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**Stable outcomes in simple cooperative games**

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**JEL Codes: D720, D710.**

**Keywords: Simple games ; Coalitions ; Median graph ; Tree ; Intermediate preferences, Condorcet**



# Stable outcomes in simple cooperative games

Gabrielle Demange<sup>†</sup>

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## Abstract

In a cooperative game, coalitions are the fundamental behavioral units. Stable outcomes (in the core) are those blocked by no coalition. This paper has two objectives. First, building on the notion of intermediate preferences indexed by a median graph, I unify and extend previous results on the existence of stable outcomes in simple games. Second, I review how and when the core approach applies in more general settings and may help to predict the stable splitting of a whole group into disjoint coalitions.

**Keywords:** simple games, coalitions, median graph, tree, intermediate preferences, Condorcet

**JEL Classification** D720, D710

## 1 Introduction

The stability of collective decisions is a prominent issue for groups, organizations, political bodies and communities. Often, instability comes from the threats of blocking coalitions. A coalition blocks a proposed decision when it has the power to impose another decision that is preferred by all its members. A stable outcome (in the core) is one that is blocked by no coalition. As illustrated by the celebrated Condorcet 'paradox', no stable outcome may exist under the majority rule. There are three individuals, three alternatives  $a, b, c$ . The first agent prefers  $a$  to  $b$  to  $c$ , the second  $b$  to  $c$  to  $a$  and the third  $c$  to  $a$  to  $b$ . 1 and 2 both prefer  $a$  to  $c$ ; since coalition  $\{1, 2\}$  has a majority of the electorate, it blocks  $c$ . Similarly,  $\{2, 3\}$  blocks  $a$  and  $\{3, 1\}$  blocks  $b$ . Such a configuration potentially gives rise to an unlimited sequence of blocking and instability. Instability is due to two ingredients: the distribution of coalition power (each of the three doubletons has the power to block any alternative) and the preferences configuration. This raises the question: under which conditions

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<sup>†</sup>Paris School of Economics, EHESS, Paris, France. E-Mail: demangegabrielle@gmail.com. I thank Andrés Carvajal for setting up this JME anniversary issue, Associate Editor Marina Nuñez-Oliva and an anonymous referee.

on coalitions power and preferences does a stable decision exist? A large part of the cooperative theory literature is devoted to answer this question in a variety of settings beyond the majority rule.

The main result of this paper provides an additional contribution that encompasses a number of known results. Consider a *simple* game, as defined by Von Neumann and Morgenstern (1944), in which coalitions are either all-powerful or ineffectual, represented by a set of *winning* coalitions as, for example, the coalitions with more than half the population in a majority game. I display families of preference that *guarantee stability*, meaning that a stable outcome exists when the preferences of all individuals involved in the game belong to such a family. Preferences in a family satisfy some interdependence structure captured by betweenness relationships, as described below. While the main result bears on a simple game, representative of voting power, the stability of collective decisions is a pervasive issue. In many political and economic situations, a group of individuals coordinate their actions while facing the threats that potential coalitions may generate by blocking or seceding. In the last section, I briefly review how the core notion sheds some lights on these situations and help to predict on the endogenous formation of stable subgroups.

The domain of simple games is quite rich in practice. Many political and organizational structures base their decisions on simple games. Unanimity, majority, weighted majority, super-majority are all simple games. More elaborate is double majority, which requires a majority of votes (or super-majority) according to two separate criteria, say the number of voters and their weights. For example, the Council of the European Union requires 55% of voting EU member states representing at least 65% of the population of voting members for a European Commission proposal to be approved (except for decisions requiring unanimity). Although simple to describe, simple games are worth studying for several reasons, methodological, didactic, and practical, as argued by Shapley (1962).<sup>1</sup> Most of the issues related to stability in games describing in detail applied situations already arise in simple games. The transparent technical tools for analyzing simple games thus shed some light on more complex situations.

Consider a group that has to choose a collective decision based on members' preferences on the set of possible decisions (alternatives). Each preference ordering is parameterized by a characteristic belonging to a connected graph, where a link between two characteristics means that they are 'close'. In a problem of locating a public facility for example, the graph represents a transportation network and individuals' characteristics are their locations on the network. In political models à la Downs or Hotelling, the graph (a line) represents an ideological space with candidates' platforms and individuals' preferences ordered from left to right. Other less familiar structures are given in the text. The graph defines a betweenness relation between characteristics as follows: A point is

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<sup>1</sup>Shapley (1962) focuses on the power and authority, i.e. the set of winning coalitions, and does not consider preferences.

between two others if it lies on a shortest path joining them.

I show that a family of preferences parameterised by characteristics on a graph guarantees stability<sup>2</sup> in any simple game if the following two conditions are met: (1) preferences are intermediate and (2) the graph linking characteristics is median. Preferences are said to be *intermediate* if the order associated to a characteristic that is between two other characteristics ranks any pair of alternatives as these two others whenever they agree on the ranking of the pair. Intermediate preferences thus imposes some interdependence structure between individual preferences but no restrictions on the shape of each one. The graph is *median* when any triple of vertices has a unique median, meaning there is a unique vertex that lies on shortest paths between each pair of vertices. Important examples of median graphs are trees, which include lines, and hypercubes, composed with all points with  $p$  coordinates equal to 1 or 0.

The result encompasses known results. For instance, it implies that the majority relation is acyclic when the sets of alternatives and characteristics are both ordered and preferences are single-crossing, as shown by Gans and Smart (1996)<sup>3</sup> because single-crossing preferences are intermediate on a line. This example is specific in several aspects. First, the majority relation is associated to a particular simple game. Second, a line describes a very specific structure of relations between characteristics. Third, the set of alternatives is ordered. Our stability result bears on any simple game and the alternative space, in particular its dimension or structure, does not matter, provided that preferences are intermediate.

The proof relies on the notion of convexity in a graph, called discrete convexity. A set is said to be convex if it contains all shortest paths between two points.<sup>4</sup> Hence intermediate preferences require that for any two alternatives  $x$  and  $y$  the subset of characteristics for which  $x$  is preferred to  $y$  is convex. The central property used in the proof is that the convex sets of a median graph satisfy the *Helly property*: for any family of pairwise intersecting convex sets, the sets in the family have a common intersection.<sup>5</sup> The fact that the space is median is essential for the result. For example, stability is not guaranteed for the majority relation when alternatives are multi-dimensional and preferences are based on the euclidean distance to their peaks (Grandmont 1978). Euclidean distances define intermediate preferences but a multi-dimensional space is not median.

The literature on median graphs is extremely vast. Median graphs have been introduced independently in different equivalent ways in several fields, as they are related to lattice theory and

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<sup>2</sup>In fact the result is stronger as the 'dominance' relation associated to blocking is acyclic.

<sup>3</sup>Single-crossing is satisfied under the Spence-Mirlees condition according to which the marginal rate of substitution for the decision -say public spending- is increasing in the characteristic -say income. See the text for definitions.

