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**Wealth in the Quadratic Loss Function of the Ramsey  
Malinvaud Cass Koopmans Model of Optimal Savings**

Jean-Bernard CHATELAIN, Kirsten RALF

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# Wealth in the Quadratic Loss Function of the Ramsey Malinvaud Cass Koopmans Model of Optimal Savings\*

Jean-Bernard Chatelain<sup>†</sup> and Kirsten Ralf<sup>‡</sup>

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## Abstract

Using the second order Taylor expansion of the Lagrangian of the Ramsey model of optimal savings, wealth is included in the quadratic loss function, and not only consumption. Its weight is given by the degree of concavity of the decreasing returns to scale production function times the marginal utility of consumption. The weight of consumption is given by the degree of concavity of the utility function. This quadratic loss function implies that the speed of convergence is explicitly driven by the trade-off between wealth smoothing (fostering convergence, related to technology) versus consumption smoothing (delaying convergence, related to preferences). By contrast, the second order Taylor expansion of the utility instead of the Lagrangian leads to a quadratic loss function with a weight of wealth equal to zero, which is false for a decreasing returns to scale production function.

**Keywords:** Linear quadratic approximation, Wealth, Consumption, Savings, Negative feedback, Speed of convergence.

**JEL classification numbers:** C61, C62, E43, E44, E47, E52, E58.

### Titre:

La richesse dans la fonction de perte quadratique du modèle d'épargne optimale de Ramsey, Malinvaud, Cass et Koopmans

### Résumé

En utilisant le développement limité de Taylor à l'ordre deux du Lagrangien du modèle d'épargne optimale de Ramsey, la richesse est prise en compte dans la fonction de perte quadratique et pas seulement la consommation. La pondération de la richesse est donnée par le degré de concavité de la fonction de production à rendements décroissants multiplié par l'utilité marginale de la consommation. La pondération de la consommation est donnée par le degré de concavité de la fonction d'utilité. Cette fonction de perte quadratique implique que la vitesse de convergence vers l'équilibre dépend de l'arbitrage entre le lissage de la richesse (qui accélère la convergence) et le lissage de la consommation (qui retarde la convergence). En revanche, le développement limité de Taylor à l'ordre deux de l'utilité (au lieu du Lagrangien) met un poids de la richesse égal à zéro dans la fonction

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<sup>†</sup>Centre d'Economie de la Sorbonne, Paris School of Economics, Université Paris I Pantheon Sorbonne, 106-112 bd de l'Hôpital 75647 Paris Cedex 13. Email: jean-bernard.chatelain@univ-paris1.fr

<sup>‡</sup>ESCE International Business School, OMNES Education, Immeuble Cœur Défense, 82 Esplanade du Général de Gaulle, 92934 Paris La Défense, Courriel: kirsten.ralf@esce.fr

de perte quadratique, ce qui est faux si la fonction de production est à rendements strictement décroissants.

**Mots Clefs:** Approximation linéaire quadratique, Richesse, Consommation, Epargne, Rétroaction négative, Vitesse de convergence.

"For two decades or more many macroeconomists 'forgot' the work of Magill (1977), and proceeded by linearizing the dynamics and taking quadratic approximations of the welfare function, which leads to wrong results." (Levine and Pearlman (2011, p.5))

## 1 Introduction

Since Kurz (1968) assumed the diminishing marginal utility of wealth and not only the diminishing marginal utility of consumption, one expects this assumption changes the economic trade-off with respect to the original version of the Ramsey model of optimal savings, where only the diminishing marginal utility of consumption is assumed (Ramsey (1928), Malinvaud (1965), Cass (1965) and Koopmans (1965), Spear and Young (2014)). The diminishing marginal utility of wealth has been first assumed by Bernoulli (1734). This assumption is currently used in new-Keynesian models (Michaillat and Saez (2019) as well as a constant marginal utility of wealth (Michau (2018)).

One may be tempted to explain the higher optimal autocorrelation of consumption and wealth in Ramsey model only by a higher consumption smoothing parameter. This autocorrelation determines the speed of convergence to the steady state. However, the fact that the higher degree of concavity of the production function also increases the speed of convergence to the steady state is not clearly related to an economic trade-off (Barro and Sala-Martin (2004)).

As soon as dynamic optimal macroeconomic models include more than a single state variable, textbooks suggest using the quadratic approximation of the loss function subject to the linear approximation of the constraints (Hansen and Sargent (2014)). This allows to apply the linear quadratic regulator algorithms for numerical simulations. The linear quadratic approximation of the Ramsey model is of pedagogical interest, because the Ramsey model is a building block for models including more than one state variable.

The linear quadratic approximation of the Ramsey model is seemingly unrelated to the assumption of the diminishing marginal utility of wealth. However, this paper shows that the approximated quadratic loss function depends not only on consumption (with a weight equal to the second derivative of the utility function at the steady state), *but also on wealth* (with a weight equal to the second derivative of the production function at the steady state times the marginal utility of consumption at the steady state). Therefore, the speed of convergence is given by the trade-off between the weight of wealth smoothing (fostering convergence) and the weight of consumption smoothing (delaying convergence). The assumption of the diminishing marginal returns of wealth in the production function (which is related to technology) is *observationally equivalent* to the assumption of the diminishing marginal utility of wealth and not only of consumption (which is related to households' preferences).

This result remained unnoticed in macroeconomics. For example, Hansen and Sargent (2014) is the benchmark book of linear quadratic model economies. In their chapter 4, they propose the seemingly exhaustive list of eight linear technologies and of six given households quadratic preferences. These six quadratic loss functions exclude wealth.

According to the result of this paper, none of their linear quadratic economies corresponds to the linear quadratic approximation of the models assuming the diminishing marginal returns of wealth in the production function. In particular, Hansen and Sargent (2014) exclude the Ramsey model from their list of linear quadratic economies.

Finally, following Judd (1996), Benigno and Woodford (2006), Levine, Pearlman and Piersse (2008), Levine and Pearlman (2011), and Benigno and Woodford (2012)), the linear quadratic approximation of optimal control programs faced a controversy between the "naive" quadratic loss function approximation found using the second order Taylor expansion of the utility function (McGrattan (1990), Chari, Christiano and Kehoe (1994)) versus the "correct" quadratic loss function approximation found using the second order Taylor expansion of the Lagrangian (Bryson and Ho (1975, p.178 and p.182), Magill (1977, p.181) and Levine, Pearlman and Piersse (2008)). This correct quadratic approximation is also mentioned as the "accessory minimum problem" or the "auxiliary variational problem" (Magill (1977)). According to Bryson (1996), this quadratic approximation of the Lagrangian was used in the calculus of variations before 1900. It was later included in the textbooks of optimal control in the 1960s (Bryson and Ho (1975), Dyer and McReynolds (1970) and Jacobson and Mayne (1970)). However, it remained unnoticed by most macroeconomists in the 1980s.

Contrasting the naive quadratic approximation of the loss function versus the correct one is related to wealth in the quadratic loss function of the Ramsey model. The second order Taylor expansion of the utility erroneously sets a weight on wealth equal to zero. It erroneously sets to zero the second derivative of the production function with respect to wealth times the Lagrange multiplier. This second derivative is only equal to zero in the case of the constant returns to scale of wealth in the production function, where output is a linear function of wealth. The naive quadratic loss approximation erroneously assumes constant returns to scale for the case of decreasing returns to scale.

