{n} \right) - c \geq f \left(\frac{\sum_{j \neq i} s_j}{n} \right).$$

Define an auxiliary function $g(x) : [t, t + \frac{1}{n}] \rightarrow \mathbb{R}$ with $g(x) = f(x) - f(x - \frac{1}{n})$. Then, g is concave⁷ and

$$\begin{aligned} & g(x) \geq c, \forall x \in \left[t, t + \frac{1}{n} \right] \\ \Leftrightarrow & \min_{x \in [t, t + \frac{1}{n}]} g(x) \geq c \\ \Leftrightarrow & \min \left\{ g(t), g\left(t + \frac{1}{n}\right) \right\} \geq c. \end{aligned}$$

Since $f(x)$ is concave for all $x \in [t, 1]$ and $t + \frac{1}{n} \leq \bar{x}$, we have $g(t + \frac{1}{n}) = f(t + \frac{1}{n}) - f(t) \geq c$.

Further, $g(t) = f(t) - f(t - \frac{1}{n})$. We define \underline{x} as

$$\underline{x} = \begin{cases} 0 & \text{if } f(x) - f(x - \frac{1}{n}) > c, \forall x \in [0, t + \frac{1}{n}], \\ x & \text{if } x \in [0, t + \frac{1}{n}] \text{ and } f(x) - f(x - \frac{1}{n}) = c. \end{cases}$$

If $\underline{x} \leq t$ uniqueness follows similarly as for convex production functions. Suppose $\underline{x} > t$. Then, since g is concave and $g(t + \frac{1}{n}) \geq c$ there exists a unique $\underline{x} \in [t, t + \frac{1}{n}]$ such that $g(\underline{x}) = f(\underline{x}) - f(\underline{x} - \frac{1}{n}) = c$ and $g(x) > c$ iff $x > \underline{x}$. Thus, \underline{x} is well-defined and unique.

Further, if $\underline{x} \leq t$, we have $g(t) = f(t) - f(t - \frac{1}{n}) \geq c$ and $g(x) \geq c$ for all $x \in [t, t + \frac{1}{n}]$. Then, $BR_i(s_{-i}) = 1$ for all $\sum_{j \neq i} s_j \in [nt - 1, nt]$. If $\underline{x} > t$, then we have $g(x) < c$ for $x < \underline{x}$ and $g(x) \geq c$ for $x \geq \underline{x}$. Thus, $BR_i(s_{-i}) = 1$ iff $\sum_{j \neq i} s_j \geq n\underline{x} - 1$.

Case 2. Suppose $nt > n\bar{x} - 1$. First, consider $\bar{x} \leq \frac{1}{n}$ and $\sum_{j \neq i} s_j \in [0, nt]$. We define $\underline{x} := 0$. By Assumption 2, we have $f'(0) \geq cn$. Further, by definition of \bar{x} we have $f'(\bar{x}) \geq cn$. Then, convexity of f on $[0, t]$ and concavity on $[t, 1]$ implies that $f'(x) \geq cn$ for all $x \in [0, \bar{x}]$. Thus,

$$\begin{aligned} & \frac{f(\bar{x}) - f\left(\frac{\sum_{j \neq i} s_j}{n}\right)}{\frac{n\bar{x} - \sum_{j \neq i} s_j}{n}} \geq cn \\ \Leftrightarrow & f(\bar{x}) - c \left(n\bar{x} - \sum_{j \neq i} s_j \right) \geq f\left(\frac{\sum_{j \neq i} s_j}{n}\right) \end{aligned}$$

and $BR_i(s_{-i}) = n\bar{x} - \sum_{j \neq i} s_j$ is the best response for all s_{-i} such that $\sum_{j \neq i} s_j \in [0, nt]$.

Now suppose $\bar{x} > \frac{1}{n}$. We consider two subcases.

Case 2.a. Suppose $\sum_{j \neq i} s_j \in [n\bar{x} - 1, nt]$.