<sup>4</sup>A convex set is thus connected, but the reverse is false, except in a line or a tree.

<sup>5</sup>The Helly property for graphs resembles, but differs, from the Helly property for convex sets in a multi-dimensional real space.

important notions used in classification. Links between various definitions appear in Mulder and Schriver (1980) and there are several surveys e.g. Klavzar and Mulder (1999) and Van de Vel (1993). I only use the Helly property for convex sets.

Intermediate preferences impose restrictions between the preferences in a family. Single-peakedness is another type of restrictions, defined with respect to some ordering on the set of alternatives and the preferences 'peak' (the most preferred alternative). Black (1948) introduces single-peaked preferences on a line as those decreasing from the peak and shows the existence of a Condorcet winner (a stable alternative under majority). In Demange (1982), I define single-peaked preferences on a tree as those decreasing from the peak on any path of the tree and extend Black's result. More recently, Nehring and Puppe (2007) define single-peaked preferences on a median space, by requiring that any alternative between the peak and another alternative is preferred to that alternative. By their very definitions, single-peakedness and intermediate preferences differ: single-peakedness assumes some ordering structure on the set of alternatives, and intermediate preferences assume some structure on a characteristic space but not on the set of alternatives. This fact is somewhat blurred because the prominent model in voting assumes both the set of alternatives and characteristics to coincide and to be ordered. However even in that case, the two notions differ (see more discussion in Saporiti (2009)). Single-peaked preferences have been mostly used in the works on the majority rule and, more generally, on strategy-proof *functions*.

More references are given in the text. The plan is as follows. Section 2 presents the framework, introduces the basic definitions of simple games, median graph and intermediate preferences, and illustrates with some examples. Section 3 gives the main result which states that the dominance relation is cycle-free for a family of intermediate preferences on a median graph, applies it to the majority relation and compares intermediate preferences and single-peaked preferences. Section 4 discusses some extensions of the core allowing to study the stability of coalition formation.

## 2 The framework

A finite set  $N = \{1, \dots, i, \dots, n\}$  of individuals -a 'society'- has to choose an alternative in a feasible set  $X$ . Individuals have preferences over these alternatives. A coalition is a nonempty subset of  $N$ . We are interested in alternatives that are stable in the sense that no coalition blocks them. A coalition blocks an alternative if it has the power to impose another alternative that all of its members prefer. Hence blocking is defined jointly by the power of coalitions and individuals' preferences.

I consider simple games, in which the power of coalitions is all or nothing, represented by the set of winning coalitions. The interpretation is that a winning coalition can block any alternative and a non-winning coalition can block none. Let  $\mathcal{W}$  denote the set of winning coalitions. The following

natural assumptions are made:

(*monotonicity*): any coalition containing a winning coalition is also winning, that is,

$$(S \in \mathcal{W} \text{ and } S \subseteq T) \text{ imply } T \in \mathcal{W}$$

(*consistency*): the complement of any winning coalition is losing, that is,<sup>6</sup>

$$S \in \mathcal{W} \text{ implies } N - S \notin \mathcal{W}.$$

An individual's preference ordering over  $X$  is represented by a binary relation  $R$  that is complete, i.e., for any pair of alternatives  $x$  and  $y$ : ( $xRy$  or  $yRx$ ) and transitive, i.e. for any triple of alternatives  $x, y, z$ :  $xRy$  and  $yRz$  imply  $xRz$ . The associated strict ordering, denoted by  $P$ , is defined by:  $xPy$  if  $xRy$  but not  $yRx$ .

Let  $\mathbf{R} = (R^1, \dots, R^n)$  denote the profile of the  $n$  individuals' preference orderings over  $X$ .

A winning coalition *blocks* alternative  $y$  if there is an alternative  $x$  that all its members prefer to  $y$ . Blocking defines a dominance relation between alternatives as follows.  $x$  is said to *dominate*  $y$  if the subset of agents preferring  $x$  to  $y$  is a winning coalition. Therefore an alternative that is dominated is blocked and conversely. Formally, given  $\mathbf{R}$ , denoting the dominance relation<sup>7</sup> by  $dom_{\mathbf{R}}$ :

$$x dom_{\mathbf{R}} y \text{ if } \{i \in N \text{ such that } xP_i y\} \in \mathcal{W}. \quad (1)$$

The *core* is the set of alternatives that are not blocked, equivalently not dominated:

$$\mathcal{C}(\mathbf{R}) = \{x \in X, \text{ for no } y \in X : y dom_{\mathbf{R}} x\}. \quad (2)$$

The set  $X$  is assumed finite for simplification. When the dominance relation is acyclic, there are alternatives that are not dominated, hence the core is nonempty.<sup>8</sup>

Many common procedures in committees and assemblies are represented by simple games. For example, qualified majority requires a specified level of support to pass a candidate. It is defined by weights on individuals, i.e.  $w_i \geq 0$  for  $i$  in  $N$  such that  $\sum_{i \in N} w_i = 1$ , and a threshold  $q \geq 1/2$  (simple majority corresponds to equal weights and  $q$  equal to  $1/2$ ).  $S$  is winning if  $\sum_{i \in S} w_i > q$ . A stable alternative is a candidate for which no other candidate is preferred by more than the proportion  $q$

<sup>6</sup>Such a condition is also called 'proper'.

<sup>7</sup>The dominance relation and the core depend on the set of winning coalitions  $\mathcal{W}$ . As the set is fixed throughout the paper, it is unnecessary to index dom by  $\mathcal{W}$ .

<sup>8</sup> The proof is as follows. Let  $x$  be in  $X$ . If  $x$  is not in the core, there is  $x_1$  that dominates  $x$ :  $x_1 dom_{\mathbf{R}} x$ . If  $x_1$  is not in the core, there is  $x_2$  such that  $x_2 dom_{\mathbf{R}} x_1$  and so on. One builds a sequence  $x_i$  of elements not in the core such that  $x_{i+1} dom_{\mathbf{R}} x_i$ . The acyclicity of  $dom_{\mathbf{R}}$  implies that all the elements are distinct. Hence, because  $X$  is finite, the sequence stops and the final element is in the core. The proof extends to an infinite set under the compactness of  $X$  and continuity of the relations.

of the weighted voters. Increasing the level  $q$  decreases the set of winning coalitions, hence increases the chances for a stable alternative to exist.

A family of preferences  $\mathcal{O}$  is said to *guarantee acyclicity* if the dominance relation  $dom_{\mathbf{R}}$  is acyclic for any profile  $\mathbf{R} = (R^1, \dots, R^n)$  in  $\mathcal{O}^n$ .<sup>9</sup> If  $\mathcal{O}$  guarantees acyclicity, then it also guarantees the existence of stable outcomes whenever all individuals' preferences belong to the family. I consider a family  $\mathcal{O}$  where preferences are parameterized by a characteristic  $\theta$  running in a set  $\Theta$ :  $\mathcal{O} = \{R_\theta, \theta \in \Theta\}$ . So  $i$ 's preference ordering is determined by  $i$ 's characteristic  $\theta^i$ , and a preference profile by the  $n$ -tuple of characteristics  $\boldsymbol{\theta} = (\theta^1, \dots, \theta^n)$  in  $\Theta^N$ .