Section two presents the linear quadratic regulator approximation of the optimal saving model and its solution. It also discusses the magnitude of the errors when using the second order Taylor expansion of the utility instead of the Lagrangian. Section three presents the relevant orders of magnitude for the parameters of the model. Section four concludes.

## 2 Wealth in the Quadratic Loss Function of the Optimal Savings Model

Let us consider the optimal control program where  $U$  and  $g$  are concave twice-differentiable functions of a unique policy target (the stock of wealth)  $k_t$  and of a unique policy instrument (the flow of consumption)  $c_t$ :

$$\max_{k_t, c_t} \sum_{t=0}^{+\infty} \beta^t U(k_t, c_t) \text{ subject to } k_t = g(k_{t-1}, c_{t-1}). \quad (1)$$

given the initial condition  $k_0$ . The discount factor is  $0 < \beta = \frac{1}{1+r} \leq 1$  with the discount rate  $r > 0$ . We seek bounded solutions:

$$\sum_{t=0}^{+\infty} \beta^t U(k_t, c_t) < +\infty.$$

The Lagrangian  $L$  of the program with the period Lagrangian  $L_t$  and the Lagrange multiplier  $\mu_t > 0$  is given by:

$$L = \sum_{t=0}^{+\infty} \beta^t L_t \text{ with } L_t = U(k_t, c_t) + \mu_t (g(k_t, c_t) - k_{t+1}) \quad (2)$$

We seek the quadratic loss function subject to a linear constraint with the same first and second order Taylor expansion as this concave program. The Hessian matrix of the second derivatives of the Lagrangian evaluated at the steady state values  $(k^*, c^*, \mu^*)$  gives the weights  $(Q, W, R)$  of the negative definite quadratic loss function to be minimized:

$$\min_{k_t, c_t} \frac{1}{2} \sum_{t=0}^{+\infty} \beta^t [Q(k_t - k^*)^2 + 2W(k_t - k^*)(c_t - c^*) + R(c_t - c^*)^2] \quad (3)$$

$$k_t - k^* = A(k_{t-1} - k^*) + B(c_{t-1} - c^*) \quad (4)$$

The steady state conditions  $(k_t = k_{t+1} = k^*, c_t = c_{t+1} = c^*, \mu_t = \mu_{t+1} = \mu^*)$  and the values of  $Q, W, R, A, B$  are given in the appendix.

In the Ramsey optimal savings model, the representative household has a strictly diminishing marginal utility of consumption and zero utility of wealth:

$$U_c > 0, U_{cc} < 0, U_k = U_{kk} = U_{ck} = 0 \quad (\text{H1})$$

The law of motion of wealth includes outflows with consumption and the linear depreciation rate of the capital stock denoted  $0 \leq \delta \leq 1$  and inflows with output produced with a decreasing or linear marginal productivity of wealth  $f(k_t) > 0, f_k > 0, f_{kk} \leq 0$ :

$$k_{t+1} = g(k_t, c_t) = f(k_t) + (1 - \delta)k_t - c_t. \quad (5)$$

$$g_c = -1, g_k = f_k + 1 - \delta > 0, g_{kk} = f_{kk} \leq 0, g_{kc} = g_{cc} = 0 \quad (\text{H2})$$

Proposition 1 shows that the discrete time linear quadratic approximation of the Ramsey model is the same as its continuous time version (Magill (1977)).

**Proposition 1** *Under hypothesis (H1) and (H2), the linear quadratic regulator approximation of the Ramsey model is given by:*

$$\begin{aligned} Q &= -U_c(c^*) f_{kk}(k^*) \geq 0 & W &= 0 & R &= -U_{cc}(c^*) > 0 \\ A &= \frac{1}{\beta} = 1 + r > 1 & B &= -1 \end{aligned}$$

**Proof.** See appendix. ■

We then solve this model in proposition 2.

**Proposition 2** (i) *The discounted scalar linear quadratic regulator solution (with the*

discounted utility or welfare denoted  $w$ ) is:

$$\begin{aligned}
k_t - k^* &= \lambda (k_{t-1} - k^*) \text{ and } k_0 \text{ given, with:} \\
0 < \lambda \left( A, B, \beta, \frac{Q}{R} \right) &\leq \min \left( A, \frac{1}{\beta A} \right) \\
c_t - c^* &= F (k_t - k^*), \text{ with } F = \frac{\lambda - A}{B} \\
w &= P (k_0 - k^*)^2 \text{ with } P = \frac{\frac{1}{2} Q + R \left( \frac{\lambda - A}{B} \right)^2}{1 - \beta \lambda^2}
\end{aligned}$$

(ii) For strictly decreasing returns to scale  $f_{kk}(k^*) < 0$  ( $Q > 0$ ), the solution is a partial adjustment model of wealth and of consumption which follows a stationary autoregressive process of order one ( $AR(1)$ ) with a parameter  $\lambda \in ]0, 1[$  and a feedback rule parameter  $F = 1 + r - \lambda \in ]r, 1 + r[$ . The speed of convergence to the steady state decreases for a higher value of  $\lambda$ .

(iii) The case of constant returns to scale  $f_{kk}(k^*) = 0$  ( $Q = 0$ ) is the maximal inertia solution of the discounted linear quadratic regulator. The solution is a unit root process for consumption and wealth with  $\lambda = 1$  and  $F = r$  as in Hall's (1978) "consumption follows a random walk".

**Proof.** For (i) and (ii), see appendix. (iii) For  $Q = 0$ ,  $\lambda = \frac{1}{2} \left( 1 + \frac{1}{\beta} - \sqrt{\left( 1 + \frac{1}{\beta} \right)^2 - \frac{4}{\beta}} \right) = \frac{1}{2} \left( 1 + \frac{1}{\beta} - \left| 1 - \frac{1}{\beta} \right| \right) = 1$ ,  $F = 1 + r - \lambda = r$  and  $w = \frac{1}{2} R (1 + r) r (k_0 - k^*)^2$ . ■

In the case of household's maximal inertia ( $Q = 0$ ), the trade-off only considers consumption smoothing excluding wealth smoothing. The consumption gap ( $c_t - c^*$ ) equals the return on wealth gap ( $r(k_t - k^*)$ ) so that the wealth gap ( $k_t - k^*$ ) remains constant over time, excluding the convergence to the steady state. The consumer lives off the interest on his own capital.

When  $\frac{Q}{R} \rightarrow +\infty$ , the trade-off tends to be fully in favor of wealth smoothing excluding consumption smoothing. The consumption gap ( $c_t - c^*$ ) tends to the sum of the return of the wealth gap and of the wealth gap ( $(1 + r)(k_t - k^*)$ ). This tends to a complete adjustment of the wealth gap to zero next period whatever the size of the shock on wealth on period  $t$ : ( $k_t - k^*$ ).