Now $BR_i(s_{-i}) = 1$ can never be a best response. Instead we have to compare the candidates $BR_i(s_{-i}) = 0$ and $BR_i(s_{-i}) = n\bar{x} - \sum_{j \neq i} s_j$. Then, $BR_i(s_{-i}) = n\bar{x} - \sum_{j \neq i} s_j$ iff

$$f(\bar{x}) - c \left(n\bar{x} - \sum_{j \neq i} s_j \right) \geq f\left(\frac{\sum_{j \neq i} s_j}{n}\right).$$

Define $h : [n\bar{x} - 1, nt] \rightarrow \mathbb{R}$ as $h(x) := f\left(\frac{x}{n}\right) + c(n\bar{x} - x)$. Then, $h(x)$ is convex for all $x \in [n\bar{x} - 1, nt]$

⁷Note, that g is not strictly increasing, since $f(x)$ is concave and $f(x - \frac{1}{n})$ is convex for all $x \in [t, t + \frac{1}{n}]$.

and $h(x) \leq f(\bar{x})$ for all $x \in [n\bar{x} - 1, nt]$ iff

$$\max\{h(n\bar{x} - 1), h(nt)\} \leq f(\bar{x}).$$

Further, since f is concave for $x \in [t, \bar{x}]$ and $f'(\bar{x}) = cn$, we have $f(t) + cn(\bar{x} - t) \leq f(\bar{x})$. Therefore, $h(nt) = f(t) + cn(\bar{x} - t) \leq f(\bar{x})$.

By Assumption 2, we have that $f'(\bar{x} - \frac{1}{n}) \geq cn$ and by definition of \bar{x} we have $f'(\bar{x}) = cn$. Therefore,

$$\begin{aligned} & \frac{f(\bar{x}) - f(\bar{x} - \frac{1}{n})}{\frac{1}{n}} \geq cn \\ \Leftrightarrow & f(\bar{x}) - f\left(\bar{x} - \frac{1}{n}\right) \geq c \\ \Leftrightarrow & f(\bar{x}) \geq f\left(\bar{x} - \frac{1}{n}\right) + c = h(n\bar{x} - 1) \end{aligned}$$

Thus, $h(x) \leq f(\bar{x})$ for all $x \in [n\bar{x} - 1, nt]$ and therefore, $BR_i(s_{-i}) = n\bar{x} - \sum_{j \neq i} s_j$ for all s_{-i} such that $\sum_{j \neq i} s_j \in [n\bar{x} - 1, nt]$. Thus, \underline{x} does not influence the best response for s_{-i} with $\sum_{j \neq i} s_j \in [n\bar{x} - 1, nt]$.

Case 2.b. Suppose $\sum_{j \neq i} s_j \in [nt - 1, n\bar{x} - 1]$.

Similar to Case 1 there exist two candidates $BR_i(s_{-i}) = 0$ or $BR_i(s_{-i}) = 1$. We define $g(x)$ as in Case 1 with $x \in [t, \bar{x}]$. Analogously to above it follows that g is concave and $g(x) \geq c$ for all $x \in [t, \bar{x}]$ iff

$$\min\{g(t), g(\bar{x})\} \geq c.$$

Then, Case 2.a. implies $g(\bar{x}) = f(\bar{x}) - f(\frac{n\bar{x}-1}{n}) \geq c$ and the definition, existence and uniqueness of \underline{x} follows analogously to Case 1. ■

Proof of Lemma 3 For any $\underline{s} \in [0, 1]$, we construct all symmetric MSCE. Then, existence follows by construction.

Part i) Suppose f is convex. Then, $\bar{x} = 1$ and the best response function of player i for a symmetric strategy profile s_{-i}^* reduces to

$$\begin{aligned} BR_i(s_{-i}^*) &= \begin{cases} 0 & \text{if } \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq \sum_{j \neq i} \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j \leq n - 1, \end{cases} \\ &= \begin{cases} 0 & \text{if } (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\underline{x} - 1, \\ 1 & \text{if } n\underline{x} - 1 \leq (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j \leq n - 1. \end{cases} \end{aligned}$$

If $\underline{x} > \frac{1}{n}$, we have $n\underline{x} - 1 > 0$. Further, if $s_j^* = 0$ for all $j \neq i$, $\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = 0$ for all $j \neq i$. Thus, the best response of player i is $s_i^* = 0$ for all $\underline{s} \in [0, 1]$.