The main result in Section 3 displays families of preferences that guarantee acyclicity. The result assumes three ingredients: (1) the set of characteristics is endowed with betweenness relationships defined by a graph (2) the betweenness relations between characteristics carry over to preferences in  $\mathcal{O}$  in a natural sense as defined by Kemeny (1959) (3) the graph on  $\Theta$  satisfies a property on triples, making  $\Theta$  a median space. Here are the formal definitions.

## 2.1 Betweenness relationships

The notion of intermediate preferences assumes there are betweenness relations on the set of characteristics. A graph structure on  $\Theta$ , meant to describe which characteristics are 'close', define betweenness relations as follows.

**Betweenness relation on characteristics** A (non-directed) graph on  $\Theta$  is specified by a set  $E$  of unordered pairs of distinct elements of  $\Theta$  called *edges*:  $(\theta, \theta')$  in  $E$  means that  $\theta$  and  $\theta'$  are close together. Denote  $\mathcal{G} = (\Theta, E)$  the graph. Following standard terminology, elements of  $\Theta$  are called *vertices* and two elements linked by an edge *neighbors*. A *path* between  $\theta$  and  $\theta'$  is a sequence of elements  $\theta_1, \dots, \theta_j, \dots, \theta_p$  in  $\Theta$  for which  $\theta = \theta_1$ ,  $\theta' = \theta_p$  and the pairs  $(\theta_j, \theta_{j+1})$  are edges for  $j \in \{1, \dots, p-1\}$ . The graph is assumed to be *connected*, meaning there is a path between any two points in  $\Theta$ . A *line* is a graph formed with a unique path; a *tree* is a graph for which there is a unique path between any two points, equivalently the graph has no cycle.

The shortest paths in graph  $\mathcal{G}$  define a betweenness relation between characteristics as follows. A shortest path linking two vertices is one with the minimal number of edges in a path joining the two vertices. This minimal number is defined as the *distance* between the two points, denoted by  $d(\theta, \theta')$  for  $\theta$  and  $\theta'$ . A vertex  $\theta''$  is *between*  $\theta$  and  $\theta''$  if it belongs to a shortest path joining  $\theta$  and  $\theta''$  or, equivalently, if the distance between  $\theta$  and  $\theta''$  is the sum of the distances between  $\theta$  and  $\theta'$

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<sup>9</sup>Other types of restrictions can be contemplated. For example, Nakamura (1975) shows the core is nonempty for any preference profile if the cardinality of the set of alternatives is less than some number, now called the Nakamura number.



and between  $\theta'$  and  $\theta''$ . The *interval* between  $\theta$  and  $\theta''$ , denoted by  $[\theta, \theta'']$ , is the set of all vertices between  $\theta$  and  $\theta''$ :

$$[\theta, \theta''] = \{\theta', d(\theta, \theta'') = d(\theta, \theta') + d(\theta', \theta'')\}. \quad (3)$$

A subset of  $\Theta$  is said to be *convex* if it includes the interval between any pair of elements, equivalently it includes *all* shortest paths between any pair. A subset is *connected* if it includes a path between any of two elements. Hence a convex set is connected. The converse is not necessarily true, except in a tree since there is a unique path between two points.

**Intermediate preferences** A preference relation is between two others if it ranks two alternatives as these two others whenever they agree on the ranking on the two alternatives (Kemeny 1959).<sup>10</sup> Preferences in  $\mathcal{O}$  are intermediate when the betweenness relationships between characteristics carry over to preferences:

**Definition 1** Let  $\Theta$  be endowed with a betweenness relationship described by a graph  $\mathcal{G}$ . The family  $\mathcal{O} = \{R_\theta, \theta \in \Theta\}$  satisfies the intermediateness property if for any  $\theta$  and  $\theta''$  in  $\Theta$  and  $\theta'$  between  $\theta$  and  $\theta''$ ,  $P_{\theta'}$  is between  $P_\theta$  and  $P_{\theta''}$  that is: for all  $x$  and  $y$  in  $X$  ( $xP_\theta y$  and  $xP_{\theta''} y$ ) imply  $xP_{\theta'} y$

The intermediateness property requires that for every pair of elements  $x$  and  $y$  in  $X$  the set  $\{\theta \in \Theta \text{ such that } xP_\theta y\}$  is a convex set. Observe that the property imposes *relations* between the preferences in a family, but does not impose restrictions on the shape of each one.

When  $X$  and  $\Theta$  are uni-dimensional, intermediate preferences appear under various related forms, such as the Spence-Mirrlees condition (monotony of marginal rates of substitution) increasing differences and ordinal single-crossing property (see Gans and Smart 1996). Grandmont (1978) introduces the notion of families of intermediate preferences indexed by a convex set in a multidimensional space. In these two cases the characteristics and the alternatives are in the same space. In Demange (1994) I define intermediate preferences for characteristics on a tree and alternatives differing from characteristics. The following examples illustrate intermediate and single crossing preferences.

**Example 1** *d-preferences*. Consider the problem of locating a public facility on a transportation network (see for example Hansen and Thisse 1981 and the references therein). Individuals are located on the network and prefer the facility to be located as close as possible to their location: An individual located at  $\theta$  prefers the public facility to be located at  $x$  rather than  $y$  if  $d(x, \theta) < d(y, \theta)$ . Call these preferences *d-preferences*. Here, the graph is the transportation network, the alternatives are the vertices and individuals' characteristics are their locations on the network: It is easy to check that the *d-preferences* are intermediate when  $\Theta$  is a line or, more generally, is a tree.

<sup>10</sup>One may also require similar conditions for the weak relations  $R_\theta$ . Theorem 1 only uses the conditions on the strict preferences.

**Example 2** *Increasing differences and Spence-Mirrlees condition.* Consider the problem of choosing the quality of a public facility. An alternative is the quality  $q$  for a public good. A characteristic belongs to  $\Theta = [\underline{\theta}, \bar{\theta}]$ . The characteristic  $\theta$  determines the valuation for quality  $q$ , represented by  $u(q, \theta)$  for some function  $u$  ( $u$  is not necessarily monotone in  $q$  if  $u$  is an indirect utility function incorporating the financing of the cost to provide quality  $q$ ).  $u$  is said to satisfy the *increasing differences* property if the incremental benefit of increasing quality is increasing in  $\theta$ : for  $q > q'$ , the function  $\theta \rightarrow u(q, \theta) - u(q', \theta)$  increases with  $\theta$ . For a twice differentiable function  $u$ , the Spence-Mirrlees condition,  $\frac{\partial^2 u}{\partial \theta \partial q} > 0$ , implies increasing differences because for  $q > q'$  the derivative of  $\theta \rightarrow u(q, \theta) - u(q', \theta)$  is positive.