Between these two extreme results, the trade-off between wealth smoothing fostering convergence versus consumption smoothing delaying convergence is summarized by the sensitivity of the optimal persistence  $\lambda$  of consumption and wealth with respect to the relative weight  $Q/R$  in the loss function, as detailed in proposition 3:

**Proposition 3** For strictly decreasing returns to scale  $f_{kk}(k^*) < 0$  ( $Q > 0$ ) and for given  $(\beta, A, B)$  with  $A > 0$  and  $0 < \beta \leq 1$ , the function of the persistence parameter  $\lambda(A, B, \beta, \frac{Q}{R})$  is continuous and strictly decreasing with  $Q/R$ . Therefore, for given  $(\beta, A, B)$ , there is an inverse function relating the preferences  $Q/R$  in the quadratic loss function to the persistence parameter  $\lambda$ .

**Proof.**

$$\text{For } A > 0, \frac{\partial \lambda}{\partial (QB^2/R)} = \frac{1}{2A} \left( 1 - \frac{A + \frac{1}{\beta A} + \frac{Q}{R} \frac{B^2}{A}}{\sqrt{\left(A + \frac{1}{\beta A} + \frac{Q}{R} \frac{B^2}{A}\right)^2 - \frac{4}{\beta}}} \right) < 0$$

■

In the Ramsey model, the parameter  $A$  is a function of the discount factor  $A = 1/\beta$ . One has  $\frac{\partial \lambda}{\partial (1/\beta)} > 0$  as well as  $\frac{\partial \lambda}{\partial \beta} < 0$ . The larger the household rate of time preference for the present  $\frac{1}{\beta} - 1$  (the more the representative household values consumption now with respect to consumption in the future), the higher the persistence parameter  $\lambda$  of consumption and wealth and, therefore, the longer the convergence process of wealth and consumption to their steady state.

For a strictly decreasing returns to scale of wealth in the production function, the phase diagram of consumption and wealth around the optimal stable equilibrium  $(k^*, c^*)$  in the plane  $(k, c)$  is given by the negative feedback policy rule  $c_t - c^* = F(k_t - k^*)$ . It consists of dots on the increasing line with a slope  $F$  converging to the stable equilibrium of a dynamic system of order one.

The interpretation of the negative-feedback rule is the one of a reservoir model (the stock of wealth) controlled by net consumption outflows in order to maintain the long run optimal level of the stock of wealth in the reservoir. At the steady state, the marginal product of capital is equal to the marginal cost of capital. If wealth is below its steady state, it is optimal to save more than the steady state level of consumption in a partial adjustment fashion in order to catch up. *If the poor representative household wishes to become rich, he consumes less and saves more.* Conversely, if wealth is above its steady state (with the marginal productivity of capital lower than its marginal cost), it is optimal to consume more than the steady state value, in a partial adjustment fashion, in order to reduce wealth.

The next proposition checks the difference between this "correct" linear quadratic approximation from the *naive* linear quadratic approximation used by Chari, Christiano and Kehoe (1994) when applied to the Ramsey model.

**Proposition 4** *For strictly decreasing returns to scale  $f_{kk}(k^*) < 0$  ( $Q > 0$ ), the naive linear quadratic approximation computes the quadratic loss function with a second order Taylor expansion of the utility instead of the Lagrangian. It erroneously finds  $Q = 0$  for the weight of wealth. The naive linear quadratic approximation erroneously approximates decreasing returns to scale with convergence ( $0 < \lambda < 1$ ) by constant returns to scale without convergence ( $\lambda = 1$ ).*

**Proof.** *The second order Taylor expansion of the utility omits the weight of wealth in the loss function  $\mu^*(k^*, c^*) f_{kk}(k^*, c^*)$  with respect to the second order Taylor expansion of the Lagrangian. This weight is erroneously computed to be zero, as if there were constant returns to scale:  $f_{kk}(k^*, c^*) = 0$ .* ■

The naive linear quadratic approximation approximates a stable equilibrium by an unstable equilibrium. If one adds additive independently and identically distributed disturbances in the wealth dynamic linear equation, it approximates a stationary AR(1) process for consumption and wealth by a non-stationary random walk process for consumption and wealth.



### 3 Economically Relevant Orders of Magnitude

The section highlights that the order of magnitude of the measures of the decreasing returns to scale of wealth and of the diminishing marginal utility of consumption implies the narrow range of high persistence values for the persistence parameter  $\lambda \in [0.95, 1]$  at the quarterly frequency. This persistence parameter is also one of the roots of real business cycles models (Campbell (1994)). In this section, we also check the identification issues of the applied econometrics of the Ramsey model, when testing for example the convergence of GDP/head across countries.

For Barro and Sala-I-Martin (2004), the relevant range of the elasticity of capital to output is  $0.3 \leq \alpha \leq 1$  in the production function  $f(k) = Ak^\alpha$  where capital includes public infrastructures, human capital and the stock of knowledge. The case  $\alpha = 1$  corresponds to the constant returns to scale production function  $y = Ak$ .

The elasticity of intertemporal substitution measures the responsiveness of the growth rate of consumption to the real interest rate. If this elasticity is low, the consumption smoothing motive is very strong ( $R$  increases). Havranek et al. (2015) meta-analysis of 169 published studies reports a mean elasticity of  $\sigma = -\frac{U_c}{cU_{cc}} = 0.5$ , subject to a publication bias to upward values. Their corresponding estimated range of the constant intertemporal elasticity of substitution for consumption is  $0 < \sigma \leq 1$ . Real business cycles calibrations (Campbell (1994)) often used the top value of this range ( $\sigma = 1$ ) corresponding to a log utility function.

Given these plausible ranges of values for  $\alpha$  and  $\sigma$ , the range of plausible values for  $Q/R$ ,  $\lambda$  and  $F$  is rather small for quarterly frequency, as shown in proposition 5 and table 1.

**Proposition 5**  $0.3 \leq \alpha \leq 1$  and  $0 < \sigma \leq 1 \Rightarrow Q/R = \frac{U_c f_{kk}}{U_{cc}} \in [0.0025, 0]$  and for quarterly frequency:  $\lambda \in [0.95, 1]$  and  $F = 1 + r - \lambda \in [r, r + 0.05]$ .

The range of plausible parameter parameters  $0.3 \leq \alpha \leq 1$  and  $0 < \sigma \leq 1$  imply that the relative weight of wealth in the quadratic loss function is very low  $Q/R = \frac{U_c f_{kk}}{U_{cc}} \in [0.0025, 0]$ . For quarterly data, this restricts the auto-correlation of consumption to the range  $\lambda \in [0.95, 1]$  with the very high persistence of consumption. Because  $\lambda = 1 + r - F$ , this implies the very low sensitivity of consumption to wealth equal or slightly above the real return on wealth:  $F \in [r, r + 0.05]$ .

The unit root case  $\lambda = 1$  where consumption follows a random walk (Hall (1978)) corresponding to the  $Ak$  linear production function and to  $F = r$ . Consumption is exactly equal to the return on wealth so that expected wealth remains constant over time.

**Table 1:** Key parameters for  $\alpha = 1/3$ ,  $\delta = 0.025$  (10% at an annual rate),  $r = 0.015$  (6% at an annual rate as in Campbell (1994)). The arrow  $\rightarrow$  indicates that the parameter tends to this value while not reaching it.