Suppose, $(n-1)\underline{s} \geq n\underline{x} - 1$ and $s_j^* = 1$ for all $j \neq i$. Then, $(n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = (n-1)\underline{s}$ and the best response of player i is $s_i^* = 1$. Thus, $s_i^* = 1$ for all $i \in N$ is a symmetric MSCE.

Part ii) Suppose f is concave. Then, the best response function is

$$BR_i(s_{-i}^*) = \begin{cases} 1 & \text{if } (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\bar{x} - 1, \\ n\bar{x} - (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j & \text{if } n\bar{x} - 1 \leq (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\bar{x}, \\ 0 & \text{if } n\bar{x} \leq (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j. \end{cases}$$

If $(n-1)\underline{s} < n\bar{x} - 1$, we have $(n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\bar{x} - 1$ for any s_j^* and therefore $s_i^* = 1$ is the unique MSCE.

If $n\bar{x} - 1 \leq (n-1)\underline{s} < (n-1)\bar{x}$, we have $(n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\bar{x}$ for any s_j^* and the last case of the best response function can never occur. Further, suppose $s_j^* = 1$ for all $j \neq i$. Then, $s_j^* \geq \underline{s}$ and $(n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = (n-1)\underline{s} \in [n\bar{x} - 1, n\bar{x})$. Thus, any symmetric MSCE has to satisfy $s_i^* = n\bar{x} - (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j$ for all $i \in n$. Remember that

$$\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = \begin{cases} s_j^* & \text{if } s_j^* < \underline{s}, \\ \underline{s} & \text{if } s_j^* \geq \underline{s}. \end{cases}$$

Suppose $s_j^* < \underline{s}$, then $s_i^* = n\bar{x} - (n-1)s_j^*$. By symmetry, we have $s_i^* = s_j^* = \bar{x}$. But this contradicts, the assumption that $(n-1)\underline{s} < (n-1)\bar{x}$. Hence, the unique MSCE has to satisfy $s_i^* = n\bar{x} - (n-1)\underline{s}$ for all $i \in n$.

For $(n-1)\bar{x} \leq (n-1)\underline{s} < n\bar{x}$, it follows analogously to above that the unique MSCE has to satisfy $s_i^* = n\bar{x} - (n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j$ for all $i \in n$. Now, since $\bar{x} < \underline{s}$, we have $\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = s_j^* = \bar{x}$ and $s_i^* = \bar{x}$ for all $i \in N$ is the unique MSCE.

If $(n-1)\underline{s} \geq n\bar{x}$, the unique symmetric MSCE equals the symmetric Nash equilibrium $s_i^* = \bar{x}$. Suppose $\bar{x} \in (0, 1)$. Then, the best response to $s_j^* = 1$ for all $j \neq i$ is $s_i^* = 0$ and the best response to $s_j^* = 0$ for all $j \neq i$ is $s_i^* > 0$. Thus, $s_i^* = 0$ for all $i \in N$ or $s_i^* = 1$ for all $i \in N$ cannot be symmetric MSCE, and any MSCE has to satisfy the second case of the best response function. For the second case of the best response function, we have $(n-1) \min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < n\bar{x} \leq (n-1)\underline{s}$. Thus, $\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j < \underline{s}$ and $\min_{s_j \in \Sigma_{ij}(s_j^*)} s_j = s_j^*$. Therefore, the unique MSCE equals the Nash equilibrium $s_i^* = \bar{x}$ for all $i \in N$. If $\bar{x} = 0$, the best response function reduces to the last case, and $s_i^* = 0 = \bar{x}$ is the unique MSCE. Similarly, if $\bar{x} = 1$, the strategy profile $s_i^* = 1 = \bar{x}$ for all $i \in N$ is the unique MSCE. Hence, for all $\bar{x} \in [0, 1]$ the unique MSCE is given by $s_i^* = \bar{x}$ for all $i \in N$.