Assume  $u$  satisfies increasing differences. Let us prove that the family  $\{u(\theta, \cdot), \theta \in \Theta\}$  satisfies the intermediate preferences property. It suffices to consider any two alternatives  $q$  and  $q'$ ,  $q > q'$ , and show that the set  $I$  of  $\theta$  for which  $q$  is preferred to  $q'$  is an interval. We have  $I = \{\theta \in [\underline{\theta}, \bar{\theta}] \text{ such that } u(q, \theta) - u(q', \theta) > 0\}$ . Increasing differences ensure that  $I$  is either empty, the whole interval, or there is a cutoff value  $\theta^*$  for which the  $\theta$  preferring  $q$  to  $q'$  are those larger than  $\theta^*$ , that is  $I = ]\theta^*, \bar{\theta}]$ . This proves the intermediateness property.

The result extends to a richer two-dimensional set of alternatives. For example, consider the problem of choosing the quality  $q$  of a public facility as well as its access price  $p$ . An alternative is thus a pair  $x = (q, p)$ . Let individual preferences be separable represented by  $u(q, \theta) - p$ . One easily checks that preferences are intermediate if  $u$  satisfies increasing differences.<sup>11</sup>

## 2.2 Median graph, Median space

A graph is median if for any triple of vertices a unique vertex belongs to a shortest path between any two points of the vertex.<sup>12</sup>

**Definition 2** *A graph  $\mathcal{G} = (\Theta, E)$  is said to be median if for every triple  $(\alpha, \beta, \gamma)$  made of elements in  $\Theta$ , a unique vertex  $\mu$  is between  $(\alpha, \beta)$ ,  $(\beta, \gamma)$  and  $(\alpha, \gamma)$ :  $\{\mu\} = [\alpha, \beta] \cap [\beta, \gamma] \cap [\alpha, \gamma]$ . The set  $\Theta$  endowed with the median graph structure  $\mathcal{G}$  is called median space.*

Familiar examples of median graphs are lines and trees. In a line, the median of the triple is always a member of the triple. In a tree, the median of the triple is not always a member of the triple, as

<sup>11</sup>Now the set  $I$  of  $\theta$  for which  $(q, p)$  is preferred to  $(q', p')$  is  $I = \{\theta \in [\underline{\theta}, \bar{\theta}] \text{ such that } u(q, \theta) - u(q', \theta) > p - p'\}$ . If  $q > q'$ , the same proof works as if there is no access price. If  $q = q'$ ,  $I$  is either empty or the whole interval  $[\underline{\theta}, \bar{\theta}]$ .

<sup>12</sup>There are different equivalent definitions of median graphs ( see the survey by Klavzar and Mulder 1999). One definition is in terms of the distance in the graph. A median is characterized by

$$d(\alpha, \beta) = d(\alpha, \mu) + d(\mu, \beta), \quad d(\beta, \gamma) = d(\beta, \mu) + d(\mu, \gamma) \quad \text{and} \quad d(\gamma, \alpha) = d(\gamma, \mu) + d(\mu, \alpha).$$

Hence the median minimizes the sum of the distance to the three elements in the triple, i.e. is a metric median. An equivalent definition thus requires the uniqueness of a metric median for each triple.

illustrated by  $(\alpha, \beta, \gamma)$  in the tree of Figure 1 on the left. A graph that contains a cycle of length different from 4 is not median. This is obvious for a cycle of length three. For a cycle of length five, as depicted in Figure 1 on the right, let us show that the triple  $\{\alpha, \beta, \delta\}$  has no median. The interval between  $\alpha$  and  $\beta$  is  $\{\alpha, \beta\}$ , the interval between  $\beta$  and  $\delta$  is  $\{\beta, \gamma, \delta\}$  and the interval between  $\alpha$  and  $\delta$  is  $\{\alpha, \varepsilon, \delta\}$ . Since the three intervals do not intersect, the triple has no median. The argument extends to cycles of length larger than 5.<sup>13</sup>

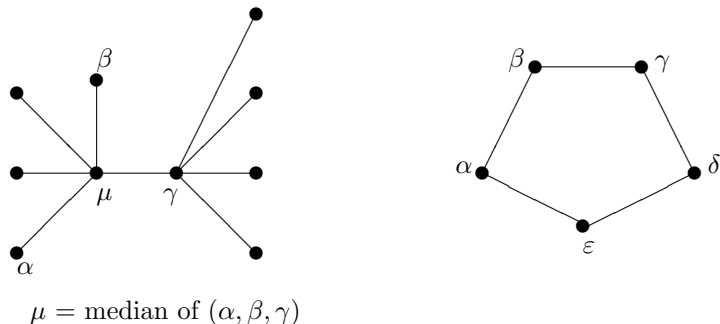


Figure 1: A tree is a median graph but not a cycle of length 5

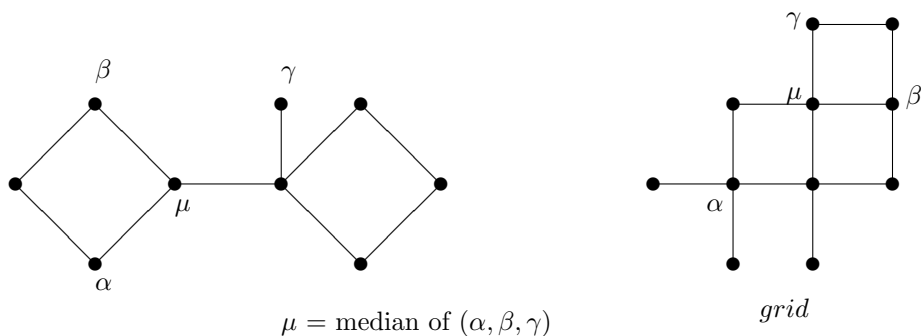


Figure 2: Median graphs with cycles of length four

The two graphs depicted in Figure 2 are median and include cycles of length four. In the grid on the right, there are three shortest paths between  $\alpha$  and  $\beta$ . More generally, the interval between two points contains all the points with coordinates between the minimum and maximum of the coordinates of the two points.

**Hypercubes** A  $p$ -cube, also called hypercube, describes points with coordinates equal to 0 or 1: the  $p$ -cube is  $\Theta = \{0, 1\}^p$ . Figure 2 on the left represents the 3-cube. One interpretation of an hypercube is in terms of *properties*. Each of the dimensions of the hypercube is associated to a property, say home ownership, marital status, worker status and so on. Each property can be

<sup>13</sup>A triple without median is one constituted of three points such that the interval for each pair is reduced to a single arc and the union of the arcs covers the cycle.

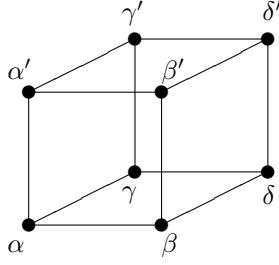


Figure 3: Median graph: a 3-cube

satisfied or not, hence is represented by 1 or 0. A family of preferences is parameterized by  $\Theta$  when the set of properties satisfied by an individual characterizes his/her preferences.