Parameters	-	-	RBC	-	BarroSIM	Hall
$\lambda = 1 + r - F$	$\rightarrow 0$	0.90	0.96	0.98	0.995	1
$F$	$\rightarrow r + 1$	$r + 0.1$	$r + 0.04$	$r + 0.02$	$r + 0.005$	$r$
$F \frac{k^*}{c^*} = F \frac{\alpha}{r+\delta-\alpha\delta}$	$\rightarrow 10.68$	1.21	0.58	0.37	0.21	0.16
$\frac{Q}{R} = (1+r) \frac{\lambda}{(1+r-\lambda)(1-\lambda)}$	$\rightarrow +\infty$	0.01297	0.00233	0.000725	0.0001	0
$\frac{Q}{R} \left(\frac{k^*}{c^*}\right)^2 = \frac{Q}{R} \left(\frac{\alpha}{r+\delta-\alpha\delta}\right)^2$	$\rightarrow +\infty$	1.437	0.259	0.080	0.011	0
$P = w \left(\frac{Q}{R}\right) / (k_0 - k^*)^2$ for $R = 1$	$\rightarrow +\infty$	0.0648	0.0291	0.0181	0.0102	0.0008
$\sigma = \frac{1}{2}$ , corresponding $\alpha$ :	$\rightarrow 0$	0.06	0.23	0.44	0.8	1
$\alpha = \frac{1}{3}$ , $\sigma = \frac{Q}{R} \frac{1}{(1-\alpha)(r+\delta)\left(\frac{r+\delta-\alpha\delta}{\alpha}\right)}$	$\rightarrow +\infty$	5.12	0.92	0.29	0.04	$\rightarrow 0$
$\alpha = \frac{3}{4}$ , $\sigma = \frac{Q}{R} \frac{1}{(1-\alpha)(r+\delta)\left(\frac{r+\delta-\alpha\delta}{\alpha}\right)}$	$\rightarrow +\infty$	45.78	8.22	2.56	0.36	$\rightarrow 0$

Table 1 also presents the weights and policy rule parameter when using percentage deviations  $\frac{c_t - c^*}{c^*}$  and  $\frac{k_t - k^*}{k^*}$  instead of deviations  $c_t - c^*$  and  $k_t - k^*$ , which are usually computed in graduate macroeconomic textbooks with the continuous time version of the Ramsey model (Barro and Sala-I-Martin (2004)). Percentage deviations mechanically increases the relative weight of wealth  $Q/R$  and the feedback rule parameter  $F$  measuring the response of the percentage deviations of consumption to the percentage deviations of wealth from their steady state. One unit increase in capital stock increases consumption by  $F$  units. When the rule is written in percentage deviations from the steady state, the rule parameter is multiplied by  $\frac{k^*}{c^*}$  as follows:  $\frac{c_t - c^*}{c^*} = F \frac{k^*}{c^*} \left(\frac{k_t - k^*}{k^*}\right)$ . A 1% increase in capital stock increases consumption by a larger percentage  $F \frac{k^*}{c^*} \%$ , because, in steady state, the capital stock is much bigger than consumption (by a factor 10). For the loss function, one has  $Q \left(\frac{k^*}{c^*}\right)^2 \left(\frac{k_t - k^*}{k^*}\right)^2 + R \left(\frac{c_t - c^*}{c^*}\right)^2$ , which multiplies  $Q/R$  by  $\left(\frac{k^*}{c^*}\right)^2$  which is around 100, because  $\frac{k^*}{c^*}$  is around a factor 10.

Table 1 compares the numerical values for three benchmark cases in the literature: Hall (1978) model of consumption with constant returns to scale of wealth ( $\lambda = 1$ ), 2% annual convergence rate in growth regressions ( $\lambda = 0.995$  for quarterly data as in Barro and Sala-I-Martin (2004)) and real business cycles calibration when the persistence of the exogenous productivity shock is set to zero ( $\lambda = 0.96$  for quarterly data which are detrended using Hodrick-Prescott filter, as in Campbell (1994)).

Hall's (1978) consumption ( $c_t - c^*$ ) is equal to the returns on wealth  $r(k_t - k^*)$  so that  $\lambda$  remains exactly equal to one. With percentage deviation model, because the steady state stock of capital is close to 10 times consumption, the policy rule parameter is multiplier by  $\frac{k^*}{c^*}$ . The model corresponds to constant returns to scale  $\alpha = 1$ , which implies  $F_{kk} = 0$  and  $Q = 0$  so that the optimal welfare function  $w$  has the lowest value of table 1.

Real business cycles theorists (Campbell 1994) focus on values around  $\lambda = 0.96$  for quarterly data, corresponding to  $Q/R = 0.002$ , with  $\alpha = 1/3$  and  $\sigma = 0.92$ , close to one which corresponds to a log utility of consumption.

Assuming a lower persistence  $\lambda = 0.9$  at the quarterly frequency implies at least an oversized intertemporal elasticity of substitution  $\sigma = 5$  for  $\alpha = 1/3$  and with larger values for  $\sigma$  for larger values of  $\alpha$ .

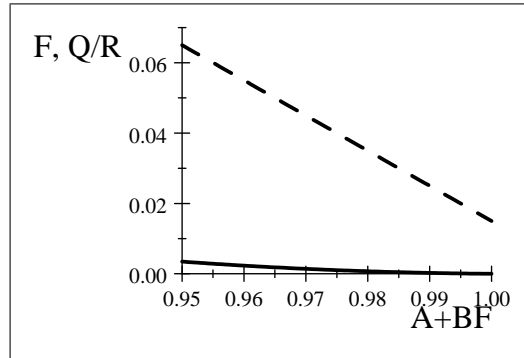
In the econometrics of the cross country growth regressions, Barro and Sala-I-Martin (2004) found empirically that  $1 - \lambda$  equals 0.02 at an annual rate or 0.005 at a quarterly frequency for the period 1960-1995 ( $\lambda = 0.995$ ) which corresponds to the ratio  $Q/R = 10^{-4}$ . They consider a larger capital share around  $\alpha = 0.8$  with the broader definition of capital including not only private capital but also public capital, human capital and

the stock of knowledge. It corresponds to a slightly lower intertemporal elasticity of substitution  $\sigma = 0.36$  than the reference of 0.5 proposed by Havranek et al. (2015).

The recent results on an increased cross country convergence for the period 2000-2020 with respect to the period 1960-2000 (Kremer, Willis and You (2022)) would suggest that the world technology shifted towards *more* decreasing returns since the last two decades (a lower share of capital  $\alpha$ ), according to the Barro and Sala-I-Martin (2004) interpretation of convergence. This would have increased the relative weight for wealth smoothing with respect to consumption smoothing.

Figure 1 illustrates proposition 3 which allows econometricians to first estimate the convergence parameter  $\lambda$  and *then* infer the preference parameters ratio  $Q/R$  (continuous line in figure 1) and the feedback rule parameter  $F$  (dash line in figure 1). See for example Salemi (1995) for the inverse control estimation of preferences in another model. Figure 1 plots the function  $(Q/R)(\lambda)$  and feedback rule function  $\frac{\lambda-A}{B}$  using Campbell (1994) real business cycles calibration  $r = 0.015$  and  $\beta = 0.985$  for quarterly data (6% at an annual rate).

**Figure 1:** Inverse control with  $Q/R$  (solid line) and  $F = 1 + r - \lambda$  (dash line) as a function of  $\lambda \in [0.95, 1]$  for  $r = 0.015$ .