Part iii) If f is s-shaped, all cases follow analogously to one of the cases for convex or concave f .

Thus, by construction, there exists at least one symmetric MSCE for any production function and any $\underline{s} \in [0, 1]$. Note that for s-shaped (or convex) production functions, $f(\frac{1}{n}) > c$ is equivalent to $\underline{x} < \frac{1}{n}$. Then, we have $n\underline{x} - 1 < 0$ and $s_i^* = 1$ is a symmetric MSCE for all \underline{s} with $(n-1)\underline{s} < n\bar{x} - 1$ (or $\underline{s} \leq 1$). ■

References

- Sophie Bade. Ambiguous act equilibria. *Games and Economic Behavior*, 71(2):246–260, 2011.
- Pierpaolo Battigalli. Comportamento razionale ed equilibrio nei giochi e nelle situazioni sociali. *unpublished undergraduate dissertation, Bocconi University, Milano*, 1987.

- Pierpaolo Battigalli, Simone Cerreia-Vioglio, Fabio Maccheroni, and Massimo Marinacci. Self-confirming equilibrium and model uncertainty. *American Economic Review*, 105(2):646–77, 2015.
- Pierpaolo Battigalli, Simone Cerreia-Vioglio, F Maccheroni, and Massimo Marinacci. Analysis of information feedback and selfconfirming equilibrium. *Journal of Mathematical Economics*, 66:40–51, 2016.
- Dirk Bergemann and Stephen Morris. Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics*, 11(2):487–522, 2016.
- Philippe Bich. Strategic uncertainty and equilibrium selection in discontinuous games. *Journal of Economic Theory*, 183:786–822, 2019.
- Andreas Diekmann. Volunteer’s dilemma. *Journal of conflict resolution*, 29(4):605–610, 1985.
- Wioletta Dziuda, A Arda Gitmez, and Mehdi Shadmehr. The difficulty of easy projects. *American Economic Review: Insights*, 3(3):285–302, 2021.
- Jürgen Eichberger and David Kelsey. Strategic complements, substitutes, and ambiguity: the implications for public goods. *Journal of Economic Theory*, 106(2):436–466, 2002.
- Ignacio Esponda. Information feedback in first price auctions. *The RAND Journal of Economics*, 39(2):491–508, 2008.
- Drew Fudenberg and David K Levine. Self-confirming equilibrium. *Econometrica: Journal of the Econometric Society*, pages 523–545, 1993.
- Boris Ginzburg. Slacktivism. *Available at SSRN 3198854*, 2022.
- Jacob K Goeree, Charles A Holt, and Angela M Smith. An experimental examination of the volunteer’s dilemma. *Games and Economic Behavior*, 102:303–315, 2017.
- Adrian Hillenbrand and Fabian Winter. Volunteering under population uncertainty. *Games and Economic Behavior*, 109:65–81, 2018.
- Philippe Jehiel. Analogy-based expectation equilibrium. *Journal of Economic Theory*, 123(2):81–104, 2005.
- Philippe Jehiel. Manipulative auction design. *Theoretical economics*, 6(2):185–217, 2011.
- Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *The American Economic Review*, 101(6):2590–2615, 2011.
- Ehud Lehrer. Partially specified probabilities: decisions and games. *American Economic Journal: Microeconomics*, 4(1):70–100, 2012.
- R. B. Myerson. Optimal coordination mechanisms in generalized principal-agent problems. *Journal of Mathematical Economics*, 10:67–81, 1982.
- Frank Riedel and Linda Sass. Ellsberg games. *Theory and Decision*, 76(4):469–509, 2014.

Ronald Stauber. A strategic product for belief functions. *Games and Economic Behavior*, 116:38–64, 2019.

Ina Taneva. Information design. *American Economic Journal: Microeconomics*, 11(4):151–85, 2019.

Muhamet Yildiz. Wishful thinking in strategic environments. *The Review of Economic Studies*, 74(1):319–344, 2007.