An edge of the graph links two vertices whose coordinates are all equal but one. It follows that the interval between two vertices  $\theta$  and  $\theta''$  is composed of all the vertices  $\theta'$  such that, for each  $k = 1, \dots, p$ , the  $k$ -th coordinate of  $\theta'$  assumes the same value as that of  $\theta$  and  $\theta''$  when there are identical:  $\theta'_k = 1$  if  $\theta_k = \theta''_k = 1$  and  $\theta'_k = 0$  if  $\theta_k = \theta''_k = 0$ . The median of three vertices is the unique point for which each coordinate is equal to the majority value in the triple: the  $k$ -th coordinate of the median is equal to 1 if at least two values in the triple assume value 1, and is equal to 0 otherwise. For example, in the 3-cube in Figure 3, the median of the triple  $\alpha' = (0, 0, 1)$ ,  $\delta' = (1, 1, 1)$  and  $\beta = (1, 0, 0)$  is  $\beta' = (1, 0, 1)$ . The intervals are  $I(\alpha', \delta') = \{\alpha', \beta', \gamma', \delta'\}$ ,  $I(\alpha', \beta) = \{\alpha', \beta, \alpha, \beta'\}$ , and  $I(\delta', \beta) = \{\delta', \beta, \delta, \beta'\}$ .

A convex set  $C$  of an hypercube is characterized by two disjoint (possibly empty) subsets  $K_0$  and  $K_1$  of  $\{1, \dots, p\}$  on which the coordinates are fixed, equal to 0 or 1:  $C = \{\theta, \text{ such that for } k \in K_0 : \theta_k = 0 \text{ and for } k \in K_1 : \theta_k = 1\}$ . It follows that preferences indexed on an hypercube are intermediate if, for each pair of alternatives  $x$  and  $y$ , whether  $xP_\theta y$  holds is characterised by two disjoint subsets  $K_0$  and  $K_1$ . Interpreting each of the  $p$  dimensions as a property,  $x$  is preferred to  $y$  by the characteristics  $\theta$  that satisfy the properties in  $K_1$  and do not satisfy those in  $K_0$ . The set of characteristics for which  $yP_\theta x$  must be also convex, hence characterized by two subsets  $K'_0$  and  $K'_1$ . Consistency imposes some requirements, for example that simultaneously  $K'_0$  is identical to  $K_0$  and  $K'_1$  to  $K_1$ . The more indifferences preferences involve, the larger the index subsets are and the less severe these requirements are. Instead, if the orderings are strict between  $x$  and  $y$ , then  $\{\theta \in \Theta \text{ such that } xP_\theta y\}$  and  $\{\theta \in \Theta \text{ such that } yP_\theta x\}$  form a partition of the cube into two convex sets. One easily checks that  $K_0$  must be empty and  $K_1$  a singleton or the reverse, i.e.  $K_0$  a singleton and  $K_1$  empty. This means that a single property determines the preferences of  $x$  over  $y$ .

Let us now consider the  $d$ -preferences (introduced in Exemple 1). Contrary to the case where the graph is a tree,  $d$ -preferences on hypercubes are not intermediate. For example, in the 3-cube in

Figure 2, both  $\alpha$  and  $\delta$  are closer to  $\beta$  than to  $\gamma'$  but  $\gamma$ , which is between  $\alpha$  and  $\delta$ , is closer to  $\gamma'$  than to  $\beta$ . More generally, let a median graph contain a cycle and  $\Theta$  has at least 5 elements. The cycle is necessarily of order four, so one can find five elements with the same structure as  $(\alpha, \beta, \gamma, \delta, \gamma')$  in the 3-cube. It follows that  $d$ -preferences are not intermediate.

### 3 The main result

The combination of a median graph structure for the set of characteristics and the property of intermediate preferences allows for a 'positive' result in the sense that the dominance relation of any profile is acyclic, hence the core is nonempty. The crucial point in the proof is that the convex sets in a median graph satisfy the following property called *Helly property*: Given a median graph, let  $\mathcal{F}$  be an arbitrary family of pairwise-intersecting convex sets. Then all sets in  $\mathcal{F}$  have a common intersection.

Applying the condition to a family with three elements, the Helly property excludes the existence of Condorcet triples of convex sets. Define a *Condorcet triple* as three coalitions that intersect each other and whose overall intersection is empty:

$$S_i, i = 1, 2, 3, \text{ with } S_i \cap S_j \neq \emptyset \text{ and } S_1 \cap S_2 \cap S_3 = \emptyset. \quad (4)$$

The following proof first shows that there are no Condorcet triples of convex sets in a median graph and then uses an induction argument to prove the Helly property.<sup>14</sup> Let  $\mathcal{F}$  be made of a Condorcet triple of convex sets. Choose for each pair a point in its intersection,  $\theta^{12}$  in  $S_1 \cap S_2$ ,  $\theta^{23}$  in  $S_2 \cap S_3$ ,  $\theta^{31}$  in  $S_3 \cap S_1$ . The median of the three points is between  $\theta^{12}$  and  $\theta^{23}$  hence belongs to  $S_2$  by convexity; similarly considering the other pairs, the median belongs to  $S_3$  and  $S_1$ . This proves that the overall intersection of the three sets is nonempty. The proof proceeds by induction on the number of elements in  $\mathcal{F}$ , using that the intersection of two convex sets is also a convex set.

**Theorem 1** *Given a set of agents  $N$ , consider a simple game on  $N$  defined by a collection of winning coalitions. Let  $\mathcal{O} = \{R_\theta, \theta \in \Theta\}$  be a family of preferences on a set of alternatives  $X$ .*

*Assume  $\Theta$  is a median space and  $\mathcal{O}$  a family of intermediate preferences. Then for each  $n$ -tuple of characteristics  $\theta$  there is a characteristic  $\bar{\theta}$ , not necessarily in the profile, such that the dominance relation is stronger than the order  $P_{\bar{\theta}}$ :*

$$x \text{ dom}_{\theta} y \text{ implies } x P_{\bar{\theta}} y. \quad (5)$$

*Hence the family  $\mathcal{O}$  guarantees that the dominance relation is acyclic and the core is nonempty.*

<sup>14</sup>The Helly property is a kind of folk theorem. Mulder and Schriber (1980) mentions the result by applying a more general result on hypergraphs. A direct proof is much simpler.

The proof is given at the end of this section. I first draw some consequences for the majority relation.

**Majority relation and majority rule** To simplify, let us assume  $n$  to be odd and preferences strict. Then two distinct alternatives are always comparable since either a majority prefers  $x$  to  $y$  or a majority prefers  $y$  to  $x$ . Theorem 1 implies that the majority relation coincides with the order  $P_{\bar{\theta}}$  hence it is transitive. Two consequences can be drawn. The majority relation aggregates well preferences and the majority rule is strategy-proof, as stated in Demange (2012) and the following corollary. Furthermore,  $\bar{\theta}$  can be qualified as a *representative characteristic* of the profile  $\theta$ .