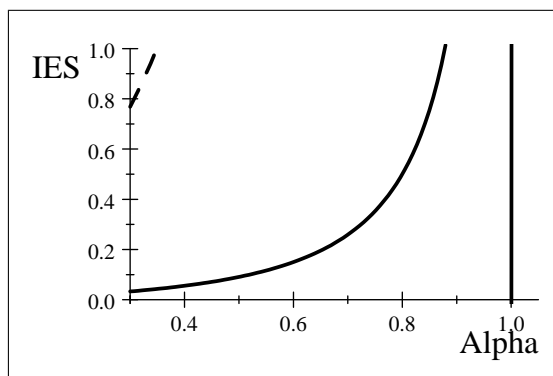
Before using inverse control, one need to estimate  $A$ ,  $B$ , and  $\lambda$  as follows. A marginal unit of consumption reduces future wealth by one unit:  $B = -1$  is known with certainty because it is a parameter of an accounting stock flow relation. The discount rate in  $A = 1 + r$  can be estimated as the average opportunity cost of the returns on wealth during the period of estimation. The speed of convergence parameter  $\lambda$  can be estimated by an autoregressive process of order one for consumption filtered by the Hodrick-Prescott filter. A test of the null hypothesis of the unit root of  $\lambda = 1$  may or may not reject the constant returns to scale regime versus a decreasing returns to scale regime.

However, the estimated autocorrelation parameter of consumption  $\lambda$  does not allow the econometrician to identify separately the intertemporal elasticity of substitution (related to parameter  $R$ ) and the elasticity of output to wealth (related to  $Q$ ), but only their ratio  $Q/R$  as shown in figure 2. *The ratio  $Q/R$  is a decreasing function of  $\alpha$  (a lower  $\alpha$  implies a larger weight  $Q$  of wealth smoothing) and an increasing function of  $\sigma$  (a lower  $\sigma$  implies a larger weight  $R$  of consumption smoothing).* For a given  $Q/R$ , there is an infinity of pairs  $(\alpha, \sigma)$  where the increase of  $\alpha$  is offset by the increase of  $\sigma$  as shown in figure 2. The econometrician may assume a linear identification restriction such as  $\alpha = 0.8$  or  $\alpha = 1/3$  or  $\sigma = 0.5$  as done in table 1.

Figure 2 presents the iso-speed of convergence curves of the intertemporal elasticity of substitution and the returns to scale parameters for  $0.3 \leq \alpha \leq 1$  and  $0 < \sigma \leq 1$ .

**Figure 2:** The pair of intertemporal elasticity of substitution ( $0 < \sigma \leq 1$ ) and decreasing or constant returns to scale parameter ( $0.3 \leq \alpha \leq 1$ ) for  $\lambda = 0.96$  (dash line

with a very restricted range for plausible values of  $(\alpha, \sigma)$  for the real business cycles model), for  $\lambda = 0.995$  (solid curve, with a much larger range of values for  $(\alpha, \sigma)$  for convergence according to Barro and Sala-I-Martin (2004)) and for Hall (1978) constant returns to scale  $\lambda = 1$  (solid vertical line,  $\alpha = 1$ ).



- Campbell (1994) real business cycle calibration  $\lambda = 0.96$  is compatible with the pairs of key parameters ranging from  $(\sigma = 0.8, \alpha = 0.3)$  to  $(\sigma = 1, \alpha = 0.33)$ . The available range for the intertemporal elasticity of substitution correspond to value  $0.8 \leq \sigma \leq 1$  higher than the preferred estimate in Havranek et al. (2015) meta-analysis:  $\sigma = 0.5$ .

- Barro and Sala-I-Martin (2004) slow cross country convergence  $\lambda = 0.995$  is compatible with the pairs of key parameters ranging from  $(\sigma = 0.02, \alpha = 0.3)$  to  $(\sigma = 1, \alpha = 0.85)$ .

- Hall (1978) unit root case  $\lambda = 1$  with the pairs of key parameters ranging from  $(\sigma = 0, \alpha = 1)$  to  $(\sigma = 1, \alpha = 1)$ .

## 4 Conclusion

Although the diminishing marginal utility of wealth is not assumed in the Ramsey model, wealth appears in its quadratic loss function approximation because of the decreasing returns to scale of wealth in the production function and of the (non-zero) marginal utility of consumption.

The linear quadratic regulator approximation of the Ramsey model allows to interpret the speed of convergence as the trade-off between wealth smoothing (fostering convergence) versus consumption smoothing (delaying convergence), summarized by the ratio  $Q/R$ .

In the case of the decreasing returns to scale of wealth in the production function, a naive second order Taylor expansion of the utility instead of the Lagrangian implies a quadratic loss function which erroneously sets a zero weight on wealth in the loss function.

## References

- [1] Barro, R., & Sala-i-Martin, X. (2004). Economic growth (second edition). The MIT Press. Cambridge, Massachusetts.
- [2] Benigno, P., & Woodford, M. (2006). Optimal taxation in an RBC model: A linear-quadratic approach. *Journal of Economic Dynamics and Control*, 30(9-10), 1445-1489.

- [3] Bernoulli D., (1738). Specimen Theoriae Novae de Mensura Sortis, in *Comentarii Academiae Scientiarum Imperialis Petropolitanae*, Tomus V, vol V, pp.175-192. [(1954): Exposition of a new theory on the measurement of risk. *Econometrica*, 22(1):23–36. Translated from Latin by Sommer, L.].
- [4] Bryson, A. E., Ho, Y. C. (1969). Applied Optimal Control. Blaisdell. Revised printing Hemisphere 1975.
- [5] Bryson, A. E. (1996). Optimal control - 1950 to 1985. IEEE Control Systems Magazine, 16(3), 26-33.
- [6] Campbell, J. Y. (1994). Inspecting the mechanism: An analytical approach to the stochastic growth model. *Journal of Monetary Economics*, 33(3), 463-506.
- [7] Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *The Review of economic studies*, 32(3), 233-240.
- [8] Chari, V.V., Christiano, L.J., Kehoe, P.J., (1994). Optimal fiscal policy in a business cycle model. *Journal of Political Economy* 102, 617–652.
- [9] Chow, G. C. (1997). *Dynamic economics: optimization by the Lagrange method*. Oxford University Press.
- [10] Dyer P. and McReynolds S.R. (1970). The Computation and Theory of Optimal Control. Academic Press.
- [11] Hall, R. E. (1978). Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence. *Journal of political economy*, 86(6), 971-987.
- [12] Hansen, L. P., & Sargent, T. J. (2014). *Recursive Models of Dynamic Linear Economics*. Princeton University Press.
- [13] Havranek, T., Horvath, R., Irsova, Z., & Rusnak, M. (2015). Cross-country heterogeneity in intertemporal substitution. *Journal of International Economics*, 96(1), 100-118.
- [14] Jacobson D., Mayne D. (1970). Differential Dynamic Programming. American Elsevier. New York.
- [15] Koopmans T.C. (1965). On the concept of optimal economic growth. In Study Week on the Econometric Approach to Development Planning, 7-13 October. Rome: Pontifical Academy of Science
- [16] Kremer, M., Willis, J., & You, Y. (2022). Converging to convergence. *NBER Macroeconomics Annual*, 36(1), 337-412.
- [17] Kurz, M. (1968). Optimal economic growth and wealth effects. *International Economic Review*, 9(3), 348-357.
- [18] Judd K.L. (1996), Approximation, perturbation and projection methods in economic analysis, in: H.M. Amman, D.A. Kendrick, J. Rust (Eds.), *Handbook of Computational Economics*, vol. 1, North-Holland, Amsterdam, 1996.

- [19] Levine, P., Pearlman, J., Piersse, R. (2008). Linear-quadratic approximation, external habit and targeting rules. *Journal of Economic Dynamics and Control*, 32(10), 3315-3349.
- [20] Levine, P., & Pearlman, J. (2011). Computation of LQ approximations to optimal policy problems in different information settings under zero lower bound constraints. *Dynare Working Paper* 10.
- [21] Malinvaud (1965). *Croissances Optimales dans un Modele Macroeconomique*. In *Study Week on the Econometric Approach to Development Planning*, 7-13 October. Rome: Pontifical Academy of Science
- [22] McCandless, G. (2008). *The ABCs of RBCs: An introduction to dynamic macroeconomic models*. Harvard University Press.
- [23] McGrattan, E. R. (1990). Solving the stochastic growth model by linear-quadratic approximation. *Journal of Business & Economic Statistics*, 8(1), 41-44.
- [24] Magill, M. J. (1977). Some new results on the local stability of the process of capital accumulation. *Journal of Economic Theory*, 15(1), 174-210.
- [25] Michaillat, P., Saez E.. (2019). Resolving New Keynesian Anomalies with Wealth in the Utility Function. *Review of Economics and Statistics*, vol. 102, no. 2, pp. 197–215.
- [26] Michau, J. B. (2018). Secular stagnation: Theory and remedies. *Journal of Economic Theory*, 176, 552-618.
- [27] Ramsey, F. P. (1928). A mathematical theory of saving. *The economic journal*, 38(152), 543-559.
- [28] Salemi, M. K. (1995). Revealed preference of the Federal Reserve: using inverse-control theory to interpret the policy equation of a vector autoregression. *Journal of Business & Economic Statistics*, 13(4), 419-433.
- [29] Sargent, T. J., Ljungqvist, L. (2018). *Recursive macroeconomic theory (fourth edition)*. MIT Press.
- [30] Spear, S. E., Young, W. (2014). Optimum savings and optimal growth: The Cass–Malinvaud–Koopmans nexus. *Macroeconomic Dynamics*, 18(1), 215-243.

## 5 Appendix

### (1) Proof of proposition 1.

We seek a linear quadratic regulator program which has the same necessary first order conditions and the same sufficient second order conditions as the concave program. This is obtained if the linear quadratic regulator program and the concave program have identical second order Taylor expansion of their Lagrangian.

The second order Taylor expansion of the period  $t$  Lagrangian around the steady state is such that the first order terms are equal to zero because of necessary first order conditions (Chow (1997), p.26-27)):

$$\begin{aligned}
& L(k_t, c_t) - L(k^*, c^*) \\
&= \left[ \frac{\partial(\beta L_t + L_{t-1})}{\partial k_t}(k^*, c^*) \right] (k_t - k^*) + \left[ \frac{\partial \beta L_t}{\partial c_t}(k^*, c^*) \right] (c_t - c^*) \\
&+ \frac{1}{2} [Q(k_t - k^*)^2 + R(c_t - c^*)^2 + 2W(k_t - k^*)(c_t - c^*)] \\
&+ \varepsilon_t [(k_t - k^*)^2 + (c_t - c^*)^2] \text{ such that } \varepsilon_t \rightarrow 0 \text{ if } \sqrt{(k_t - k^*)^2 + (c_t - c^*)^2} \rightarrow 0.
\end{aligned}$$

With:

$$\begin{aligned}
Q &= \frac{\partial^2(\beta L_t + L_{t-1})}{\partial k_t^2}(k^*, c^*) = U_{kk}(k^*, c^*) + U_c(k^*, c^*) g_{kk}(k^*, c^*) \\
W &= \frac{\partial^2(\beta L_t + L_{t-1})}{\partial k_t \partial c_t}(k^*, c^*) = U_{kc}(k^*, c^*) + U_c(k^*, c^*) g_{kc}(k^*, c^*) \\
R &= \frac{\partial^2 \beta L_t}{\partial c_t^2}(k^*, c^*) = U_{cc}(k^*, c^*) + U_c(k^*, c^*) g_{cc}(k^*, c^*)
\end{aligned}$$

where  $U_{kk}$ ,  $U_{kc}$  and  $U_{cc}$  denote second-order derivatives.

The sufficient second order condition for a local maximum is such that the Hessian matrix of the Lagrangian is *negative definite*, with at least one eigenvalue which is strictly negative. A sufficient condition is to assume the Hessian matrices of  $U$  and  $g$  are negative at the steady state, with at least one of them negative definite ( $U$  or  $g$  is strictly concave in  $k_t$  and  $c_t$  in the neighborhood of the steady state, and both  $U$  and  $g$  are concave)

The necessary first order conditions are:

$$\begin{aligned}
\frac{\partial(\beta L_t + L_{t-1})}{\partial k_t}(k_t, c_t) = 0 &\Rightarrow U_k(k_t, c_t) + \mu_t g_k(k_t, c_t) - \frac{1}{\beta} \mu_{t-1} = 0 \\
\frac{\partial \beta L_t}{\partial c_t}(k_t, c_t) &\Rightarrow U_c(k_t, c_t) + \mu_t g_c(k_t, c_t) = 0
\end{aligned}$$

The steady state condition ( $k_t = k_{t+1} = k^*$ ,  $c_t = c_{t+1} = c^*$ ,  $\mu_t = \mu_{t+1} = \mu^*$ ) in the policy transmission mechanism implies:

$$g(k^*, c^*) = k^*.$$

The steady state values  $k^*$ ,  $c^*$  and  $\mu^*$  verify the first order conditions, assuming  $g_c(k^*, c^*) \neq 0$  for controllability.

$$\begin{aligned}
U_k(k^*, c^*) + \mu^* \left( g_k(k^*, c^*) - \frac{1}{\beta} \right) &= 0 \\
U_c(k^*, c^*) &= -\mu^* g_c(k^*, c^*)
\end{aligned}$$

The Hessian matrix of the Lagrangian is the negative definite quadratic loss function

to be minimized. With this definition, the second order sufficient conditions are exactly the same for the concave program and for the linear quadratic approximation:

$$\frac{1}{2} \sum_{t=0}^{+\infty} \beta^t L_t \text{ with}$$

$$L_t = Q (k_t - k^*)^2 + 2W (k_t - k^*) (c_t - c^*) + R (c_t - c^*)^2 + 2\mu_t [A (k_{t-1} - k^*) + B (c_{t-1} - c^*)]$$