**Corollary 1** *Let  $\Theta$  be a median space and let  $\mathcal{O} = \{R_{\theta}, \theta \in \Theta\}$  be a family of intermediate strict preferences on a set of alternatives  $X$ . Let  $n$  be odd. The majority rule, which assigns to each  $(\theta^1, \dots, \theta^n)$  its Condorcet winner, i.e. the top alternative of  $P_{\bar{\theta}}$ , is strategy-proof.*

Theorem 1 readily applies to variations of the majority relation obtained by adding 'phantom' voters in a similar way as in Moulin (1980). Fix  $(\tau^1, \dots, \tau^p)$ , the profile of  $p$  phantom voters. Given a characteristics' profile  $(\theta^1, \dots, \theta^n)$ , consider the majority relation for the profile  $(\tau^1, \dots, \tau^p, \theta^1, \dots, \theta^n)$ . Corollary 1 applies hence the aggregation and strategy-proof results hold for these modified majority relations.

In Demange (2012), I prove the result in a different way by relying on the *local* Condorcet characteristic of the profile  $\theta$  for the graph distance. By definition, a local Condorcet characteristic  $\bar{\theta}$  beats all its neighbors for the  $d$ -preferences, meaning that, for each neighbor, a majority of voters is closer to  $\bar{\theta}$  than to that neighbor. Bandelt and Barthélemy (1984) prove the existence and uniqueness of a local Condorcet characteristic when the graph is median.

In a tree,  $\bar{\theta}$  is also a global Condorcet characteristic, meaning that, for each other characteristic, a majority of voters is closer to  $\bar{\theta}$  than to that characteristic. This follows from Theorem 1: since the  $d$ -preferences are intermediate, a global Condorcet characteristic exists, necessarily equal to the local one. In an hypercube,  $\bar{\theta}$  may not be a global Condorcet characteristic, since  $d$ -preferences are not intermediate. To illustrate, consider  $\theta = (\alpha', \beta, \beta, \delta', \gamma)$  in the 3-cube of Figure 2. Let us prove that  $\beta$  is the local Condorcet characteristic but is not a global one. Since  $\beta$  is the location of agents 2 and 3, let us check that, for each neighbor, at least one agent among 1, 4 and 5 prefers  $\beta$  to that neighbor. The neighbors of  $\beta$  are  $\alpha$ ,  $\delta$  and  $\beta'$ .  $\beta$  is closer to  $\alpha$  for agent 4 (since  $d(\delta', \beta) = 2$  and  $d(\delta', \alpha) = 3$ ), closer to  $\delta$  for agent 1 (since  $d(\alpha', \beta) = 2$  and  $d(\alpha', \delta) = 3$ ) and closer to  $\beta'$  for agent 5 (since  $d(\gamma, \beta) = 2$  and  $d(\gamma, \beta') = 3$ ). This proves that  $\beta$  is the local Condorcet characteristic. But  $\gamma'$  beats  $\beta$ : Agents 1,4 and 5 all prefer  $\gamma'$  to  $\beta$  because  $\alpha', \delta'$  and  $\gamma$  are at a distance of 1 to  $\gamma'$  and at a distance of 2 to  $\beta$ . Hence  $\beta$  is not a global Condorcet characteristic. This example also shows that the assumption of intermediate preferences in Theorem 1 is crucial.

**Comparison between intermediate preferences and single-peaked preferences** It is useful to clarify the differences between single-peaked and intermediate preferences and the types of associated results. In that purpose, let the set of alternatives  $X$  be endowed with a tree structure. A strict preference ordering on  $X$  is single-peaked if the ordering decreases along any path starting at the peak. A Condorcet winner always exists for single-peaked preferences on a tree (Demange 1982). The majority relation however may have cycles as illustrated in Figure 1 on the left. Let  $X = \{\alpha, \beta, \gamma, \mu\}$  and three individuals have preferences  $\mu P^1 \alpha P^1 \beta P^1 \gamma$ ,  $\mu P^2 \beta P^2 \gamma P^2 \alpha$  and  $\mu P^3 \gamma P^3 \alpha P^3 \beta$ . These preferences are single-peaked and  $\mu$  is obviously the Condorcet winner. But the majority relation has a cycle on  $(\alpha, \beta, \gamma)$ . Hence, the set of single-peaked preferences on a tree guarantees a Condorcet winner but does not guarantee the transitivity of the majority relation.

Consider now a family of intermediate preferences for which the characteristic represents the peak. The above example is excluded since a single preference relation has  $\mu$  as the peak. From Theorem 1 the majority relation is transitive.

Although this example may suggest that the framework with intermediate preferences is more restrictive than the one with single-peaked preferences, this is not true for at least two reasons. First, even in uni-dimensional spaces, preferences satisfy the intermediate property but not the single-peaked one in several applications to political economy and models of redistribution (see Saporiti (2009) for various examples and references and a thorough discussion on the differences between the two properties). The basic reason is that intermediate preferences do not impose restrictions on the shape of individual preferences. Second, intermediate preferences are not necessarily parameterised by their peaks or by any alternatives. They allow for any set of alternatives different from the set of characteristics. This fact has been somewhat blurred because the prominent model in applications assume a uni-dimensional set of alternatives and characteristics specified by their peaks.

A different viewpoint asks the following question: Given observed preferences, does it exist some set  $\Theta$  endowed with a betweenness structure such that these preferences are intermediate? or some ordering of the alternatives such that these preferences are single-peaked? Computational social choice addresses these questions, with a focus on the design and complexity of algorithms for recognizing whether a family of preferences satisfies a particular property. Elkind, Lackner, and Peters (2022) survey the results on the single-crossing and single-peaked properties.

**Proof of Theorem 1** For a pair  $w, x$  of alternatives, denote  $\Theta_{w>x}$  the subset of characteristics for which  $w$  is preferred to  $x$ :  $\Theta_{w>x} = \{\theta \in \Theta \text{ s.t. } w P_\theta x\}$ . Let us define the family  $\mathcal{F}$  of subsets  $\Theta_{w>x}$  for which  $w \text{ dom}_\theta x$ . Let us first prove that the assumptions imply that the intersection of all the sets in  $\mathcal{F}$  is nonempty. Because of intermediate preferences,  $\Theta_{w>x}$  is a convex set of the median graph  $\mathcal{G}$ . From Lemma 1, each pair of sets in  $\mathcal{F}$  has a nonempty intersection. Hence, by the Helly

property, their overall intersection is nonempty.