With

$$A = g_k (k^*, c^*) = \frac{1}{\beta} + \frac{U_k (k^*, c^*)}{U_c (k^*, c^*)} g_c (k^*, c^*)$$

$$B = g_c (k^*, c^*) \neq 0$$

This Lagrangian is the one of this linear quadratic regulator program, with the above defined solutions for  $Q$ ,  $W$ ,  $R$ ,  $A$  and  $B$ :

$$\min \frac{1}{2} \sum_{t=0}^{+\infty} \beta^t [Q (k_t - k^*)^2 + 2W (k_t - k^*) (c_t - c^*) + R (c_t - c^*)^2] \text{ s.t.}$$

$$k_t - k^* = A (k_{t-1} - k^*) + B (c_{t-1} - c^*)$$

In the case of the Ramsey model, the utility is increasing and strictly concave in consumption ( $U_c > 0$ ,  $U_{cc} < 0$ ). It does not depend on wealth:  $U_{kc} = U_k = U_{kk} = 0$ . The function in the dynamic constraint is:

$$k_{t+1} = g(k_t, c_t) = f(k_t) + (1 - \delta) k_t - c_t.$$

It is separable with respect to consumption and wealth ( $g_{kc} = 0$ ). It is linear for consumption ( $g_c = -1$ ,  $g_{cc} = 0$ ). It is concave for wealth ( $g_{kk} = f_{kk} < 0$ ).

$$Q = U_{kk} (k^*, c^*) + U_c (k^*, c^*) g_{kk} (k^*, c^*) = U_c (k^*, c^*) f_{kk} (k^*, c^*)$$

$$W = U_{kc} (k^*, c^*) + U_c (k^*, c^*) g_{kc} (k^*, c^*) = 0$$

$$R = U_{cc} (k^*, c^*) + U_c (k^*, c^*) g_{cc} (k^*, c^*) = U_{cc} (k^*, c^*)$$

$$A = g_k (k^*, c^*) = \frac{1}{\beta} + \frac{U_k (k^*, c^*)}{U_c (k^*, c^*)} g_c (k^*, c^*) = \frac{1}{\beta} = 1 + r \geq 1.$$

$$B = g_c (k^*, c^*) = -1 \neq 0.$$

## (2) Proof of proposition 2.

We minimize the Lagrangian of the linear quadratic regulator:

$$\frac{1}{2} \sum_{t=0}^{+\infty} \beta^t L_t \text{ with}$$

$$L_t = Q (k_t - k^*)^2 + 2W (k_t - k^*) (c_t - c^*) + R (c_t - c^*)^2 + 2\mu_t [A (k_{t-1} - k^*) + B (c_{t-1} - c^*)]$$

The first order necessary condition with respect to the policy instrument (consump-



tion) implies this linear relation between the Lagrange multiplier and the policy instrument:

$$0 = R(c_t - c^*) + B(\mu_t - \mu^*) \Rightarrow \mu_t - \mu^* = \frac{R}{B}(c_t - c^*)$$

The first order necessary condition with respect to the policy target (the stock of capital) is the Euler equation of consumption. It is a non-stationary equation with an autoregressive parameter larger than one ( $1 + \beta\frac{Q}{R} > 1$ ):

$$\begin{aligned} 0 &= Q(k_t - k^*) + A(\mu_t - \mu^*) - \frac{1}{\beta}(\mu_{t-1} - \mu^*) = 0 \\ &\Rightarrow c_t - c^* = c_{t-1} - c^* - \beta\frac{Q}{R}\left(\frac{1}{\beta}(k_{t-1} - k^*) - (c_{t-1} - c^*)\right) \\ c_t - c^* &= \left(1 + \beta\frac{Q}{R}\right)(c_{t-1} - c^*) - \frac{Q}{R}(k_{t-1} - k^*) \end{aligned}$$

The necessary conditions of the transmission mechanism and the consumption Euler equation is the Hamiltonian dynamic system of the discounted linear quadratic regulator:

$$\begin{pmatrix} k_{t+1} - k^* \\ c_{t+1} - c^* \end{pmatrix} = \begin{pmatrix} A & B \\ -\frac{Q}{R} & 1 + \beta\frac{Q}{R} \end{pmatrix} \begin{pmatrix} k_t - k^* \\ c_t - c_t^* \end{pmatrix} = \mathbf{H} \begin{pmatrix} k_t - k^* \\ c_t - c_t^* \end{pmatrix}.$$

The Hamiltonian matrix is denoted  $\mathbf{H}$ . The determinant of the Hamiltonian matrix, denoted  $P$ , is the product of the two roots of the characteristic polynomial  $p(\lambda)$  of this matrix. Hence, these two roots are denoted  $\lambda^*$  and  $\frac{1}{\beta\lambda^*}$  with  $\lambda^* \leq \frac{1}{\beta\lambda^*}$ . The sum of the main diagonal elements is the trace of the Hamiltonian matrix, denoted  $S$ . It is the sum of the two roots of the characteristic polynomial  $p(\lambda)$  of this matrix:

$$\begin{aligned} 0 &= p(\lambda) = \lambda^2 - S\lambda + P = (\lambda - \lambda^*)\left(\lambda - \frac{1}{\beta\lambda^*}\right) = \det(\mathbf{H} - \lambda\mathbf{I}) \\ P &= \frac{1}{\beta} = \lambda^* \left(\frac{1}{\beta\lambda^*}\right) > 1, \\ S &= A + \frac{1}{\beta A} + \frac{Q}{R} \frac{B^2}{A} = \lambda^* + \frac{1}{\beta\lambda^*} > 2. \end{aligned}$$

The inverse function with  $Q/R$  function of  $\lambda^*$ ,  $A$ ,  $B$ ,  $\beta$  is obtained from the characteristic polynomial (we check the interval for  $\lambda^*$  in what follows).

$$p(\lambda) = 0 \Rightarrow \frac{Q}{R} = \frac{A}{B^2} \frac{(A - \lambda)\left(\frac{1}{\beta A} - \lambda\right)}{\lambda} > 0 \text{ for } \lambda \in \left] 0, \min\left(A, \frac{1}{\beta A}\right) \right[$$

The characteristic polynomial can also be obtained as the first order condition  $\frac{\partial W}{\partial \lambda} = 0$ , minimizing the utility or welfare denoted  $w$ , guessing the optimal solution is a proportional feedback rule  $c_t - c^* = \frac{\lambda - A}{B}(k_t - k^*)$ :

$$w = \frac{1}{2} \sum_{t=0}^{+\infty} \beta^t (Q (k_t - k^*)^2 + R (c_t - c^*)^2) = \left( Q + R \left( \frac{\lambda - A}{B} \right)^2 \right) \sum_{t=0}^{+\infty} \beta^t \lambda^{2t} (k_0 - k^*)^2$$

$$w = \frac{Q + R \left( \frac{\lambda - A}{B} \right)^2}{1 - \beta \lambda^2} (k_0 - k^*)^2 \text{ if } |\lambda| < \frac{1}{\sqrt{\beta}} \text{ and } W = +\infty \text{ if } |\lambda| \geq \frac{1}{\sqrt{\beta}} \text{ (or } \beta \lambda^2 \geq 1).$$

The discriminant is positive so that both roots are real numbers. It is strictly positive with distinct roots except in the case where it is exactly zero with a single root with an order of multiplicity equal to two for the parameters:  $Q = 0$  and  $A = \frac{1}{\sqrt{\beta}} = \lambda^* = \frac{1}{\beta \lambda^*}$ . But in this case,  $\beta \lambda^{*2} = 1 \geq 1$ , the geometric series of the loss function is not bounded, so that we rule out this particular case and consider only the case with two distinct real roots.