Let  $\bar{\theta}$  belong to the overall intersection of subsets in  $\mathcal{F}$ . By definition, for each pair such that  $w \text{dom}_{\theta} x$ ,  $\bar{\theta}$  belongs to the set  $\Theta_{w>x} = \{\theta \in \Theta \text{ s.t. } wP_{\theta}x\}$ . Hence  $w \text{dom}_{\theta} x$  implies  $wP_{\bar{\theta}}x$ , which proves (5). A cycle of  $\text{dom}_{\theta}$  would be a cycle of  $P_{\bar{\theta}}$ , which is impossible. ■

**Lemma 1** *Consider a simple game. Let two pairs  $w, x$  and  $y, z$  of elements of  $X$  such that  $w \text{dom}_{\theta} x$  and  $y \text{dom}_{\theta} z$ . Then there is  $i$  in  $N$  for which  $wP_{\theta_i}x$  and  $yP_{\theta_i}z$ , hence  $\Theta_{w>x} \cap \Theta_{y>z} \neq \emptyset$ .*

**Proof.**  $w \text{dom}_{\theta} x$  if the set  $\{i, wP_{\theta_i}x\}$  is a winning coalition. Hence  $w \text{dom}_{\theta} x$  and  $y \text{dom}_{\theta} z$  imply that both coalitions  $\{i, wP_{\theta_i}x\}$  and  $\{i, yP_{\theta_i}z\}$  are in  $\mathcal{W}$ . By consistency of the game, the two coalitions intersect, and for  $i$  in their intersection,  $wP_{\theta_i}x$  and  $yP_{\theta_i}z$ . This proves the lemma. ■

## 4 Extending the approach

In many political and economic situations a group of individuals coordinate their actions while facing the threats that potential coalitions may generate by blocking or seceding. A main issue is whether these threats hamper the group to coordinate efficiently. Stability and the core notion as investigated in the previous section in the voting context extend to these situations. The approach is not limited to games where the whole society must coordinate. Indeed, in some problems, coordination brings benefits but, at the same time, it is efficient for the whole group to split into smaller self-sufficient groups. It is then natural to consider partitions of the society and to investigate which ones are stable to blocking. This insight was pioneered by Aumann and Dreze (1974), who introduced *stable coalition structures*. The notion of the core applied to partitions thus produces predictions on the endogenous formation of subgroups. Before presenting such approach, I make two observations.

First, the notion of stable coalition structures assumes the absence of externalities across coalitions, meaning that the actions that are feasible for a coalition if it forms as well as the payoffs to its members are not affected by the organization of agents outside the coalition. Without externalities, a coalition can predict the payoffs its members can achieve by splitting from the rest of the society. Externalities across groups, negative or positive, have an important impact in some contexts, those concerning environmental agreements for example. Their analysis however raises a number of conceptual and technical difficulties and there are no general results.<sup>15</sup>

Second, alternative approaches to blocking have been proposed to address the stability to coalitions deviations. Blocking can be perceived as myopic, as a coalition does not account for further

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<sup>15</sup>One difficulty is that the blocking behavior depends of which prediction a coalition makes on how the remaining agents organize themselves. See e.g. Chander and Tulkens (1997), Carraro and Siniscalco (1997), Bloch (1996), Yi (1997), Ray and Vohra (1999), Currarini (2007).



blocking. These critics have led to consider different types of deviations, for example Chwe (1995) introduce farsighted coalitions and Bernheim, Peleg, and Whinston (2014) introduce credible and self-enforcing deviations. Considering coalition moves that differ from blocking leads to different equilibria. The literature is too vast to give a faithful account of these approaches. An argument to justify considering alternatives in the core is similar to the one that justifies considering Nash equilibria: if a stable alternative is outstanding, it is common knowledge that no coalition benefits from an unilateral move.

**Modeling a (cooperative) game** As in the previous section, the analysis is based on the payoffs that coalitions can achieve on their own. Two ingredients are fundamental to describe these payoffs: the power of coalitions and individuals' preferences.

The power of a coalition is specified by the decisions that its members can make independently of the organization of outsiders. Let  $X(S)$  denote the set of feasible decisions for coalition  $S$ . The description of the feasible sets depends on the problem at hand. In a simple game,  $X(S)$  is empty for a non-winning coalition and the whole set  $X$  for a winning one. In general, a feasible set incorporates the technological, institutional or informational constraints that a coalition faces if it forms and restrict the way decisions it can make. For example, a coalition of citizens can choose personalized taxes when perfectly informed about all citizens' characteristics but must rely on unit taxes or consumption taxes when less information is available.

Individuals' preferences bear on the decisions, represented by a utility function,  $u_i$  for  $i$ . If  $S$  forms and chooses  $a$ ,  $i$  member of  $S$  obtains utility level  $u_i(a)$ .

**Super-additive games** Super-additive games represent situations where each coalition is always at least as efficient as any of its partitions. Formally, given two disjoint coalitions,  $S$  and  $T$ , for any  $a$  feasible for  $S$  and  $b$  feasible for  $T$ , there is  $c$  feasible for the union  $S \cup T$  under which all members are at least as well off:

$$\left\{ \begin{array}{l} \text{for any } a \text{ in } X(S) \text{ and } b \text{ in } X(T), \text{ there is } c \text{ in } X(S \cup T) \text{ such that} \\ \text{for each } i \in S : u_i(c) \geq u_i(a) \text{ and for each } i \in T : u_i(c) \geq u_i(b). \end{array} \right. \quad (6)$$

These conditions state that the payoffs  $S$  and  $T$  can achieve by acting separately can be achieved by their union. A game in which (6) holds for each disjoint pair of coalitions is called *super-additive*. A simple game is super-additive since at least one of two disjoint coalitions has no power, hence its alternative set is empty. Under strict inequalities in (6), *efficiency* requires the whole set  $N$  to form and make a decision that applies to each member of the society.

Since a coalition  $S$  can achieve on its own the vector of utility levels  $(u_i(a))_{i \in S}$  for  $a$  in  $X(S)$ , the blocking and core notions follow: A decision  $a \in X(N)$  is blocked by coalition  $S$  if there is a

decision  $b$  feasible for  $S$  that makes every member of  $S$  better off:

$$\text{for some } b \text{ in } X(S) \ u_i(b) > u_i(a) \text{ each } i \text{ in } S.$$

$a$  is stable (or in the core) if it is not blocked. We already know that the core of a super-additive game can be empty. Working on the set of feasible payoffs for each coalition,<sup>16</sup> Bondareva (1963), Shapley (1967) and Scarf (1967) provide conditions under which the game has a non-empty core. Roughly speaking, these conditions require some form of super-additivity for families of coalitions that are not disjoint but form with some probability (called balanced family of coalitions). These conditions apply to a *given* game, that is to fixed feasible sets and preferences. In the spirit of the previous section, it is interesting to investigate stability for a range of games by varying preferences or the feasible sets. I present results along this line in games that are not necessarily super-additive.

**Stable coalition structures** In games that are not super-additive, the whole society may benefit from splitting into a partition of smaller self-sufficient groups. A coalition structure describes such an organization.

A *coalition structure* of  $N$  is a family  $(a_\ell, S_\ell)_{\ell=1, \dots, L}$  where  $(S_\ell)_{\ell=1, \dots, L}$  is a partition of  $N$  and  $a_\ell$  is feasible for  $S_\ell$ ,  $\ell = 1, \dots, L$ .

The interpretation is that each group  $S_\ell$  forms and adopts decision  $a_\ell$ , which is feasible for it independently of the remaining agents. Letting  $\ell(i)$  denote the unique coalition to which  $i$  belongs,  $i$ 's achieves the utility level equal to  $u_i(a_{\ell(i)})$ . The blocking notion follows:

The structure  $(a_\ell, S_\ell)_{\ell=1, \dots, L}$  is *blocked* by coalition  $S$  if there is a decision  $b$  feasible for  $S$  that makes every member of  $S$  better off:

$$\text{for some } b \text{ in } X(S) \ u_i(b) > u_i(a_{\ell(i)}) \text{ each } i \text{ in } S.$$

The structure is *stable* if it is blocked by no coalition. If the game is strictly super-additive (i.e. the inequalities (6) hold strictly)  $L$  is necessarily equal to 1 and we fall back on the previous notion. Otherwise, the number of subgroups  $L$  is not fixed a priori and depends on the problem. Observe that, even though there are no externalities across coalitions, the full organization of the society determines the benefit from blocking.<sup>17</sup>

The splitting of the society as captured by coalition structures may be due to exogenous restrictions to coalition formation or stem from preferences diversity. I examine these factors in turn.

<sup>16</sup>The set of payoffs for  $S$  is defined by  $V(S) = \{(x_i)_{i \in S} \text{ such that } x_i \leq u_i(a) \text{ for } a \in X(S)\}$ .

<sup>17</sup>In particular, stability is related to the notions of free individual mobility and free entry (i.e. free formation of coalitions). See Chapters 7 and 8 in Demange and Wooders (2005).

**Essential coalitions** In some situations, some coalitions play an essential role and stability depends only on them. Non-essential coalitions are not permitted to form or have a blocking power dominated by the essential ones (as is the case for a non-winning coalition in a simple game). The more limited the set of essential coalitions, the more likely there are stable coalition structures. One question is to which extent the set of essential coalitions must be limited to *guarantee* stability, i.e. the existence of stable structures *whatever* the preferences.

Assignment games or matching markets are well known examples where the set of essential coalitions is restricted. The society is partitioned into two sides, say men and women, buyers and sellers, colleges and students and only singletons and couples (pairs composed of one member of each side) are essential. A coalition structure (called matching or assignment) describes which couples are formed and possibly the transfers of money within each couple. When there are no transfers, a matching is stable if (a) no new couple can form in which both the man and the woman prefer each other to their standing partner (no couple blocks) (b) no person would prefer to stay single (no individual blocks). Gale and Shapley (1962) define the deferred-acceptance algorithm and show its convergence to a stable matching, hence proving its existence. This algorithm serves as a basis for a huge amount of assignment procedures used in practice, especially in the academic world. Similarly, when agents in a couple can exchange money as considered by Shapley and Shubik (1971), stable outcomes exist. These positive results have a common root: the family of singletons and couples in a two-sided setting guarantees stability. The two-sided aspect is crucial: in the so-called roommate problem where there are no sides, a stable outcome may not exist: a family of singletons and pairs does not guarantee stability.

Restrictions in communication or exchanges also induce restricted sets of essential formation, as introduced by Myerson (1977). Let individuals be connected through a (social) network. The links in the network represent the possibility of exchanges or communication among the individuals, so that only the connected coalitions can form. When the network is a line, the game is called a consecutive game and the essential coalitions are interval; stability is guaranteed (Greenberg and Weber 1993). When the network is a tree, stability is guaranteed and furthermore stable outcomes, called hierarchical,<sup>18</sup> are easy to compute and to interpret as I show in Demange (2004).<sup>19</sup> When the network contains a cycle, stability is no longer guaranteed. Such a result may provide a rationale for organizations such as tree-hierarchy structures. For a more recent treatment of organizations using coalition structures, see Morelli and Park (2016).

More generally, which families of essential coalitions guarantee stability? Let  $\mathcal{C}$  be the set of

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<sup>18</sup>Recent works study hierarchical outcomes and related concepts as an allocation mechanism, see e.g. Herings, van der Laan and Talman (2008).

<sup>19</sup>The problem of 'sharing' a river studied in Ambec and Sprumont (2002) shares some features of a hierarchy.

essential coalitions. It is easy to see that a set  $\mathcal{C}$  of essential coalitions that contains a Condorcet triple (defined by (4)) does not guarantee stability. For example, letting  $S_i, i = 1, 2, 3$  be such a triple, consider the transferable utility game<sup>20</sup> defined by the values:  $v(N) = 1, v(S) = b$  if  $S$  is a strict subset of  $N$  that contains at least one of the  $S_i$  and  $v(S) = 0$  otherwise. For  $b$  smaller than 1 the game is super-additive but, for  $b$  strictly larger than  $2/3$ , any feasible payoff is blocked by one of the  $S_i$ . But the absence of Condorcet triples in  $\mathcal{C}$  is only necessary for guaranteeing stability. Mathematical characterizations for the guarantee are provided by Kaneko and Wooders (1982) in terms of the extreme solutions to a system of linear equations (see also Le Breton, Owen, and Weber 1992).

**Preferences diversity** Splitting may stem from the diversity in individual tastes. For example, let the society decide on the quality of a pure public good. If the good is financed by a tax on income, with sufficient heterogeneity in taste or in income, the society may split into two communities, each producing a public good of different quality with a different tax level. Making accessible the high quality good to anyone would be more efficient but destabilised by the secession of part of the population. In general, two opposite forces -increasing returns to size and to coordination on the one hand and heterogeneity of preferences on the other- arise in a variety of contexts, including public goods choices, industrial organization, and jurisdictions boundaries (see the survey Demange (2017) for more detailed examples and references). The number of subgroups in a stable coalition structure then depends on the relative strength of these two forces. To name a few works, see Wooders (1978) and Guesnerie and Oddou (1981) for public good models, Greenberg and Weber (1993) for a uni-dimensional space and single-peaked preferences, Demange and Henriet (1991) on oligopolies and Demange (1994) and (2004) on games on trees when preferences are intermediate. In these works, a technique is to show that the assumptions on the preferences restrict the set of blocking coalitions one needs to consider.

To conclude, let us mention empirical works that build on the coalitional approach. Robbett (2015) conducts laboratory experiments on how agents partition themselves into groups, Melatos and Woodland (2016) investigate trade bloc formation in a general equilibrium model, Desmet, Le Breton, Ortuno-Ortin and Weber (2011) present a model of nations where agents vote on the level of public spending, Weese (2015) analyzes the interaction between the national governmental transfers and the merger of municipalities in Japan.

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<sup>20</sup>In a transferable utility game. all transfers among coalition's members are allowed and payoffs are linear in those transfers. The set of feasible payoffs to a coalition  $S$  is determined by its maximal attainable aggregate payoff,  $v(S)$ , called its *value*:  $\{(x_i)_{i \in S} \text{ such that } \sum_{i \in S} x_i \leq v(S)\}$ .

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