$$A > 0, R > 0, 0 < \beta \leq 1, Q \geq 0, B \neq 0 \Rightarrow$$

$$\Delta = S^2 - 4P = \left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta} \geq \left( A + \frac{1}{\beta A} \right)^2 - \frac{4}{\beta} = \left( A - \frac{1}{\beta A} \right)^2 \geq 0.$$

Because we use algebraic identities:

$$\left( A + \frac{1}{\beta A} \right)^2 - \frac{4}{\beta} = A^2 + 2 \frac{A}{\beta A} + \left( \frac{1}{\beta A} \right)^2 - \frac{4}{\beta} = A^2 - 2 \frac{A}{\beta A} + \left( \frac{1}{\beta A} \right)^2 = \left( A - \frac{1}{\beta A} \right)^2$$

For the first inequality, the equality is obtained for  $Q = 0$ . For the second inequality leading to a unique root of order of multiplicity equal to two, the equality is only obtained for  $Q = 0$  and  $A = \frac{1}{\sqrt{\beta}}$  because:

$$\text{For } A > 0, A - \frac{1}{\beta A} = 0 \Rightarrow A^2 = \frac{1}{\beta} \Rightarrow A = \frac{1}{\sqrt{\beta}}$$

However, in this case of a single root of order of multiplicity equal to two, the loss function is not bounded so that  $Q = 0$  and  $A = \frac{1}{\sqrt{\beta}} = \lambda^* = \frac{1}{\beta \lambda^*}$  is the unique case without a bounded optimal solution.

Else, the product of the two distinct roots is strictly positive, so that both eigenvalues have the same sign. Since their sum is also strictly positive and they have both the same sign, both eigenvalues are strictly positive. Because of the discount factor, for  $1 \leq A \leq \frac{1}{\beta}$ ,  $A \neq \frac{1}{\sqrt{\beta}}$ , there are cases where the utility is bounded although  $k_t$  is not bounded  $1 \leq \lambda^* \leq \min \left( A, \frac{1}{\beta A} \right)$ . The optimal parameter of the persistence of consumption and wealth is:

$$0 < \lambda^* = \frac{1}{2} \left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} - \sqrt{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta}} \right) \leq \min \left( A, \frac{1}{\beta A} \right).$$

Knowing the optimal value of  $\lambda^*$ , the optimal stationary auto-regressive process for wealth and consumption is implemented by the unique optimal proportional negative-feedback rule, which is a consumption function where consumption depends only on wealth:  $c_t - c^* = F (k_t - k^*)$ . One has  $A + BF = \lambda^*$  so that  $F = \frac{\lambda^* - A}{B}$ .

In the case of the Ramsey model,  $A = \frac{1}{\beta} = 1 + r$  and  $B = -1$ :

$$0 < \lambda^* = \frac{1}{2} \left( 1 + \frac{1}{\beta} + \beta \frac{Q}{R} - \sqrt{\left( 1 + \frac{1}{\beta} + \beta \frac{Q}{R} \right)^2 - \frac{4}{\beta}} \right) = 1 + r - F \leq 1$$

$$r \leq F = 1 + r - \lambda^* < 1 + r$$

For  $Q > 0$ , i.e. for a strictly concave production function  $f_{kk}(k^*) < 0$  around the equilibrium, the eigenvalue  $\lambda^*$  is strictly below one. For  $Q = 0$ , one has  $\lambda^* = 1$  and  $F = r$ . The case  $A = \frac{1}{\beta}$  belongs to the cases:  $1 \leq A \leq \frac{1}{\beta}$  and  $A \neq \frac{1}{\sqrt{\beta}}$  where welfare is bounded although it is possible that  $k_t$  is not bounded:  $\lambda = 1$ , because of the discount factor in welfare.

Finally, if  $A^2 < \frac{1}{\beta} + \frac{QB^2}{R}$

$$\text{sign} \frac{\partial \lambda}{\partial A} = \text{sign} \frac{1}{A^2} \left( A^2 - \frac{1}{\beta} - \frac{QB^2}{R} \right) \underbrace{\left( 1 - \frac{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)}{\sqrt{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta}}} \right)}_{<0} > 0$$

Else, if  $A^2 > \frac{1}{\beta} + \frac{QB^2}{R}$

$$\text{sign} \frac{\partial \lambda}{\partial A} = \text{sign} \frac{1}{A^2} \left( A^2 - \frac{1}{\beta} - \frac{QB^2}{R} \right) \underbrace{\left( 1 - \frac{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)}{\sqrt{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta}}} \right)}_{<0} < 0$$

And also:

$$\begin{aligned} & \text{sign} \frac{\partial \lambda}{\partial (1/\beta)} \\ &= \text{sign} \frac{1}{A} - \frac{\frac{1}{A} \left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right) - 2}{\sqrt{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta}}} \\ &= \text{sign} \frac{1}{A} \left( A - \frac{1}{\beta A} - \frac{Q B^2}{R A} \right) \underbrace{\left( 1 - \frac{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)}{\sqrt{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta}}} \right)}_{<0} + \frac{2}{\sqrt{\left( A + \frac{1}{\beta A} + \frac{Q B^2}{R A} \right)^2 - \frac{4}{\beta}}} > 0. \end{aligned}$$

Then if  $A^2 > \frac{1}{\beta} + \frac{QB^2}{R}$ ,  $\text{sign} \frac{\partial \lambda}{\partial (1/\beta)} < 0$ .

### (3) Formulas for table 1.

The constant intertemporal elasticity of substitution function and the concave production ( $0 < \alpha \leq 1$ ) have the following properties:

$U(c_t, h_t) = \ln(c_t)$  for  $\sigma \neq 1$  and  $U(c_t, h_t) = \frac{1}{1 - \frac{1}{\sigma}} c_t^{1 - \frac{1}{\sigma}}$  for  $\sigma \neq 1$ , and  $\sigma > 0$ .

$$U_c = c_t^{-\frac{1}{\sigma}}, U_{cc} = -\frac{1}{\sigma} c_t^{-\frac{1}{\sigma} - 1}, \frac{U_{cc}}{U_c} = -\frac{1}{\sigma c_t}$$

$$f(k^*) = k^{*\alpha} = \frac{k^*(r + \delta)}{\alpha} \Rightarrow \frac{y^*}{k^*} = \frac{f(k^*)}{k^*} = \frac{r + \delta}{\alpha},$$

$$f_k(k^*) = \alpha k^{\alpha-1} = r + \delta \Rightarrow f_{kk}(k_t) = \alpha(\alpha - 1) k^{*(\alpha-2)} = \frac{(\alpha - 1)(r + \delta)}{k^*}$$

$$\frac{c^*}{k^*} = \frac{y^* - \delta k^*}{k^*} = \frac{r + \delta}{\alpha} - \delta,$$

$$\frac{Q}{R} = \frac{U_c f_{kk}}{U_{cc}} = \sigma c^* \frac{(1 - \alpha)(r + \delta)}{k^*} = \sigma(1 - \alpha)(r + \delta) \left( \frac{r + \delta}{\alpha} - \delta \right).$$